



Comparison of Starting Values for Implicit Iterative Solutions to Hyperbolic Orbits Equation

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Abstract: In the implicit method, the indefinite integral is approximated by a trapezoid instead of a rectangle, thereby reducing the error in the approximation. The purpose of this work is applied an implicit method to hyperbolic orbit equation using several starting iteration. The solution is examined with $1.5 \leq e \leq 6$ and $0.5 \leq M_h \leq 3$ with the grid size $\Delta m = \Delta e = 0.5$, tables and graphs are included to show that rapid convergence is always reached with less iterations if the first guess hyperbolic eccentric anomaly is $H_o = \ln\left(\frac{2M_h+1.5e}{e}\right)$ and $H_o = \ln\left(\frac{2m+2e}{e}\right)$ where e is eccentricity and M_h is the hyperbolic mean anomaly.

Keywords: implicit method, starting values, hyperbolic mean anomaly, hyperbolic orbit.

I. Introduction

Many instances of hyperbolic orbits occur in the solar system and recently, among the artificial satellites, lunar and solar probes. Moreover, in some cases of orbit determination for an elliptic orbit, it may very well happen that during the solution process (usually iteration), the eccentricity e becomes greater than unity and the orbit becomes hyperbolic. Also, in the interplanetary transfer, the escape from the departure planet and the capture by the target planet involves hyperbolic orbits. On the other hand, in orbit determination of visual binaries provisional hyperbolic orbits are used to represent the periastron section of high-eccentricity orbits of long and indeterminate period. In fact, we should handle hyperbolic orbits frequently when integrating a perturbed motion with the initial condition of nearly parabolic orbits [1-3].

In a nutshell, the so-called Kepler's problem consists of determining the radial and angular coordinates, r and, respectively of an object about the sun as a function of time. For the case of a hyperbolic orbit, characterized by $e > 1$, we have [4]

$$e \sinh H - H = M_h, \quad 1 \leq e < \infty; 0 \leq M_h \leq \infty \quad (1)$$

where e (eccentricity) and M_h (hyperbolic mean anomaly) are assumed to be known and H (hyperbolic eccentric anomaly) is to be determined. Since the range of e is from unity (rectilinear hyperbolic orbit) to infinity (uniform linear motion, for zero attractive force), so eq. (1) can be written as:

$$\sinh H - gH = L, \quad 0 \leq g < 1; 0 \leq L \leq \infty \quad (2)$$

where $g = \frac{1}{e}$ and $L = \frac{M_h}{e}$,

Eq. (2) is a transcendental equation which is most easily solved numerically which in turn need: (a) initial guess and (b) an iterative scheme. Burkarat [5] considered starting values for the iteration solution of Kepler's equation for hyperbolic orbits, eq. (2) and for generalized versions of the equation, while; Gooding [6] developed a procedure to solve the hyperbolic Kepler's equation.

An efficient iterative method of arbitrary integer order of convergence ≥ 2 was established by Sharaf [7] for solving the hyperbolic form of Kepler's equation, error formulae for solving a hyperbolic form of Kepler's equation using the Homotopy method presented by [8]. Other works concerning Kepler's equation for

hyperbolic case can be found in [9-12]. In this paper, an implicit method will be applied to solve hyperbolic form of Kepler's equation, eq. (2) with different values of (eccentricity, e) and (hyperbolic mean anomaly, M_h) with some initial starting values.

II. Procedure Solving Hyperbolic Equation

The hyperbolic equation, eq. (2) can be written as

$$f(H) = \sinh H - gH - h = 0 \quad (3)$$

In the implicit scheme, the indefinite integral involved in the following equation

$$f(H) = f(H_n) + \int_{H_n}^H \hat{f}(\lambda) d\lambda \quad (4)$$

Is approximated by the trapezoidal rule to obtain

$$\int_{H_n}^H \hat{f}(\lambda) d\lambda = \frac{H-H_n}{2} [\hat{f}(H_n) + \hat{f}(H)] \quad (5)$$

Put (5) into (4), yields

$$f(H) = f(H_n) + \frac{H-H_n}{2} [\hat{f}(H_n) + \hat{f}(H)] \quad (6)$$

Since $f(H) = 0$, therefore eq. (6) can be written as

$$H = H_n - \frac{2f(H_n)}{[\hat{f}(H_n) + \hat{f}(H)]} \quad (7)$$

Take the next iterative point, that is put $H = H_{n+1}$ to obtain

$$f(H_{n+1}) = f(H_n) + \frac{1}{2}(H_{n+1} - H_n)(\hat{f}(H_n) + \hat{f}(H_{n+1})) \text{ or } H_{n+1} = H_n - \frac{2f(H_n)}{[\hat{f}(H_{n+1}) + \hat{f}(H_n)]} \quad (8)$$

Eq. (8) is an implicit scheme, which requires having the derivative of the function at the $(n+1)^{\text{th}}$ iterative step to calculate the $(n+1)^{\text{th}}$ iterative itself.

Thus, the resulting scheme is

$$H_{n+1} = H_n - \frac{2f(H_n)}{[\hat{f}(H_{n+1}) + \hat{f}(H_n)]} \quad (9)$$

where

$$\hat{f}(H_{n+1}) = H_n - \frac{f(H_n)}{\hat{f}(H_n)}, \quad n = 0, 1, 2, 3, \dots \quad (10)$$

and H_0 is the starting value.

III. Starting Iterative Value H_0

Burkardt [5], show that the results for the starting iteration $H_0 = M_h$ is not generally efficient, and the results can be improved by substitution for $\sinh H$, eq. (1) can be written as

$$e \left(\frac{e^H - e^{-H}}{2} \right) - H - M_h = 0$$

If H is positive and too small, the term $\frac{e \cdot e^H}{2}$ will dominate the term $\frac{e \cdot e^{-H}}{2}$ and H , then $\frac{e \cdot e^H}{2} - M_h \cong 0$ or $e^H = \frac{M_h}{2}$, that is the initial guess can be suggested as $H_0 = \ln(\frac{M_h}{e})$ (11)

For a modification of eq. (11), consider

$$H_0 = \ln \left(\frac{2M_h + K e}{e} \right), \quad K > 0 \quad (12)$$

In this paper, we will take the values of $K=1, 1.5, 1.8, 2, 3, \pi$.

Results and Discussion

To compare In this section it is shown that the implicit method, eq. (9) and (10) will always converge with less iteration than the standard Newton-Raphson method given by

$$H_{n+1} = H_n - \frac{e \sinh H - H - M_h}{e \cosh H - 1}, \quad n = 0, 1, 2, \dots \quad (13)$$

For any given initial first guess, H_0 . The solution of (2) will be examined

with the values of e and M_h such that $e \in [1.5, 6]$ and $M_h \in [0.5, 3]$ with the grid size $\Delta M_h = \Delta e = 0.5$. The results of making a first guess $H_0 = \ln(\frac{2M_h}{e})$ for both Newton method (NM) and the implicit method (I M) is illustrated in table (1).

Table (1) Number of iterations to convergence with $H_0 = \ln(\frac{2M_h}{e})$

e	M _h											
	0.5		1		1.5		2		2.5		3	
	NM	IM	NM	IM	NM	IM	NM	IM	NM	IM	NM	IM
1.5	7	5	9	6	8	5	8	5	8	5	7	4
2	7	4	8	5	8	5	8	5	7	4	7	4
2.5	7	4	7	4	7	4	7	4	7	4	7	4
3	7	4	7	4	7	4	7	4	7	4	7	4
3.5	9	6	7	4	6	5	8	5	7	4	7	4
4	7	4	7	4	6	5	7	4	7	4	7	4
4.5	7	4	7	4	7	4	7	4	8	5	7	4
5	7	4	7	4	7	4	6	5	7	4	7	4
5.5	7	4	7	4	7	4	6	5	6	5	7	4
6	8	5	7	4	7	4	7	4	6	5	7	4

The number of iterations to 15 digit convergence, for various combinations of (e, M_h) when the first guess is $H_0 = \ln(\frac{2^{M_h+k}e}{e})$ with k = 1,1.5,1.8,2,3, π is listed in tables (2-6).

Table (2) number of iterations with $H_0 = \ln(\frac{2^{M_h+1.8}e}{e})$

e	M _h											
	0.5		1		1.5		2		2.5		3	
	NM	IM	NM	IM	NM	IM	NM	IM	NM	IM	NM	IM
1.5	4	4	4	4	4	4	4	4	4	4	4	4
2	4	4	4	4	4	4	4	4	4	4	4	4
2.5	4	4	4	4	4	4	4	4	4	4	4	4
3	4	4	4	4	4	4	4	4	4	4	4	4
3.5	4	4	4	4	4	4	4	4	4	4	4	4
4	4	4	4	4	4	4	4	4	4	4	4	4
4.5	4	4	4	4	4	4	4	4	8	4	4	4
5	4	4	4	4	4	4	4	4	4	4	4	4
5.5	4	4	4	4	4	4	4	4	4	4	4	4
6	4	4	4	4	4	4	4	4	4	4	4	4

Table (3) number of iterations with $H_0 = \ln(\frac{2^{M_h+e}}{e})$

e	M _h											
	0.5		1		1.5		2		2.5		3	
	NM	IM	NM	IM	NM	IM	NM	IM	NM	IM	NM	IM
1.5	4	4	5	4	5	4	5	4	5	4	4	4
2	4	4	5	4	5	4	5	4	4	4	4	4
2.5	4	4	4	4	4	4	4	4	4	4	4	4
3	4	4	4	4	4	4	4	4	4	4	4	4
3.5	5	4	4	4	4	4	5	4	4	4	4	4
4	4	4	4	4	4	4	4	4	4	4	4	4
4.5	4	4	4	4	4	4	4	4	5	4	4	4
5	4	4	4	4	4	4	4	5	4	4	4	4
5.5	4	4	4	4	4	4	4	5	4	5	4	4
6	5	4	4	4	4	4	4	4	4	5	4	4

Table (4) number of iterations with $H_0 = \ln(\frac{2^{M_h+1.5}e}{e})$

e	M _h											
	0.5		1		1.5		2		2.5		3	
	NM	IM	NM	IM	NM	IM	NM	IM	NM	IM	NM	IM
1.5	5	4	4	3	4	3	4	3	4	3	5	4
2	5	4	4	3	4	3	4	3	5	4	5	4
2.5	5	4	5	4	5	4	5	4	5	4	5	4
3	5	4	5	4	5	4	5	4	5	4	5	4
3.5	4	3	5	4	5	4	4	3	5	4	5	4
4	5	4	5	4	5	4	5	4	5	4	5	4
4.5	5	4	5	4	5	4	5	4	4	3	5	4
5	5	4	5	4	5	4	5	4	5	4	5	4
5.5	5	4	5	4	5	4	5	4	5	4	5	4
6	4	3	5	4	5	4	5	4	5	4	5	4

Table (5) number of iterations with $H_0 = \ln\left(\frac{2M_h + 2e}{e}\right)$

e	M_h											
	0.5		1		1.5		2		2.5		3	
	NM	IM	NM	IM	NM	IM	NM	IM	NM	IM	NM	IM
1.5	5	4	4	3	4	3	4	3	4	3	5	4
2	5	4	4	3	4	3	4	3	5	4	5	4
2.5	5	4	5	4	5	4	5	4	5	4	5	4
3	5	4	5	4	5	4	5	4	5	4	5	4
3.5	4	3	5	4	5	4	4	3	5	4	5	4
4	5	4	5	4	5	4	5	4	5	4	5	4
4.5	5	4	5	4	5	4	5	4	4	3	5	4
5	5	4	5	4	5	4	5	4	5	4	5	4
5.5	5	4	5	4	5	4	5	4	5	4	5	4
6	4	3	5	4	5	4	5	4	5	4	5	4

Table (6) number of iterations with $H_0 = \ln\left(\frac{2M_h + 3e}{e}\right)$

e	M_h											
	0.5		1		1.5		2		2.5		3	
	NM	IM	NM	IM	NM	IM	NM	IM	NM	IM	NM	IM
1.5	5	5	5	4	5	4	5	4	5	4	5	5
2	5	5	5	4	5	4	5	4	5	5	5	5
2.5	5	5	5	5	5	5	5	5	5	5	5	5
3	5	5	5	5	5	5	5	5	5	5	5	5
3.5	5	6	5	5	5	5	5	4	5	5	5	5
4	5	5	5	5	5	5	5	5	5	5	5	5
4.5	5	5	5	5	5	5	5	5	5	4	5	5
5	5	5	5	5	5	5	5	5	5	5	5	5
5.5	5	5	5	5	5	5	5	5	5	5	5	5
6	5	5	5	5	5	5	5	5	5	5	5	5

Table (7) number of iterations with $H_0 = \ln\left(\frac{2M_h + \pi e}{e}\right)$

e	M_h											
	0.5		1		1.5		2		2.5		3	
	NM	IM	NM	IM	NM	IM	NM	IM	NM	IM	NM	IM
1.5	5	5	5	4	5	4	5	4	5	4	5	5
2	5	5	5	4	5	4	5	4	5	5	5	5
2.5	5	5	5	5	5	5	5	5	5	5	5	5
3	5	5	5	5	5	5	5	5	5	5	5	5
3.5	5	4	5	5	5	5	5	4	5	5	5	5
4	5	5	5	5	5	5	5	5	5	5	5	5
4.5	5	5	5	5	5	5	5	5	5	4	5	5
5	5	5	5	5	5	5	5	5	5	5	5	5
5.5	5	5	5	5	5	5	5	5	5	5	5	5
6	5	4	5	5	5	5	5	5	5	5	5	5

The following stopping criteria is used for computer programs

$$|H_{n+1} - H_n| < 10^{-15}$$

And so, when the stopping criterion is satisfied H_{n+1} is taken as the exact root. The approximate roots H_n for $e \in [1.5, 3]$ and $M_h \in [0.5, 3]$ with $\Delta e = \Delta M_h = 0.5$ are displayed in table (8).

Table (8) approximate roots H_n

e	M					
	0.5	1	1.5	2	2.5	3
1.5	0.767343174954097	1.161635444504607	1.419318997767211	1.612685809758494	1.768471980642198	1.899455945779613
2	0.465918338092022	0.814096796302133	1.068949515655137	1.266466394761583	1.427248348233055	1.562846184058930
2.5	0.323849090576097	0.604253077612260	0.833482624172061	1.021455837209285	1.178878455389913	1.313659736077374
3	0.246255329197959	0.473210512943636	0.672297648451068	0.844160895220278	0.992920932830292	1.122965790420873
3.5	0.198180254127233	0.386434195697187	0.558673705730567	0.713181111482953	0.850935018666102	0.973963134170166
4	0.165655092505051	0.325620320153691	0.475792271167117	0.614197943662934	0.740568312317250	0.855618644559227
4.5	0.142239842774321	0.250943771306711	0.413312072960204	0.527641782954010	0.653259942666596	0.760231909836225
5	0.124596711715797	0.246856484955990	0.364817047438061	0.477114133702127	0.583008513011944	0.682276119458112
5.5	0.110833600247207	0.220046547437196	0.326223563606965	0.428293238691502	0.525571491411331	0.617726098649502
6	0.099801092042564	0.198434206048714	0.294850971404480	0.388210290545513	0.477917546576672	0.563618464084191

Depending on the results of table (8), will illustrate the relation between hyperbolic eccentric anomaly H and eccentricity e when the values of hyperbolic mean anomaly $M_h = 0.5, 1, 1.5, 2, 2.5, 3$ by the figures (1-6).

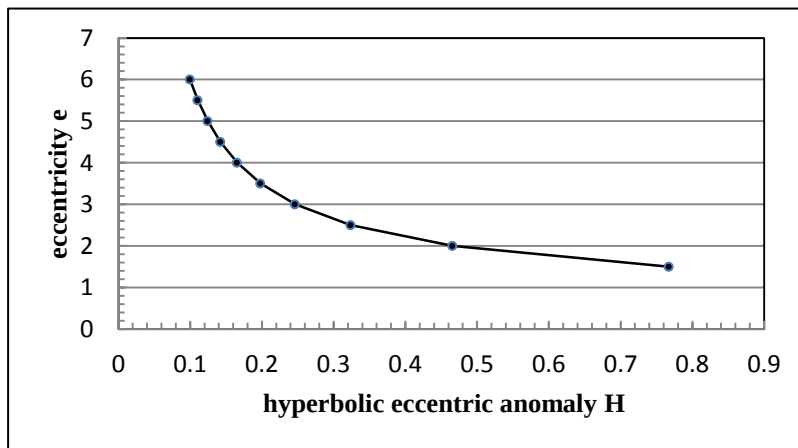


Fig. (1) the relation between e and H with $M_h = 0.5$

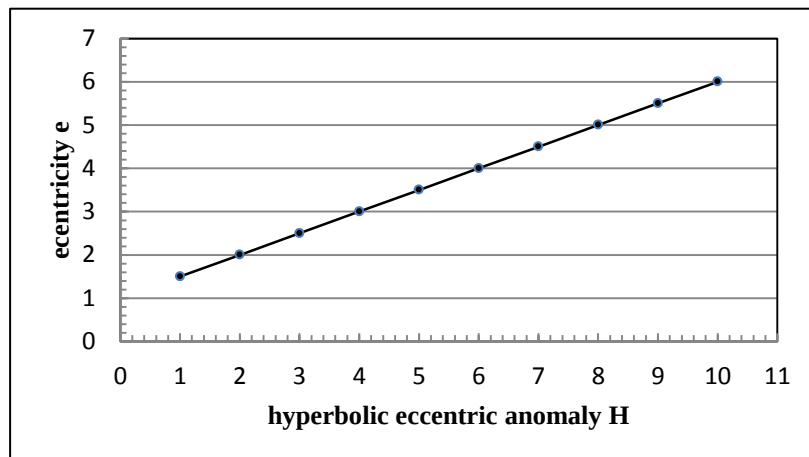


Fig. (2) the relation between e and H with $M_h = 1$

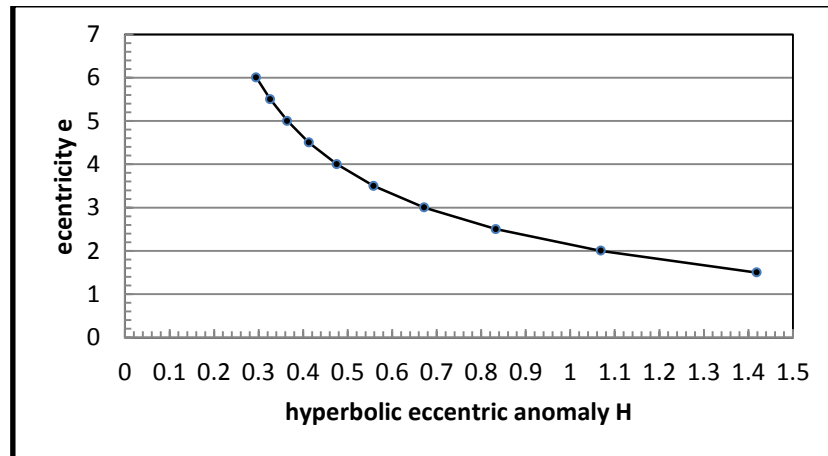


Fig. (3) the relation between e and H with $M_h = 1.5$

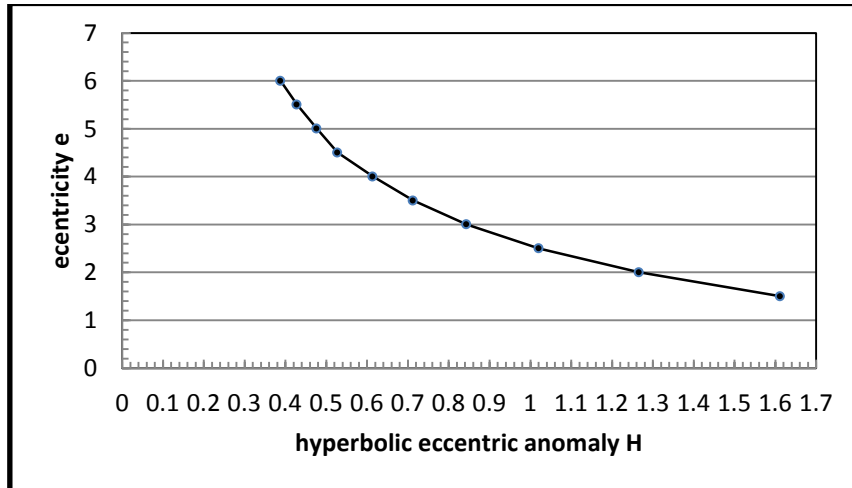


Fig. (4) the relation between e and H with $M_h = 2$

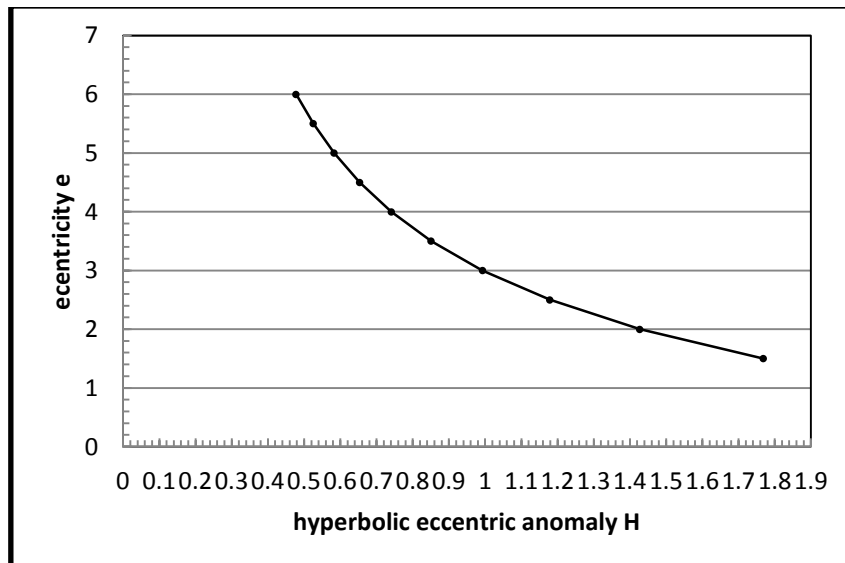


Fig. (5) the relation between e and H with $M_h = 2.5$

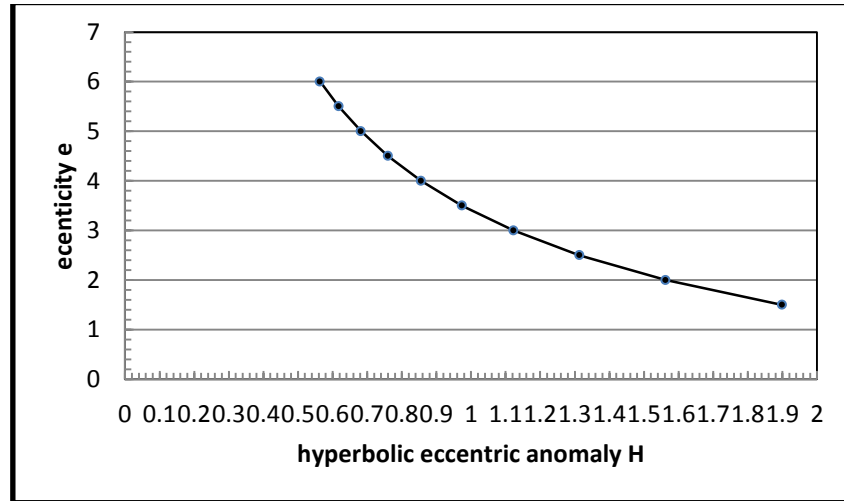


Fig. (6) the relation between e and H with $M_h = 3$

IV. Conclusions

We have applied an implicit method to the combinations of e and M_h , with the results shown in tables (1-7), which indicates that convergence was achieved in every case. It is clear from the results of tables that implicit method is more suitable and efficient than Newton-Raphson method and rapid convergence is always reached if the first guess hyperbolic mean anomaly is $H_o = \ln\left(\frac{2M_h + 1.5e}{e}\right)$ and $H_o = \ln\left(\frac{2M_h + 2e}{e}\right)$.

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