



On The Biquadratic Equation With Four Unknowns

$$x^4 - y^4 = 10(z + w)P^2$$

R.Anbuselvi¹, K.S.Araththi²

Associate Professor¹ Assistant Professor²

Department of Mathematics, A.D.M. College for Women, Nagapattinam, India ^{1,2}

Abstract: The Binary Quadratic Equation $x^4 - y^4 = 10(z + w)P^2$ is analyzed for its non-zero distinct integer solutions. Six different patterns of non-zero distinct integer solutions to the equation under consideration are obtained. A few interesting relation between the solutions and special numbers are exhibited.

Keywords: Integral solutions, Nasty number, Binary Quadratic

I. INTRODUCTION

The Binary Quadratic Equation offers an unlimited field for research because of their variety [1-2]. For an extensive review of various problems, one may refer [3-10]. This communication concerns with yet another interesting Binary quadratic equation $x^4 - y^4 = 10(z + w)P^2$ representing a homogenous cone for determining its infinitely many non-zero integral solutions. Also a few interesting relations among the solutions have been presented.

II. METHOD OF ANALYSIS

The Biquadratic Diophantine equation with four unknowns to be solved for getting non-zero integral solution is

$$x^4 - y^4 = 10(z + w)P^2 \quad (1)$$

On substituting the linear transformations

$$x = u + v, y = u - v, z = 2uv + 1, w = 2uv - 1 \quad (2)$$

Then $x^4 - y^4 = 10(z + w)P^2$ becomes

$$(u + v)^4 - (u - v)^4 = 10(2uv + 1 + 2uv - 1)P^2 \quad (3)$$

We obtain six different patterns of integral solutions to (1) through solving (2) which are illustrated as follows :

Pattern 1:

$$\text{Let } P = a^2 + b^2$$

$$u^2 + v^2 = 5(a^2 + b^2)$$

$$(u + iv)(u - iv) = (2 + i)(2 - i)(a + ib)(a - ib)$$

$$u = 2a - b$$

$$v = 2b + a \quad (4)$$

Substituting (4) in (2), the distinct integral solutions of (1) are expressed by,

$$x = x(a, b) = u + v = 3a + b$$

$$y = y(a, b) = a - 3b$$

$$z = z(a, b) = 2uv + 1 = 4a^2 - 4b^2 + 6ab + 1$$

$$w = w(a, b) = 2uv - 1 = 4a^2 - 4b^2 + 6ab - 1$$

Properties:

- (i) $x(1, n) + y(1, n) \equiv 0 \pmod{2}$
- (ii) $z(n, 1) + w(n, 1) - 4y(n, 1) + 12 = t_{18, n} + 15n$
- (iii) $z(n, 1) + w(n, 1) - 12n \equiv 0 \pmod{8}$
- (iv) $x(n, 1) - y(1, n)$ is a nasty number.
- (v) $x(n, 1) + w(n, 1) - t_{10, n} \equiv 0 \pmod{4}$

- (vi) $40x(n, 1) + 20y(1, n) \equiv 0 \pmod{10}$
 (vii) $100z(n, 1) + 200w(1, n) + 400n^2 \equiv 0 \pmod{30}$

Pattern 2:

From (3), $\Rightarrow u^2 + v^2 = 5P^2 - (3)(u + iv)(u - iv) = (1 + 2i)(1 - 2i)(a + ib)(a - ib)$

Equating real and imaginary parts,

$$x = -a + 3b$$

$$y = 3a + b$$

$$z = -4a^2 + 4b^2 - 6ab + 1$$

$$w = -4a^2 + 4b^2 - 6ab - 1$$

Properties:

- i) $x(n, 1) + y(1, n)$ is a nasty no
 ii) $2z(n, 1) - 2w(1, n) - 20 \in$ is a square number
 iii) $y(1, n) + z(1, n) - t_{10, n} \equiv 0 \pmod{2}$
 iv) $x(k, k) - w(k, k) - 7k + 1 = t_{14, k}$
 v) $y(n, 1) + x(n, 1) - Gno_n \equiv 0 \pmod{6}$
 vi) $30x(n, 1) + 60y(n, 1) \equiv 0 \pmod{30}$
 vii) $x(2^n, 1) + 60y(n, 1) \equiv 0 \pmod{30}$
 viii) $60w(1, n) + 90z(n, 1) \equiv 0 \pmod{100}$

Pattern :3

$$u^2 + v^2 = 5p^2$$

$$(u + iv)(u - iv) = \frac{(3 + 4i)(3 - 4i)}{25} \cdot (a^2 + b^2)$$

$$(u + iv) = \frac{3 + 4i}{25} (a + ib)$$

$$= \frac{1}{5} [3a + 3ib + 4ia - 4b]$$

$$u = \frac{1}{5} [3a - 4b]$$

$$v = \frac{1}{5} [3b - 4a]$$

Taking $a = 5A, \quad b = 5B$

$$X = \frac{1}{5} [3(5A) - 4(5B)]$$

$$= 3A - 4B$$

$$Y = \frac{1}{5} [-a - 7b] = \frac{1}{5} [-5A - 7(5B)]$$

$$= -A - 7B$$

$$Z = \frac{2}{25} [9ab + 12a^2 - 12b^2 - 16ab] + 1$$

$$= 24A^2 - 24B^2 - 14AB + 1$$

$$W = \frac{1}{25} [24a^2 - 24b^2 - 14ab] - 1$$

$$= 24A^2 - 24B^2 - 14AB - 1$$

Properties:

- i) $x(1, n) + y(n, 1) + 3n \equiv 0 \pmod{10}$
- ii) $x(n, 1) - y(1, n) - 10n \equiv 0 \pmod{3}$
- iii) $z(n, 1) - 3x(n, 1) + 11 = t_{50, n}$
- iv) $w(n, 1) - 3x(n, 1) - t_{50, n}$ is a prime integer
- v) $w(n, 1) + 8y(n, 1) + 22n \equiv 0 \pmod{3}$
- vi) $25x(n, 1) + 50y(1, n) \equiv 0 \pmod{5}$
- vii) $5z(n, 1) + 10y(1, n) + 125 \equiv 0 \pmod{20}$

Pattern: 4

$$\begin{aligned}
 u^2 + v^2 &= 5p^2 = 4p^2 + p^2 \\
 u^2 + v^2 &= 4p^2 + p^2 \\
 \Rightarrow u^2 - 4p^2 &= p^2 - v^2 \\
 \Rightarrow (u + 2p)(u - 2p) &= (p + v)(p - v) \\
 \Rightarrow \frac{u + 2p}{p + v} &= \frac{p - v}{u - 2p} = \frac{A}{B}, B \neq 0
 \end{aligned}$$

$$\begin{aligned}
 x &= x(a, b) = -b^2 + a^2 + 6ab \\
 y &= y(a, b) = -3b^2 + 3a^2 - 2ab \\
 z &= z(a, b) = 2[-2b^2 + 2a^2 + 2ab][b^2 - a^2 + 4ab] + 1 \\
 w &= w(a, b) = 2[-2b^2 + 2a^2 + 2ab][b^2 - a^2 + 4ab] - 1
 \end{aligned}$$

Properties

- i) $x(1, n) + y(n, 1) \equiv 0 \pmod{2}$
- ii) $6y(1, n) + 5x(1, n) + 5n + t_{48, n} \equiv 0 \pmod{23}$
- iii) $x(1, n) + z(n, 1) - 23n^2 \equiv 0 \pmod{2}$
- iv) $w(n, 1) + 2y(n, 1) - t_{30, n} - 13n + 1$ is cubic integer
- v) $3x(1, n) - y(n, 1) - 20n = 6(1 - n^2)$ is a nasty no
- vi) $4x(1, n) + 8z(n, 1) - 64cp_n^6 \equiv 0 \pmod{4}$
- vii) $w(n, 1) + y(1, n) - 21n^2 + 2 = 8cp_n^6 + t_{21, n}$

Pattern : 5

$$\begin{aligned}
 x &= x(a, b) = 3b^2 - 3a^2 + 2ab \\
 y &= y(a, b) = b^2 - a^2 - 6ab \\
 p &= p(a, b) = -a^2 - b^2 \\
 z &= z(a, b) = 2[2b^2 - 2a^2 - 2ab][b^2 - a^2 + 4ab] + 1 \\
 w &= w(a, b) = 2[2b^2 - 2a^2 - 2ab][b^2 - a^2 + 4ab] - 1
 \end{aligned}$$

Properties:

- i) $x(1, n) + y(1, n) \equiv 0 \pmod{4}$
- ii) $x(1, n) + 3w(1, n) + 79n^2$ is a nasty no
- (iii) $y(n, 1) + z(n, 1) + t_{20, n} + 1 \equiv 0 \pmod{2}$
- iv) $x(n, 1) + y(n, 1) \equiv 0 \pmod{2}$
- v) $50x(1, n) + 150y(n, 1) \equiv 0 \pmod{25}$
- vi) $152z(1, n) + 114x(1, n) \equiv 0 \pmod{38}$
- vii) $2y(n, 1) + 2w(1, n) + Gno_n - 24CP_n^6 \equiv 0 \pmod{8}$

Pattern:6

$$x = x(a, b) = -a^2 + 3b^2 - 2ab$$

$$y = y(a, b) = -3a^2 + b^2 + 6ab$$

$$z = z(a, b) = 2[2b^2 - 2a^2 + 2ab][a^2 + b^2 - 4ab] + 1$$

$$w = w(a, b) = 2[2b^2 - 2a^2 + 2ab][a^2 + b^2 - 4ab] - 1$$

Properties:

- i) $x(1, n) + y(1, n) - Gno_n - t_{10, n} - 5n \equiv 0 \pmod{3}$
- ii) $x(n, 1) + z(1, n) + 1 \equiv 0 \pmod{4}$
- iii) $y(n, 1) + z(1, n) + t_{40, n} \equiv 0 \pmod{2}$
- iv) $x(1, n) + w(1, n) + t_{13, n} + 4n^3(n - 3)$ is a nasty number
- v) $x(n, 1) + y(1, n) + n^2 + 1 \equiv 0 \pmod{8}$
- vii) $3x(1, n) - 7y(n, 1) + 1$ is a nasty number
- viii) $140x(1, n) + 280z(1, n) \equiv 0 \pmod{28}$

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