



Solving Linear Algebraic Equations Using Inverse with Identity Matrix and Gauss Elimination Methods

Olopade I. A.¹, Adesanya A.O.², Akinwumi T.O.³, Mohammed I.T.⁴ and Afolabi M.A.⁵

^{1,2,3}Department of Mathematics and Computer Science, Elizade University, P.M.B. 002,
Ilara-Mokin, Ondo State, NIGERIA

^{4,5}Department of Statistics, Osun State Polytechnic, P.M.B. 301, Iree, Osun State, NIGERIA

Abstract: The application of matrix used to solve linear system of equations can give more information about the nature of systems of linear equations than the numerical values of the unknowns. Solving linear equations simultaneously using substitution or elimination method is a simple task, but it becomes difficult most especially when the unknown variables are beyond two (2) or three (3). In this research work, application of matrices shall be employed to solve linear algebraic equations using inverse method. The results shall be validated by Gauss elimination method.

Keyword: Matrices, Inverse, Identity, Gauss Elimination, Elements.

I. Introduction

A matrix is an organized cluster of data, or a collection of numbers arranged into a fixed number of rows and columns [1]. When describing a matrix we name it by the number of rows it has “by” the number of columns that it has. For example, a matrix with four rows and 3 columns would be referred to as a “four by three matrix” [12]. Usually the numbers of a matrix are real numbers. Matrices are helpful when dealing with systems of equations and large amounts of data [1]. The word “matrix” comes from the Latin word for “womb [13]” because of the way that the matrix acts as a womb for the data that it holds [13]. The matrix is frequently used in many different modern career fields, but we have seen displays of the importance of matrices throughout history. Matrix is usually written in box brackets and parenthesis. As discussed earlier, a matrix is a rectangular array of numbers expressed only in rows and columns, matrix is represented with upper case letters, symbolically a matrix say ‘M’ can be expressed as follows;

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

This is a 2X2 matrix read as ‘2 by 2 matrix’, which implies we have 2 rows and 2 columns. Similarly, a matrix

$$M = \begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix}$$

The above matrix is called 3X2 matrix, meaning a 3 rows and 2 columns matrix. Generally, the above representation is referred to as the SIZE OF A MATRIX.

The work of many mathematicians, researchers and their development of matrices have allowed modern society to solve and learn about large systems of equations, which appear in a multitude of fields. They will continue to enrich our study of applied mathematics for a long time to come [14] Explained pairwise comparison matrices are widely used in solving multi-attribute decision making problems. Their research focused on testing various characteristics of pairwise comparison (PC) matrices in controlled experiments.

In this research work, method of Inverse with identity matrix and Gauss elimination shall be adopted to solve linear algebraic equations with four unknowns, A linear equation is an algebraic equation in which each term is either a constant or the product of a constant and a single variable [6]. A linear algebraic equation is an algebraic equation of the first degree in all its unknowns [7]. It is the branch of mathematics concerning linear equation such as $a_1x_1 + a_2x_2 + \dots + a_nx_n = b_i$ and central to almost all areas of mathematics because it allows modeling many natural phenomena [3].

II. The Methods

In matrices there are cases where we have a single row; this is often referred to as row vector. Similarly there are cases of single column; this is also referred to as column vector. Meanwhile, a matrix that has the same number of rows and column is known as square matrix.

For example

$$M = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

This is a square matrix as it contains 3 rows and 3 columns

In general Matrix;

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1j} \\ a_{21} & a_{22} & \dots & a_{2j} \\ \cdot & \cdot & & \cdot \\ a_{i1} & a_{i2} & \dots & a_{ij} \end{pmatrix}$$

Here, the elements $a_{11}, a_{12}, \dots, a_{1j}$ are the elements of the first row, and the elements $a_{11}, a_{21}, \dots, a_{i1}$ are the elements of the first column. Similarly, the elements $a_{i1}, a_{i2}, \dots, a_{ij}$ are the elements of the m-th row and the elements $a_{1j}, a_{2j}, \dots, a_{ij}$ are the elements of the n-th column. Consider again the matrix A above, the elements a_{11}, a_{22} and a_{33} are the elements in the main diagonal.

Also in so many cases, we have a matrix with infinite number of rows and infinite number of columns; this is referred to as infinite matrix

On rare occasions but common in contexts such as computing and programming, we consider a matrix with no row or column and it is called empty matrix.

Diagonal and Triangular Matrix

A matrix that has one part of the diagonal all zeroes is called a triangular matrix. Meanwhile, upper triangular matrix is a matrix that has all the entries below the diagonal as zeroes. Similarly, the lower triangular matrix is a matrix that has all entries above the diagonal all zeroes

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 2 & 1 & 3 \end{pmatrix}$$

Lower triangular matrix

$$M = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}$$

Upper triangular matrix

Diagonal (Square) Matrix

Definition: A matrix is called a diagonal matrix if non-zero element exists along the main diagonal and all other elements are zero. e.g.

$$(1) \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} \quad (2) \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

Identity Matrix

This is a matrix that has all the entries in the main diagonal as one '1' and all other entries are zeroes. It is usually represented with I.

$$(1) I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (2) I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Elements of a Matrix

The entries in the rows and columns of a matrix are called the elements of a matrix.

Consider the following;

$$M = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

a, b, c, d, e, f, g, h and i are all the elements in the matrix 'M'.

It is very important to Note that in cases of a zero as one of the rows or column, then zero is an element.

$$M = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$$

The elements of the above matrix are 1, 2, 1 and 0.

The elements of a matrix can be real or complex numbers. The most beautiful thing about the elements of a matrix is that it is the basis on which all other operations depend on. i.e. it make up the matrix itself.

Zero Matrixes

Definition: It is a matrix where all elements are 0 e.g.

$$(1) \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad (2) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Symmetric Matrix

A square matrix $A = (a_{ij})_{n \times n}$ is said to be a symmetric if $a_{ij} = a_{ji}$ for all i and j.

$$\text{Example: } A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{23} \end{pmatrix}, \quad B = \begin{pmatrix} 8 & -2 & 7 \\ -2 & -9 & 3 \\ 7 & 3 & 5 \end{pmatrix}$$

Kronecker Delta δ_{ij}

Leopold Kronecker (a German Mathematician) introduced a symbol δ_{ij} called Kronecker delta. Delta is a Greek letter.

Kronecker Delta δ_{ij} is defined as follows;

$$\delta = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$$

e.g. $\delta_{11}, \delta_{22}, \delta_{33}, \delta_{44}, \delta_{55}$ are all 1(s) whereas, $\delta_{12}, \delta_{21}, \delta_{31}, \delta_{41}, \delta_{51}$ are all 0(s)

E.g. Let us write δ_{ij}

$$\begin{bmatrix} \delta_{11} & \delta_{12} \\ \delta_{21} & \delta_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

i. e. $i = j = 1 \quad i \neq j = 0$

Transpose of a Matrix

If the rows and columns of matrix A are interchanged, the resulting matrix written as A^T is called the Transpose of A.

e.g. let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

If

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 9 \\ 2 & 3 & 7 \end{bmatrix}, A^T = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 9 \\ 1 & 3 & 7 \end{bmatrix}$$

Equal Matrix

Definition: Two or more matrices which are of the same order (the number of rows and columns of a matrix) are said to be equal if and only if their corresponding entries are equal.

e.g. Let

$$A = \begin{pmatrix} 3 & 1 \\ 2 & 6 \end{pmatrix} \text{ and } B = \begin{pmatrix} 3 & 1 \\ 2 & 6 \end{pmatrix}$$

Then, matrix A equals matrix B

Hermitian Matrix

A matrix A is called an Hermitian matrix if $A = \bar{A}^T$, where $\bar{A}$ is the conjugate of matrix A.

$$A = \begin{bmatrix} 1 & 1-i \\ 1+i & 3 \end{bmatrix}, \bar{A} = \begin{bmatrix} 1 & 1+i \\ 1-i & 3 \end{bmatrix}$$

$$\bar{A}^T = \begin{bmatrix} 1 & 1-i \\ 1+i & 3 \end{bmatrix}$$

$A = \bar{A}^T$ Is called an hermitian matrix.

Determinant of Matrices

In linear algebra, the determinant is a useful value that can be computed from the elements of a square matrix.

The determinant of a matrix A is denoted $\det(A)$, $\det A$, or $|A|$

Let A be a square matrix and its determinant. Then, each element in A is associated with its minor who is found by omitting the row and column containing the elements concerned. First of all the matrix must be square (i.e. have the same number of rows as columns). Then it is just basic arithmetic.

If

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$|A| = a_{11}a_{22} - a_{12}a_{21}$$

i.e.

$$B = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

$$|B| = 2 \times 1 - 1 \times 1 = 1$$

For a 3×3 matrix

$$A = \begin{bmatrix} + & - & + \\ 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$|A| = 1 \begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix} - 2 \begin{bmatrix} 4 & 6 \\ 7 & 9 \end{bmatrix} + 3 \begin{bmatrix} 4 & 5 \\ 7 & 8 \end{bmatrix}$$

$$1(5 \times 9 - 6 \times 8) - 2(4 \times 9 - 6 \times 7) + 3(4 \times 8 - 5 \times 7)$$

$$1(45 - 48) - 2(36 - 42) + 3(32 - 35)$$

$$-3 - 2(-6) + 3(-3)$$

$$-3 + 12 - 9$$

$$= 0$$

Therefore, since the above determinant is zero, the above matrix is called a Singular matrix.

Properties of a Matrix

The properties of a matrix are the major characteristics with which all the possible operations depend on. This simply means that with these properties, we are left with the full knowledge on how a matrix will behave.

1. Commutativity of addition: for any matrices A and B, we can always have

$$A + B = B + A$$

2. Associativity of Addition: for any matrices A, B and C, we can always have

$$A + (B + C) = (A + B) + C$$

3. Associativity of scalar multiplication: for any matrices A and for any scalar b and c, we can always have

$$(B \times C) \times A = B \times (C \times A)$$

4. Scalar identity: for any matrix A and for any scalar identity I, we can always have

$$IA = AI = I$$

5. Distributive: for any matrices A and B and for any scalar c, we can always have

$$c(A + B) = cA + cB \quad \text{For any scalar d, } (c + d)A = cA + dA$$

6. Additive identity: for any matrix A and for any scalar zero or for any empty matrix, we can always have;

$$A + 0 = 0 + A = A$$

7. Associativity of Multiplication: for any matrices A, B and C, we can always have

$$(A \times (B \times C)) = (A \times B) \times C$$

8. Left Distributive and Right Distributive: for any matrices A, B and C, we can always have

$$A(B + C) = AB + AC$$

Left distributive

$$(A + B)C = AC + BC$$

Right distributive

9. Multiplicative Identity: for any matrices A and for any identity matrix I, we can always have

$$IA = AI = A$$

10. Scalar Associativity/commutativity: for any matrices A and B and for any scalar c, we can always have;

$$C(AB) = (cA)B = A(cB) = (AB)c$$

The above listed properties are the proven basis on which matrix algebra depends. This will lead us to basic operations in matrix algebra.

Basic Operations in Matrix Algebra

The operations in matrices are in varieties; this depends on the reason of formulating the matrix. The operations also come as a result of the field it is being applied on. Some of the lists of operations in matrices are as follows;

Addition and Subtraction of matrices

Multiplication of matrices

Transposition of matrices

Reduction of matrices

Inverse of a matrix

Solving algebraic equation with matrices

Inverse of a Matrix

The inverse of a square matrix A sometimes called a reciprocal matrix is a matrix A^{-1} such that $AA^{-1} = I$, where I is the identity matrix [8]. When A is multiplied by A^{-1} the result is the identity matrix. Non square matrix does not have inverse. A square matrix A has an inverse denoted as A^{-1} if and only if determinants of A denoted as $|A|$ is not equal to zero [11]. A square matrix which has an inverse is called invertible or non-singular while a square matrix without an inverse is called noninvertible or singular.

Gauss Elimination Method

Gaussian elimination is a method for solving matrix equation of the form $Ax = b$. Gaussian elimination also known as row reduction is an algorithm for solving system of linear equation. It is usually understand as a sequence of operations performed on the matrix of coefficients [2].

To perform Gaussian elimination starting with the system of equations

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1k} \\ a_{12} & a_{22} & \dots & a_{2k} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_{k1} & a_{k2} & \dots & a_{kk} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_k \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ \cdot \\ b_k \end{pmatrix}$$

Compose the [augmented matrix](#) equation

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1k} \\ a_{12} & a_{22} & \dots & a_{2k} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_{k1} & a_{k2} & \dots & a_{kk} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ \cdot \\ b_k \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_k \end{pmatrix}$$

Here, the [column](#) vector in the variables X is carried along for labeling the matrix rows. Now, perform elementary row operation to put the augmented matrix into the upper triangular form

$$\left| \begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1k} & b_1 \\ 0 & a_{22} & \dots & a_{2k} & b_2 \\ \cdot & \cdot & & \cdot & \cdot \\ 0 & 0 & \dots & a_{kk} & b_k \end{array} \right|$$

Solve the equation of the K th row for X_k , then substitute back into the equation of the $(K-1)$ st row to obtain a solution for (X_{k-1}) etc.[4],[5],[9] and [10].

III. Solving the System Using Inverse with an Identity Matrix Method

Consider the system of equations

$$a - b - c + d = 0 \quad (1)$$

$$2a + 2c = 8 \quad (2)$$

$$-b - 2c = -8 \quad (3)$$

$$3a - 3b - 2c + 4d = 7 \quad (4)$$

The coefficient of a , b , c and d are i.e. $a = (1, 2, 0, 3)$, $b = (-1, 0, -1, -3)$,

$c = (-1, 2, -2, -2)$ and $d = (1, 0, 0, 4)$ in equations (1) – (4) respectively. NOTE: The system of equation can be written in matrix form as

$$A = \begin{bmatrix} 1 & -1 & -1 & 1 \\ 2 & 0 & 2 & 0 \\ 0 & -1 & -2 & 0 \\ 3 & -3 & -2 & 4 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ -8 \\ 7 \end{bmatrix} \quad (5)$$

Matrix A has 4 rows and 4 columns. Therefore matrix A is called 4x4 matrix.

$$\begin{bmatrix} 1 & -1 & -1 & 1 \\ 2 & 0 & 2 & 0 \\ 0 & -1 & -2 & 0 \\ 3 & -3 & -2 & 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ -8 \\ 7 \end{bmatrix} \quad (6)$$

$$\begin{pmatrix} 1 & -1 & -1 & 1 \\ 2 & 0 & 2 & 0 \\ 0 & -1 & -2 & 0 \\ 3 & -3 & -2 & 4 \end{pmatrix} \begin{pmatrix} V_{11} & V_{12} & V_{13} & V_{14} \\ V_{21} & V_{22} & V_{23} & V_{24} \\ V_{31} & V_{32} & V_{33} & V_{34} \\ V_{41} & V_{42} & V_{43} & V_{44} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (7)$$

From equation (8), we have;

$$\left. \begin{aligned} V_{11} - V_{21} - V_{31} + V_{41} &= 1 \\ V_{12} - V_{22} - V_{32} + V_{42} &= 0 \\ V_{13} - V_{23} - V_{33} + V_{43} &= 0 \\ V_{14} - V_{24} - V_{34} + V_{44} &= 0 \end{aligned} \right\} \text{ROW 1}$$

$$\left. \begin{aligned} 2V_{11} + 2V_{31} &= 0 \\ 2V_{12} + 2V_{32} &= 1 \\ 2V_{13} + 2V_{33} &= 0 \\ 2V_{14} + V_{34} &= 0 \end{aligned} \right\} \text{ROW 2}$$

$$\left. \begin{aligned} -V_{21} - 2V_{31} &= 0 \\ -V_{22} - 2V_{32} &= 0 \\ -V_{23} - 2V_{33} &= 1 \\ -V_{24} - 2V_{34} &= 0 \end{aligned} \right\} \text{ROW 3}$$

$$\left. \begin{aligned} 3V_{11} - 3V_{21} - 2V_{31} + 4V_{41} &= 0 \\ 3V_{12} - 3V_{22} - 2V_{32} + 4V_{42} &= 0 \\ 3V_{13} - 3V_{23} - 2V_{33} + 4V_{43} &= 0 \\ 3V_{14} - 3V_{24} - 2V_{34} + 4V_{44} &= 1 \end{aligned} \right\} \text{ROW 4}$$

Hence;

$$\begin{aligned} V_{11} &= 4, & V_{12} &= 0, & V_{13} &= -1, & V_{14} &= -1 \\ V_{21} &= 8, & V_{22} &= -1, & V_{23} &= -3, & V_{24} &= -2 \\ V_{31} &= -4, & V_{32} &= -\frac{1}{2}, & V_{33} &= 1, & V_{34} &= 1 \\ V_{41} &= 1, & V_{42} &= -\frac{1}{2}, & V_{43} &= -1, & V_{44} &= 0 \end{aligned}$$

Therefore;

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 4 & 0 & -1 & -1 \\ 8 & -1 & -3 & -2 \\ -4 & 1/2 & 1 & 1 \\ 1 & -1/2 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 8 \\ -8 \\ 7 \end{pmatrix}$$

$a = 1, b = 2, c = 3$ and

$d = 4$

Therefore;

$a = 1, b = 2, c = 3$ and $d = 4$.

The results shall be validated by another method called Gauss elimination.

A. Gauss Elimination Method

Consider the same system of equations (1)-(4) above

$$a - b - c + d = 0 \quad (1)$$

$$2a + 2c = 8 \quad (2)$$

$$-b - 2c = -8 \quad (3)$$

$$3a - 3b - 2c + 4d = 7 \quad (4)$$

The system of equation can be written in matrix form as

$$A = \begin{bmatrix} 1 & -1 & -1 & 1 \\ 2 & 0 & 2 & 0 \\ 0 & -1 & -2 & 0 \\ 3 & -3 & -2 & 4 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ -8 \\ 7 \end{bmatrix}$$

Using Gauss elimination, we have;

$$\left| \begin{array}{cccc|c} 1 & -1 & -1 & 1 & 0 \\ 2 & 0 & 2 & 0 & 8 \\ 0 & -1 & -2 & 0 & -8 \\ 3 & -3 & -2 & 4 & 7 \end{array} \right|$$

$$R_2 = -2R_1 + R_2$$

$$\left| \begin{array}{cccc|c} 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & 4 & -2 & 8 \\ 0 & -1 & -2 & 0 & -8 \\ 3 & -3 & -2 & 4 & 7 \end{array} \right|$$

$$R_4 = -3R_1 + R_4$$

$$\left| \begin{array}{cccc|c} 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & 4 & -2 & 8 \\ 0 & -1 & -2 & 0 & -8 \\ 0 & 0 & 1 & 1 & 7 \end{array} \right|$$

$$R_3 = \frac{1}{2}R_2 + R_3$$

$$\left| \begin{array}{cccc|c} 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & 4 & -2 & 8 \\ 0 & 0 & 0 & -1 & -4 \\ 0 & 0 & 1 & 1 & 7 \end{array} \right|$$

$$R_3 = R_4$$

$$\left[\begin{array}{cccc|c} 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & 4 & -2 & 8 \\ 0 & 0 & 1 & 1 & 7 \\ 0 & 0 & 0 & -11 & -4 \end{array} \right]$$

$$\left[\begin{array}{cccc|c} 1 & -1 & -1 & 1 & a \\ 0 & 2 & 4 & -2 & b \\ 0 & 0 & 1 & 1 & c \\ 0 & 0 & 0 & -1 & d \end{array} \right] = \left[\begin{array}{c} 0 \\ 8 \\ 7 \\ -4 \end{array} \right]$$

Therefore;

$$d = 4, c = 3, b = 2 \text{ and } a = 1$$

IV. Discussion of Results and Conclusion

Application of matrices can be used to solve linear algebraic equations, especially when the unknown variables are more than three (3).

Inverse with identity matrix method has been used to solved linear algebraic equation of four unknown variables; the results have been validated using Gauss elimination method which shows the method of inverse with identity matrix is fast, good and accurate.

Conclusively, solving linear algebraic equations of more than three unknown is no longer a problem in the mathematical society, application of matrix can be applied to solve the unknown variables using matrix inverse with identity method, and the results can be validated using Gauss elimination method or any other analytical method. The method of inverse with identity matrix is straight forward, accurate and better.

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