



Sum Square Difference Product Prime Labeling of Some Cycle Related Graphs

Sunoj B S* and Mathew Varkey T K**

*Department of Mathematics, Government Polytechnic College, Attingal, Kerala 695103, INDIA.

**Department of Mathematics, T K M College of Engineering, Kollam, Kollam, Kerala 691005, INDIA.

Abstract: Sum square difference product prime labeling of a graph is the labeling of the vertices with $\{0, 1, 2, \dots, p-1\}$ and the edges with absolute difference of the square of the sum of the labels of the incident vertices and product of the labels of the incident vertices. The greatest common incidence number of a vertex (**gcin**) of degree greater than one is defined as the greatest common divisor of the labels of the incident edges. If the **gcin** of each vertex of degree greater than one is one, then the graph admits sum square difference product prime labeling. Here we identify some cycle related graphs for sum square difference product prime labeling.

Keywords: Graph labeling, greatest common incidence number, sum square, cycles.

I. Introduction

All graphs in this paper are simple, finite, connected and undirected. The symbol $V(G)$ and $E(G)$ denotes the vertex set and edge set of a graph G . The graph whose cardinality of the vertex set is called the order of G , denoted by p and the cardinality of the edge set is called the size of the graph G , denoted by q . A graph with p vertices and q edges is called a (p, q) - graph.

A graph labeling is an assignment of integers to the vertices or edges. Some basic notations and definitions are taken from [2],[3] and [4]. Some basic concepts are taken from [1] and [2]. In this paper we investigated sum square difference product prime labeling of some cycle related graphs.

Definition: 1.1 Let G be a graph with p vertices and q edges. The greatest common incidence number (**gcin**) of a vertex of degree greater than or equal to 2, is the greatest common divisor (gcd) of the labels of the incident edges.

II. Main Results

Definition 2.1 Let $G = (V(G), E(G))$ be a graph with p vertices and q edges. Define a bijection

$f: V(G) \rightarrow \{0, 1, 2, 3, \dots, p-1\}$ by $f(v_i) = i-1$, for every i from 1 to p and define a 1-1 mapping

$f_{ssdapl}: E(G) \rightarrow$ set of natural numbers N by

$f_{ssdapl}(uv) = |f(u) + f(v)|^2 - f(u)f(v)$. The induced function f_{ssdapl}^* is said to be sum square difference product prime labeling, if for each vertex of degree at least 2, the greatest common incidence number is 1.

Definition 2.2 A graph which admits sum square difference product prime labeling is called a sum square difference product prime graph.

Theorem 2.1 Cycle C_n admits sum square difference product prime labeling.

Proof: Let $G = C_n$ and let $v_1, v_2, \dots, v_n$ are the vertices of G

Here $|V(G)| = n$ and $|E(G)| = n$

Define a function $f: V \rightarrow \{0, 1, 2, 3, \dots, n-1\}$ by

$$f(v_i) = i-1, i = 1, 2, \dots, n$$

Clearly f is a bijection.

For the vertex labeling f , the induced edge labeling f_{ssdapl}^* is defined as follows

$$f_{ssdapl}^*(v_i v_{i+1}) = 3i^2 - 3i + 1, \quad i = 1, 2, \dots, n-1$$

$$f_{ssdapl}^*(v_1 v_n) = (n-1)^2$$

Clearly f_{ssdapl}^* is an injection.

$$\text{gcin of } (v_i) = \gcd \text{ of } \{f_{ssdapl}^*(v_1 v_2), f_{ssdapl}^*(v_1 v_n)\}$$

$$= \gcd \text{ of } \{1, (n-1)^2\} = 1.$$

$$\text{gcin of } (v_{i+1}) = \gcd \text{ of } \{f_{ssdapl}^*(v_i v_{i+1}), f_{ssdapl}^*(v_{i+1} v_{i+2})\}$$

$$\begin{aligned}
 &= \gcd \text{ of } \{ 3i^2-3i+1, 3i^2+3i+1 \} \\
 &= \gcd \text{ of } \{ 6i, 3i^2-3i+1 \} \\
 &= \gcd \text{ of } \{ 3i, 3i^2-3i+1 \} \\
 &= \gcd \text{ of } \{ 3i, 3i(i-1)+1 \} \\
 &= 1, \quad i = 1, 2, \dots, n-2 \\
 \text{gcin of } (v_n) &= \gcd \text{ of } \{ f_{ssdapl}^*(v_1 v_n), f_{ssdapl}^*(v_{n-1} v_n) \} \\
 &= \gcd \text{ of } \{ (n-1)^2, 3n^2-9n+7 \}, \\
 &= \gcd \text{ of } \{ (n-1), (n-1)(3n-6)+1 \} = 1.
 \end{aligned}$$

So, **gcin** of each vertex of degree greater than one is 1.

Hence C_n , admits sum square difference product prime labeling. ■

Theorem 2.2 Middle graph of cycle C_n admits sum square difference product prime labeling, when $(n+3) \equiv 0 \pmod{6}$.

Proof: Let $G = M(C_n)$ and let $v_1, v_2, \dots, v_{2n}$ are the vertices of G

Here $|V(G)| = 2n$ and $|E(G)| = 3n$

Define a function $f : V \rightarrow \{0, 1, 2, 3, \dots, 2n-1\}$ by

$$f(v_i) = i-1, \quad i = 1, 2, \dots, 2n$$

Clearly f is a bijection.

For the vertex labeling f , the induced edge labeling f_{ssdapl}^* is defined as follows

$$\begin{aligned}
 f_{ssdapl}^*(v_i v_{i+1}) &= 3i^2-3i+1, & i = 1, 2, \dots, 2n-1 \\
 f_{ssdapl}^*(v_{2i} v_{2i+2}) &= 12i^2+1, & i = 1, 2, \dots, n-1 \\
 f_{ssdapl}^*(v_1 v_{2n}) &= (2n-1)^2. \\
 f_{ssdapl}^*(v_2 v_{2n}) &= 4n^2-2n+1
 \end{aligned}$$

Clearly f_{ssdapl}^* is an injection.

$$\text{gcin of } (v_{i+1}) = 1, \quad i = 1, 2, \dots, 2n-2$$

$$\begin{aligned}
 \text{gcin of } (v_1) &= \gcd \text{ of } \{ f_{ssdapl}^*(v_1 v_2), f_{ssdapl}^*(v_1 v_{2n}) \} \\
 &= \gcd \text{ of } \{ 1, (2n-1)^2 \} = 1.
 \end{aligned}$$

$$\begin{aligned}
 \text{gcin of } (v_{2n}) &= \gcd \text{ of } \{ f_{ssdapl}^*(v_1 v_{2n}), f_{ssdapl}^*(v_{2n-1} v_{2n}) \} \\
 &= \gcd \text{ of } \{ (2n-1)^2, 12n^2-18n+7 \}, \\
 &= \gcd \text{ of } \{ (2n-1), (2n-1)(6n-6)+1 \} = 1.
 \end{aligned}$$

So, **gcin** of each vertex of degree greater than one is 1.

Hence $M(C_n)$, admits sum square difference product prime labeling. ■

Theorem 2.3 Total graph of cycle C_n admits sum square difference product prime labeling, when $(n+3) \equiv 0 \pmod{6}$.

Proof: Let $G = T(C_n)$ and let $v_1, v_2, \dots, v_{2n}$ are the vertices of G

Here $|V(G)| = 2n$ and $|E(G)| = 4n$

Define a function $f : V \rightarrow \{0, 1, 2, 3, \dots, 2n-1\}$ by

$$f(v_i) = i-1, \quad i = 1, 2, \dots, 2n$$

Clearly f is a bijection.

For the vertex labeling f , the induced edge labeling f_{ssdapl}^* is defined as follows

$$\begin{aligned}
 f_{ssdapl}^*(v_i v_{i+1}) &= 3i^2-3i+1, & i = 1, 2, \dots, 2n-1 \\
 f_{ssdapl}^*(v_{2i} v_{2i+2}) &= 12i^2+1, & i = 1, 2, \dots, n-1 \\
 f_{ssdapl}^*(v_{2i-1} v_{2i+1}) &= 12i^2-12i+4, & i = 1, 2, \dots, n-1 \\
 f_{ssdapl}^*(v_1 v_{2n}) &= (2n-1)^2. \\
 f_{ssdapl}^*(v_2 v_{2n}) &= 4n^2-2n+1 \\
 f_{ssdapl}^*(v_1 v_{2n-1}) &= (2n-2)^2.
 \end{aligned}$$

Clearly f_{ssdapl}^* is an injection.

$$\text{gcin of } (v_i) = 1, \quad i = 1, 2, \dots, 2n-2$$

$$\text{gcin of } (v_{i+1}) = 1,$$

$$\text{gcin of } (v_{2n}) = 1,$$

So, **gcin** of each vertex of degree greater than one is 1.

Hence $T(C_n)$, admits sum square difference product prime labeling. ■

Theorem 2.4 2-tuple graph of cycle C_n admits sum square difference product prime labeling

Proof: Let $G = T^2(C_n)$ and let $v_1, v_2, \dots, v_{2n}$ are the vertices of G

Here $|V(G)| = 2n$ and $|E(G)| = 3n$

Define a function $f : V \rightarrow \{0, 1, 2, 3, \dots, 2n-1\}$ by

$$f(v_i) = i-1, i = 1, 2, \dots, 2n$$

Clearly f is a bijection.

For the vertex labeling f , the induced edge labeling f_{ssdapl}^* is defined as follows

$$\begin{aligned} f_{ssdapl}^*(v_{2i-1} v_{2i}) &= 12i^2 - 18i + 7, & i = 1, 2, \dots, n \\ f_{ssdapl}^*(v_{2i-1} v_{2i+1}) &= 12i^2 - 12i + 4, & i = 1, 2, \dots, n-1 \\ f_{ssdapl}^*(v_{2i} v_{2i+2}) &= 12i^2 + 1, & i = 1, 2, \dots, n-1 \\ f_{ssdapl}^*(v_1 v_{2n-1}) &= (2n-2)^2. \\ f_{ssdapl}^*(v_2 v_{2n}) &= 4n^2 - 2n - 1. \end{aligned}$$

Clearly f_{ssdapl}^* is an injection.

$$\begin{aligned} \text{gcin of } (v_1) &= 1, \\ \text{gcin of } (v_2) &= 1, \\ \text{gcin of } (v_{2n}) &= \gcd \{ f_{ssdapl}^*(v_{2n-1} v_{2n}), f_{ssdapl}^*(v_{2n-2} v_{2n}) \} \\ &= \gcd \{ 12n^2 - 24n + 13, 12n^2 - 18n + 7 \} \\ &= \gcd \{ (6n-6), 12n^2 - 24n + 13 \} \\ &= \gcd \{ (n-1), (n-1)(12n-12)+1 \} \\ &= 1 \\ \text{gcin of } (v_{2n-1}) &= \gcd \{ f_{ssdapl}^*(v_{2n-1} v_1), f_{ssdapl}^*(v_{2n-1} v_{2n}) \} \\ &= \gcd \{ (2n-2)^2, 12n^2 - 18n + 7 \} \\ &= \gcd \{ (2n-2), 12n^2 - 24n + 13 \} \\ &= \gcd \{ (n-1), (n-1)(12n-6)+1 \} \\ &= 1 \\ \text{gcin of } (v_{2i+1}) &= \gcd \{ f_{ssdapl}^*(v_{2i+1} v_{2i+3}), f_{ssdapl}^*(v_{2i+1} v_{2i+2}) \} \\ &= \gcd \{ 12i^2 + 12i + 4, 12i^2 + 6i + 1 \} \\ &= \gcd \{ (6i+3), 12i^2 + 6i + 1 \} \\ &= \gcd \{ (2i+1), (6i)(2i+1)+1 \} \\ &= 1, & i = 1, 2, \dots, n-2 \\ \text{gcin of } (v_{2i+2}) &= \gcd \{ f_{ssdapl}^*(v_{2i} v_{2i+2}), f_{ssdapl}^*(v_{2i+2} v_{2i+4}) \} \\ &= \gcd \{ 12i^2 + 1, 12i^2 + 24i + 13 \} \\ &= \gcd \{ (24i+12), 12i^2 + 1 \} \\ &= \gcd \{ (2i+1), 12i^2 + 1 \} \\ &= \gcd \{ (2i+1), 4 \} \\ &= 1, & i = 1, 2, \dots, n-2 \end{aligned}$$

So, **gcin** of each vertex of degree greater than one is 1.

Hence $T^2(P_n)$, admits sum square difference product prime labeling .

Theorem 2.5 Corona of cycle C_n admits sum square difference product prime labeling.

Proof: Let $G = C_n \odot K_1$ and let $v_1, v_2, \dots, v_{2n}$ are the vertices of G

Here $|V(G)| = 2n$ and $|E(G)| = 2n$

Define a function $f : V \rightarrow \{0, 1, 2, 3, \dots, 2n-1\}$ by

$$f(v_i) = i-1, i = 1, 2, \dots, 2n$$

Clearly f is a bijection.

For the vertex labeling f , the induced edge labeling f_{ssdapl}^* is defined as follows

$$\begin{aligned} f_{ssdapl}^*(v_{2i} v_{2i+2}) &= 12i^2 + 1, & i = 1, 2, \dots, n \\ f_{ssdapl}^*(v_{2i-1} v_{2i}) &= 12i^2 - 18i + 7, & i = 1, 2, \dots, n-1 \\ f_{ssdapl}^*(v_2 v_{2n}) &= 4n^2 - 2n + 1 \end{aligned}$$

Clearly f_{ssdapl}^* is an injection.

$$\begin{aligned} \text{gcin of } (v_{2i}) &= \gcd \{ f_{ssdapl}^*(v_{2i-1} v_{2i}), f_{ssdapl}^*(v_{2i} v_{2i+2}) \} \\ &= \gcd \{ 12i^2 - 18i + 7, 12i^2 + 1 \} \\ &= \gcd \{ (6i-6), 12i^2 - 18i + 7 \} \\ &= \gcd \{ (6i-6), (6i-6)(2i-1)+1 \} \\ &= 1, & i = 1, 2, \dots, n-1 \\ \text{gcin of } (v_{2n}) &= \gcd \{ f_{ssdapl}^*(v_{2n-1} v_{2n}), f_{ssdapl}^*(v_{2n} v_{2n-2}) \} \\ &= \gcd \{ 12n^2 - 18n + 7, 12n^2 - 24n + 13 \} \\ &= \gcd \{ (6n-6), 12n^2 - 24n + 13 \} \\ &= \gcd \{ (6n-6), (6n-6)(2n-2)+1 \} \\ &= 1. \end{aligned}$$

So, **gcin** of each vertex of degree greater than one is 1.

Hence $C_n \odot K_1$, admits sum square difference product prime labeling.

Theorem 2.6 Corona of cycle C_n with K_2 admits sum square difference product prime labeling.

Proof: Let $G = C_n \odot K_2$ and let $v_1, v_2, \dots, v_{3n}$ are the vertices of G

Here $|V(G)| = 3n$ and $|E(G)| = 4n$

Define a function $f : V \rightarrow \{0, 1, 2, 3, \dots, 3n-1\}$ by

$$f(v_i) = i-1, i = 1, 2, \dots, 3n$$

Clearly f is a bijection.

For the vertex labeling f , the induced edge labeling f_{ssdapl}^* is defined as follows

$$\begin{aligned} f_{ssdapl}^*(v_{3i-2} v_{3i-1}) &= 27i^2 - 45i + 19, & i = 1, 2, \dots, n \\ f_{ssdapl}^*(v_{3i} v_{3i-1}) &= 27i^2 - 27i + 7, & i = 1, 2, \dots, n \\ f_{ssdapl}^*(v_{3i} v_{3i-2}) &= 27i^2 - 36i + 13, & i = 1, 2, \dots, n \\ f_{ssdapl}^*(v_{3i-1} v_{3i+2}) &= 27i^2 - 4i + 3, & i = 1, 2, \dots, n-1 \\ f_{ssdapl}^*(v_2 v_{2n-1}) &= 9n^2 - 9n + 3 \end{aligned}$$

Clearly f_{ssdapl}^* is an injection.

$$\begin{aligned} \text{gcin of } (v_{3i-2}) &= \gcd \{ f_{ssdapl}^*(v_{3i-2} v_{3i-1}), f_{ssdapl}^*(v_{3i-2} v_{3i}) \} \\ &= \gcd \{ 27i^2 - 45i + 19, 27i^2 - 36i + 13 \} \\ &= \gcd \{ (9i-6), 27i^2 - 45i + 19 \} \\ &= \gcd \{ (9i-6), (9i-6)(3i-3)+1 \} \\ &= 1, & i = 1, 2, \dots, n \end{aligned}$$

$$\begin{aligned} \text{gcin of } (v_{3i}) &= \gcd \{ f_{ssdapl}^*(v_{3i-2} v_{3i}), f_{ssdapl}^*(v_{3i-1} v_{3i}) \} \\ &= \gcd \{ 27i^2 - 27i + 7, 27i^2 - 36i + 13 \} \\ &= \gcd \{ (9i-6), 27i^2 - 36i + 13 \} \\ &= \gcd \{ (9i-6), (9i-6)(3i-2)+1 \} \\ &= 1, & i = 1, 2, \dots, n \end{aligned}$$

$$\begin{aligned} \text{gcin of } (v_{3i-1}) &= \gcd \{ f_{ssdapl}^*(v_{3i-2} v_{3i-1}), f_{ssdapl}^*(v_{3i-1} v_{3i}) \} \\ &= \gcd \{ 27i^2 - 45i + 19, 27i^2 - 27i + 7 \} \\ &= \gcd \{ (18i-12), 27i^2 - 45i + 19 \} \\ &= \gcd \{ (9i-6), (9i-6)(3i-3)+1 \} \\ &= 1, & i = 1, 2, \dots, n \end{aligned}$$

So, **gcin** of each vertex of degree greater than one is 1.

Hence $C_n \odot K_2$, admits sum square difference product prime labeling. ■

Definition 2.1 Sun flower graph $SF(n)$ is defined as a graph obtained by starting with an n – cycle with consecutive vertices $v_1, v_2, \dots, v_n$ and creating new vertices with $w_1, w_2, \dots, w_n$ with w_i connected to v_i and v_{i+1} .

Theorem 2.7 Sun flower graph admits, sum square difference product prime labeling.

Proof: Let $G = SF(n)$ and let $v_1, v_2, \dots, v_{2n}$ are the vertices of G

Here $|V(G)| = 2n$ and $|E(G)| = 3n$

Define a function $f : V \rightarrow \{0, 1, 2, 3, \dots, 2n-1\}$ by

$$f(v_i) = i-1, i = 1, 2, \dots, 2n$$

Clearly f is a bijection.

For the vertex labeling f , the induced edge labeling f_{ssdapl}^* is defined as follows

$$\begin{aligned} f_{ssdapl}^*(v_i v_{i+1}) &= 3i^2 - 3i + 1, & i = 1, 2, \dots, 2n-1 \\ f_{ssdapl}^*(v_1 v_{2n}) &= (2n-1)^2 \\ f_{ssdapl}^*(v_{2i-1} v_{2i+1}) &= 12i^2 - 12i + 4, & i = 1, 2, \dots, n-1 \\ f_{ssdapl}^*(v_1 v_{2n-1}) &= (2n-2)^2 \end{aligned}$$

Clearly f_{ssdapl}^* is an injection.

$$\begin{aligned} \text{gcin of } (v_1) &= \gcd \{ f_{ssdapl}^*(v_1 v_2), f_{ssdapl}^*(v_1 v_3) \} \\ &= \gcd \{ 1, 4 \} = 1. \end{aligned}$$

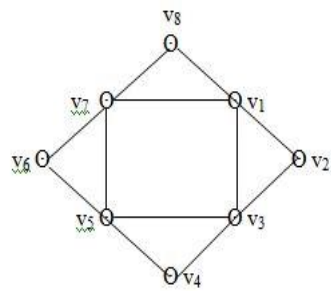
$$\text{gcin of } (v_{i+1}) = 1, \quad i = 1, 2, \dots, 2n-2$$

$$\begin{aligned} \text{gcin of } (v_n) &= \gcd \{ f_{ssdapl}^*(v_1 v_{2n}), f_{ssdapl}^*(v_{2n-1} v_{2n}) \} \\ &= \gcd \{ (2n-1)^2, 12n^2 - 18n + 7 \} = 1. \end{aligned}$$

So, **gcin** of each vertex of degree greater than one is 1.

Hence $SF(n)$, admits sum square difference product prime labeling. ■

Example 2.1 Illustration of the theorem with flower graph FL(4)



III. References

- [1] Apostol. Tom M, Introduction to Analytic Number Theory, Narosa, (1998).
- [2] F Harary, Graph Theory, Addison-Wesley, Reading, Mass, (1972)
- [3] Joseph A Gallian, A Dynamic Survey of Graph Labeling, The Electronic Journal of Combinatorics(2016), #DS6, pp 1 – 408.
- [4] T K Mathew Varkey, Some Graph Theoretic Generations Associated with Graph Labeling, PhD Thesis, University of Kerala 2000.