



Minimax Estimation of the Scale Parameter of the Laplace Distribution under Different Loss Function

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Abstract: In this paper, we obtained Minimax estimators of the scale parameter θ for the Laplace distribution under different loss function by applying the theorem of Lehmann [1950]. Some Bayes estimators for the unknown scale parameter θ of Laplace distribution have been obtained, using Non-informative prior (Jeffreys) and Informative priors (Gumbel Type II) under different loss function, represented by (Quadratic loss function, Squared – log error loss function, Entropy loss function).

According to Monte-Carlo simulation study, the performance of these estimates is compared depending on the mean squared errors (MSE's).

Key words: Minimax estimator; Laplace distribution; Bayes estimator; Quadratic loss function; Entropy loss function; Squared-log error loss function; Jeffery prior; Gumbel Type II prior; Mean squared error.

I. Introduction

Laplace distribution has been used for modeling data that have heavier tails than those of the normal distribution and analyze engineering, financial, industrial, environmental, and biological data (Kotz et al., 2001)[5]. It is used for modeling data that have heavier tails than those of the normal distribution.

The probability density functions of a Laplace distributed random variable is given by: [9]

$$f(x|a, \theta) = \frac{1}{2\theta} \exp\left[-\frac{|x-a|}{\theta}\right] \quad -\infty < x < \infty \quad (1)$$

Where $a \in (-\infty, \infty)$ and $\theta > 0$ are location and scale parameters, respectively.

The cumulative distribution function is given by:

$$F(x|a, \theta) = \begin{cases} 1 - \frac{1}{2} \exp\left[-\frac{a-x}{\theta}\right] & \text{for } x \geq a \\ \frac{1}{2} \exp\left[-\frac{x-a}{\theta}\right] & \text{for } x < a \end{cases}$$

With moment –generating function defined as:

$$M_x(t) = \frac{e^{t\mu}}{1-b^2t^2}$$

II. Bayes Estimators

A. Jefferys Prior Information [9]

Assume that, θ has non-information prior density, defined

$$g \propto \sqrt{I(\theta)} \quad (2)$$

Where $I(\theta)$ represented Fisher information which defined as follows:

$$I(\theta) = -nE\left[\frac{\partial^2 \ln f(x;a,\theta)}{\partial \theta^2}\right]. \text{ Hence}$$

$$g(\theta) = k \sqrt{-nE\left(\frac{\partial^2 \ln f(x;a,\theta)}{\partial \theta^2}\right)} \quad (3)$$

$$\ln f(x; a, \theta) = -\ln(2\theta) - \frac{|x-a|}{\theta}$$

$$\frac{\partial \ln f(x;a,\theta)}{\partial \theta} = -\frac{1}{2\theta} - \frac{|x-a|}{\theta^2}$$

$$\frac{\partial^2 \ln f(x;a,\theta)}{\partial \theta^2} = \frac{1}{2\theta^2} - \frac{2|x-a|}{\theta^3}$$

$$E\left[\frac{\partial^2 \ln f(x;a,\theta)}{\partial \theta^2}\right] = \frac{1}{2\theta^2} - \frac{2}{\theta^3} E[|x-a|]$$

Let $X \sim \text{Laplace}(a, \theta)$ then, [5]

$$|x_i - a| = \theta W$$

Where (W) is standard exponential with mean and variance equal to one.

Hence, $|x_i - a| \sim \text{Gamma}(1, \theta^{-1})$

$$\text{Then, } E \left[\frac{\partial^2 \ln f(x; a, \theta)}{\partial \theta^2} \right] = \frac{-3}{2\theta^2} \quad (4)$$

After Substitution (4) into (3), we get

$$g(\theta) = k \sqrt{-n \left(-\frac{3}{2\theta^2} \right)} = \frac{k}{\theta} \sqrt{\frac{3n}{2}} \quad (5)$$

Now, combining the prior (5) with the likelihood function, we have the posterior distribution of θ with Jeffreys prior information given by $h_1(\theta|\underline{x})$:

$$h_1(\theta|\underline{x}) = \frac{g(\theta)L(\theta; x_1, x_2, \dots, x_n)}{\int_0^\infty g(\theta)L(\theta; x_1, x_2, \dots, x_n)d\theta}$$

$$h_1(\theta|\underline{x}) = \frac{\frac{1}{\theta^{n+1}} \exp \left[-\frac{\sum_{i=1}^n |x_i - a|}{\theta} \right]}{\int_0^\infty \frac{1}{\theta^{n+1}} \exp \left[-\frac{\sum_{i=1}^n |x_i - a|}{\theta} \right] d\theta}$$

On simplification, we have

$$h_1(\theta|\underline{x}) = \frac{\left(\sum_{i=1}^n |x_i - a| \right)^n e^{-\frac{\sum_{i=1}^n |x_i - a|}{\theta}}}{\theta^{n+1} \Gamma(n)} = \frac{T^n e^{-\frac{T}{\theta}}}{\theta^{n+1} \Gamma(n)} \quad (6)$$

The posterior density $h_1(\theta|\underline{x})$ is recognized as the density of the Inverse Gamma (IG) distribution: $(\theta|\underline{x}) \sim \text{IG}(n, T)$

B. Gumbel Type II prior Information

Assume that, the unknown scale parameter θ has Gumbel type II prior with hyper-parameter (b), defined as:^[1]

$$g_2(\theta) = b \left(\frac{1}{\theta} \right)^2 \text{Exp} \left[\frac{-b}{\theta} \right], \quad b, \theta > 0 \quad (7)$$

Now, combining the prior (7) with the likelihood function

$L(x_1, x_2, \dots, x_n; \theta) = \left(\frac{1}{2\theta} \right)^n \exp \left[-\frac{\sum_{i=1}^n |x_i - a|}{\theta} \right]$ yields the posterior distribution of θ with Gumbel Type II prior distribution information denoted by $h_2(\theta|\underline{x})$, where:

$$h_2(\theta|\underline{x}) = \frac{g_2(\theta)L(\theta; x_1, x_2, \dots, x_n)}{\int_0^\infty g_2(\theta)L(\theta; x_1, x_2, \dots, x_n)d\theta} = \frac{\frac{1}{\theta^{n+2}} e^{-(\sum_{i=1}^n |x_i - a| + b)/\theta}}{\int_0^\infty \frac{1}{\theta^{n+2}} e^{-(\sum_{i=1}^n |x_i - a| + b)/\theta} d\theta} \quad (8)$$

After simplification, we get:

$$h_2(\theta|\underline{x}) = \frac{\left(\sum_{i=1}^n |x_i - a| + b \right)^{n+1} e^{-(\sum_{i=1}^n |x_i - a| + b)/\theta}}{\theta^{n+2} \Gamma(n+1)} \quad (9)$$

Which implies that, $(\theta|\underline{x}) \sim \text{Inv-Gamma}(n+1, (\sum_{i=1}^n |x_i - a| + b))$

Where, $E(\theta|\underline{x}) = \frac{\sum_{i=1}^n |x_i - a| + b}{n}$, $\text{Var}(\theta|\underline{x}) = \frac{(\sum_{i=1}^n |x_i - a| + b)^2}{n^2(n-1)}$, $n > 1$

III. Bayes Estimators under Different Loss Function

Bayesian estimates of the scale parameter of the Laplace distribution has been obtained under different loss functions and as follows:

A. Baye Estimator under Quadratic Loss Function

De Groot (1970) discussed different types of loss function and obtained the Bayes estimates under these loss function which is a non-negative symmetric and continuous loss function and is defined as:[9]

$$L(\hat{\theta}, \theta) = \left(\frac{\hat{\theta} - \theta}{\theta} \right)^2 = \left(1 - \frac{\hat{\theta}}{\theta} \right)^2 \quad (10)$$

Hence, Bayes estimator under the Quadratic loss function will be:[9]

$$\hat{\theta}_Q = \frac{E \left(\frac{1}{\theta} \right)}{E \left(\frac{1}{\theta^2} \right)} \quad (11)$$

(i) With Jeffreys prior information:

From (6) posterior density function, the Bayes estimator of θ of Laplace distribution under Quadratic loss function can be obtained as follows:

$$E \left(\frac{1}{\theta^m} | \underline{x} \right) = \int_0^\infty \frac{1}{\theta^m} h_1(\theta|\underline{x}) d\theta$$

$$= \int_0^\infty \frac{1}{\theta^m} \frac{\left(\sum_{i=1}^n |x_i - a| \right)^n}{\theta^{n+1} \Gamma(n)} e^{-\frac{\sum_{i=1}^n |x_i - a|}{\theta}} d\theta$$

Therefore,

$$E \left(\frac{1}{\theta^m} | \underline{x} \right) = \frac{\Gamma(n+m)}{\Gamma(n) \left(\sum_{i=1}^n |x_i - a| \right)^m}, \quad m = 0, 1, 2, \dots \quad (12)$$

After substituting, we get the Bayes estimator of parameter θ under Quadratic loss function with Jeffreys prior information as:

$$\hat{\theta}_{QJ} = \frac{\sum_{i=1}^n |x_i - a|}{(n+1)} \quad (13)$$

(ii) With Gumbel Type II prior information:

Using the posterior density function (9), the Bayes estimator of θ of Laplace distribution under Quadratic loss function is obtained as:

$$\hat{\theta}_{QG} = \frac{E\left(\frac{1}{\theta} | \underline{x}\right)}{E\left(\frac{1}{\theta^2} | \underline{x}\right)} \quad (14)$$

$$\begin{aligned} E\left(\frac{1}{\theta} | \underline{x}\right) &= \int_0^\infty \frac{1}{\theta} h_2(\theta | \underline{x}) d\theta \\ &= \int_0^\infty \frac{(\sum_{i=1}^n |x_i - a| + b)^{n+1}}{\theta^{n+m+2} \Gamma(n+1)} e^{-\frac{(\sum_{i=1}^n |x_i - a| + b)}{\theta}} d\theta \\ &= \frac{\Gamma(n+m+1)}{\Gamma(n+1)(\sum_{i=1}^n |x_i - a| + b)^m} \int_0^\infty \frac{(\sum_{i=1}^n |x_i - a| + b)^{n+m+1}}{\Gamma(n+m+1)\theta^{n+m+2}} e^{-\frac{(\sum_{i=1}^n |x_i - a| + b)}{\theta}} d\theta \end{aligned}$$

Hence

$$E\left(\frac{1}{\theta} | \underline{x}\right) = \frac{\Gamma(n+m+1)}{\Gamma(n+1)(\sum_{i=1}^n |x_i - a| + b)^m}$$

After substituting, we get the Bayes estimator of parameter θ under Quadratic loss function with Gumbel Type II information as:

$$\hat{\theta}_{QG} = \frac{(\sum_{i=1}^n |x_i - a| + b)}{n+2} \quad (15)$$

B. Bayes Estimator Under Squared – Log Error Loss Function

Farsipour and Zakerzadeh (2006) [4] studied the estimation of generalized variance under an asymmetric loss function named the squared log error loss function, where is of the form:

$$L(\hat{\theta}, \theta) = (\ln \hat{\theta} - \ln \theta)^2 = \left(\ln \frac{\hat{\theta}}{\theta}\right)^2 \quad (16)$$

This is balanced with $\lim L(\hat{\theta}, \theta) \rightarrow \infty$ as $\hat{\theta} \rightarrow 0$ or ∞

A balanced loss function takes both error of estimation and goodness of fit in to account but the unbalanced loss function considers only error of estimation. This loss function is convex for $\frac{\hat{\theta}}{\theta} \leq e$ and concave otherwise, it's risk function has a unique minimum with respect to $\hat{\theta}$. [2]

According to the above mentioned loss function, we derive the corresponding Bayes estimators for θ using Risk function $R(\hat{\theta}, \theta)$, which minimizes the posterior risk.

$$\begin{aligned} R(\hat{\theta}, \theta) &= E[L(\hat{\theta}, \theta)] \\ &= \int_0^\infty (\ln \hat{\theta} - \ln \theta)^2 h(\theta | \underline{x}) d\theta \end{aligned} \quad (17)$$

The Bayes estimator for the parameter θ of Laplace distribution under the Squared-log error loss function is:[9]

$$\hat{\theta}_{SL} = \text{Exp}[E(\ln \theta | \underline{x})] \quad (18)$$

(i) With Jeffreys prior information

According to the posterior density function (6), the Bayes estimator of θ of Laplace distribution under Squared – log error loss function can be obtained as

$$\hat{\theta}_{SLJ} = \text{Exp}[E(\ln \theta | \underline{x})_J] \quad (19)$$

$$\begin{aligned} E(\ln \theta | \underline{x})_J &= \int_0^\infty \ln \theta h_1(\theta | \underline{x})_J d\theta \\ E(\ln \theta | \underline{x})_J &= \int_0^\infty \ln \theta \frac{(\sum_{i=1}^n |x_i - a|)^n}{\theta^{n+1} \Gamma(n)} e^{-\frac{\sum_{i=1}^n |x_i - a|}{\theta}} d\theta = \frac{(\sum_{i=1}^n |x_i - a|)^n}{\Gamma(n)} \int_0^\infty \ln \theta \frac{e^{-\frac{\sum_{i=1}^n |x_i - a|}{\theta}}}{\theta^{n+1}} d\theta \end{aligned}$$

let $y = \frac{\sum_{i=1}^n |x_i - a|}{\theta}$, which implies that

$$\theta = \frac{\sum_{i=1}^n |x_i - a|}{y}, \quad \left| \frac{d\theta}{dy} \right| = \left| -\frac{\sum_{i=1}^n |x_i - a|}{y^2} \right|, \quad d\theta = \frac{\sum_{i=1}^n |x_i - a|}{y^2} dy$$

Hence,

$$\begin{aligned}
 E(\ln \theta | \underline{x})_J &= \frac{(\sum_{i=1}^n |x_i - a|)^n}{\Gamma(n)} \int_0^\infty \frac{(\ln(\sum_{i=1}^n |x_i - a|) - \ln(y)) e^{-y} \frac{(\sum_{i=1}^n |x_i - a|)}{y^2}}{\frac{(\sum_{i=1}^n |x_i - a|)^{n+1}}{y^{n+1}}} dy \\
 &= \frac{\ln(\sum_{i=1}^n |x_i - a|)}{\Gamma(n)} \int_0^\infty y^{n-1} e^{-y} dy - \int_0^\infty \frac{\ln(y) y^{n-1} e^{-y}}{\Gamma(n)} dy \\
 &= \ln(\sum_{i=1}^n |x_i - a|) - \frac{\Gamma'(n)}{\Gamma(n)} \\
 E(\ln \theta | \underline{x})_J &= \ln(\sum_{i=1}^n |x_i - a|) - \psi(n)
 \end{aligned} \tag{20}$$

Such that $\psi(n) = \frac{\Gamma'(n)}{\Gamma(n)}$, where $\psi(n)$ is a digamma function.[10]

After substituting (20) in (19), we get the Bayes estimator of parameter θ under Squared-log error loss function with Jeffreys prior information as:

$$\begin{aligned}
 \hat{\theta}_{SLJ} &= \text{Exp}[\ln(\sum_{i=1}^n |x_i - a|) - \psi(n)] \\
 \hat{\theta}_{SLJ} &= \frac{\sum_{i=1}^n |x_i - a|}{e^{\psi(n)}}
 \end{aligned} \tag{21}$$

(ii) **With Gumbel Type II prior information:**

From the posterior density function (9), the Bayes estimator of θ of Laplace distribution under Squared – log error loss function is obtained as:

$$\hat{\theta}_{SLG} = \text{Exp} \left[E(\ln \theta | \underline{x})_G \right] \tag{22}$$

$$\begin{aligned}
 E(\ln \theta | \underline{x})_G &= \int_0^\infty \ln \theta h_2(\theta | \underline{x})_G d\theta \\
 E(\ln \theta | \underline{x})_G &= \int_0^\infty \ln \theta \frac{(\sum_{i=1}^n |x_i - a|)^{n+1} e^{-\frac{(\sum_{i=1}^n |x_i - a| + b)}{\theta}}}{\theta^{n+2} \Gamma(n+1)} d\theta \\
 E(\ln \theta | \underline{x})_G &= \frac{(\sum_{i=1}^n |x_i - a|)^{n+1}}{\Gamma(n+1)} \int_0^\infty \ln \theta \frac{e^{-\frac{(\sum_{i=1}^n |x_i - a| + b)}{\theta}}}{\theta^{n+2}} d\theta
 \end{aligned}$$

let $y = \frac{(\sum_{i=1}^n |x_i - a| + b)}{\theta}$, then:

$$\theta = \frac{(\sum_{i=1}^n |x_i - a| + b)}{y}, \left| \frac{d\theta}{dy} \right| = \left| -\frac{(\sum_{i=1}^n |x_i - a| + b)}{y^2} \right|, d\theta = \frac{(\sum_{i=1}^n |x_i - a| + b)}{y^2} dy$$

Hence

$$\begin{aligned}
 E(\ln \theta | \underline{x})_G &= \frac{\ln(\sum_{i=1}^n |x_i - a|)}{\Gamma(n+1)} \int_0^\infty y^n e^{-y} dy - \int_0^\infty \frac{\ln(y) y^n e^{-y}}{\Gamma(n+1)} dy \\
 &= \ln(\sum_{i=1}^n |x_i - a| + b) - \frac{\Gamma'(n+1)}{\Gamma(n+1)} \\
 E(\ln \theta | \underline{x})_G &= \ln(\sum_{i=1}^n |x_i - a| + b) - \psi(n+1)
 \end{aligned} \tag{23}$$

Substituting (23) into (22), we get the Bayes estimator of parameter θ under Squared-log error loss function with Gumbel Type II prior information as

$$\begin{aligned}
 \hat{\theta}_{SLG} &= \text{Exp} \left[\ln \left(\sum_{i=1}^n |x_i - a| + b \right) - \psi(n+1) \right] \\
 \hat{\theta}_{SLG} &= \frac{(\sum_{i=1}^n |x_i - a| + b)}{e^{\psi(n+1)}}
 \end{aligned} \tag{24}$$

C. Bayesian Estimator under Entropy Loss Function

In many practical situations, it appears to be more realistic to express the loss in terms of the ratio $\frac{\hat{\theta}}{\theta}$. In this case used the Entropy loss function which is defined as:[8]

$$l(\hat{\theta}, \theta) = z \left[\left(\frac{\hat{\theta}}{\theta} \right) - \ln \left(\frac{\hat{\theta}}{\theta} \right) - 1 \right], \quad z > 0 \tag{25}$$

Then the Risk function under the Entropy loss function, denoted by $R_{EN}(\hat{\theta}, \theta)$ is [12].

$$\begin{aligned}
 R_{EN}(\hat{\theta}, \theta) &= E[L_{EN}(\hat{\theta}, \theta)] \\
 &= \int_0^\infty L_{EN}(\hat{\theta}, \theta) h(\theta | \underline{x}) d\theta \\
 &= \int_0^\infty z \left[\left(\frac{\hat{\theta}}{\theta} \right) - \ln \left(\frac{\hat{\theta}}{\theta} \right) - 1 \right] h(\theta | \underline{x}) d\theta \\
 &= \int_0^\infty z \left[\left(\frac{\hat{\theta}}{\theta} \right) - \ln(\hat{\theta}) - \ln(\theta) - 1 \right] h(\theta | \underline{x}) d\theta \\
 &= z \hat{\theta} E \left(\frac{1}{\theta} | \underline{x} \right) - z \ln(\hat{\theta}) - z E(\ln(\theta)) - z
 \end{aligned}$$

$$\frac{\partial R_{EN}(\hat{\theta}, \theta)}{\partial \hat{\theta}} = z E \left(\frac{1}{\theta} | \underline{x} \right) - z \frac{1}{\hat{\theta}}$$

$$\text{Let } \frac{\partial R_{EN}(\hat{\theta}, \theta)}{\partial \hat{\theta}} = 0, \text{ then}$$

$$\frac{1}{\hat{\theta}_{EN}} = E\left(\frac{1}{\theta} | \underline{x}\right)$$

$$\hat{\theta}_{EN} = \left[E\left(\frac{1}{\theta} | \underline{x}\right)\right]^{-1} \quad (26)$$

Provided that, $E\left(\frac{1}{\theta} | \underline{x}\right)$ exist and is finite.

(i) With Jeffreys prior information

From the posterior density function (6), the Bayes estimator of θ under Entropy loss function with Jeffreys prior information, is derived as follows:

$$\hat{\theta}_{ENJ} = \left[E\left(\frac{1}{\theta} | \underline{x}\right)_J\right]^{-1}$$

$$E\left(\frac{1}{\theta} | \underline{x}\right)_J = \int_0^{\infty} \frac{1}{\theta} h_1(\theta | \underline{x}) d\theta$$

Putting $m=1$ in (14), we get

$$E\left(\frac{1}{\theta} | \underline{x}\right) = \frac{n}{\sum_{i=1}^n |x_i - a|} \quad (27)$$

Therefore,

$$\hat{\theta}_{ENJ} = \frac{\sum_{i=1}^n |x_i - a|}{n} \quad (28)$$

(ii) With Gumbel Type II prior information

According to the posterior density function (9), the Bayes estimator of θ under Entropy loss function with Gumbel type II prior information, is derived as follows:

$$\hat{\theta}_{ENG} = \left[E\left(\frac{1}{\theta} | \underline{x}\right)_G\right]^{-1} \quad (29)$$

Recall that,

$$E\left(\frac{1}{\theta} | \underline{x}\right) = \frac{n+1}{(\sum_{i=1}^n |x_i - a| + b)} \quad (30)$$

Therefore, Bayes estimator of θ under entropy loss function with Gumbel Type II prior information is:

$$\hat{\theta}_{ENG} = \frac{(\sum_{i=1}^n |x_i - a| + b)}{n+1} \quad (31)$$

IV. Minimax Estimators

The derivation of Minimax estimators depends primarily on a theorem due to Lehmann, which can be stated as follows:

Lehmann's Theorem: [3]

Let $\tau = \{F\theta; \theta \in \Theta\}$ be a family of distribution functions and D a class of estimators of θ . Suppose that $d^* \in D$ is a Bayes' estimator against a prior distribution $\xi^*(\theta)$ on the parameter space Θ and the risk function $R(d^*, \theta) = \text{constant on } \Theta$; then d^* is a minimax estimator of θ .

The main results are contained in the following Theorem.

(A) Definition [7]

An estimator t^* is defined to be a Minimax estimator if and only if:

$$\sup_{\theta} R_{t^*}(\theta) \leq \sup_{\theta} R_t(\theta) \text{ for every estimator } t.$$

(B) Theorem:

Let $X = (X_1, X_2, \dots, X_n)$ be n independently and identically distributed random variables drawn from the density (1), then:

$$\hat{\theta}_{MQJ} = \frac{\sum_{i=1}^n |x_i - a|}{(n+1)}$$

Is the Minimax estimator of the scale parameter θ for the Laplace distribution for the Quadratic loss function for the type (10) [9]

Proof :

In order to prove the Theorem (4.2), it will be sufficient to show that, $\hat{\theta}_{MQJ} = \frac{\sum_{i=1}^n |x_i - a|}{(n+1)}$ is the minimax estimator of θ for Quadratic loss function (10). If we can show that, the risk of $\hat{\theta}_{MQJ}$ is constant, then, the Theorem (4.2) will be proved. Since,

$$R_{\hat{\theta}_{MQJ}}(\theta) = E[L(\theta | \hat{\theta}_{MQJ})]$$

$$\begin{aligned}
 &= E \left[\left(\frac{\theta - \hat{\theta}_{QJ}}{\theta} \right)^2 \right] \\
 &= \frac{1}{\theta^2} E \left[\theta^2 - \frac{2\theta}{n+1} E(\sum_{i=1}^n |x_i - a|) + \frac{1}{(n+1)^2} E(\sum_{i=1}^n |x_i - a|)^2 \right]
 \end{aligned} \quad (32)$$

Recall that, $|x_i - a| \sim \text{Gamma}(1, \theta^{-1})$

Hence, $T = \sum_{i=1}^n |x_i - a| \sim \text{Gamma}(n, \theta^{-1})$

With $E(T) = n\theta$, $\text{Var}(T) = n\theta^2$

Then, substituting $E(T)$ and $E(T^2)$ into (35), we get

$$\begin{aligned}
 R_{\hat{\theta}_{QJ}}(\theta) &= \frac{1}{\theta^2} \left[\theta^2 - \frac{2\theta}{n+1} (n\theta) + \frac{n\theta^2(n+1)}{(n+1)^2} \right] \\
 &= 1 - \frac{2n}{(n+1)} + \frac{n}{(n+1)}, \text{ which is a constant}
 \end{aligned}$$

So, according to the Lehmann's theorem it follows that:

$$\hat{\theta}_{MQJ} = \frac{\sum_{i=1}^n |x_i - a|}{n+1}$$

is minimax estimator of the unknown parameter θ under Quadratic loss function

C. Theorem:

Let $X = (X_1, X_2, \dots, X_n)$ be n independently and identically distributed random variables drawn from the density (1), then:

$$\hat{\theta}_{SLJ} = \frac{\sum_{i=1}^n |x_i - a|}{e^{\psi(n)}}$$

Is minimax estimator of the scale parameter θ for the Laplace distribution for the Squared-log error loss function (22), where $\hat{\theta}_{SLJ}$ is the estimate of θ .

Proof:

In order to prove the Theorem (4.3), it will be sufficient to show that, $\hat{\theta}_{SLJ} = \frac{\sum_{i=1}^n |x_i - a|}{e^{\psi(n)}}$, is the minimax estimator of θ for Squared-log error loss function (16).

If we can show that, the risk of $\hat{\theta}_{SLJ}$ is a constant, then, the Theorem (4.3) will be proved. Since,

$$\begin{aligned}
 R_{\hat{\theta}_{SLJ}}(\theta) &= E[L(\theta|\hat{\theta}_{SLJ})] \\
 &= \int_0^\infty (\ln \hat{\theta}_{SLJ} - \ln \theta)^2 h_1(\theta|x) dx \\
 &= E(\ln \hat{\theta}_{SLJ})^2 - 2(\ln \theta)E(\ln \hat{\theta}_{SLJ}) + (\ln \theta)^2
 \end{aligned} \quad (33)$$

Now,

$$\begin{aligned}
 E[\ln \hat{\theta}_{SLJ}] &= E \left(\ln \frac{\sum_{i=1}^n |x_i - a|}{e^{\psi(n)}} \right) \\
 E[\ln \hat{\theta}_{SLJ}] &= E[\ln(\sum_{i=1}^n |x_i - a|) - \psi(n)]
 \end{aligned} \quad (34)$$

Now, from the conclusion, we get:

$$T = \sum_{i=1}^n |x_i - a| \sim \Gamma(n, \theta^{-1})$$

Thus, we have:

$$E[\ln(\sum_{i=1}^n |x_i - a|)] = \psi(n) + \ln \theta \text{ and } \text{var}[\ln(\sum_{i=1}^n |x_i - a|)] = \psi'(n)$$

After substituting in (34), yields

$$E[\ln \hat{\theta}_{SLJ}] = \psi(n) + \ln \theta - \psi(n) = \ln \theta \quad (35)$$

$$\begin{aligned}
 E[\ln \hat{\theta}_{SLJ}]^2 &= E \left[\ln \frac{\sum_{i=1}^n |x_i - a|}{e^{\psi(n)}} \right]^2 \\
 E[\ln \hat{\theta}_{SLJ}]^2 &= E[\ln(\sum_{i=1}^n |x_i - a|) - \psi(n)]^2 \\
 &= E(\ln(\sum_{i=1}^n |x_i - a|))^2 - 2\psi(n)E(\ln(\sum_{i=1}^n |x_i - a|)) + \psi^2(n)
 \end{aligned} \quad (36)$$

Since $\text{var}[\ln(\sum_{i=1}^n |x_i - a|)] = E(\ln(\sum_{i=1}^n |x_i - a|))^2 - [E(\ln(\sum_{i=1}^n |x_i - a|))]^2$

Then

$$\begin{aligned}
 E(\ln(\sum_{i=1}^n |x_i - a|))^2 &= \text{var}[(\sum_{i=1}^n |x_i - a|)] + [E(\ln(\sum_{i=1}^n |x_i - a|))]^2 = \psi'(n) + (\psi(n) + \ln \theta)^2 \\
 &= \psi'(n) + \psi^2(n) + 2\psi(n) \ln \theta + (\ln \theta)^2
 \end{aligned} \quad (37)$$

Substitution (37) and (35) into (36), we get

$$\begin{aligned}
 E[\ln \hat{\theta}_{SLJ}]^2 &= \psi'(n) + \psi^2(n) + 2\psi(n) \ln \theta + (\ln \theta)^2 - 2\psi(n)(\psi(n) + \ln \theta) + \psi^2(n) \\
 &= \psi'(n) + 2\psi^2(n) + 2\psi(n) \ln \theta + (\ln \theta)^2 - 2\psi^2(n) - 2\psi(n) \ln \theta \\
 &= \psi'(n) + (\ln \theta)^2
 \end{aligned} \quad (42)$$

Substitution (35) and (38) into (33), we get

$$R_{\hat{\theta}_{SLJ}}(\theta) = \psi(n) + (\ln \theta)^2 - 2 \ln(\theta) (\ln \theta) + (\ln \theta)^2 = \psi(n) + (\ln \theta)^2 - 2(\ln \theta)^2 + (\ln \theta)^2 = \psi(n)$$
(43)

Then, $R_{\hat{\theta}_{SLJ}}(\theta)$ is a constant

So, according to the Lehmann's theorem it follows that:

$$\hat{\theta}_{SLJ} = \frac{\sum_{i=1}^n |x_i - a|}{e^{\psi(n)}}$$

Is minimax estimator of the parameter θ of the Laplace distribution under the Squared – log error loss function.

D. Theorem

Let $X = (x_1, x_2, \dots, x_n)$ be independently and identically distributed random variable drawn from the density (1). Then

$$\hat{\theta}_{ENJ} = \frac{\sum_{i=1}^n |x_i - a|}{n}$$

Is minimax estimator of the scale parameter θ for the Laplace distribution for the Entropy loss functions:

$$l(\hat{\theta}, \theta) = z \left[\left(\frac{\hat{\theta}}{\theta} \right) - \ln \left(\frac{\hat{\theta}}{\theta} \right) - 1 \right], \quad z > 0$$

Where θ is the parameter to be estimated and $\hat{\theta}$ is the estimator of θ .

Proof :

In order to prove the Theorem (4.4), it will be sufficient to show that, $\hat{\theta}_{ENJ} = \frac{\sum_{i=1}^n |x_i - a|}{n}$, is the minimax estimator of θ for Entropy loss function (25).

If we can show that, the risk of $\hat{\theta}_{ENJ}$ is a constant, then, the Theorem (4.4) will be proved. Since,

$$\begin{aligned} R_{\hat{\theta}_{ENJ}}(\theta) &= E[L(\hat{\theta}_{ENJ}, \theta)] \\ &= \int_0^\infty \left[\left(\frac{\hat{\theta}}{\theta} \right) - \ln \left(\frac{\hat{\theta}}{\theta} \right) - 1 \right] h_1(\theta|x) d\theta \\ &= E \left[\frac{\sum_{i=0}^n |x - a|}{n\theta} - \ln \frac{\sum_{i=0}^n |x - a|}{n\theta} + 1 \right] \\ &= \frac{1}{n\theta} E[\sum_{i=0}^n |x - a|] - E[\ln(\sum_{i=0}^n |x - a|)] + \ln(\theta) + \ln(n) + 1 \end{aligned}$$

Since $E[\sum_{i=0}^n |x - a|] = n\theta$ and $E[\ln(\sum_{i=0}^n |x - a|)] = \ln(\theta)$

$$\text{Then } R_{\hat{\theta}_{ENJ}}(\theta) = \frac{n\theta}{n\theta} - \ln(\theta) + \ln(\theta) + \ln(n) + 1$$

$$R_{\hat{\theta}_{ENJ}}(\theta) = 2 + \ln(n)$$
(44)

Which is a constant.

So, according to the Lehmann's theorem it follows that:

$\hat{\theta}_{ENJ} = \frac{\sum_{i=1}^n |x_i - a|}{n}$ is minimax estimator of the parameter θ of the Laplace distribution under the Entropy loss function.

V. Simulation Results

In this section, Monte-Carlo simulation study is performed to compare the methods of estimation by using mean squared errors (MSE's) as an index for precision to compare the efficiency of each of estimators, where:

$$MSE(\hat{\theta}) = \frac{\sum_{i=1}^I (\hat{\theta}_i - \theta)^2}{I}$$

In our simulation study, we generated $I = 5000$ samples of size $n = 5, 10, 20, 50$ and 100 from Laplace distribution to represent small, moderate and large sample size with $\theta = 0.5$ and 3 .

The default values of the hyper-parameter of Gumbel type II prior (b) chosen are (b= 0.9, 2, 5).

The results were summarized and tabulated in the tables (1-6) for each estimator and for all sample sizes.

VI. Discussion

Based on the results of estimating the scale parameter of Laplace distribution, the following important conclusion could be reached:

1. Generally, the performance of Bayes estimator under different loss functions using Gumbel Type II prior is more appropriate than using Jefferys prior.
2. The values of MSE's for Bayes estimators using Gumbel Type II informative prior, are decreasing with using the value of the parameter of Gumbel prior (b), nearly, twice the value of the scale parameter θ .
3. In most cases, it is obvious that, the Bayes estimator under Quadratic loss function with Gumbel prior (QG) is the best estimator for θ .

Finally, we can say that, the best estimator of the scale parameter (θ) for the Laplace distribution is the Bayes minimax estimator under Quadratic loss function with Gumbel prior using hyper parameter (b) for Gumbel prior, nearly twice the value of θ in comparing with other minimax estimators under study

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Table (1): Estimated Value and MSE of Different Estimates of the Scale parameter of the Laplace Distribution when $\theta = 0.5$ and $b = 0.9$

Estimator	$\frac{n}{\text{Criteria}}$	5	10	20	50	100
QJ	EXP	0.41807	0.45551	0.47807	0.49019	0.49553
	MSE	0.04081	0.02254	0.01205	0.00490	0.00247
QG	EXP	0.48691	0.49255	0.49725	0.49807	0.49949
	MSE	0.02522	0.01733	0.01055	0.00462	0.00240
SLJ	EXP	0.55630	0.52721	0.51482	0.50514	0.50318
	MSE	0.06354	0.02828	0.01363	0.00512	0.00253
SLG	EXP	0.61887	0.56272	0.53362	0.51295	0.50712
	MSE	0.05460	0.02649	0.01327	0.00507	0.00252
ENJ	EXP	0.50168	0.50106	0.50197	0.50000	0.50048
	MSE	0.04910	0.02488	0.01275	0.00499	0.00249
ENG	EXP	0.56807	0.53733	0.52092	0.50784	0.50444
	MSE	0.03873	0.02195	0.01200	0.00486	0.00247
Best Estimator		QG	QG	QG	QG	QG

Table (2): Estimated Value and MSE of Different Estimates of the Scale parameter of the Laplace Distribution when $\theta = 0.5$ and $b = 2$

Estimator	$\frac{n}{\text{Criteria}}$	5	10	20	50	100
QJ	EXP	0.41807	0.45551	0.47807	0.49019	0.49553
	MSE	0.04081	0.02254	0.01205	0.00490	0.00247
QG	EXP	0.64406	0.58422	0.54725	0.51923	0.51028
	MSE	0.04580	0.02437	0.01277	0.00499	0.00250
SLJ	EXP	0.55630	0.52721	0.51482	0.50514	0.50318
	MSE	0.06354	0.02828	0.01363	0.00512	0.00253
SLG	EXP	0.81860	0.66745	0.58728	0.53473	0.51807
	MSE	0.14198	0.05059	0.01975	0.00610	0.00280
ENJ	EXP	0.50168	0.50106	0.50197	0.50000	0.50048
	MSE	0.04910	0.02488	0.01275	0.00499	0.00249
ENG	EXP	0.75140	0.63733	0.57331	0.52941	0.51533
	MSE	0.09730	0.03942	0.01694	0.00567	0.00268
Best Estimator		QJ	QJ	QJ	QJ	QJ

Table (3): Estimated Value and MSE of Different Estimates of the Scale parameter of the Laplace Distribution when $\theta = 0.5$ and $b = 5$

Estimator	$\frac{n}{\text{Criteria}}$	5	10	20	50	100
QJ	EXP	0.418068	0.455509	0.478068	0.490193	0.495528
	MSE	0.040811	0.022541	0.012047	0.004897	0.002465
QG	EXP	1.072630	0.834215	0.683610	0.576920	0.539689
	MSE	0.352956	0.128978	0.044251	0.010535	0.003973
SLJ	EXP	0.556297	0.527207	0.514822	0.505137	0.503179
	MSE	0.063544	0.028284	0.013632	0.005124	0.002532
SLG	EXP	1.363324	0.953068	0.733613	0.594149	0.547929
	MSE	0.785799	0.227820	0.066711	0.013762	0.004769
ENJ	EXP	0.501681	0.501060	0.501971	0.499997	0.500484
	MSE	0.049104	0.024880	0.012755	0.004995	0.002495
ENG	EXP	1.251401	0.910055	0.716162	0.588232	0.545034
	MSE	0.598702	0.188706	0.058292	0.012586	0.004473

Best Estimator	QJ	QJ	QJ	QJ	QJ
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Table (4): Estimated Value and MSE of Different Estimates of the Scale parameter of the Laplace Distribution when $\theta = 3$ and $b = 0.9$

Estimator	$\begin{matrix} n \\ \text{Criteria} \end{matrix}$	5	10	20	50	100
QJ	EXP	2.50841	2.73305	2.868405	2.94116	2.97318
	MSE	1.46920	0.81146	0.433674	0.17629	0.08876
QG	EXP	2.27863	2.58029	2.778936	2.90191	2.95284
	MSE	1.42223	0.79812	0.428237	0.17587	0.08854
SLJ	EXP	3.33779	3.16324	3.088928	3.03082	3.01908
	MSE	2.28757	1.01821	0.490744	0.18447	0.09114
SLG	EXP	2.89616	2.94792	2.982201	2.98857	2.99793
	MSE	1.46772	0.81454	0.437212	0.17645	0.08898
ENJ	EXP	3.01009	3.00636	3.01183	2.99998	3.00290
	MSE	1.76775	0.89569	0.459174	0.17981	0.08981
ENG	EXP	2.65840	2.81487	2.911269	2.95881	2.98209
	MSE	1.34422	0.77448	0.42423	0.17452	0.08836
Best Estimator		ENG	ENG	ENG	ENG	ENG

Table (5): Estimated Value and MSE of Different Estimates of the Scale parameter of the Laplace Distribution when $\theta = 3$ and $b = 2$

Estimator	$\begin{matrix} n \\ \text{Criteria} \end{matrix}$	5	10	20	50	100
QJ	EXP	2.50841	2.73305	2.86841	2.94116	2.97318
	MSE	1.46920	0.81146	0.43367	0.17629	0.08876
QG	EXP	2.43578	2.67197	2.82893	2.92305	2.97343
	MSE	1.22021	0.72958	0.40863	0.17216	0.08702
SLJ	EXP	3.33779	3.16324	3.08893	3.03082	3.01908
	MSE	2.28757	1.01821	0.49074	0.18447	0.09114
SLG	EXP	3.09590	3.05264	3.03585	3.01035	3.01882
	MSE	1.46613	0.81460	0.43818	0.17643	0.08933
ENJ	EXP	3.01009	3.00636	3.01183	2.99998	3.00290
	MSE	1.76775	0.89569	0.45917	0.17981	0.08981
ENG	EXP	2.84174	2.91487	2.96364	2.98038	3.00287
	MSE	1.25258	0.74745	0.41768	0.17321	0.08804
Best Estimator		QG	QG	QG	QG	QG

Table (6): Estimated Value and MSE of Different Estimates of the Scale parameter of the Laplace Distribution when $\theta = 3$ and $b = 5$

Estimator	$\begin{matrix} n \\ \text{Criteria} \end{matrix}$	5	10	20	50	100
QJ	EXP	2.50841	2.73305	2.86841	2.94116	2.97318
	MSE	1.46920	0.81146	0.43367	0.17629	0.08876
QG	EXP	2.86435	2.92197	2.96530	2.98075	2.99304
	MSE	0.92027	0.62807	0.38057	0.16661	0.08637
SLJ	EXP	3.33779	3.16324	3.08893	3.03082	3.01908
	MSE	2.28757	1.01821	0.49074	0.18447	0.09114
SLG	EXP	3.64062	3.33826	3.18220	3.06977	3.03874
	MSE	1.86733	0.92625	0.47009	0.18119	0.09047
ENJ	EXP	3.01009	3.00636	3.01183	2.99998	3.00290
	MSE	1.76775	0.89569	0.45917	0.17981	0.08981
ENG	EXP	3.34174	3.18760	3.10651	3.03920	3.02267
	MSE	1.34432	0.77540	0.42770	0.17436	0.08855
Best Estimator		QG	QG	QG	QG	QG