

Analysis of effects of an Earthquake using Differential Equations

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Abstract: Mechanical and electrical vibrations are an important component of ordinary differential equations (ODE's) because they exemplify mathematical models of the real world. The differential equations are simple, yet the applications are extremely important to physics and engineering. Earthquakes cause great destruction and result in the loss of human life and property. Thus understanding more about earthquakes and simulating them to understand them better is essential in today's world. This paper is a modest beginning to study the effects of horizontal seismic motion on buildings and simulate the motion of a tall building. The software Matlab is used for this purpose. It is meant to provide an insight into the different types of mechanical vibrations that affect a building and the response on the building when an earthquake strikes it.

Keywords: Mathematical model of buildings; Earthquakes; Newton's laws

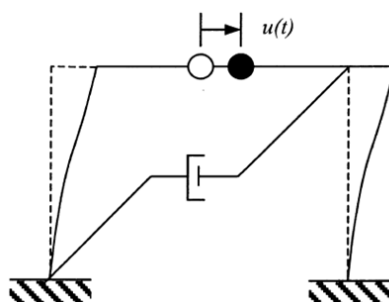
I. Introduction

The goal is to analyze the effects of an earthquake on a building. Although it is impossible to completely protect a building from the effects of an earthquake, it is possible to effect building designs which minimize the damage that a building is likely to sustain. In this paper, the lateral motion of a building is modeled, and the cause of excessive displacements of a given floor, leading to significant damage of a building is examined.

As an introduction, the vibrations of a single story building are modeled. After this, the behavior of multi-story buildings was considered. In either case, catastrophe strikes when a floor of the building experiences excessive lateral deflection, causing permanent damage to the structure. It is too difficult to model the fine details of the motion of a building. Instead, buildings as idealized structures consisting of relatively heavy, inextensional floors and light, elastic walls are modeled.

Considering each floor of the building as a point mass located at the center of mass of the floor, the equations of motion for each floor of the building are formulated. The walls provide elastic forces which act opposite to the direction of motion when each floor is displaced from its equilibrium position.

The model of a single story building can be represented as:



The first task is to derive the equation of motion for a single-story building. Then both free and damped vibrations of the building are examined. Next, the effects of a sinusoidal forcing function are considered in both the absence and presence of damping. With an understanding of the basic behavior of a single-story building, multi-story buildings are analyzed. Both forced and unforced vibrations are considered. With this understanding, we proceed to model the behavior of an idealized tall building undergoing horizontal seismic motion, using the software Matlab.

II. Mathematical formulations and solutions.

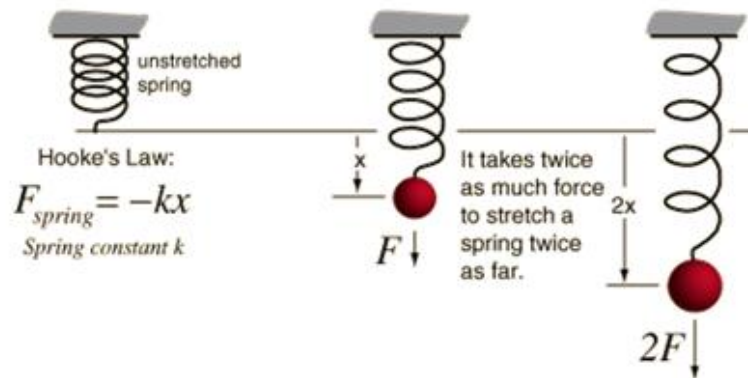
We begin by first stating the two important laws that we have used in this paper:

1. Hooke's Law

Hooke's law is a principle of physics that states that the force needed to extend or compress a spring by some distance is proportional to that distance. That is,

$$F = kx,$$

Where k is a constant factor characteristic of the spring, its stiffness.



2. Newton's second law of motion

The acceleration a of a body is parallel and directly proportional to the net force F acting on the body, is in the direction of the net force, and is inversely proportional to the mass m of the body, i.e., $F = ma$.

Another important concept we learned is called the single degree of freedom. The deformed shape of the system at any instant can be represented in terms of a single dynamic coordinate $u(t)$, called the single degree of freedom. A car with mass m connected to a fixed end with a spring with stiffness k , which is free to move on rollers only in the lateral direction is a typical SDOF system. An inverted pendulum type of structure where the concentrated mass m is connected to the fixed base with a massless cantilever column is also an ideal SDOF system.

With this, we began our analysis of an idealized one-story building as an ordinary spring-mass-damper system. Ignoring damping effects, standard arguments yield the following initial value problem (IVP) for the displacement of the center of mass of the roof:

$$mu'' + ku = 0, \quad u(0) = u_0, \quad u'(0) = u_1$$

The constants m and k represent the total mass of the roof and the overall stiffness of the walls, respectively. In the present setting, $k = EI/h^3$, where E is the elastic modulus of the supporting columns, I is the moment of inertia of the columns about the bending axis, and h is the length of the columns.

By solving the above equation, the natural frequency of oscillation is square root (k/m). The earthquake force E is given by Newton's second law: $E(t) = -mF''(t)$. If $\omega = \text{square root}(k/m)$, then the amplitude of $x(t)$ is large compared to the amplitude of the force E . The amplitude increase in $x(t)$ means that a small-amplitude earthquake wave can resonant with the building and possibly demolish the structure.

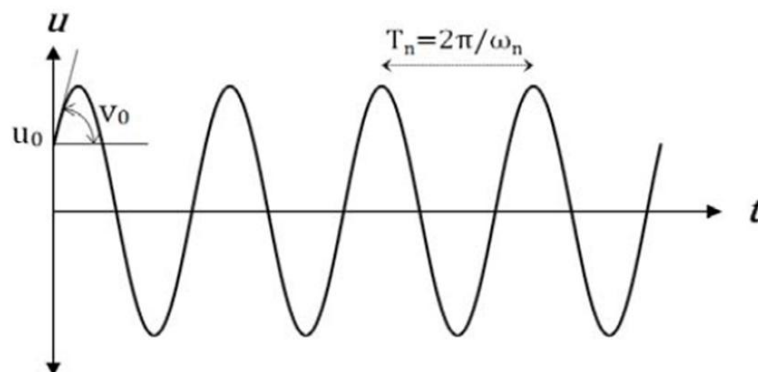
The solution is given by:

$$u(t) = A \sin \omega_n t + B \cos \omega_n t$$

A and B are determined by introducing the initial conditions $u(0) = u_0$ and $u'(0) = v_0$, leading to

$$u(t) = u_0 \cos \omega_n t + \frac{v_0}{\omega_n} \sin \omega_n t$$

This solution can be graphically represented as:



With this basic understanding of the movement of a 1 storeyed building, we now consider the effects of free vibration responses. The equation of free vibration is given as:

$$m\ddot{u} + c\dot{u} + ku = 0$$

Dividing all terms by the mass m gives

$$\ddot{u} + \frac{c}{m} \dot{u} + \frac{k}{m} u = 0$$

Let

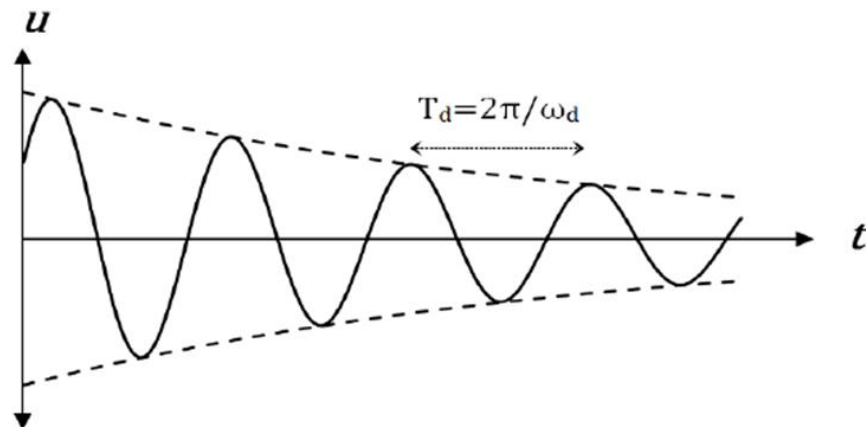
$$\frac{c}{m} = 2\xi\omega_n$$

Thus,

$$\ddot{u} + \xi\omega_n\dot{u} + \omega_n^2 u = 0$$

The dimensionless parameter ξ is the critical damping ratio, and ω_n is the natural frequency (rad/s). Vibration occurs only if $\xi < 1$. The presence of damping in free vibration imposes a decaying envelope on the undamped free vibration cycles presented above. Decay is exponential, and decay rate depends on ω_n . The solution is given by:

$$u(t) = e^{-\xi\omega_n t} \left[u_0 \cos \omega_d t + \frac{v_0 + u_0 \xi \omega_n}{\omega_d} \sin \omega_d t \right]$$



Now we have learnt about the equations of motion and their responses without harmonic oscillation. Harmonic excitation can either be applied as an external harmonic force, or an effective harmonic force due to a harmonic base excitation. Equation of motion under harmonic excitation can then be written as

$$m\ddot{u} + c\dot{u} + ku = F_0 \sin \bar{\omega} t \equiv -ma_0 \sin \bar{\omega} t$$

Now let us consider a 5 storeyed building. A horizontal earthquake oscillation $F(t) = F_0 \sin \omega t$ affects each floor of a 5-floor building. The effect of the earthquake depends upon the natural frequencies of oscillation of the floors.

The following assumptions and symbols are used to quantize the oscillation of the 5-floor building.

- Each floor is considered a point mass located at its center-of-mass. The floors have masses $m_1, \dots, m_5$.
- Each floor is restored to its equilibrium position by a linear restoring force or Hooke's force $-k(\text{elongation})$.

The Hooke's constants are $k_1, \dots, k_5$.

- The locations of masses representing the 5 floors are $x_1, \dots, x_5$. The equilibrium position is $x_1 = \dots = x_5 = 0$.

- Damping effects of the floors are ignored. This is a frictionless system. The differential equations for the model are obtained by competition: the Newton's second law force is set equal to the sum of the Hooke's forces and the external force due to the earthquake wave. This results in the following system, where $k_6 = 0$, $E_j = m_j F''$ for $j = 1, 2, 3, 4, 5$ and $F = F_0 \cos \omega t$.

$$\begin{aligned}m_1 x_1'' &= -(k_1 + k_2)x_1 + k_2 x_2 + E_1, \\m_2 x_2'' &= k_2 x_1 - (k_2 + k_3)x_2 + k_3 x_3 + E_2, \\m_3 x_3'' &= k_3 x_2 - (k_3 + k_4)x_3 + k_4 x_4 + E_3, \\m_4 x_4'' &= k_4 x_3 - (k_4 + k_5)x_4 + k_5 x_5 + E_4, \\m_5 x_5'' &= k_5 x_4 - (k_5 + k_6)x_5 + E_5.\end{aligned}$$

In particular, the equations for a floor depend only upon the neighboring floors. The bottom floor and the top floor are exceptions: they have just one neighboring floor. Using matrices the second order differential equations can be represented as:

$$M = \begin{pmatrix} m_1 & 0 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 & 0 \\ 0 & 0 & m_3 & 0 & 0 \\ 0 & 0 & 0 & m_4 & 0 \\ 0 & 0 & 0 & 0 & m_5 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}, \quad \mathbf{H} = \begin{pmatrix} E_1 \\ E_2 \\ E_3 \\ E_4 \\ E_5 \end{pmatrix},$$

$$K = \begin{pmatrix} -k_1 - k_2 & k_2 & 0 & 0 & 0 \\ k_2 & -k_2 - k_3 & k_3 & 0 & 0 \\ 0 & k_3 & -k_3 - k_4 & k_4 & 0 \\ 0 & 0 & k_4 & -k_4 - k_5 & k_5 \\ 0 & 0 & 0 & k_5 & -k_5 - k_6 \end{pmatrix}$$

In the last row, $k_6 = 0$, to reflect the absence of a floor above the fifth. The second order system is

$$M \mathbf{x}''(t) = K \mathbf{x}(t) + \mathbf{H}(t)$$

The matrix M is called the mass matrix and the matrix K is called the Hooke's matrix. The external force $\mathbf{H}(t)$ can be written as a scalar function $E(t) = -F''(t)$ times a constant vector:

$$\mathbf{H}(t) = -\omega^2 F_0 \cos \omega t \begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ m_5 \end{pmatrix}$$

$$K = \begin{pmatrix} -k_1 - k_2 & k_2 & 0 & 0 & 0 \\ k_2 & -k_2 - k_3 & k_3 & 0 & 0 \\ 0 & k_3 & -k_3 - k_4 & k_4 & 0 \\ 0 & 0 & k_4 & -k_4 - k_5 & k_5 \\ 0 & 0 & 0 & k_5 & -k_5 - k_6 \end{pmatrix}$$

Now, let us assume that all floors have the same mass m and the same Hooke's constant k . Then $M = mI$ and the equation becomes^[3]:

$$\mathbf{x}'' = m^{-1} \begin{pmatrix} -2k & k & 0 & 0 & 0 \\ k & -2k & k & 0 & 0 \\ 0 & k & -2k & k & 0 \\ 0 & 0 & k & -2k & k \\ 0 & 0 & 0 & k & -k \end{pmatrix} \mathbf{x} - F_0 \omega^2 \cos(\omega t) \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

III. Discussion

Let $k/m = 10$.

$$\frac{1}{m}K = \begin{pmatrix} -20 & 10 & 0 & 0 & 0 \\ 10 & -20 & 10 & 0 & 0 \\ 0 & 10 & -20 & 10 & 0 \\ 0 & 0 & 10 & -20 & 10 \\ 0 & 0 & 0 & 10 & -10 \end{pmatrix}$$

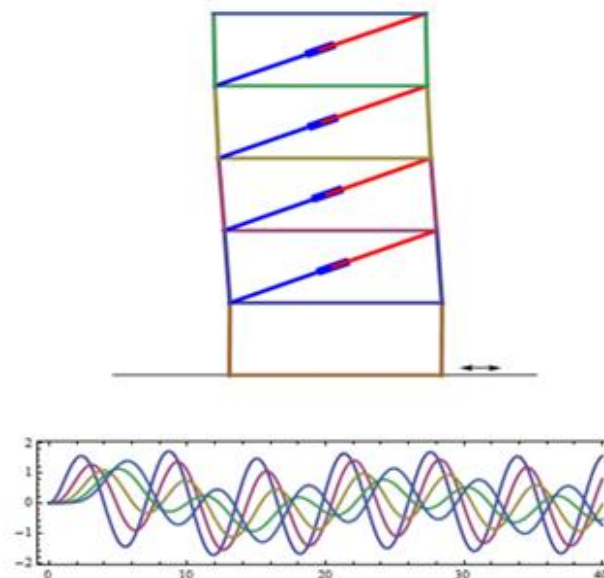
And the values $\omega_1, \dots, \omega_5$ are found by solving the determinant equation $\det((1/m)K + \omega^2 I) = 0$, to obtain the values in the Table.

Frequency	Value
ω_1	0.900078068
ω_2	2.627315231
ω_3	4.141702938
ω_4	5.320554507
ω_5	6.068366391

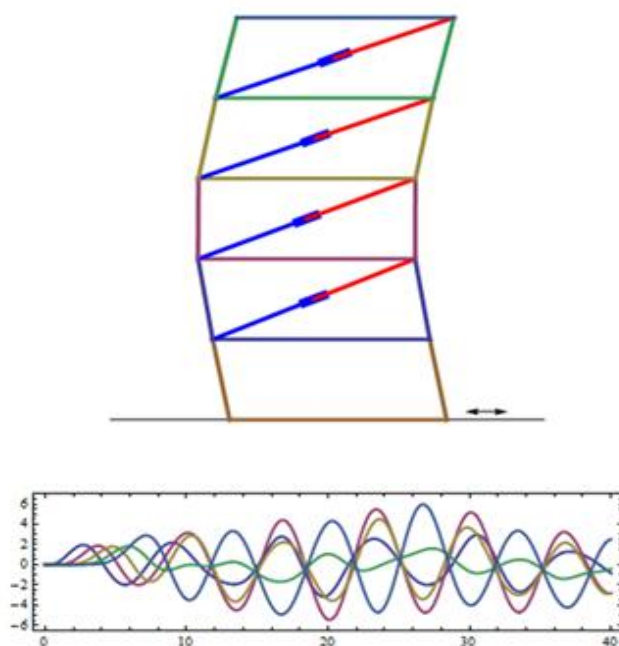
When ω is close to one of the frequencies $\omega_1, \dots, \omega_5$, then the amplitude of a component of $\mathbf{x}$ can be very large, causing the floor to take an excursion that is too large to maintain the structural integrity of the floor. The physical interpretation is that an earthquake wave of the proper frequency, having a time duration sufficiently long, can demolish a floor and hence demolish the entire building. The amplitude of the earthquake wave does not have to be large: a fraction of a centimeter might be enough to start the oscillation of the floors.

IV. Simulations

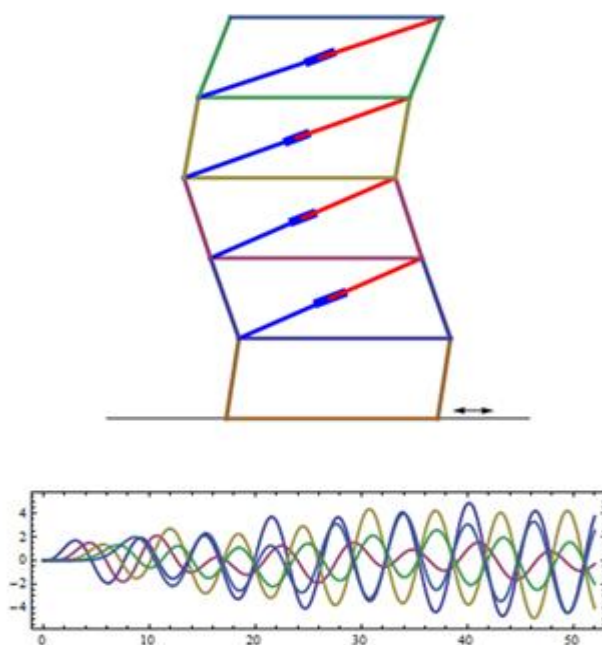
The output obtained for a high damping factor, gamma is:



The diagonal lines represent shock absorbers that introduce a damping factor γ
The output obtained for a low gamma value, that is, low damping factor, is:



The output obtained for a high mass value is as follows:



V. Practical Application

A building which is made of a material with a large damping coefficient is highly resistant to earthquakes, even when all the floors have large masses. Typical damping ratios of common materials are as follows:

Structural type	Damping ratio
Steel	0.02-0.03
Reinforced concrete	0.05-0.08
Masonry	0.10-0.20

VI. Conclusions

Through the research, it is concluded that earthquakes typically shake the ground with primary periods between 0.5 and 3 seconds. Thus we have to make sure that the natural periods of our buildings are outside this range. An earthquake wave of the proper frequency, having the time duration sufficiently long, can demolish the ground floor and hence demolish the entire building. The amplitude of the earthquake wave does not have to be large: a fraction of a centimeter might be enough to start the oscillation of the floors.

VII. References

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