



An Analytical Study of Weakly Nonlinear Convection in a Horizontal Mushy Layer

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Abstract: *The present analytical study deals with the problem of weakly nonlinear convective instability in a horizontal mushy layer under suitable assumptions and approximations. By applying a modified perturbation technique, the basic state solutions, solutions to the first and the second orders are determined. The influence of the governing parameters on the profiles of total Rayleigh number, velocity and local solid-fraction are studied. The computations are done for a feasible set of values of the parameters. An excellent correlation is obtained between the present and the available results for the limiting cases.*

Keywords: *Mushy layer, Convection, Chimney formation, local solid fraction, solvability condition.*

I. Introduction

During the solidification of a binary or multicomponent melt, a region called ‘mush layer’ in which the solid and the liquid phases co-exist, will be formed due to the morphological instability of a solid-liquid interface[1]. Thus a mushy layer can be defined as a reactive porous two – phase medium that comprises the solid-matrix as well as the residual liquid. The understanding of the complex interactions between the flow of melt and the solidification is interesting and is of great importance since the transfer of heat and mass associated with the fluid flow has a profound influence on the process of solidification and becomes responsible for the chimney formation which causes imperfections in the resulting solid. The interesting phenomenon that occurs within the solidifying melt is the formation of chimneys which are narrow dendrite-free cylindrical regions or narrow cylindrical regions of zero solid fraction and are very much similar to the imperfections called ‘freckles’ that appear in the casting of metallic alloys [2][3].

The formation of chimneys during the solidification of a binary or a multicomponent alloy constitutes three stages viz., finger, plume and chimney convections[4][5]. Actually, during the solidification process the solidification front or the interface between the solid and the liquid becomes highly dendritic due to the morphological instability. As a consequence there will be a formation of a region called ‘mushy layer’ consisting of a partially solidified melt, the dendritic structure of which is quite complex [1]. Then the system becomes unstable due to the density gradient that results from the rejected materials and there will be a transition to convection. Further, as a result of the interaction of the thermal fields and the generated convective motions, chimneys which are responsible for the imperfections in the resulting solid will be formed [2][6][7][8][9]. The review article by [10] discusses the striking fluid-mechanical events that take place during the solidification process.

Especially in metallurgy, dynamics of sea and geophysics, the mechanism and the process of formation of chimneys which spoil the quality, physical properties and the internal structure of the resulting solid, are important study areas [11][12][13]. In the past three decades the study pertaining to the development of different convective models and analysis for the case of convection in mushy layers has attracted researchers [14][8]. The works connected with the formulation of the governing equations in the study of convection in mush layers, the development of mathematical models and the solution procedure are available [15][16].

The mathematical models describing the characteristics of mushy layer and the related phenomena are exclusively based on the key feature that the length scale of the internal boundaries are extremely small when compared to the macroscopic dimensions of the mushy layer. Linear and weakly nonlinear convective instability in a mushy layer has been studied by quite a number of researchers under different types of assumptions and approximations [17][6][18][19][20][21][22]. Quite a number of works on convective flow in a mushy layer is available. A detailed review on convection in mushy layers is given by [6][22]. Recently [23][24][25] have applied weakly nonlinear evolution approach to study two-dimensional convective motions in a mushy layer with impermeable solidification front under different situations. Finally [26] have studied numerically the effects of inertia on convection in a mushy layer with constant permeability. Analytically [27] have studied the effects of inertia on convection in a passive mushy layer. The main objective behind these studies is to study the history of the solidification process that could yield pure solid and facilitate the suppression of the freckles which have catastrophic effects on the internal structure of the resulting solid. A thorough survey of the

literature pertaining to the subject reveals that no in depth analytical work is available for the case of convection in a mushy layer with and without constraints. Therefore the present analytical work is carried out to study the effects of large Stefan number, Concentration ratio and the permeability function on the total Rayleigh number, velocity and local solid fraction profiles in a mushy layer near eutectic temperature. The boundaries are assumed to be impervious so that the Darcy's equation is valid. The paper is organised as follows: Section 1 deals with a brief introduction to the study together with the literature survey. Sections 2, discusses the mathematical formulation. In sections 3 and 4, the basic state, the linear stability and weakly nonlinear analyses are presented. Finally, section 5, presents the results and discussions of the study.

II. Mathematical formulation

The physical configuration consists of a horizontal mushy layer formed during the solidification of a binary alloy as shown in fig 1. The process of uniform cooling from below of the system results in the upward advancement of the solid – mush and mush – liquid interface with a constant solidification speed V_0 . In other words the mushy layer is sandwiched between the solid and the liquid regions. The study is carried out in a moving frame of reference.

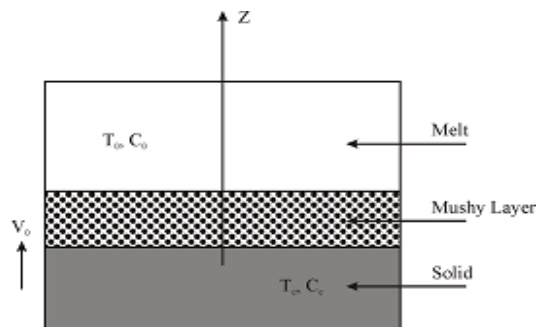


Figure 1. The schematic diagram of the physical system

where the bottom boundary of the mushy layer $z = 0$ is kept at the eutectic temperature $T = T_E$, while the top boundary $z=d$ is kept at the liquidus temperature $T_L(C_0)$ and the mixture of the composition C_0 is supplied through the surface.

Following are the assumptions made for the study:

1. The top and the bottom boundaries of the mushy layer are assumed to be isothermal non-deformable and impermeable to the fluid flow, so that the mushy layer is kept dynamically isolated from the other components of the system [18].
2. The solidification front(the frame of reference) is moving upwards with a velocity V_0 relative to the solid formed and the solid dendrites within the mushy layer. This makes the basic state to be steady.
3. The temperature T and the composition C of the liquid in the mushy layer are required to satisfy a linear liquidus relationship $T = T_L(C) = T_L(C_0) + \Gamma (C - C_0)$, Where Γ is a constant. The liquid is assumed to be Newtonian with a linearized equation of state $\rho = \rho_0 [1 + \beta(C - C_0)]$ where ρ_L is the density of the liquid, ρ_0 is a reference density, $\beta = \beta^* - \alpha^* \Gamma$, α^* and β^* are constant expansion coefficients for heat and solute respectively.
4. First following [18], we study a limit in which the thickness of the mushy layer is much less than the diffusion length scale by letting $\delta \leq 1$.

5. Sec
only we assume that the compositional ratio is large by writing $C_R = \frac{C_R}{\delta}$ with $C_R = O(1)$ as $\delta \rightarrow 0$ which corresponds to the near eutectic approximation introduced by [28].

6. we consider the limit in which the Stefan number is large[29] by taking $S = S/\delta$ with $S = O(1)$ as $\delta \rightarrow 0$ which corresponds to the situation in which the latent heat liberated during the local phase change is much larger than the heat associated with the typical variations of temperature across the mushy layer. Note that the particular scaling allows the destabilisation of the system to an oscillatory mode of convection [20][29].

7. However, that a key implication of the near-eutectic approximation($C=O(\delta^{-1})$) is that the solid fraction is small, and hence the permeability is uniform to the lowest order. As a consequence, we follow [18] and expand the permeability in terms of the small solid fraction Φ ,

$$K(\Phi) = 1 + K_1\Phi + K_2\Phi^2 + \dots \quad (1)$$

where, on physical grounds, we demand that K_1, K_2 , etc. are non-negative.

Under the above assumptions and approximations the governing equations of the systems are Conservation of momentum, Conservation of mass, Conservation of heat and solute

$$\frac{\mu}{\pi} \vec{q} = -\nabla P - (\rho - \rho_0)g \vec{k} \quad (2)$$

$$\nabla \cdot \vec{q} = 0 \quad (3)$$

$$\frac{\partial T}{\partial t} + \vec{q} \cdot \nabla T = k \nabla^2 T + \frac{l_h}{\gamma} \frac{\partial \Phi}{\partial t} \quad (4)$$

$$\chi \cdot \frac{\partial C}{\partial t} + \vec{q} \cdot \nabla C = (C - C_0) \frac{\partial \Phi}{\partial t} \quad (5)$$

The above set of equations are cast in the dimensionless perturbed form by using the following scale viz., $\frac{V_0}{\pi_0}$, ΔT , $\frac{k}{V_0^2}$, $\frac{\kappa \mu}{\pi_0}$, $\frac{k}{V_0}$ for the variables velocity, temperature, time, pressure and length respectively :

$$(\partial_t - \partial_z)(\theta - S\Phi) + \vec{q} \cdot \nabla \theta = \nabla^2 \theta, \quad (6)$$

$$(\partial_t - \partial_z)((1 - \Phi)\theta + C\Phi) + \vec{q} \cdot \nabla \theta = 0, \quad (7)$$

$$K(\Phi) \vec{q} = -\nabla p - Ra \theta \vec{k}, \quad (8)$$

$$\nabla \cdot \vec{q} = 0. \quad (9)$$

Equation (8) in the component form is given by

$$K(\Phi)u = -\frac{\partial p}{\partial x} \quad (10a)$$

$$K(\Phi)v = -\frac{\partial p}{\partial y} \quad (10b)$$

$$K(\Phi)w = -\frac{\partial p}{\partial z} - R \theta \quad (10c)$$

Next, we eliminate the pressure in the above set of equations by applying the following transformation:

To start with, the application of the transformation

$$\frac{\partial}{\partial z} \left[\frac{\partial(10a)}{\partial x} + \frac{\partial(10b)}{\partial y} \right] - \nabla_1^2(10c) \quad (11)$$

yields

$$\nabla^2(Kw) - \frac{\partial}{\partial z}(u \cdot \nabla K) = -R \nabla_1^2 \theta \quad (11a)$$

Next by applying the transformation,

$$\frac{\partial}{\partial x} \left[\frac{\partial(10b)}{\partial y} + \frac{\partial(10c)}{\partial z} \right] - \nabla_1^2(10a) \quad (12)$$

we get

$$\nabla^2(Ku) - \frac{\partial}{\partial x}(u \cdot \nabla K) = +R \frac{\partial}{\partial x} \frac{\partial}{\partial z} \theta \quad (12b)$$

Finally by applying the transformation,

$$\frac{\partial}{\partial y} \left[\frac{\partial(10c)}{\partial z} + \frac{\partial(10a)}{\partial x} \right] - \nabla_1^2(10b) \quad (13)$$

we get,

$$\nabla^2(Kv) - \frac{\partial}{\partial y}(u \cdot \nabla K) = R \frac{\partial}{\partial y} \frac{\partial}{\partial z} \theta \quad (13a)$$

The associated boundary conditions are

$$\theta = -1, w = 0 \text{ @ } z = 0,$$

$$\theta = 0, w = 0, \Phi = 0 \text{ @ } z = \delta. \quad (14)$$

The above boundary conditions correspond to impermeable rigid boundaries of the mushy layer. The lower plate, between the solid and the mush, is maintained at the eutectic temperature T_E , while the upper boundary between the liquid and the mush (that is, at zero solid fraction Φ), is maintained at the far-field liquidus temperature $T_L(C_0)$. The porous medium is such that Darcy's law holds good.

Dimensionless parameters

The function $K(\Phi)$ appearing in equation (8) measures the variation of permeability with respect to some zero solid-fraction permeability $\pi(0)$, assumed to be finite, such that

$$K(\Phi) = \frac{\pi(0)}{\pi(\Phi)} \quad (15)$$

The dimensionless parameters appearing in (6-8) are

$$\text{the Stefan number } S = \frac{L}{C(T_L(C_0) - T_E)}$$

$$\text{the concentration ratio } C = \frac{C_S - C_0}{C_0 - C_E} \text{ and}$$

$$\text{the Rayleigh number } R = \frac{\beta \Delta C \pi(0) g}{\mu V_0} \text{ (Rayleigh number)} \quad (16)$$

where

$\Phi = 1 - \chi$ Local solid fraction, χ – Local liquid fraction, P – Dynamic pressure, $\pi = \pi(\Phi)$, permeability is a function of the local solid fraction, μ : Dynamic viscosity, $t, T, \kappa, l_h, \gamma$ are time, temperature, thermal diffusivity, specific heat, latent heat/unit mass, C_s : Composition of the solid phase, C_0 : Composition of the liquid phase, ρ, ρ_0 : densities, $\vec{g} = (0, 0, g)$ acceleration due to gravity, $\pi(0)$: The reference permeability, $\vec{q} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$, $\vec{q}$ is the Darcy velocity vector and (u, v, w) are the horizontal and vertical components of $\vec{q}$, $\mathbf{i}, \mathbf{j}, \mathbf{k}$: unit vectors along the x, y and z axes. d : mushy layer thickness, β : Expansion coefficient, T_∞ : Far-field temperature, C_E : Eutectic composition. A fifth dimensionless parameter, appearing in the study is the dimensionless mushy layer thickness $\delta = d/(\kappa V_0)$, appears in the boundary conditions. Before analysing the linear stability of the system the basic state analysis is carried out as follows:

III. Basic state Analysis

The steady motionless basic state system is considered here where each of the corresponding dependent variables is designated by a subscript “B”. The basic state variables are assumed to be functions of z only.

$$\begin{aligned}\theta &= \theta_B(z) + \varepsilon \hat{\theta}(x, y, z, t) \\ \Phi &= \Phi_B(z) + \varepsilon \hat{\Phi}(x, y, z, t) \\ \vec{q} &= 0 + \varepsilon \vec{\hat{q}}(x, y, z, t) \\ P &= P_B(z) + \varepsilon P(x, y, z, t)\end{aligned}\quad (17)$$

where the perturbation parameter $\varepsilon \ll 1$ and the perturbed quantities can vary with respect to spatial and time variables. Using (17) in (18) we rescale the variables as

$$(\partial_t - \delta \partial_z)(\theta - \frac{\bar{S}}{\delta} \Phi) + \delta(\vec{q} \cdot \nabla)\theta = \nabla^2 \theta, \quad (18)$$

$$(\partial_t - \delta \partial_z)((1 - \Phi)\theta + \frac{C}{\delta} \Phi) + \delta(\vec{q} \cdot \nabla)\theta = 0 \quad (19)$$

$$\Delta K(\Phi) \vec{q} = -\nabla p - Ra \theta \hat{k}, \quad (20)$$

$$\nabla \cdot \vec{q} = 0. \quad (21)$$

Equations (18 -20) allow a motionless steady basic state solution, depending only on the vertical position. By setting $\vec{q} = 0$ and $\partial_t = 0$, we get

$$-\delta D \theta_B + \bar{S} D \Phi_B = \nabla^2 \theta_B, \quad (22)$$

$$-\delta D \theta_B + \delta D(\theta_B \Phi_B) - \bar{C} D \theta_B \quad (23)$$

$$0 = -D P_B - R \theta_B \quad (24)$$

From (22 -24), we have the following equations:

$$\delta D \theta_B (\Phi_B - 1) + \delta D \Phi_B (\theta_B - \frac{C}{\delta}) = 0 \quad (25)$$

$$\delta (1 - \Phi_B) D \theta_B + D \Phi_B (\bar{C} - \delta \theta_B) = 0 \quad (26)$$

$$D^2 \theta_B + \delta D \theta_B - S D \Phi_B = 0 \quad (27)$$

$$D P_B + R \theta_B = 0 \quad (28)$$

Boundary conditions are $\theta_B = -1$ @ $z = 0$, $\theta_B = 0$, $\Phi_B = 0$ @ $z = 1$

On multiplying (27) by $(\bar{C} - \delta \theta_B)$ and (26) by $\bar{S}$ and adding, we get

$$(\bar{C} - \delta \theta_B)(D^2 \theta_B + \delta D \theta_B - \bar{S} D \Phi_B) + \bar{S}(\bar{C}(1 - \Phi_B) D \theta_B + (\bar{C} - \delta \theta_B) D \Phi_B) = 0 \quad (29)$$

Equations (26) - (28) are solved asymptotically by applying the following expansions

$$\begin{aligned}\theta_B &= \theta_{B0} + \delta \theta_{B1} + \dots \\ \Phi_B &= \delta \Phi_{B0} + \delta^2 \Phi_{B1} + \dots\end{aligned} \quad (30)$$

On substituting (30) in (29) and on equating the like powers of δ^0 , we get

$$D^2 \theta_{B0} = 0 \quad (31)$$

The solution of (31) is given by applying the boundary conditions

$$\theta_{B0} = -1 \text{ @ } z = 0 \text{ and } \theta_{B0} = 0 \text{ @ } z = 1 \quad (31a)$$

$$\theta_{B0} = z - 1 \quad (32)$$

Now, on equating the like powers of δ

$$\bar{C} D^2 \theta_{B1} - \theta_{B0} D^2 \theta_{B0} + (\bar{C} + \bar{S}) D \theta_{B0} = 0$$

$$\bar{C} D^2 \Theta_{B1} + (\bar{C} + \bar{S}) = 0 \text{ (since } D^2 \Theta_{B0} = 0).$$

The solution is given by

$$\Theta_{B1} = -\frac{\Omega}{2} (z^2 - z) \Rightarrow D \Theta_{B1} = \frac{\Omega}{2} (2z - 1) \Rightarrow D^2 \Theta_{B1} = \Omega \quad (33)$$

On equating the like powers of δ^2 :

$$\bar{C} D^2 \Theta_{B2} - \Theta_{B0} D^2 \Theta_{B1} - \Theta_{B1} D^2 \Theta_{B0} + (\bar{C} + \bar{S}) D \Theta_{B1} - [\Theta_{B0} + \bar{S} \Phi_{B0}] D \Theta_{B0} = 0 \quad (34)$$

Substituting, and solving, we get

$$\therefore \Theta_{B2} = \frac{2(\Omega-1)}{c} \left[\frac{z^3}{6} - \frac{z^2}{2} \right] + \frac{\Omega^2}{2} \left[\frac{z^3}{3} - \frac{z^2}{2} \right] + \left[1 - \frac{2(\Omega-1)}{3c} + \frac{\Omega^2}{12} \right] z - 1 \quad (35)$$

on multiplying (27) by $(1 - \Phi_B)$ and on subtracting from (26), we get

$$(1 - \Phi_B) D^2 \Theta_B - [S(1 - \Phi_B) + (\bar{C} - \delta \Theta_B)] D \Phi_B = 0 \quad (36)$$

Collecting the coefficients of δ^0 , and solving we get

$$\Phi_{B0} = \frac{-1}{\bar{C}} [z - 1] \quad (37)$$

Again by equating the like powers of δ^2 : we get,

$$D^2 \Theta_{B2} - \Phi_{B0} D^2 \Theta_{B1} - \Phi_{B1} D^2 \Theta_{B0} - (\bar{C} + \bar{S}) D \Phi_{B1} - [\bar{S} \Phi_{B0} + \bar{C} \Phi_{B0}] D \Phi_{B0} = 0 \quad (38)$$

On solving we get,

$$\Phi_{B1} = -\frac{(z-1)^2}{\bar{C}} + \frac{\Omega}{2c} (z-1)^2 \bar{C} \quad (39)$$

Finally by substituting the above values in (30), we get

$$\Theta_B = (z-1) - \delta \frac{\Omega}{2} (z^2 - z) + \delta^2 \left[\frac{2(\Omega-1)}{c} \left[\frac{z^3}{6} - \frac{z^2}{2} \right] + \frac{\Omega^2}{2} \left[\frac{z^3}{3} - \frac{z^2}{2} \right] + \left[1 - \frac{2(\Omega-1)}{3c} + \frac{\Omega^2}{12} \right] z - 1 \right] + \dots \quad (40) \Phi_B$$

$$= -\delta \frac{1}{\bar{C}} [z - 1] + \delta^2 \left[-\frac{(z-1)^2}{\bar{C}} + \frac{\Omega}{2c} (z-1)^2 \right] + \dots \quad (41)$$

IV. Weakly nonlinear analysis

In this section we first perform the linear stability analysis of the system to find the critical conditions and the growth rate σ and next consider a finite – amplitude perturbation expansion of the equations in order to predict the nonlinear effects on the system. For this purpose we consider the expansions of the perturbation quantities in the form

$$\theta = (\theta_{00} + \delta \theta_{01} + \dots) + \epsilon (\theta_{10} + \delta \theta_{11} + \dots) + \epsilon^2 (\theta_{20} + \delta \theta_{21} + \dots)$$

$$\Omega \Phi = (\Phi_{00} + \delta \Phi_{01} + \dots) + \epsilon (\Phi_{10} + \delta \Phi_{11} + \dots) + \epsilon^2 \left(\frac{1}{\delta} \Phi_{2(-1)} + \delta \Phi_{20} + \dots \right)$$

$$\bar{q}_{\Omega^2} = (\bar{q}_{00} + \delta \bar{q}_{01} + \dots) + \epsilon (\bar{q}_{10} + \delta \bar{q}_{11} + \dots) + \epsilon^2 (\bar{q}_{20} + \delta \bar{q}_{21} + \dots)$$

$$\Omega^{1/2} R = (R_{00} + \delta R_{01} + \dots) + \epsilon (R_{10} + \delta R_{11} + \dots) + \epsilon^2 (R_{20} + \delta R_{21} + \dots) \quad (42)$$

From (42) it is observed that the equation is singular at the order of ϵ^2 as $\delta > 0$. In fact this is the contribution of the forcing term Φ_{00} from the equation for Φ at the order of ϵ^2 . Therefore the balance is made by adding the term $(1/\delta) \Phi_{2(-1)}$ at order of ϵ^2 .

Further, the asymptotic expansion of (42) is meaningful only when $\epsilon^2 \delta$ is very much less than 1. In order to study the stability characteristics of the problem, we substitute (42) into (6), (7), (11a), (12a) and (13a) and collecting the terms of order δ^{-1} and δ^0 , we get the following results:

$$\sigma_{00} = 0 \quad (43)$$

$$S(\sigma_{01} - D) \Phi_{00} - R_{00} w_{00} + \nabla^2 \theta_{00} = 0 \quad (44)$$

$$(\sigma_{01} - D) C \Phi_{00} + R_{00} W_{00} = 0 \quad (45)$$

$$\nabla^2 w_{00} - \alpha^2 R_{00} \theta_{00} = 0 \quad (46)$$

$$\nabla^2 u_{00} - ik R_{00} D \theta_{00} = 0 \quad (47)$$

$$\text{Let } \theta_{00} = -f_k \sin(\pi z) \quad (48)$$

From (43) - (47) and on solving, we get

$$w_{00} = \frac{(\pi^2 + \alpha^2) \sin(\pi z)}{\Omega R_{00}} \quad (49)$$

$$u_{00} = \frac{ik R_{00} \pi \cos(\pi z)}{(\pi^2 + \alpha^2)} \quad (50)$$

$$R_{00} = \frac{(\pi^2 + \alpha^2)}{\alpha \sqrt{\Omega}} \quad (51)$$

$$\Phi_{00} = \frac{-\pi f_k (\pi^2 + \alpha^2)}{\Omega C (\pi^2 + \frac{\sigma^2}{\delta^2})} \left[e^{\frac{\alpha}{\delta}(z-1)} + \frac{\sigma}{\delta \pi} \sin(\pi z) + \cos(\pi z) \right] \quad (52)$$

Next we consider the system of δ^0 :

$$\nabla^2 \theta_{01} - R_{00} (w_{01} + w_{00} D \theta_{01}) - R_{01} w_{00} + D \theta_{00} - S D \theta_{01} = 0 \quad (53)$$

$$D \theta_{01} - \frac{R_{00}}{C} w_{01} - \frac{1}{C} (R_{00} w_{00} D \theta_{01} + R_{01} w_{00}) + \frac{1}{C} D \theta_{00} - D \theta_{00} \Phi_{00} = 0 \quad (54)$$

$$\nabla^2 w_{01} - \alpha^2 R_{00} \theta_{01} - \alpha^2 R_{01} \theta_{00} + K_1 \nabla^2 (w_{00} \Phi_{B0}) = K_1 D(D\Phi_{B0} w_{00}) \quad (55)$$

In (53) to (55), we eliminate w_{01} and Φ_{01} so that

$$[(D^2 - \alpha^2)^2 - \alpha^2 (R_{00})^2 \Omega] \theta_{01} - \alpha^2 R_{00} R_{01} \theta_{00} \cdot \Omega + R_{00} K_1 \Omega (D^2 - \alpha^2)(\Phi_{B0} w_{00}) - K_1 R_{00} \Omega D(D\Phi_{B0} w_{00}) - (D^2 - \alpha^2) \{ R_{00} w_{00} D\theta_{01} \} \Omega - (D^2 - \alpha^2) R_{01} w_{00} \Omega + (D^2 - \alpha^2) D\theta_{00} \Omega - (D^2 - \alpha^2) S D((\theta_{B0} \Phi_{00})) = 0 \quad (56)$$

Multiplying the above equation by θ_{00} and applying the solvability and orthogonality property, we obtain: R_{00}^2

$$= \frac{(\pi^2 + \alpha^2)^2}{\Omega \alpha^2} + \frac{\Omega \alpha^2 R_{01} R_{00}^2}{2(\pi^2 + \alpha^2)} + \frac{R_{01}(\pi^2 + \alpha^2)}{2} - \frac{K_1(\pi^2 + \alpha^2)}{4C} R_{00} + \frac{S\pi}{4} R_{00} \quad (57)$$

where $\Omega = 1 + \frac{S}{C}$

On Substituting for R_{00} and on simplifying we get,

$$R_{01} = \left[\frac{K_1}{4C} - \frac{S\pi}{4(\pi^2 + \alpha^2)} \right] R_{00} \quad (58)$$

Further we have $R = R_{00} + \delta R_{01}$ and

$$R_C = \frac{2\pi}{\sqrt{\Omega}} \left[1 + \frac{\delta}{4} \left(\frac{K_1}{C} - \frac{S}{\pi} \right) \right], \alpha_c = \pi; \quad (59)$$

V. Results and discussion

In the present study, the analytical solutions for the basic state, which is necessary for the determination of the linearized as well as the weakly nonlinear systems are determined by setting $\sigma = 0$. Further, the zeroth order solutions w_{00} , u_{00} , θ_{00} and Φ_{00} are analytically determined for the general case of variable permeability in convection in a horizontal mushy layer. Then considering the asymptotic expansions for the variables in terms of two parameters $\varepsilon \ll 1$ and $\delta < 1$ (the mushy layer thickness, the weakly nonlinear system of order δ^0 , $\delta^{(1)}$, $\delta^{(2)}$) are obtained and R_{01} is determined from the inhomogeneous system of differential equations by applying the solvability conditions. The growth rate σ_{01} is also determined in terms of the governing parameters. The graphs of total $R = R_{00} + \delta R_{01}$, w_{00} (vertical velocity) and Φ_{00} (local solid fraction) are plotted for the experimental set [30] of values viz., $K_1 = 2.5, 3.5, 4.5$; $C = 1.5$; $\Omega = 1.2, 1.35556$ and 1.5 respectively in figs. (2 - 6). In fig. 2, R vs α is plotted, for $K_1 = 2.5, 3.5, 4.5$. It is found that R increases with K_1 i.e., K_1 has a stabilizing influence on the convective system. From figs 3 and 4 it is observed that total R decreases with the increase in the values of Ω and C . In other words, these parameters have a destabilizing influence on the system. In figs 5 and 6, the profiles of Φ_{00} and w_{00} are presented for values of $C = 0.5, 1.2, 4$ and $\Omega = 1.35556, 2.3, 4$ respectively. The input values for z are 0.0 to 1.0. The profiles clearly indicate the reduction in the nonlinearity as the parameters C and Ω decrease i.e., the system exhibits the destabilizing nature as C and Ω increase. From the above results, it is evident that by a suitable choice of the governing parameters K_1 , C and Ω , it is possible to have a good control over the formation of chimneys which are responsible for the formation of 'freckles' causing imperfections in the resulting solid formed during the solidification process. The present analytical results are in excellent agreement with those of [28][20] in the limiting cases.

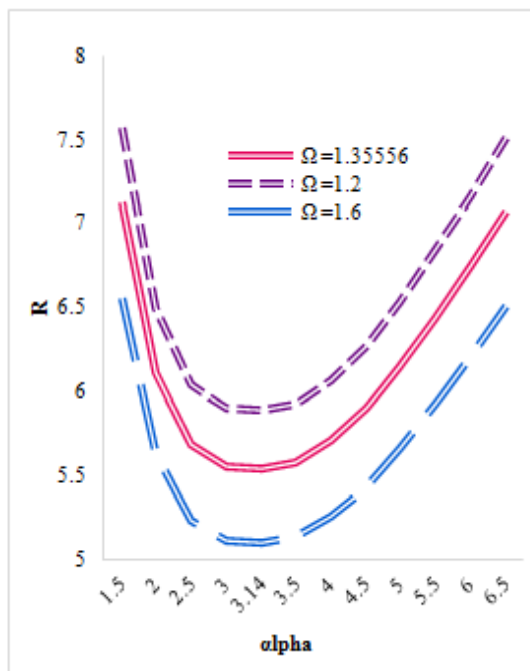


Fig. 2 Total R vs α for $\Omega = 1.2, 1.35556, 1.5$

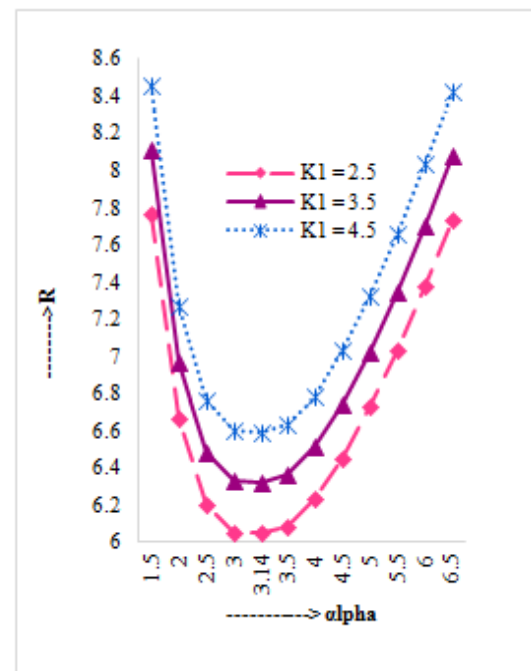


Fig. 3 Total R vs α for $K_1 = 2.5, 3.5, 4.5$

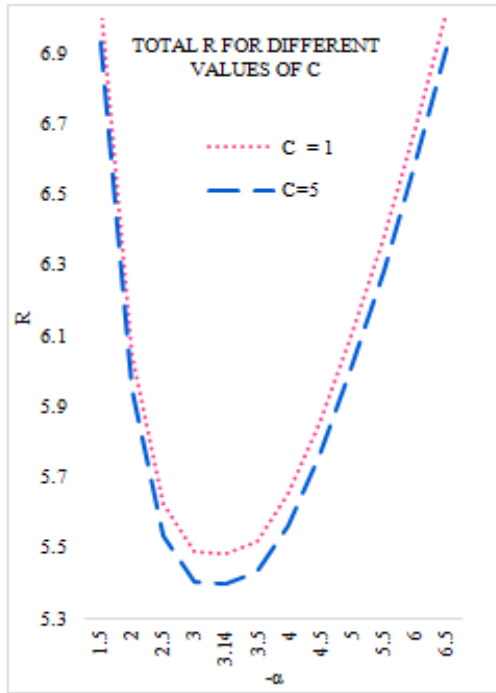


Fig. 4 Total R vs α for C = 1 and 5

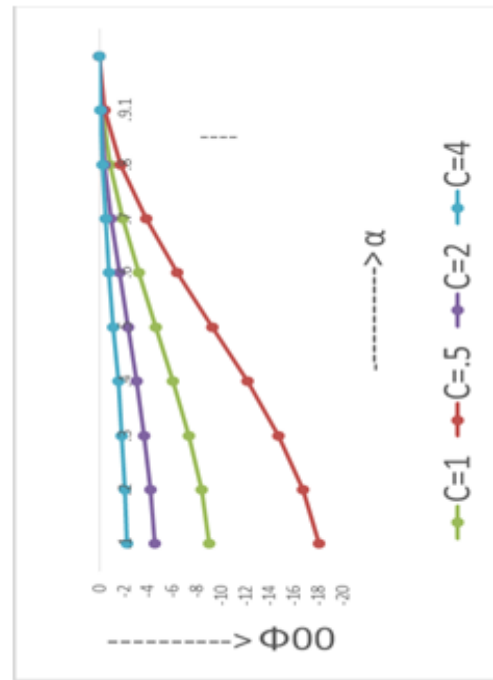


Fig. 5 Φ_{00} vs α for C = 0.5, 1, 2, 4

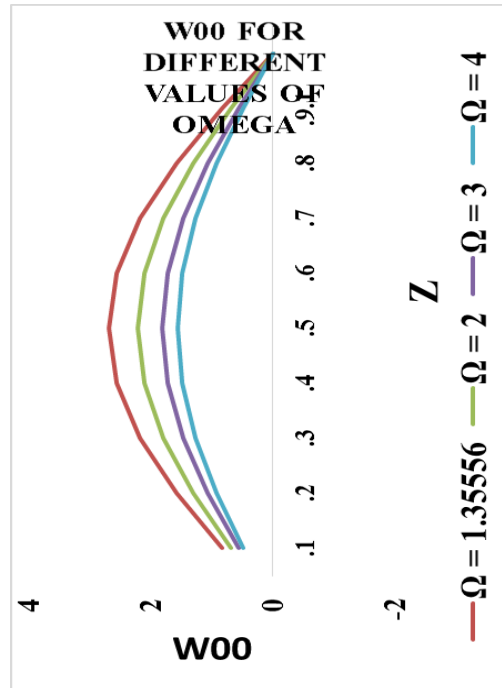


Fig. 6 W_{00} vs α for $\Omega = 1.35556, 2, 3, 4$

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