



Lebesgue Convergence of Modified Complex Trigonometric Sums

Jatinderdeep Kaur, Sandeep Kaur and S.S. Bhatia
School of Mathematics and Computer Applications,
Thapar University, Patiala (Punjab)-147004, INDIA.

Abstract: In this paper, some estimates for Dirichlet kernel has been proved. Also, we have obtained the complex form of modified trigonometric cosine and sine sums of X.Z. Krasniqi [10] and have proved the convergence of complex trigonometric series under class J^* of complex coefficients in metric space L^1 .

Keywords: L^1 -convergence, Dirichlet Kernel, Fejer Kernel.

I. Introduction

It is well known that if a trigonometric series converges in L^1 -norm to a function $\hat{f} \in L^1$, then it is the Fourier series of the function $\hat{f}$. But in 1932, Riesz { [5], Vol. II, Ch.VIII } gave an example $\sum_{n=2}^{\infty} \frac{\cos nx}{\log n}$ which shows

that it is a Fourier series but it does not converges in L^1 -metric. This enhance many authors to study the Lebesgue convergence of trigonometric series. During literature survey, one can found that many authors introduced modified cosine and sine sums and use these modified sums as a tool to study the Lebesgue convergence of trigonometric series as these sums approximate their limits better than the classical trigonometric series.

Lebesgue convergence of complex trigonometric series was studied by C.V. Stanojevic and V.B. Stanojevic [2] Sheng Shu Yun [6], F. Moricz [3], C.P Chen [1] and S.S. Bhatia and B. Ram [8] for various classes of complex sequences by considering additional conditions to control the sine and mixed part of complex trigonometric series. Some of authors like S.S. Bhatia and B. Ram [7], Ž Tomovski [9], J. Kaur and S.S. Bhatia [4] have obtained the complex form of modified cosine and sine sums and obtained necessary and sufficient condition for Lebesgue convergence of complex trigonometric series.

Xhevat Z. Krasniqi [10] introduced new modified cosine and sine sums as

$$\hat{H}_n(x) = \frac{-1}{2 \sin x} \sum_{k=0}^n \sum_{j=k}^n \Delta[(\hat{a}_{j-1} - \hat{a}_{j+1}) \cos jx] \quad (1.1)$$

and

$$\hat{G}_n(x) = \frac{1}{2 \sin x} \sum_{k=1}^n \sum_{j=k}^n \Delta[(\hat{a}_{j-1} - \hat{a}_{j+1}) \sin jx] \quad (1.2)$$

and studied their Lebesgue convergence with semi-convex coefficients.

Keeping this idea, we have obtained the complex form of modified sums (1.1) and (1.2) as:

$$\hat{g}_n(\hat{C}, t) = \hat{S}_n(\hat{C}, t) + \frac{i}{2 \sin t} \left[\hat{c}_n e^{i(n+1)t} - \hat{c}_{-n} e^{-i(n+1)t} + \hat{c}_{n+1} e^{int} - \hat{c}_{-(n+1)} e^{-int} + (n+1)(\hat{c}_n - \hat{c}_{n+2}) e^{i(n+1)t} \right. \\ \left. + (n+1)(\hat{c}_{-(n+2)} - \hat{c}_{-n}) e^{-i(n+1)t} \right]$$

and studied its L^1 -convergence under a class J^* of complex coefficients.

II. Basic notations and background

Let the partial sums of the complex trigonometric series $\sum_{|n| \leq \infty} \hat{c}_n e^{int}$ be denoted by

$$\hat{S}_n(\hat{C}, t) = \sum_{|k| \leq n} \hat{c}_k e^{ikt}, t \in T = R / 2\pi Z.$$

If a trigonometric series is the Fourier series of some function $\hat{f} \in L^1$, we shall write $\hat{c}_n = \hat{f}(n)$ for all n and $\hat{S}_n(\hat{C}, t) = \hat{S}_n(\hat{f}, t) = \hat{S}_n(\hat{f})$. Let

$$D_n(t) = \frac{1}{2} + \cos t + \cos 2t + \dots + \cos nt = \frac{\sin\left(n + \frac{1}{2}\right)t}{2\sin\frac{t}{2}}$$

$$\tilde{D}_n(t) = \sin t + \sin 2t + \dots + \sin nt = \frac{\cos\frac{t}{2} - \cos\left(n + \frac{1}{2}\right)t}{2\sin\frac{t}{2}}$$

and

$$\tilde{K}_n(t) = \frac{1}{n+1} \sum_{j=0}^n \tilde{D}_j(t) = \frac{1}{4\sin^2\frac{t}{2}} \left[\sin t - \frac{\sin(n+1)t}{n+1} \right]$$

denote the Dirichlet's kernel, the conjugate Dirichlet's kernel and the conjugate Fejer's kernel respectively. Let

$E_n(t) = \sum_{k=0}^n e^{ikt}$, then the first differentials $D'_n(t)$ and $\tilde{D}'_n(t)$ of $D_n(t)$ and $\tilde{D}_n(t)$ can be written as

$$2D'_n(t) = E'_n(t) + E'_{-n}(t)$$

$$2i\tilde{D}'_n(t) = E'_n(t) - E'_{-n}(t)$$

where $E'_n(t)$ denotes the first differential of $E_n(t)$.

Definition. [4] A null sequence $\{\hat{c}_n\}$ of complex numbers belongs to class J^* if there exists a sequence $\{\hat{A}_n\}$ such that

$$\hat{A}_n \downarrow 0, \text{ as } n \rightarrow \infty \quad (2.1)$$

$$\sum_{n=1}^{\infty} n \hat{A}_n < \infty \quad (2.2)$$

$$\left| \Delta \left(\frac{\hat{c}_n - \hat{c}_{-n}}{n} \right) \right| \leq \frac{\hat{A}_n}{n}, \quad \text{for all } n = 1, 2, 3, \dots \quad (2.3)$$

III. Lemmas

The proof of main result is based upon the following lemmas.

Lemma 3.1. [8] Let r be a non-negative integer and $0 < \epsilon < \pi$. Then there exists $M_{r\epsilon} > 0$ such that for all $\epsilon \leq |t| \leq \pi$ and for all $n \geq 1$,

$$(i) \quad |E_n^r(t)| \leq A_{r\epsilon} n^r / |t|,$$

$$(ii) \quad |E_{-n}^r(t)| \leq A_{r\epsilon} n^r / |t|,$$

$$(iii) \quad |D_n^r(t)| \leq A_{r\epsilon} n^r / |t|,$$

$$(iv) \quad |\tilde{D}_n^r(t)| \leq A_{r\epsilon} n^r / |t|.$$

Lemma 3.2. For $n \geq 1$, we have

$$(i) \quad \left\| \frac{E_n(t)}{2\sin t} \right\| = o(n), \quad n \rightarrow \infty$$

$$(ii) \quad \left\| \frac{E_{-n}(t)}{2\sin t} \right\| = o(n), \quad n \rightarrow \infty$$

$$(iii) \quad \left\| \frac{e^{int}}{2\sin t} \right\| = o(\log n), \quad n \rightarrow \infty$$

Proof. For $t \neq 0$, we note that $\frac{\sin t}{t} \geq \frac{2}{\pi}$ in $(0, \pi/2)$ and using Lemma 1, we get

$$\left\| \frac{E_n(t)}{2\sin t} \right\| = \int_0^\pi \left| \frac{E_n(t)}{2\sin t} \right| dt \leq \int_0^\pi \frac{M_\epsilon}{2|t\sin t|} dt \leq 2 \int_0^{\pi/2} \frac{M_\epsilon}{2|t||\sin t|} dt$$

$$\text{Therefore, } \left\| \frac{E_n(t)}{2\sin t} \right\| \leq \int_0^{\pi/2} \frac{M_\epsilon}{t^2} dt = \lim_{n \rightarrow \infty} \left[\frac{-M_\epsilon}{t} \right]_{\pi/n}^{\pi/2} = o(n), \quad n \rightarrow \infty,$$

Similarly,

$$\left\| \frac{E_{-n}(t)}{2\sin t} \right\| = o(n),$$

and to prove (iii), we consider,

$$\left\| \frac{e^{int}}{2\sin t} \right\| = \int_0^\pi \left| \frac{e^{int}}{2\sin t} \right| dt = 2 \int_0^{\pi/2} \left| \frac{1}{2\sin t} \right| dt = \int_0^{\pi/2} \left| \frac{1}{t} \right| dt$$

Therefore

$$\left\| \frac{e^{int}}{2\sin t} \right\| = \lim_{n \rightarrow \infty} [\log t]_{\pi/n}^{\pi/2} = o(\log n), \quad n \rightarrow \infty.$$

IV. Main Result

The main result of this paper read as follows:

Theorem 4.1. Let $\hat{c}_n \in J^*$. Then there exists $\hat{f}(t)$ such that

- (i) $\lim_{n \rightarrow \infty} \hat{g}_n(\hat{C}, t) = \hat{f}(t)$ for $|t| \in (0, \pi]$,
- (ii) $\hat{f}(t) \in L^1(0, \pi]$ and $\|\hat{g}_n(\hat{C}, t) - \hat{f}(t)\|_{L^1} = o(1)$ as $n \rightarrow \infty$,
- (iii) $\|\hat{S}_n(\hat{f}, t) - \hat{f}(t)\|_{L^1} = o(1)$ as $|n| \rightarrow \infty$.

Proof. Consider,

$$\hat{g}_n(\hat{C}, t) = \hat{S}_n(\hat{C}, t) + \frac{i}{2\sin t} \left[\hat{c}_n e^{i(n+1)t} - \hat{c}_{-n} e^{-i(n+1)t} + \hat{c}_{n+1} e^{int} - \hat{c}_{-(n+1)} e^{-int} + (n+1)(\hat{c}_n - \hat{c}_{n+2}) e^{i(n+1)t} + (n+1)(\hat{c}_{-(n+2)} - \hat{c}_{-n}) e^{-i(n+1)t} \right]$$

Since $\frac{e^{int}}{2\sin t}$ is bounded in $(0, \pi]$. Also $\{\hat{c}_n\}$ is a null sequence. Therefore, we can write

$$\lim_{n \rightarrow \infty} \hat{g}_n(\hat{C}, t) = \lim_{n \rightarrow \infty} \hat{S}_n(\hat{C}, t) + \lim_{n \rightarrow \infty} \frac{iA_\epsilon}{2} [(n+1)(\hat{c}_n - \hat{c}_{-n}) + (n+1)(\hat{c}_{-(n+2)} - \hat{c}_{n+2})]$$

But for all $n \geq 1$, we note that

$$\begin{aligned} \frac{n(n+1)(\hat{c}_n - \hat{c}_{-n})}{n} &= n(n+1) \sum_{k=n}^{\infty} \Delta \left(\frac{\hat{c}_k - \hat{c}_{-k}}{k} \right) \\ &\leq \sum_{k=n}^{\infty} k(k+1) \frac{\hat{A}_k}{k} \\ &= o(1), \end{aligned}$$

By hypothesis of the theorem. Therefore,

$$\lim_{n \rightarrow \infty} \hat{g}_n(\hat{C}, t) = \lim_{n \rightarrow \infty} \hat{S}_n(\hat{C}, t) = \hat{f}(t)$$

Now, we show that $\hat{f}(t)$ exists in $(0, \pi]$. Consider,

$$\hat{S}_n(\hat{C}, t) = \sum_{k=-n}^n \hat{c}_k e^{ikt} = \hat{c}_0 + \sum_{k=1}^n \left[\frac{\hat{c}_k}{k} k e^{ikt} + \frac{\hat{c}_{-k}}{k} k e^{-ikt} \right]$$

By the use of Abel's transformation, we get

$$\begin{aligned} \hat{S}_n(\hat{C}, t) &= \hat{c}_0 + \sum_{k=1}^{n-1} \Delta \left(\frac{\hat{c}_k}{k} \right) (-iE'_k(t)) + \frac{\hat{c}_n(-iE'_n(t))}{n} + \sum_{k=1}^{n-1} \Delta \left(\frac{\hat{c}_{-k}}{k} \right) (iE'_{-k}(t)) + \frac{\hat{c}_{-n}(iE'_{-n}(t))}{n} \\ &= \hat{c}_0 - \sum_{k=1}^{n-1} \left[\Delta \left(\frac{\hat{c}_k}{k} \right) (iE'_k(t)) - \Delta \left(\frac{\hat{c}_{-k}}{k} \right) (iE'_{-k}(t)) \right] - \frac{\hat{c}_n(iE'_n(t)) - \hat{c}_{-n}(iE'_{-n}(t))}{n} \end{aligned}$$

By using Lemma 3.1 and (2.3), we get

$$\begin{aligned} \sum_{k=1}^{\infty} \left| \Delta \left(\frac{\hat{c}_k}{k} \right) (iE'_k(t)) - \Delta \left(\frac{\hat{c}_{-k}}{k} \right) (iE'_{-k}(t)) \right| &\leq \left(\frac{A_{\infty}}{|t|} \right) \sum_{k=1}^{\infty} k \left| \Delta \left(\frac{\hat{c}_k - \hat{c}_{-k}}{k} \right) \right| \\ &\leq \left(\frac{A_{\infty}}{|t|} \right) \sum_{k=1}^{\infty} \frac{k \hat{A}_k}{k} \\ &= \left(\frac{A_{\infty}}{|t|} \right) \sum_{k=1}^{\infty} \hat{A}_k < \infty, \end{aligned}$$

Since $\hat{A}_n \leq n \hat{A}_n$ for all $n \geq 1$ and $\sum n \hat{A}_n$ converges, by hypothesis of the theorem.

Therefore $\sum \hat{A}_n$ is also convergent.

Also,

$$\left| \frac{\hat{c}_n(iE'_n(t)) - \hat{c}_{-n}(iE'_{-n}(t))}{n} \right| \leq \frac{A_{\infty} n}{|t|} \left| \frac{\hat{c}_n - \hat{c}_{-n}}{n} \right| = o(1), \text{ as } n \rightarrow \infty.$$

Hence, $\hat{f}(t) = \lim_{n \rightarrow \infty} \hat{S}_n(\hat{C}, t)$ exists and thus (i) follows.

Further, for $t \neq 0$, Consider

$$\hat{f}(t) - \hat{g}_n(\hat{C}, t) = \sum_{|k| > n} \hat{c}_k e^{ikt} - \frac{i}{2 \sin t} \left[\hat{c}_n e^{i(n+1)t} - \hat{c}_{-n} e^{-i(n+1)t} + \hat{c}_{n+1} e^{int} - \hat{c}_{-(n+1)} e^{-int} + (n+1)(\hat{c}_n - \hat{c}_{n+2}) e^{i(n+1)t} + (n+1)(\hat{c}_{-(n+2)} - \hat{c}_{-n}) e^{-i(n+1)t} \right]$$

Using Abel's transformation, we get

$$\begin{aligned} \hat{f}(t) - \hat{g}_n(\hat{C}, t) &= \sum_{k=n+1}^{\infty} \left[\Delta \left(\frac{\hat{c}_k}{k} \right) (-iE'_k(t)) + \Delta \left(\frac{\hat{c}_{-k}}{k} \right) (iE'_{-k}(t)) \right] - \frac{\hat{c}_{n+1}(-iE'_n(t))}{n+1} - \frac{\hat{c}_{-(n+1)}(iE'_{-n}(t))}{n+1} \\ &\quad - \frac{i}{2 \sin t} \left[\hat{c}_n e^{i(n+1)t} - \hat{c}_{-n} e^{-i(n+1)t} + \hat{c}_{n+1} e^{int} - \hat{c}_{-(n+1)} e^{-int} + (n+1)(\hat{c}_n - \hat{c}_{n+2}) e^{i(n+1)t} + (n+1)(\hat{c}_{-(n+2)} - \hat{c}_{-n}) e^{-i(n+1)t} \right] \end{aligned}$$

$$\begin{aligned} \|\hat{f}(t) - \hat{g}_n(\hat{C}, t)\|_{L^1} &\leq \int_0^{\pi} \sum_{k=n+1}^{\infty} k \left| \Delta \left(\frac{\hat{c}_k - \hat{c}_{-k}}{k} \right) \right| \frac{A_{\infty}}{|t|} dt + \int_0^{\pi} \frac{A_{\infty} n}{|t|} \left| \frac{\hat{c}_{n+1} - \hat{c}_{-(n+1)}}{n} \right| dt \\ &\quad + \int_0^{\pi} \left| \frac{(\hat{c}_{n+1} - \hat{c}_{-(n+1)}) + (n+2)(\hat{c}_n - \hat{c}_{-n}) - (n+1)(\hat{c}_{n+2} - \hat{c}_{-(n+2)})}{\sin t} \right| dt \end{aligned}$$

Therefore,

$$\begin{aligned} \|\hat{f}(t) - \hat{g}_n(\hat{C}, t)\|_{L^1} &\leq \int_0^{\pi} \sum_{k=n+1}^{\infty} k \frac{\hat{A}_k}{k} \frac{A_{\infty}}{|t|} dt + \int_0^{\pi} \frac{A_{\infty}}{|t|} (\hat{c}_{n+1} - \hat{c}_{-(n+1)}) dt + \int_0^{\pi} \left| \frac{(\hat{c}_n - \hat{c}_{-n}) + (\hat{c}_{n+1} - \hat{c}_{-(n+1)})}{|\sin t|} \right| dt \\ &\quad + \int_0^{\pi} \left| \frac{(n+1)(\hat{c}_n - \hat{c}_{-n}) - (n+1)(\hat{c}_{n+2} - \hat{c}_{-(n+2)})}{|\sin t|} \right| dt \\ &= O \left(\sum_{k=n+1}^{\infty} \hat{A}_k \log k \right) + o(\log n (\hat{c}_n - \hat{c}_{-n})) + o(\log n (\hat{c}_n - \hat{c}_{-n})) + o(\log n (\hat{c}_n - \hat{c}_{-n})) \end{aligned} \quad (4.1)$$

But, for all $n \geq 1$, we note that $\log n (\hat{c}_n - \hat{c}_{-n}) \leq n^2 \frac{(\hat{c}_n - \hat{c}_{-n})}{n}$

$$\begin{aligned}
 &\leq \sum_{k=n}^{\infty} k^2 \Delta \left(\frac{\hat{c}_k - \hat{c}_{-k}}{k} \right) \\
 &\leq \sum_{k=n}^{\infty} k^2 \left(\frac{\hat{A}_k}{k} \right) \\
 &= o(1)
 \end{aligned} \tag{4.2}$$

Thus, $\|\hat{f}(t) - \hat{g}_n(\hat{C}, t)\|_{L^1} = o(1)$, $n \rightarrow \infty$ and since $\hat{g}_n(\hat{C}, t)$ is a polynomial, it follows that $\hat{f} \in L^1(0, \pi]$, which proves the assertion (ii).

Further, we notice that

$$\begin{aligned}
 \|\hat{f} - \hat{S}_n\|_{L^1} &= \|\hat{f} - \hat{g}_n + \hat{g}_n - \hat{S}_n\|_{L^1} \\
 &\leq \|\hat{f} - \hat{g}_n\|_{L^1} + \|\hat{g}_n - \hat{S}_n\|_{L^1} \\
 &\leq \|\hat{f} - \hat{g}_n\|_{L^1} + \left\| \frac{i}{2 \sin t} \left[\hat{c}_n e^{i(n+1)t} - \hat{c}_{-n} e^{-i(n+1)t} + \hat{c}_{n+1} e^{int} - \hat{c}_{-(n+1)} e^{-int} + (n+1)(\hat{c}_n - \hat{c}_{n+2}) e^{i(n+1)t} \right. \right. \\
 &\quad \left. \left. + (n+1)(\hat{c}_{-(n+2)} - \hat{c}_{-n}) e^{-i(n+1)t} \right] \right\|
 \end{aligned}$$

Using (4.1), (4.2) and assertion (ii), we get the conclusion of assertion (iii).

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Acknowledgment

This work is the part of UGC sponsored Major Research Project entitled "On L^1 -convergence of trigonometric series with special coefficients