



SCORE OF A TOURNAMENT

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Abstract: In this paper, a theorem on Tournament scores to oriented graph scores is proved to calculate and check the score of a tournament and it can be also used as the necessary and sufficient condition for a sequence to be the score of an oriented graph.

I. Introduction

The theory of Di-graphs is one of the important theories in Graph Theory and has developed enormously within the last three decades. There are numerous applications of directed graphs in many areas of science and technology. Algorithms on graphs often play an important role in problems arising in several areas, including computer science and operations research. Secondly, many problems on Directed graphs are inherently algorithmic. The concept of degrees and degree sequences in graphs has been extended to di-graphs in many ways, like out degrees, in degrees, weights, scores, imbalances, marks, lengths, etc. This concept of attaching a non-negative integer to the vertices of a digraph is interesting for research as it finds applications in many ways like in the investigation of the structure of the digraphs. The tournament theory is one of the interesting areas of research in digraphs, and an earlier collection of results in tournaments is given by Moon. One of the important aspects of tournaments is the score structure in which much work has been done and some of the results can be seen in the survey article by Reid. The score sequence problem of an r -tournament and the score sequence pair problem of an (r_{11}, r_{12}, r_{22}) -tournament are applied to the theoretical frame work of the communication network central technique. The name tournament originates from such a graph's interpretation as the outcome of a round-robin tournament in which every player encounters every other player exactly once, and in which no draws occur. In the tournament digraph, the vertices correspond to the players. The edge between each pair of players is oriented from the winner to the loser. If player a beats player b , then it is said that a dominates b .

Key words: Oriented graphs, Di-graph, Score sequence, and Tournament.

We mention here some definitions which have been used in the theorem.

II. Preliminary Results

Definition 1.1 Oriented graph: An oriented graph is a digraph with no symmetric pairs of directed arcs and without loops.

Definition 1.2 A tournament $T=(V,A)$ is a complete oriented graph with vertex set $V(T)=V=\{v_1, v_2, v_3, \dots, v_n\}$ and arc set A , that is for any pair of vertices v_i, v_j either (v_i, v_j) is an arc or (v_j, v_i) is an arc, but not both. In other words, a tournament is an orientation of a complete simple graph.

Definition 1.3 The score of a vertex v_i is denoted by s_i , is the out degree of v_i .

Clearly, $0 \leq s_i \leq n-1$.

The sequences $= \{s_1, s_2, s_3, \dots, s_n\}$ in non decreasing order is the score sequence or the score structure of a tournament T . A sequence S of non-negative integers in non-decreasing order is said to be realizable if there exists a tournament with score sequence S .

i.e., $s_1 \leq s_2 \leq s_3 \leq \dots \leq s_{n-1}$

A tournament can be considered as a result of a competition where n

$$I \subseteq [n] = \{1, 2, \dots, n\}, \sum_{i \in I} s_i \geq \binom{|I|-1}{2}, \quad (1)$$

With equality when $|I|=n$, where $|I|$ is the cardinality of the set I .

Since $s_1 \leq s_2 \leq \dots \leq s_n$ the inequality (1.1), called Landau inequalities, are equivalent to $\sum_{i \in I} s_i \geq \binom{k}{2}$, for $k=1, 2, \dots, n-1$, and equality for $k=n$.

Definition 1.4. A directed graph D is said to be an (r_{11}, r_{12}, r_{22}) -tournament if the vertex set of D is partitioned into two disjoint sets A and B such that there are r_{11} directed arcs between every pair of vertices in A , r_{22} directed arcs between every pair of vertices in B , and r_{12} directed arcs between each vertex of A and each vertex of B . The score of the vertex is the out degree of the vertex.

Definition 1.5. An oriented graph D is a di graph with no symmetric pairs of directed arcs and with no loops. In D , let d_i^+ and d_i^- be the out degree and in degree of the vertex v_i . Define a_i of a vertex v_i as follows.

$$a_i = n-1 + d_i^+ - d_i^-.$$

Evidently, $0 \leq a_i \leq 2n-2$. The list of scores $\{a_1, a_2, \dots, a_n\}$ in non-decreasing or non-increasing order is called the score sequence of D .

Let d_i^+, d_i^- and d_i^* respectively be out degree, indegree and non-arcs incident at v_i . Then $d_i^+ + d_i^- + d_i^* = n - 1 = a_i - d_i^+ + d_i^-$

Or,

$$a_i = 2d_i^+ + d_i^*$$

This shows that $a_i = n-1 + d_i^+ - d_i^- = 2(\text{wins}) + (\text{draws})$.

Avery extended Landau's theorem on tournament scores to oriented graph scores.

Theorem: A sequence $A = [a_i]_1^n$ of non-negative integers in non-decreasing order is a score of an oriented graph if and only if each subset I ,

$$\sum_{i \in I} a_i \geq k(k-1) \quad \text{if } I \subseteq [n] = \{1, 2, \dots, n\}.$$

$$\text{and } \sum_{i \in I} a_i = k(k-1) \quad \text{if } |I| = n.$$

Theorem: A sequence $A = [a_i]_1^n$ of non-negative integers in non-decreasing order is a score of an oriented graph if and only for $1 \leq k \leq n-1$,

$$\sum_{i=1}^k a_i \geq k(k-1) \tag{2}$$

With equality when $k=n$.

Proof

Necessity:

Let $A = [a_i]_1^n$ be a score sequence of some oriented graph D .

Let W be the oriented sub graph induced by any k vertices $w_1, w_2, w_3, \dots, w_k$ of D .

Let α denote the number of arcs of D that start in W and end outside W and let β denote the number of arcs of D that start outside of W and end in W .

Clearly, $\beta \leq k(n-k)$.

Thus

$$\begin{aligned} \sum_{i=1}^k a_{w_i} &= \sum_{i=1}^k \{(n-1) + d_D^+(w_i) - d_D^-(w_i)\} \\ &= nk - k + \sum_{i=1}^k d_D^+(w_i) - \sum_{i=1}^k d_D^-(w_i) \\ &= nk - k + [\sum_{i=1}^k d_w^+(w_i) + \alpha] - [\sum_{i=1}^k d_w^-(w_i) + \beta] \\ &= nk - k + (\text{number of arcs of } W) + \alpha - (\text{number of arcs of } W) - \beta \\ &= nk - k + \alpha - \beta. \\ &\geq nk - k - \beta \\ &\geq nk - k - k(n-k) \\ &= k(k-1). \end{aligned}$$

Sufficiency:

Let n denote the least integer so that there is a non-decreasing sequence of non-negative integers satisfying conditions (1) that is not a score sequence of any oriented graph.

Among all such sequences of length n , consider $A = [a_i]_1^n$, in which the smallest term a_1 is as small as possible.

We consider two cases (i) equality in (1) holds for some $k < n$ and (ii) Each inequality in (1) is strict for all $k < n$.

Case (i):

Assume $k (< n)$ is the smallest integer such that

$$\sum_{i=1}^k a_i = k(k-1).$$

Clearly the sequence $\{a_1, a_2, a_3, \dots, a_k\}$ satisfies condition (1) and is a sequence of length less than n .

Therefore by the given assumption $\{a_1, a_2, a_3, \dots, a_k\}$ is a score sequence of some oriented graph, say D_1

Now

$$\begin{aligned} \sum_{i=1}^p (a_{k+i} - 2k) &= \sum_{i=1}^{p+k} a_i - \sum_{i=1}^k a_i - 2pk \\ &\geq (p+k)(p+k-1) - k(k-1) - 2pk \\ &= p(p-1) \end{aligned}$$

I.e., for each p ,

$$1 \leq p \leq n-k, \quad \sum_{i=1}^p (a_{k+i} - 2k) \geq p(p-1)$$

With equality when $p=n-k$.

Since $p < n$, the minimality of n implies that the sequence $\{a_{k+1}-2k, a_{k+2}-2k, \dots, a_n-2k\}$ is the score sequence of some oriented graph D_2 .

The oriented graph D of order n , consisting of disjoint copies of D_1 and D_2 , such that there is an arc from each vertex of D_2 to every vertex of D_1 , has score sequence $A = [a_i]_1^n$, a contradiction.

Case (ii):

Assume that each inequality in conditions (1) is strict for all $k < n$.

Clearly, $a_1 > 0$. Consider the sequence $A' = [a'_i]_1^n$, where $a'_i = a_i - 1$, or $a'_i = a_i + 1$, or a_i according as $i=1$, or $i=n$, or otherwise.

Then, $\sum_{i=1}^k a'_i = \sum_{i=1}^k a_i - 1$
 $> k(k-1) - 1$, for all $k, 1 \leq k \leq n$.

Therefore, $\sum_{i=1}^k a'_i \geq k(k-1)$, for all $k, 1 \leq k \leq n$.

Also, $\sum_{i=1}^n a'_i = \sum_{i=1}^n a_i - 1 + 1$
 $= n(n-1)$.

Thus the sequence $A' = [a'_i]_1^n$ satisfies the conditions (1) and therefore is a score sequence of some oriented graph d .

Let u and v , respectively denote the vertices with score $a'_i = a_i - 1$ and $a'_n = a_n + 1$.

If in D either $v(1-0)u$, or $v(0-0)u$, then transforming them respectively to $v(0-0)u$, or $v(0-1)u$, we get an oriented graph with score sequence A , a contradiction.

Now let $u(1-0)v$.

We claim that there exists at least one vertex w so that the triple formed by the vertices u, v and w intransitive, that is, of the form $u(1-0)v(1-0)w(1-0)u$, or $u(1-0)v(1-0)w(0-0)u$, or $u(1-0)v(0-0)w(1-0)u$.

Assume to the contrary that for each vertex $w \in V - \{u, v\}$, the triple formed by the vertices u, v and w are transitive, that is, of the form $u(1-0)v(1-0)w(1-0)u$, or $u(1-0)v(0-1)w(1-0)u$, or $u(1-0)v(0-1)w(0-1)u$, or $u(1-0)v(0-0)w(0-0)u$, or $u(1-0)v(0-1)w(0-0)u$, or $u(0-0)v(0-0)w(0-0)u$.

Then in all such cases, $d^+(u) > d^+(v)$ and $d^-(u) > d^-(v)$.

This shows that $a_u < a_v$.

This proves the claim.

Hence transforming the intransitive triples respectively to $u(1-0)v(0-0)w(0-0)u$ or $u(1-0)v(0-0)w(0-1)u$ or $u(1-0)v(0-1)w(0-0)u$, we obtain an oriented graph with score sequence A . This contradicts the assumption.

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