



On finding $\sqrt[n]{m}$ leading to Newton Raphson's improved method

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Abstract: New iterative algorithms for finding the n th root of a positive number m , to any degree of accuracy, are discussed. The convergence of these methods is also analyzed and the factors affecting the rate of convergence are studied analytically, as well as graphically. The parameters involved in the iterative schemes are studied. Expressions are derived for the optimal values of these parameters. The rates of convergence of these new methods can be accelerated through these parameters, which prove to be much faster than the Newton Raphson method for finding $\sqrt[n]{m}$ in some cases. Several examples are given for clarity purposes. Also, numerical comparative study is made between the improved Newton Raphson's method and the third order Halley's method.

Keywords: Iterative algorithm; Fixed point iteration; Fixed-b method; Adaptive-b method; Simplified Adaptive-b method; Halley's method; Newton Raphson Method; Newon-Raphson Improved.

I. Introduction

The n^{th} root of a positive m is a number p satisfying $pn = m$. Any real number m has n such n^{th} roots. In this paper, we are concerned with the numerical approximation of $\sqrt[n]{m}$. There are numerical methods, such as, the bisection method, the regula-falsi method, the Newton-Raphson method and many more. Any standard textbook on numerical analysis explains these methods [1]. All these methods use the function $f(x) = xn - m$. In [2], an iterative algorithm for finding the $\sqrt[n]{m}$ is discussed, which involves generating a sequence of approximations to $\sqrt[n]{m}$. The method is also directly related to the continued fraction representation of $\sqrt[n]{m}$. The convergence of this method is established by studying the eigenvalues and eigenvectors of a matrix, directly related to the algorithm itself. The approximations are then obtained from the following sequence of fractions:

$$\frac{a}{b} \rightarrow \frac{a + mb}{a + b}, \quad (1)$$

which can also be viewed as a sequence generated from

$$\gamma \rightarrow \frac{\gamma + m}{\gamma + 1}, \quad (2)$$

where a is replaced with γ . If we consider any fraction as a two dimensional vector, then a can be represented by $\begin{bmatrix} a \\ b \end{bmatrix}$, and vice-versa. The right hand expression in relation (1) is then equivalent to the matrix product

$$\begin{bmatrix} 1 & m \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}. \quad (3)$$

Therefore, the successive generation of the sequence of approximations to $\sqrt[n]{m}$, involve the multiplication of higher powers of the square matrix in (3). The convergence of the iterative algorithm directly depends on the nature of the eigenvalues and eigenvectors of the matrix.

In this paper, we have discussed four numerical methods in sequence: the Fixed-b method, the Adaptive-b method, a simplified form of the Adaptive-b method, called the Simplified Adaptive-b method leading to the Newton-Raphson Improved (NRI) method. Several numerical examples are worked out for a clear understanding of these methods. The Newton-Raphson's improved method is also compared with the third order Halley's method [3], [4].

In section 2, we have discussed the Fixed-b method and in section 3 we have analyzed this method in a greater detail. Section 4 discusses the Adaptive-b method and its analysis. In section 5, an improved version of Newton-Raphson method, called the NRI method is explained, which is derived from the Simplified Adaptive-b method (SA-b). Several examples are included for clarity purposes. Although m is any positive number, we have chosen $m = 2012, 2013$ and 2014 , commemorating the years of research.

II. Fixed-b method

Intuitively, one can generalize the relation (2) as

$$\gamma \longrightarrow \frac{\gamma^2 + \gamma + m}{\gamma^2 + \gamma + 1}, \quad (4)$$

for finding the cube-root of a number m . The relation (4) converges to $\sqrt[3]{m}$, for $m < 2(5 + 3\sqrt{3})$. But, it is found that for higher values of m the above sequence oscillates and never converges. In order to eliminate these oscillations, we perturbed the right hand expression of (4) to obtain

$$\gamma \longrightarrow \frac{b\gamma^2 + \gamma + m}{\gamma^2 + b\gamma + 1}, \quad (5)$$

where b is a real parameter. For a given m , the convergence of the sequence in (5) depends on b . This idea can be further generalized to obtain an approximate value for the sequence $\sqrt[n]{m}$. We consider

$$\gamma \longrightarrow \frac{b\gamma^{n-1} + \gamma^{n-2} + \gamma^{n-3} + \dots + m}{\gamma^{n-1} + b\gamma^{n-2} + \gamma^{n-3} + \dots + \gamma^0}. \quad (6)$$

This iterative sequence for $n = 2$ will take the form

$$\gamma \longrightarrow \frac{b\gamma + m}{\gamma + b},$$

which is different from relation (2). By varying the parameter b in (6), we can achieve convergence and improve the rate of convergence as explained in section 3.

The sequence in (6) can be rewritten as an iterative formula as given below

$$\gamma_{k+1} = Fb(b, \gamma_k), k = 0, 1, \dots, \quad (7)$$

where we have defined $Fb(b, \gamma)$ as

$$Fb(b, \gamma) := \frac{b\gamma^{n-1} + \gamma^{n-2} + \gamma^{n-3} + \dots + m}{\gamma^{n-1} + b\gamma^{n-2} + \gamma^{n-3} + \dots + \gamma^0}. \quad (8)$$

It is to be mentioned that, b and γ_0 is chosen initially.

A. Example

To illustrate this new method we choose $m = 2013$ and $n = 3$. Using the formula in (7), with $\gamma_0 = 1, b = 10$, we get $\sqrt[3]{2013} = 12.6265$, up to 3 decimal accuracy, in 29 iterations.

III. Analysis of Fixed-b method

The convergence of the Fixed-b method depends on the choice of b and γ_0 . In this section, we analyze the rate of convergence, by alternately varying b and γ_0 , for the examples (i) $\sqrt[4]{2014}$ and (ii) $\sqrt[3]{2012}$. In both examples, we have fixed four decimal places of accuracy. In this example, we have studied how b affects the rate of convergence by fixing $\gamma_0 = 1$. Choosing $n = 4$ and $m = 2012$ in relation (7), we obtain:

$$\gamma \longrightarrow \frac{b\gamma^3 + \gamma^2 + \gamma + 2012}{\gamma^3 + b\gamma^2 + \gamma + 1}.$$

We have plotted the number of iterations t against b , which is varied from 1 to 100. The iteration diverges whenever $b < 6.5258$ and converges for $b > 6.5258$. A formula for b is derived later in this section, for which the iteration starts to converge, as shown later in this section.

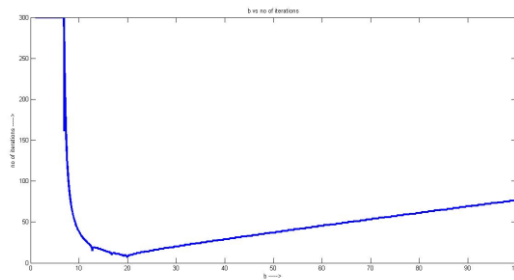


Figure 1: No. of Iterations against b

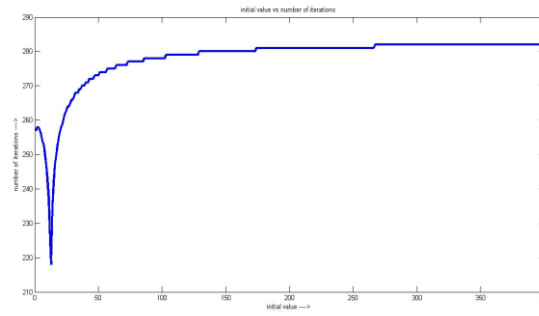


Figure 2: No. of Iterations against γ_0

Figure 1 clearly shows that the number of iterations reaches a minimum at $b_{min,t} \approx 3\sqrt[4]{2012}$ and it slowly increases after this value. At this b the number of iterations is observed to be $t = 7$. The region of divergence is shown by a horizontal line parallel to the b -axis.

(i) It can be observed that $b_{min,t} \approx (n-1)\sqrt[n]{m}$ and that whenever $b > b_{min,t}$ the rate of convergence decreases or in other words t increases.

(ii) In this example, we have studied how γ_0 affects the rate of convergence by fixing $b = 500 > b_{min,t}$. From relation (7) we obtain for $n = 3$ and $m = 2012$:

$$\gamma \rightarrow \frac{500\gamma^2 + \gamma + 2012}{\gamma^2 + 500\gamma + 1}.$$

We have plotted the number of iterations t against γ_0 , which is varied up to 400. Figure 2 shows that, the number of iterations vary much more when γ_0 is chosen closer to the actual root, than, when chosen far away from the actual root.

B. Error Analysis

Let the error in the k^{th} stage of iteration be $\gamma_k - \sqrt[n]{m}$. Thus, the error ratio in the $(k+1)^{th}$ stage so from relation (7), $\gamma_{k+1} = Fb(b, \gamma)$. Thus, the above inequality reduces to

$$-1 < \frac{Fb(b, \gamma) - \sqrt[n]{m}}{\gamma - \sqrt[n]{m}} < 1. \quad (9)$$

We define a new function $Fbs(b, x) = Fb(b, x + \sqrt[n]{m}) - \sqrt[n]{m}$, which eases the analysis. This function translates the origin to $(\sqrt[n]{m}, \sqrt[n]{m})$. The inequality (9) can be cast into the form:

$$-1 < ER(b, x) < 1, \quad (10)$$

where $ER(b, x) = \frac{Fbs(b, x)}{x}$. It is to be noted that, relation (9) can be obtained from (10), by setting $\gamma = x + \sqrt[n]{m}$. Also,

$$ER(b, 0) = Fb'_s(b, x)|_{x=0} = Fb'(b, x + \sqrt[n]{m})|_{x=0}.$$

Therefore, the criteria

$$-1 < ER(b, 0) < 1, \quad (11)$$

becomes a necessary condition for convergence.

The exact value of $b_{min,t}$ is derived to be

$b_{min,t} = (n-1)\sqrt[n]{m} - \frac{1}{\sqrt[n]{m}} - \frac{1}{\sqrt[n]{m^2}} - \frac{1}{\sqrt[n]{m^3}} - \dots - \frac{1}{\sqrt[n]{m^{n-2}}}$ solving $ER(b, 0) = 0$. The expression for $b_{min,t}$ is derived based on the pattern observed for the values of $n = 1, 2, 3, 4$ and 5 . The general form of $b_{min,t}$ can be validated by directly substituting in the expression of $ER(b, 0)$, yielding zero.

C. Range of b for the convergence of the Fixed- b method

It is verified that $ER(b, x)$ is an increasing function of b , when ever $x \geq 0$. Also, $ER(b, 0) < 1$ and that $ER(b, 0) = 1$. Now, solving for b from $ER(b, 0) = -1$, we obtain the threshold value b_{TH} , which can be derived to be

$$b_{TH} = \frac{\frac{n-2}{2} \sqrt[n]{m^{n-1}} - 1 - \sqrt[n]{m} - \sqrt[n]{m^2} - \dots - \sqrt[n]{m^{n-3}}}{\sqrt[n]{m^{n-2}}}. \quad (12)$$

The expression is derived by noticing the pattern and then the general expression is validated by back substituting in $ER(b, 0)$. The inequality (11) is satisfied and the iteration converges, whenever $b > b_{TH}$. In all the earlier worked examples, we have chosen $b > b_{TH}$. It is to be noted that for a fixed x and for any $b_2 > b_{TH}$, the iteration formula using b_1 shows a faster rate of convergence, whenever $|ER(b_1, x)| < |ER(b_2, x)|$.

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IV. Adaptive-b method and its analysis

In section 3, we have mentioned that the best choice of b is $b_{min,t} \approx (n-1)\sqrt[n]{m}$. Since, this choice of b involves the original problem of finding $\sqrt[n]{m}$, the best approximate to $\sqrt[n]{m}$ is the k^{th} iterate γ_k itself. Thus, after each iteration, b can be updated to $(n-1)\gamma_k$. Note that b now depends on γ . Hence, we set $b(\gamma) = \beta\gamma$, where β is a parameter. Defining b as a linear function of γ gives a new variant of the formula (8). The method is now called as the Adaptive- b method and the function is named as $Ab(\beta, x)$. A detailed analysis of the effect of the parameter β on the rate of convergence has been discussed later in this section. Setting $b = b(\gamma) = \beta\gamma$, the iteration formula in (8) takes a new form, given by

$$Ab(\beta, \gamma) = \frac{\beta\gamma^n + \gamma^{n-2} + \gamma^{n-3} + \dots + m}{(\beta+1)\gamma^{n-1} + \gamma^{n-3} + \gamma^{n-4} + \dots + 1}. \quad (13)$$

Table 1. Comparison between Adaptive-b and Fixed-b methods

Iterations	Adaptive-b	Fixed-b
γ_0	10	10
γ_1	13.3688	13.3688
γ_2	12.6701	12.5293
γ_3	12.6288	12.6446
γ_4	12.6285	12.6260
γ_5		12.6289
γ_6		12.6285

The sequence formed by (13) converges faster than that formed by (7). For example, let $m = 2014$ and $n = 3$. Set $b = 20 (= \beta\gamma_0)$, so that $\gamma_0 = 10$, $\beta = 2$. Table 1 shows that the Adaptive- b method requires 4 iterations, whereas Fixed- b method requires 6 iterations, for four decimal places of accuracy.

A. Comparison of Adaptive-b method with Newton Raphson method

The Newton Raphson iteration formula for finding the n^{th} root of a number m is

$$\gamma_{k+1} = NR(\gamma_k),$$

where

$$NR(\gamma) = \frac{(n-1)\gamma^n + m}{n\gamma^{n-1}}. \quad (14)$$

Comparing the rates of convergence of Adaptive- b method and Newton-Raphson method, we observe that the two curves $Ab(n-1, x)$ and $NR(x)$ almost coincide. In particular, for $n = 2$, the two curves $Ab(1, x)$ and $NR(x)$ coincide and for $n > 2$, and at an appropriate β , the two curves are in close proximity. For example, choosing $m = 2012.91$ and $n = 3$, we observe that the $Ab(\beta, x)$ curve tends to $NR(x)$ curve, as $\beta \approx n-1$. Figure 3 explains this fact, where β is chosen to be 2.25 and 3.

Intuitively, if $b(\gamma) = (n-1)\gamma - \frac{1}{\gamma} - \frac{1}{\gamma^2} - \frac{1}{\gamma^3} - \dots - \frac{1}{\gamma^{n-2}}$, then $Fb(b(\gamma), \gamma) = NR(\gamma)$. The terms after the first are insignificant if $\gamma > 1$ and therefore are negligible. Thus, the convergence of Adaptive- b and N R methods are almost the same, if we choose $b(\gamma) = \beta\gamma = (n-1)\gamma$.

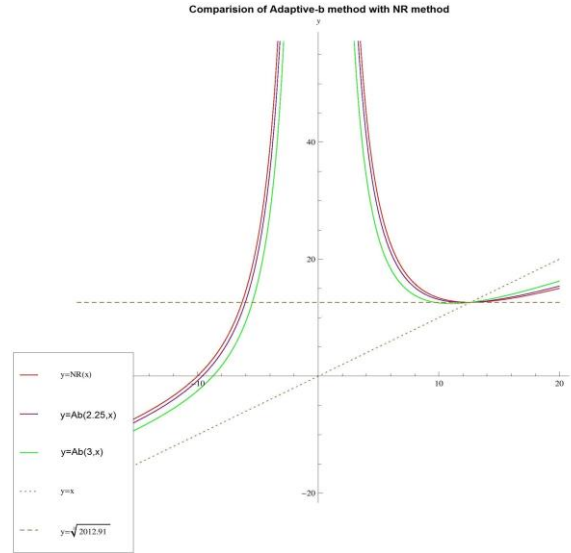


Figure 3: Comparison between Adaptive-b method and Newton Raphson method

B. Convergence analysis of Adaptive-b method

The analysis of the rate of convergence of the Adaptive-b method is similar to that of Fixed-b method, wherein we replace the function $Fb(b, x)$ with $Ab(\beta, x)$. In figure 4, we observe that the graph of $ER\beta(\beta, x)$ for $\beta \approx (n - 1)$, is almost equal to that of the Newton-Raphson method. In this example, we have chosen $n = 5$, $\beta = 3.5$ and $m = 2012$.

The graph for any $m > 0$ is similar to that shown in Figure 4. The necessary condition for convergence is $-1 < ER\beta(\beta, 0) < 1$ and it is found that $ER\beta(\beta, 0)$ has a point of discontinuity at $\beta_\infty = -1 - \sqrt[n]{m^{-2}} - \sqrt[n]{m^{-3}} - \dots - \sqrt[n]{m^{-(n-1)}}$. It is also verified that $ER\beta(\beta, 0)$ is an increasing function of β , $\forall \beta = \beta_\infty$ and that $ER\beta(\beta, 0) < 1$, for $\beta > \beta_\infty$. This can be observed in figure 5. We can derive the threshold value β_{TH} , similar to that of b_{TH} as,

$$\beta_{TH} = \frac{n}{2} - 1 - \sqrt[n]{m^{-2}} - \sqrt[n]{m^{-3}} - \dots - \sqrt[n]{m^{-(n-1)}}. \quad (15)$$

For $\beta > \beta_{TH}$, the necessary condition is satisfied and that the iteration starts to converge. It is found that β_{TH} monotonically increases with m and an upper bound is given to be $\beta_{TH,max} = \frac{n}{2} - 1$. Thus for $\beta > \frac{n}{2} - 1$, the iterative scheme of Adaptive-b always converge.

V. Newton Raphson Improved method

A slight variation in Adaptive-b method results in Simplified Adaptive-b method or SA-b method, wherein we drop the terms $\gamma^{n-2}, \gamma^{n-3}, \dots, \gamma$ from the numerator and the terms $\gamma^{n-3}, \dots, \gamma, 1$ from the denominator, of relation (13). This gives rise to a new iteration formula after replacing β by α , as given below

$$\gamma_{k+1} := SAb(\alpha, \gamma_k), \quad (16)$$

where the function $SAb(\alpha, \gamma)$ is defined as

$$SAb(\alpha, \gamma) = \frac{\alpha\gamma^n + m}{(\alpha + 1)\gamma^{n-1}}. \quad (17)$$

Clearly, when $\alpha = n - 1$, the two functions $SAb(\alpha, \gamma)$ and $NR(\gamma)$ coincide. Also, the threshold value of $\alpha_{TH} = \frac{n}{2} - 1$. This can be derived on similar lines of deriving β_{TH} . For $\alpha > \alpha_{TH}$ this method converges.

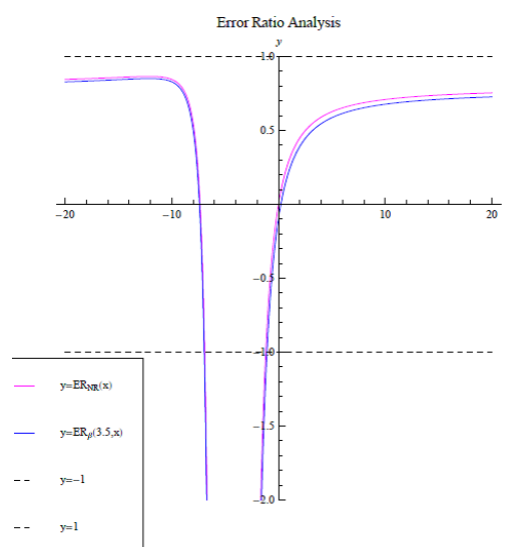


Figure 4: Error ratios of Adaptive-b and NR methods

Now, we compare the two methods SA-b and Adaptive-b and this is illustrated in Figure 6. We have chosen $m = 2013$, $n = 4$ and $\alpha = \beta = 2.5$. We observed that both the Adaptive-b and the SA – b methods can be faster than the Newton-Raphson method, if we update the parameters β and α at every iteration. Table 2 and Table 3 illustrates this fact by choosing $n = 3$ and $m = 2013$. In tables 2 and 3, α is chosen such that α is less than $n - 1$ when $\gamma \gg \sqrt[n]{m}$, greater than $n - 1$ when $\gamma \ll \sqrt[n]{m}$ and $\alpha \approx n - 1$ when $\gamma \approx \sqrt[n]{m}$. The method of choosing an optimal value of α at every iteration is given by the below formula, which characterizes the above behavior

$$\alpha_k = (n - 2) + \frac{m}{\gamma_k^n}. \quad (18)$$

Notice that, for $\gamma > 0$, (18) gives values of α such that $\alpha > \alpha_{TH}$ at every iteration stage, thus choosing $\gamma > 0$ ensures convergence and unlike earlier methods, no other parameter needs to be chosen. Now, substituting (18) in (16) gives the iteration formula for Newton-Raphson Improved method or NRI method as given below:

$$\gamma_{k+1} := NRI(\gamma_k) \quad (19)$$

where the function $NRI(\gamma_k)$ is defined as,

$$NRI(\gamma) = \gamma \left[\frac{(n - 2)\gamma^n + 2m}{(n - 1)\gamma^n + m} \right]. \quad (20)$$

For γ closer to $\sqrt[n]{m}$, NRI method tends to Newton-Raphson method. At $\gamma_0 = \sqrt[n]{m}$, $\alpha_k = n - 1$ and $NRI(\gamma_k) = NR(\gamma_k)$.

Table 2. Comparison of the SA-b, the Adaptive-b, (updating α and β) and the Newton Raphson method when $\gamma_0 > \sqrt[n]{m}$ is chosen

Iterations	α	SA-b	β	Adaptive-b	NR
γ_0		100		100	100
γ_1	1	50.1007	1	50.1031	66.7338
γ_2	1	25.413	1	25.4574	44.6398
γ_3	1	14.2794	1.5	16.5224	30.0966
γ_4	1.5	12.5166	1.5	12.8684	20.8052
γ_5	2	12.6274	2	12.6314	50.4203
γ_6	2	12.6264	2	12.6265	13.1021
γ_7	2		2	12.6264	12.6435
γ_8	2		2		12.6265
γ_9	2		2		12.6264

In terms of computational complexity for implementing on a computer, both the NRI and the NR methods require $(n + 2)$ multiplications and one division, per iteration. Although the NRI method needs two addition operations more than the NR method, there is a significant reduction in the number of iterations in the NRI method, for a fixed decimal of accuracy.

Table 3. Comparison of the SA-b, the Adaptive-b, (updating α and β) and the Newton Raphson method when $\gamma_0 > \sqrt[n]{m}$ is chosen

Iterations	α	SA-b	β	Adaptive-b	NR
γ_0		1		1	1
γ_1	200	11.0100	200	10.9604	671.6667
γ_2	20	11.2764	20	11.2363	447.7793
γ_3	15	11.5611	15	11.5304	298.5229
γ_4	10	11.8792	10	11.8584	199.0228
γ_5	5	12.2768	5	12.2674	132.6988
γ_6	4	12.4926	4	12.4889	88.5040
γ_7	3	12.5941	3	12.5930	59.0883
γ_8	2	12.6265	2	12.6265	39.5844
γ_9	2	12.6264	2	12.6264	26.8178
γ_{10}	2		2		18.8115
γ_{11}	2		2		14.4372
γ_{12}	2		2		12.8441
γ_{13}	2		2		12.6301
γ_{14}	2		2		12.6265
γ_{15}	2		2		12.6264

Thus, the NRI method is superior to the NR method for computation of $\sqrt[n]{m}$. For example, if $m = 35^5$, $n = 5$ and $\gamma_0 = 1$, the NRI method takes 8 iterations, whereas, the Newton-Raphson method takes 61 iterations. Table 4 illustrates the number of iterations “i” required by the two methods, in achieving four decimal place of accuracy, for different n, m and γ_0

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Finally, let us compare NRI method with the Halley’s method, which is a third order method. As shown in Table 4, NRI method can be faster than Halley’s method. In some cases, NRI method is slightly slower than Halley’s method but NRI method needs three lesser multiplication operations per iteration. The formula for computing $\sqrt[n]{m}$ in by Halley’s method can be derived in the form as

$$H(\gamma) = \gamma \left[\frac{\gamma^n(n-1) + m(n+1)}{\gamma^n(n+1) + m(n-1)} \right]. \quad (21)$$

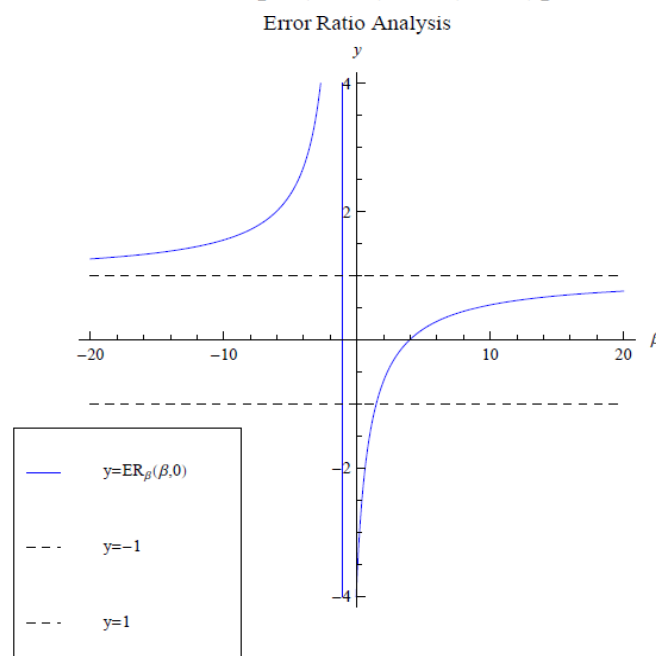


Figure 5: $ER_{\beta}(\beta, 0)$ against β

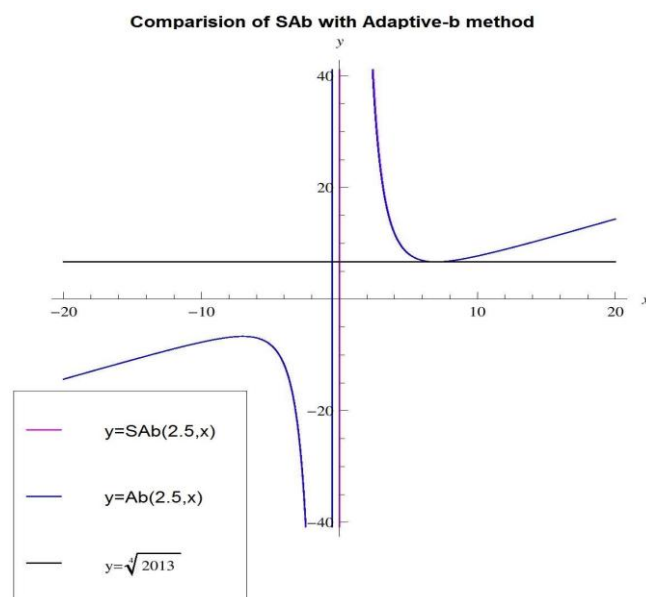


Figure 6: Comparison of $SA - b$ with Adaptive-b Method

VI. Conclusions

New iterative methods named as, the Fixed- b method, the Adaptive- b method, the Simplified Adaptive- b method and the Newton Raphson Improved NRI method, have been studied and analyzed, for finding the n th root of a number m . The convergence of these methods has also been explained. The parameters that affect the rate of convergence have been explained. Further, we also have discussed about choosing an optimum value of these parameters. These iterative methods have been compared to the well-known Newton-Raphson method. It is evident from the examples that the NRI method is much faster than the Newton-Raphson method, for finding $\sqrt[n]{m}$. We conclude by stating that the NRI method is a better alternative for finding $\sqrt[n]{m}$. Although, Halley's method is third order method, the numerical examples prove that, at times, Newton Raphson's improved method can be still better.

Table 4. Comparison of the NRI, the NR and the Halley's methods, for various n , m and γ_0 .

m, n	γ_0	i	NRI	NR	Halley
2014.91, 5	1	1	1.9975	403.7820	1.4994
		2	3.8468	323.0256	2.2421
		3	4.6845	258.4205	3.2875
		4	4.5822	206.7364	4.3222
		5	4.5798	165.3891	4.5781
		6		132.3113	4.5798
		.		.	.
		24		4.5799	.
		25		4.5798	.
		.		.	.
	100	1	75.0000	80.0000	44.4445
		2	56.2500	64.0000	29.63
		3	42.1876	51.2000	19.7548
		.		.	.
		10		10.7559	4.5798
		11	5.8783	8.6348	.
		12	4.9008	6.9803	.
		13	4.6021	5.7540	.
		14	4.5800	4.9708	.
		.	4.5798	.	.

	γ_0	i	NRI	NR
1		1	0.7505	0.8003
		2	0.6750	0.6984
		3	0.6698	0.6715
		4	0.6697	0.6697
0.2012		1	0.3960	6.3266
		2	0.6503	4.7451
		3	0.6700	3.5593
		4	0.6697	2.6706
		.		...
		10		0.6793
		11		0.6699

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