



Study of the Maxwell-Boltzmann Distribution Asymmetry

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Abstract: It was shown that the asymmetry of the Maxwell-Boltzmann distribution is its invariant property, which does not depend on the distribution parameters. The analytical expressions for the symmetrical and asymmetrical parts of the distribution were obtained. The asymmetry parameters were calculated. It is concluded that the asymmetry of the Maxwell-Boltzmann distribution can be considered as a particular manifestation of the Nature's inherent property of asymmetry.

Keywords: Maxwell-Boltzmann distribution; asymmetry measures; approximation

I. Introduction

Maxwell-Boltzmann distribution (MBD) function, which is one of the most widespread in statistics, describes processes occurring in nature and in society [1]. For more than a century, this distribution has been widely used in scientific researches and practical applications such as spectroscopy (e.g., distribution of particles over energy levels [2]), economy (e.g., financial data [3]), communication (e.g., traffic measurements in communication networks [4]), and sociology [1]. Since the "bell-like" shape of the MBD has a long right-hand tail, it can be considered as belonging to a general group that includes "long-tail" distributions (e.g., Pareto distribution, Zipf's, and Newcomb–Benford's laws) [5]. All the distributions of this group were shown to be a consequence of the maximum entropy principle, which states that the entropy of a statistical system tends to reach its maximum [5, 6]. According to a theoretical study [5], all of the above distributions can be viewed as the boundaries of the same probability distribution. The mathematical expression of the MBD is the product of a parabolic and an exponential functions, whose arguments include the universal Boltzmann constant. This constant relates the energy of single particles with the temperature characterizing the ensemble of particles (individual level vs collective level). The particles are distributed over a continuous range of energy states according to the MBD. Due to its long right-hand tail, the distribution is asymmetric, which is clearly demonstrated graphically. The asymmetry of the "long-tail" distributions demonstrated by numerous empirical results, e.g., "the probability to live in a big city is higher than the probability to live in a small village" or "the probability to be poor is higher than the probability to be rich" (cited from [5]).

In view of the above, the goal of the present research was finding out whether the MBD asymmetry depends on the parameters of the statistical system and evaluating the MBD asymmetry parameters quantitatively.

Standard algebraic notations are used throughout the paper. All calculations were performed and the plots were built using the MATLAB program.

II. Theory

Let us consider a container of N molecules of ideal gas with absolute temperature T . The mass of each molecule is m . Then the relative number of the molecules with the speed in the $[v + dv]$ interval is [7]:

$$\frac{N_v}{N} dv = F dv = \frac{4}{\sqrt{\pi}} \sqrt{k} y \exp(-y) dv, \quad (1)$$

where $k = m/2k_B T$, k_B is the Boltzmann constant, and $y = kv^2$. Function F is called the MBD. The mean speed $\bar{v}$ is equal to the first statistical moment of F :

$$\bar{v} = \int_0^\infty F v dv = \frac{4}{\sqrt{\pi k}} \quad (2)$$

The n -order central moment of F

$$M_n = \int_0^\infty (F - \bar{v})^n dv \quad (3)$$

can be obtained using the complementary error function (Erfc) derivatives [8]. Therefore the skewness

$$\gamma_s = M_3/M_2^{3/2}, \quad (4)$$

as the asymmetry measure of function F , can be evaluated only numerically. We have found that, with increasing the integration limits, the second moment increases, while the third moment decreases. For very large velocity values (v_{lim}), the skewness keeps increasing slowly (Fig. 1). The dependence of the skewness on the integration limits makes this measure inconvenient to apply in practice. However, we found that the curve shown in Fig. 1 does not depend on the MBD parameters.

Earlier, as a measure of spectral line asymmetry, we used the ratio of the absolute values of the minimum and the maximum amplitude of the first-order line derivative [9]. Using this measure for the MBD is illustrated in Fig. 2. The first-order (F') and the second-order (F'') derivatives of the MBD are:

$$F' = \frac{4}{\sqrt{\pi}} \sqrt{k} 2kv(1-y)\exp(-y) \quad (5)$$

$$F'' = \frac{4}{\sqrt{\pi}} \sqrt{k} 2k\exp(-y)(2y^2 - 5y + 1) \quad (6)$$

Substituting the roots of Eq. 6 ($1.25 + \sqrt{17/16}$ and $1.25 - \sqrt{17/16}$) into Eq. 5, the asymmetry parameter is easily obtained :

$$\gamma_{d1} = |F'_{min}/F'_{max}| = 0.6733 \quad (7)$$

It should be emphasized that γ_{d1} does not depend either on mass or on temperature. The drawback of Eq. 7 is characterizing the asymmetry only at one point. The full asymmetry measure is obtained by decomposing the distribution into the symmetrical and asymmetrical parts [9]. First of all, the speed will be transformed to a dimensionless variable:

$$x = (v - v_m)/v_m, \quad (8)$$

where v_m is the most probable speed (the speed corresponding to the MBD maximum) and $v_m = 1/\sqrt{k}$ is obtained at the zero point of the first-order derivative (Eq. 5). Eq. (8) includes translating the MBD along X-axis and transforming its scale. It can be seen from Fig. 3a that the temperature increase causes the MBD broadening, but, due to the translation, the MBD maximum does not shift. The use of a new coordinate system (Eq. 8) demonstrates the invariance of the MBD form (Fig. 3b), which does not depend on the MBD parameters.

Substituting Eq. 8 into Eq. 1, we obtain

$$F = \frac{4}{\sqrt{\pi}} \sqrt{k} [\exp(-x^2 - 1)][(x + 1)^2 \exp(-2x)] \quad (9)$$

Term $(x + 1)^2$ can be approximated by the exponent

$$\exp(a_5 x^5 + a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0). \quad (10)$$

The initial values of coefficients a_k ($k = 0 - 5$) were obtained by polynomial approximation in the interval of $x = [-0.9, 2.25]$.

The final expression for the MBD has the form:

$$F_{appr} = F^{sym} F^{asym}, \quad (11)$$

where $F^{sym} = \frac{4}{\sqrt{\pi}} \sqrt{k} \exp(-1) \exp[(a_2 - 1)x^2 + a_4 x^4]$ and $F^{asym} = \exp[(a_1 - 2)x + a_3 x^3 + a_5 x^5]$. More precise values of coefficients a_k , calculated by the least squares fitting of Eq. 11 to Eq. 9, are: $a_0 = 0$, $a_1 = 1.96$, $a_2 = -0.96$, $a_3 = 0.95$, $a_4 = -0.84$, and $a_5 = 0.27$. The dependences of $F(x)$ and $F_{appr}(x)$ are shown in Fig. 3b. The relative error of the approximation is less than 0.91%.

The maximum relative error of the approximation of the right-hand side of the MBD, calculated according to Eq. 11 is less than 0.51%.

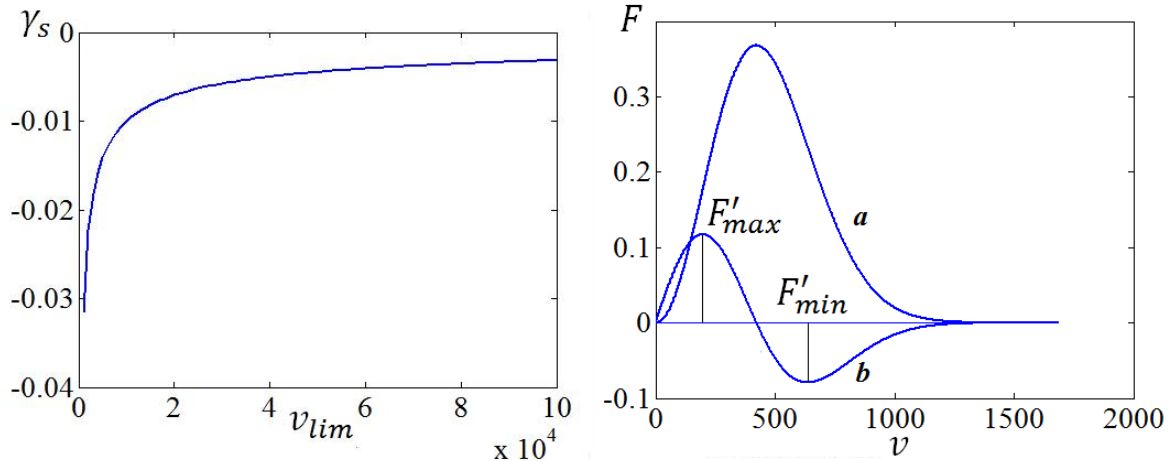
It is interesting to note that the asymmetrical part of Eq. 11 is close to that of the log-normal distribution (function $F_{LN}(t)$ [10]) if asymmetry coefficient $\tau_{LN} = -1$ [9] (the minus sign is explained by the “left” asymmetry of the log-normal distribution studied in [8]):

$$F_{LN}(t) = \begin{cases} \exp(-\frac{1}{\tau_{LN}} \ln^2 t) \\ 0 \end{cases} \approx \begin{cases} \exp[-\frac{1}{\tau_{LN}} (x^2 + \frac{11}{12} x^4) + \frac{1}{\tau_{LN}} (x^3 + \frac{1}{2} x^5)] \\ 0 \end{cases}, \quad \begin{matrix} v < a \\ v \geq a \end{matrix} \quad (12)$$

where $t = \frac{a-v}{a-v_m}$, v is abscissa, $\tau_{LN} = \ln^2 \rho / \ln 2$, v_m is the position of the peak maximum, ρ is the asymmetry parameter, and a is the value that determines the curve limits on the abscissa.

Figure 1 Skewness of the MBD of nitrogen molecule velocities first-order derivative.

Figure 2 Extremes of the MBD



$T = 300K$.

(a) MBD; (b) MBD first-order derivative.

Thus it can be seen that function $F_{LN}(t)$ is also a "long-tailed" distribution [11].

Precise decomposition of the MBD to symmetrical and asymmetrical parts was performed numerically [9] (Fig. 4a):

$$F^{sym}(x) = \sqrt{F(x)F(-x)} \quad (13)$$

$$F^{asym}(x) = \sqrt{F(x)/F(-x)} \quad (14)$$

The plots of precise (Eqs. 13, 14) and approximate (Eq. 11) decompositions are shown in Fig. 4, panels a and b, respectively. It can be seen that the central parts of the plots are similar, while the wings are different.

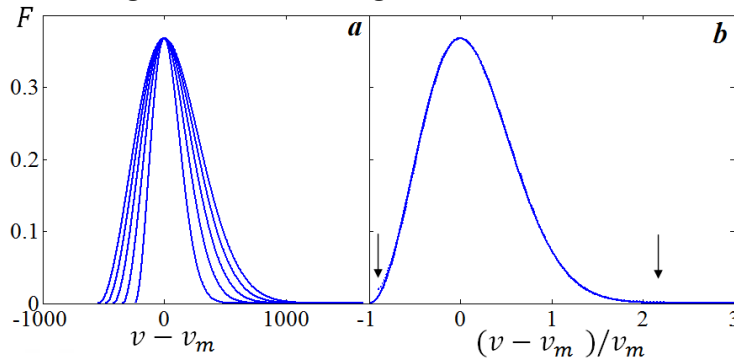
It was shown earlier [9] that the difference between the left- and the right-hand components (relative to the peak maximum) of the integral angular function, which is expressed as

$$I_\varphi(x_m) = 0.5 - (2/\pi x_m) \int_0^{x_m} \text{atan}(F^{asym}(x)) dx, \quad (15)$$

is a sensitive indicator of the line asymmetry. It can be seen from the plot of $I_\varphi(x_m)$ (Fig. 5) that this function slowly decreases in the range $x = [0.1-0.9]$. The $I_\varphi(x_m)$ plot for F_{LN} (Eq. 12) is also shown for comparison. It can be concluded that the impact of the MBD right wing is the crucial factor of the distribution asymmetry.

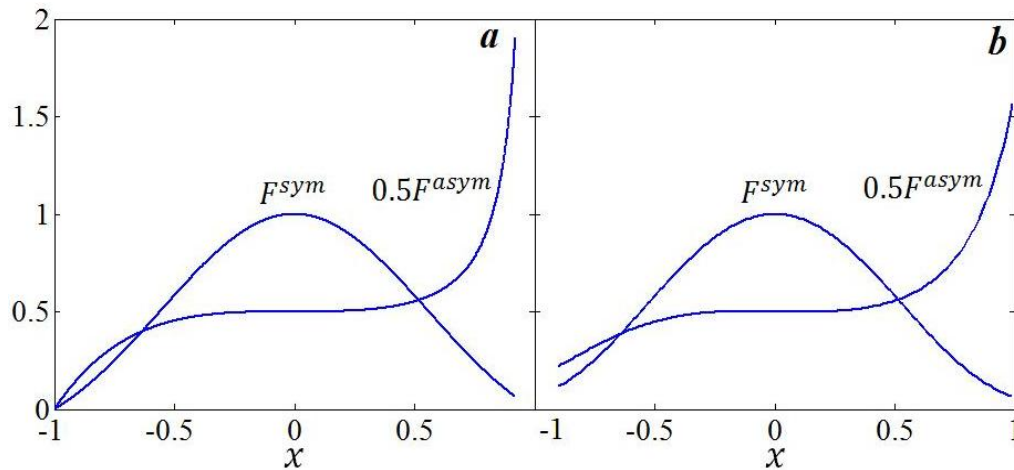
In summary, the above analysis shows that the MBD asymmetry does not depend on its parameters, being, thus, an internal property of this distribution. It can be suggested that this property reflects the Nature's inherent asymmetry. In other words, as Prof. T. D. Lee, a Nobel prize winner in physics, said, "An asymmetrical law implies an asymmetrical world" [12].

Figure 3 MBD of nitrogen molecule velocities.



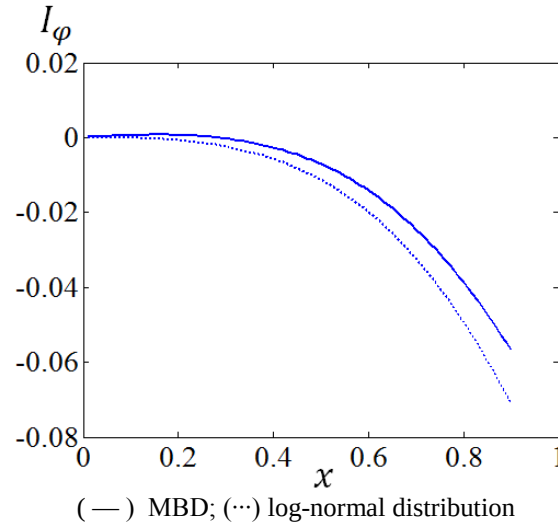
(a) MBD ($T = 500, 400, \dots, 100K$ for the curves from the narrowest to the widest, respectively).
(b) Normalized plot. The arrows show the approximation limits (Eq. 11).

Figure 4 Decomposition of MBD into the product of its symmetrical and asymmetrical parts.



Precise (a) and approximate (b) decomposition.

Figure 5. The integral angular functions of the MBD and of the log-normal distribution.



References

- [1] O. Kafri and H. Kafri, "Entropy - God's Dice Game", CreateSpace Independent Publishing Platform (August 1, 2013).
- [2] B. K. Sharma, Spectroscopy. 19th Ed. India, Meerut-Delhy: Goel Publishing House, 2007.
- [3] H. Kleinert and X.J. Chen, "Boltzmann Distribution and Market Temperature", Physica A: Statistical Mechanics and its Applications, vol. 383, 2007, pp. 513-518.
- [4] A. Feldmann and W. Whitt, "Fitting mixtures of exponentials to long-tail distributions to analyze network performance models", Performance Evaluation, vol. 31, 1998, pp. 245-279.
- [5] O. Kafri, "The Distributions in Nature and Entropy Principle", <http://arxiv.org/ftp/arxiv/papers/0907/0907.4852.pdf>.
- [6] T. Gro'nholm and A. Annala, "Natural distribution", Mathematical Biosciences, vol. 210, 2007, pp. 659-667.
- [7] F. Mandl, Statistical Physics. 2nd Ed., Manchester, England: John Wiley & Sons, 2008.
- [8] S. Gradshteyn, I. M. Ryznik, Table of Integrals, Series, and Products. 7th Ed., San Diego, USA: Elsevier, 2007.
- [9] J. Dubrovkin, "Mathematical analysis of asymmetrical spectral lines", Journal of Emerging Technologies in Computational and Applied Sciences, vol. 1-8, 2014, pp. 27-36.
- [10] E. A. Burstein and V. I. Emelyanenko, "Log-normal description of fluorescence spectra of organic fluorophores", Photochem. Photobiol., vol. 64, 1996, pp. 316-320.
- [11] M. Mitzenmacher, "A brief history of generative models for power law and lognormal distribution", Internet Mathematics, vol. 1, 2003, pp. 226-251.
- [12] T. D. Lee, Symmetries, asymmetries and the world of particles, USA: University of Washington Press, 1988.