



Mixed Quadrature Rules for Numerical Integration of Real Definite Integrals

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Abstract: Some mixed quadrature rules of degrees of precision seven and nine have been formulated for the approximate evaluation of real definite integrals in this paper. Rules constructed are found as preferable than the compound form of basic rules. The convergence of proposed rules have been studied analytically and their asymptotic error bounds determined.

Keywords: mixed quadrature rule, degree of precision, asymptotic error estimate, error bound.

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I. Introduction

Newton-Cote's type or Gauss type of quadrature rules usually used to evaluate the real define integrals of the type:

$$I(f) = \int_{-1}^1 f(x)dx. \quad (1)$$

where $f(x)$ is continuously differentiable function without having any kind of singularity defined in the interval $[-1, 1]$. It is well known that a Gauss- type of rule is more accurate than a Newton-Cote' s type of rule, although both of them evaluate the integral (1) with same number of functional evaluations. This is due to the degree of precision of n - point Gauss type rule (i.e. $(2n - 1)$) is far more than the n - point Newton-Cote's type of rule:

$$\begin{cases} n; & \text{for } n \text{ is odd} \\ (n-1); & \text{for } n \text{ is even.} \end{cases}$$

But when two rules, one from each class having the same degree of precision are mixed, a rule of higher degree of precision is developed. The developed rule is defined as the Mixed quadrature rule by Das and Pradhan [3] and Das and Hota[4]. As it is observed that the desired accuracy of an integral may not be ascertained by any of the standard basic quadrature rule, however the same can be achieved by applying the mixed quadrature rule [3] and [4]. Also, since a mixed quadrature rule is the weighted mean of two basic quadrature rules, thus it does not require further evaluation of function at any of the nodes involve in it. As a result there is no further occurrence of any type of errors like truncation error, round-off error or machine error due to the finite precision of computing machine. The basic objective of this paper is to formulate some more mixed quadrature rules with the degree of precision seven and nine which integrates an integral with less number of functional evaluations than the compound form of basic rules. Thus, a mixed quadrature rule may be preferable than the former and this fact is very much noticed in the numerical approximation of some standard test integrals examined in the section-4.

II. Formulation of Mixed Quadrature Rules

In this section, we have formulated mixed quadrature rules of precision seven and nine from the basic quadrature rules:

Gauss-Labatto 4-point rule:

$$R_1(f) = \frac{1}{6} \left[f(-1) + f(1) + 5 \left\{ f\left(-\frac{1}{\sqrt{5}}\right) + f\left(\frac{1}{\sqrt{5}}\right) \right\} \right] \quad (2)$$

along with two more quadrature rules Weddle's Rule:

$$R_2(f) = \frac{1}{10} \left[f(-1) + 6f(0) + f(1) + \left\{ f\left(-\frac{1}{3}\right) + f\left(\frac{1}{3}\right) \right\} + 5 \left\{ f\left(-\frac{2}{3}\right) + f\left(\frac{2}{3}\right) \right\} \right] \quad (3)$$

and Gauss-Legendre 3-point rule:

$$R_3(f) = \frac{1}{9} \left[8f(0) + 5 \left\{ f\left(-\sqrt{\frac{3}{5}}\right) + f\left(\sqrt{\frac{3}{5}}\right) \right\} \right] \quad (4)$$

each of which is of degree of precision five (Ref. [1,5]). At first in subsection-A, we have formulated three mixed quadrature rules each of precision seven and then following the same technique three more quadrature rules of precision nine of the same class is derived in the subsection -B from the basic rules given in equations (2) to (4).

A. **Mixed Quadrature Rules of Precision Seven**

We assume here that the function $f(x)$ is continuously differentiable in the interval $[-1,1]$. Under this assumption expanding $f(x)$ about $x = 0$ and using the Taylor's theorem we obtain:

$$f(x) = \sum_{n=0}^{\infty} c_n f^{(n)}(0) \quad (5)$$

where $c_n = \frac{f^{(n)}(0)}{(n)!}$; $n=1,2,3,\dots$ are the Taylor's coefficients.

Since the above series (5) is uniformly convergent in $[-1,1]$; thus by integrating term by term we obtain:

$$I(f) = 2c_0 + \frac{2}{3}c_2 + \frac{2}{5}c_4 + \dots \quad (6)$$

Now by substituting $x = -1$ and $x = 1$ in succession in the series given in equation (5) and then adding the results we have:

$$f(-1) + f(1) = 2c_0 + 2c_2 + 2c_4 + \dots \quad (7)$$

Further, substituting $x = -\frac{1}{\sqrt{5}}$ and $x = \frac{1}{\sqrt{5}}$ successively in the Taylor's expansion of $f(x)$ given in equation (5)

and reusing the results in equation (2) we get:

$$R_1(f) = 2c_0 + \frac{2}{3}c_2 + \frac{2}{5}c_4 + \frac{26}{75}c_6 + \frac{126}{375}c_8 + \dots \quad (8)$$

Denoting

$$E_1(f) = I(f) - R_1(f);$$

as the truncation error associated with the Labatto 4-point quadrature rule $R_1(f)$ in the approximation of the integral $I(f)$ given in (1) and (5) we obtain:

$$I(f) = R_1(f) + \frac{1}{6!} \left(-\frac{32}{525} \right) f^{(6)}(0) + \frac{1}{8!} \left(-\frac{128}{1125} \right) f^{(8)}(0) + \dots \quad (9)$$

Similarly if $E_2(f)$ and $E_3(f)$ are being the truncation errors associated with the quadrature rules $R_2(f)$ and $R_3(f)$ (Equations: (3) and (4)); proceeding as above we get:

$$I(f) = R_2(f) + \frac{1}{6!} \left(-\frac{4}{1701} \right) f^{(6)}(0) + \frac{1}{8!} \left(-\frac{184}{10935} \right) f^{(8)}(0) + \dots \quad (10)$$

and

$$I(f) = R_3(f) + \frac{1}{6!} \left(\frac{8}{175} \right) f^{(6)}(0) + \frac{1}{8!} \left(\frac{88}{1125} \right) f^{(8)}(0) + \dots \quad (11)$$

Now by multiplying $\left(\frac{1}{81}\right)$ in equation (9) and $\left(-\frac{8}{25}\right)$ in equation (10) and further adding the results obtain with subsequent simplifications we have:

$$I(f) = \frac{1}{623} [648R_1(f) - 25R_2(f)] + \frac{1}{623} [648E_1(f) - 25E_2(f)] \quad (12)$$

By denoting the first term of equation (12) as:

$$R_{12}(f) = \frac{1}{623} [648R_1(f) - 25R_2(f)] \quad (13)$$

we claim here that $R_{12}(f)$ is the mixed quadrature rule of precision seven associated with the truncation error

$$E_{12}(f) = \frac{1}{623} [648E_1(f) - 25E_2(f)] \quad (14)$$

Further, by suitably combining any two of the three equations (9) to (11) we get two more mixed quadrature rules:

$$R_{23}(f) = \frac{1}{511} [486R_2(f) + 25R_3(f)] \quad (15)$$

and

$$R_{13}(f) = \frac{1}{7} [3R_1(f) + 4R_3(f)] \quad (16)$$

associated with the corresponding error terms

$$E_{23}(f) = \frac{1}{511} [486E_2(f) + 25E_3(f)] \quad (17)$$

and

$$E_{13}(f) = \frac{1}{7} [3E_1(f) + 4E_3(f)] \quad (18)$$

respectively meant for the numerical approximation of the integral (1). Next we consider:

B. Mixed Quadrature Rules of Precision Nine

Proceeding in the same way we construct here three more mixed quadrature rules of precision nine by combining the rules $R_{12}(f)$, $R_{23}(f)$ and $R_{13}(f)$ as formulated earlier in the above subsection. To avoid the repetition of our work we simply state the rules as:

$$Q_1(f) = \frac{1}{43339} [45475R_{13}(f) - 2136R_{12}(f)] \quad (19)$$

$$Q_2(f) = \frac{1}{114097} [132787R_{23}(f) - 18690R_{12}(f)] \quad (20)$$

and

$$Q_3(f) = \frac{1}{583} [875R_{13}(f) - 292R_{23}(f)] \quad (21)$$

with their corresponding truncation errors

$$E_{Q1}(f) = \frac{1}{43339} [45475E_{13}(f) - 2136E_{12}(f)] \quad (22)$$

$$E_{Q2}(f) = \frac{1}{114097} [132787E_{23}(f) - 18690E_{12}(f)] \quad (23)$$

and

$$E_{Q3}(f) = \frac{1}{583} [875E_{13}(f) - 292E_{23}(f)] \quad (24)$$

for the numerical integration of the real definite integrals of the type(1).

III. Error Analysis

In this section we have obtained the asymptotic error bounds of the truncation errors associated with the mixed quadrature rules of precision seven and nine along with their convergence have been studied by following the techniques due to Lether [6]. The analytical convergence and the error bound of both the class of rules are given in the Theorem-1 and Theorem-2 respectively.

A. Asymptotic Error Bounds of Degree Seven Rules

The first leading terms of the truncation errors $E_{12}(f)$, $E_{23}(f)$ and $E_{13}(f)$ associated with the degree seven rules $R_{12}(f)$, $R_{23}(f)$ and $R_{13}(f)$ are given in Table-1.

Table-1

Rules	First Leading Term of Error Expressions
$R_{12}(f)$	$-2.15 \times 10^{-6} f^{(8)}(0)$
$R_{23}(f)$	$-3.02 \times 10^{-7} f^{(8)}(0)$
$R_{13}(f)$	$-1.08 \times 10^{-7} f^{(8)}(0)$

From Table-1, it is further evident that all the rules as referred here are of degree of precision seven and the rule $R_{13}(f)$ may provided a better accuracy among the rules of same class when each of the rules is to be applied for the numerical evaluation of an integral of the type (1).

Theorem-1:

If $f(z)$ is the analytic continuation of $f(x)$ in a closed disc:

$$\Omega = \{z \in \mathbb{C} : |z| \leq r, r > 1\};$$

$$|E_{pq}(f)| \leq Me_{pq}(r); \text{ for } p \neq q; p, q = 1, 2, 3.$$

$$M = \max_{|z|=r} |f(z)|;$$

$$e_{12}(r) = \left| r \ln \left(\frac{r+1}{r-1} \right) - \frac{1}{623} \left[\frac{1819}{15} \left(\frac{r^2}{r^2-1} \right) - \frac{625}{3} \left(\frac{r^2}{5r^2-1} \right) + \left(\frac{31498r^4 - 52488r^2}{405r^4 - 225r^2 + 20} \right) + \frac{1944}{5} \right] \right|;$$

$$e_{23}(r) = \left| r \ln \left(\frac{r+1}{r-1} \right) - \frac{1}{511} \left[\frac{486}{5} \left(\frac{r^2}{r^2-1} \right) + \frac{4374}{5} \left(\frac{r^2}{9r^2-1} \right) + 4374 \left(\frac{r^2}{9r^2-4} \right) + \frac{1250}{9} \left(\frac{r^2}{5r^2-3} \right) + \frac{14122}{45} \right] \right|; \text{ and}$$

$$e_{13}(r) = \left| r \ln \left(\frac{r+1}{r-1} \right) - \frac{1}{7} \left[\left(\frac{r^2}{r^2-1} \right) + \left(\frac{25r^2}{5r^2-1} \right) + \left(\frac{360r^2-96}{45r^2-27} \right) \right] \right|;$$

each of which $\rightarrow 0$; as $r \rightarrow \infty$. The quantities $e_{pq}(r)$ are defined as error constant by Lether [6]. Since the derivation of this theorem for the given values of p and q are similar, to avoid the repetition here we prove only for the case $p=1$ and $q=3$. The rest of the case of the theorem for different values of p and q can be derived in the same way.

Proof:

Let

$$f(z) = f(x); \text{ for } z \in [-1, 1].$$

Since $E_{13}(f)$ denotes the truncation error in approximation of integral $I(f)$ by the rule $R_{13}(f)$ i.e.

$$I(f) = R_{13}(f) + E_{13}(f) \quad (25)$$

and $E_{13}(f)$ being a linear operator, we obtain from equations (5), (16), and (25) the following:

$$E_{13}(f) = \sum_{k=0}^{\infty} c_k E_{13}(x^k) \quad (26)$$

where

$$E_{13}(x^k) = \int_{-1}^1 x^k dx - R_{13}(x^k); k \geq 0.$$

Further since the rule $R_{13}(f)$ is a fully symmetric quadrature rule of precision seven, thus

$$E_{13}(x^k) = 0;$$

for $k=0(1)7$ and k as odd. Hence equation (26) further simplifies to

$$E_{13}(f) = \sum_{\mu=4}^{\infty} c_{2\mu} E_{13}(x^{2\mu}).$$

However since

$$E_{13}(x^{2\mu}) = 2 \left[\frac{1}{2\mu+1} - \frac{1}{14} \left\{ 1 + \left(\frac{1}{5} \right)^{\mu-1} - \frac{40}{63} \left(\frac{3}{5} \right)^{\mu} \right\} \right] < 2 \left[\frac{1}{2\mu+1} - \frac{1}{14} \right] < 0; \text{ for } \mu \geq 4;$$

thus by using Cauchy- inequality [2];

$$|E_{13}(f)| \leq M \sum_{\mu=4}^{\infty} \frac{1}{r^{2\mu}} |E(x^{2\mu})| \quad (27)$$

Now by following the technique due to Lether [6],

$$|E_{13}(f)| \leq M e_{13}(r);$$

where

$$e_{13}(r) = \left| E_{13} \left[\left(1 - \frac{x}{r} \right)^{-1} \right] \right|.$$

But,

$$\left| E_{13} \left[\left(1 - \frac{x}{r} \right)^{-1} \right] \right| = \left| r \ln \left(\frac{r+1}{r-1} \right) - \frac{1}{7} \left[\left(\frac{r^2}{r^2-1} \right) + \left(\frac{25r^2}{5r^2-1} \right) + \left(\frac{360r^2-96}{45r^2-27} \right) \right] \right|.$$

From the above expression, it is observed that $e_{13}(r) \rightarrow 0$; for $r \rightarrow \infty$; which in turn implies that,

$E_{13}(f) \rightarrow 0$; for $r \rightarrow \infty$. This completes the proof of the theorem.

B. Comparative Analysis of Error Constants

The numerical values of the error constants $e_{12}(r)$, $e_{23}(r)$ and $e_{13}(r)$ associated with the degree seven rules $R_{12}(f)$, $R_{23}(f)$ and $R_{13}(f)$ are determined for different values of $r > 1$ which is depicted in Table-2. Also based on these tabulated values, respective graphs of these error constants have been drawn and is given in Fig.-1. From the table and the graph as well, it is evident that:

$$e_{13}(r) < e_{23}(r) < e_{12}(r) \quad (28)$$

The above empirical relation further signifies that the rule $R_{13}(f)$ is the rule of maximum accuracy in its class.

Next we consider the error analysis of mixed quadrature rules of degree nine as formulated in this paper.

Table-2

r	$e_{12}(r)$	$e_{23}(r)$	$e_{13}(r)$
1.1	0.18956629	0.18166269	0.09725224
1.3	0.01095945	0.01040365	0.00446771
1.5	0.00187596	0.00177489	0.000695488
1.9	0.00016063	0.00015159	0.00005500
2.5	0.00001275	0.00001201	0.00000418
3.3	0.00000115	0.00000109	0.00000037
4.4	0.00000010	0.00000010	0.00000003
5.8	0.00000001	0.00000001	0.00000000

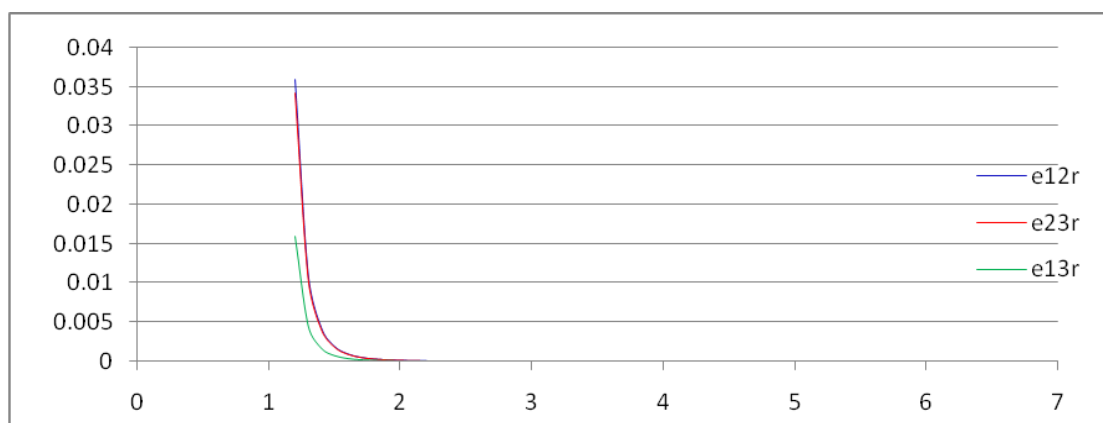


Fig.-1

C. Asymptotic Error Estimates of Degree Nine Rules

The first leading term of the truncation errors $E_{Q1}(f)$, $E_{Q2}(f)$ and $E_{Q3}(f)$ associated with the degree nine rules $Q_1(f)$, $Q_2(f)$ and $Q_3(f)$ respectively are presented in Table-3. From the table of values it is vividly

observed that the rule $Q_3(f)$ will give better approximation to an integral among all the degree nine rules constructed in this paper for the numerical integration of real definite integrals.

Table-3

Rules	First Leading Term of Error Expressions
$Q_1(f)$	$-3.2 \times 10^{-8} f^{(10)}(0)$
$Q_2(f)$	$1.3 \times 10^{-8} f^{(10)}(0)$
$Q_3(f)$	$1.7 \times 10^{-9} f^{(10)}(0)$

Theorem-2:

If $f(z)$ is the analytic continuation of $f(x)$ in a closed disc:

$$\Omega = \{z \in \mathbb{C} : |z| \leq r, r > 1\};$$

then

$$|E_{Q_k}(f)| \leq M e_{Q_k}(r); k=1, 2, 3$$

where

$$e_{Q_1}(r) = \frac{1}{43339} [45475e_{13}(r) - 2136e_{12}(r)]$$

$$e_{Q_2}(r) = \frac{1}{114097} [132787e_{23}(r) - 18690e_{12}(r)] \text{ and}$$

$$e_{Q_3}(r) = \frac{1}{583} [875e_{13}(r) - 292e_{23}(r)].$$

each of which $\rightarrow 0$; for $r \rightarrow \infty$. The quantities $e_{Q_k}(r)$; $k=1,2,3$ are defined as error constant by Lether [6]. Theorem-2 can be proved as likely to the proof of Theorem-1. For which, the proof of Theorem-2 is avoided here.

D. Comparative Analysis of Error Constants

Different values of the error constants $e_{Q_k}(r)$; for $k=1,2,3$ associated with the degree nine mixed quadrature rules $Q_k(f)$ for $k=1,2,3$ are depicted in Table-4 and the graph drawn based on these tabulated values are presented in Fig.-2. From the graph as well as from the Table-4 it is observed that:

$$e_{Q_3}(r) < e_{Q_2}(r) < e_{Q_1}(r) \quad (29)$$

Further from Table-3 and equation (29) it may be taken as a conclusion that the degree nine rule $Q_3(f)$ is the rule of maximum accuracy in its class of rules constructed in this paper for the approximate evaluation of real definite integrals. This observation is also very much noticed in the numerical integration of standard integrals considered in this paper and the results of numerical integrations are given in next section.

Table-4

r	$e_{Q_1}(r)$	$e_{Q_2}(r)$	$e_{Q_3}(r)$
1.2	0.033873725	0.014940489	0.006792361
1.5	0.001758337	0.000637307	0.000154859
1.7	0.000455826	0.000156323	0.000027429
2.2	0.000037689	0.000012135	0.000001138
2.7	0.000006040	0.000001895	0.000000111
3.3	0.000001081	0.000000333	0.000000013
4.1	0.000000175	0.000000053	0.000000001
5.2	0.000000025	0.000000008	0.000000000

IV. Numerical Experiments

In this section the real definite integral

$$I = \int_{-1}^1 e^x dx$$

has been numerically integrated by applying the mixed quadrature rules of precision seven and nine as constructed in this paper. Further, this integrals has been also evaluated by the compound form of the basic rules which are used for the formulation of mixed quadrature rules in this paper. The result of the numerical integration is given in Table-5.

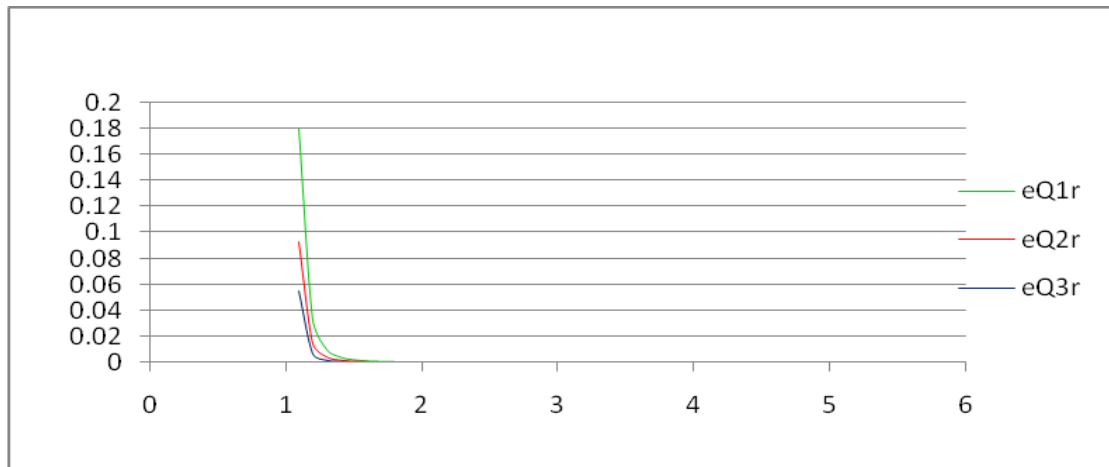


Fig.2
Table-5

Rules	Approx. of $I = \int_{-1}^1 e^x dx$	Number (n) of Functional Evaluations Required by Compound Lobatto 4-pointRule	Number (n) of Functional Evaluations Required by Compound Weddle's Rule	Number (n) of Functional Evaluations Required by Compound Gauss-Legendre 3-point Rule
$R_1(f)$	2.350489908	4	-	-
$R_2(f)$	2.350406081	-	7	-
$R_3(f)$	2.350336929	-	-	3
$R_{12}(f)$	2.350402717	-	-	-
$R_{23}(f)$	2.350402697	-	-	-
$R_{13}(f)$	2.350402491	12	14	9
$Q_1(f)$	2.350402494	20	21	12
$Q_2(f)$	2.350402480	-	-	-
$Q_3(f)$	2.350402387	36	35	24
Exact Value	2.350402387	**	**	**

V. Conclusion

It is observed that the mixed quadrature rules of precision seven and nine are by far superior than all the compound rules with respect to the number of functional evaluations. Also since a mixed quadrature rule of higher precision reuses the values of function already evaluated in case of the rules of lower precision, but it is not so in case of compound rules with subsequent subdivision of interval of integration thus a mixed quadrature rules of higher precisions are preferable than the compound form of basic rules.

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