



Mathematical applications of mechanics and analyze its solutions by using Laplace and Inverse Laplace Transform

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Abstract: Mathematics is a part of science in which the properties and interactions of idealized objects are examined. It always played a special role in scientific thought, serving since ancient times as a model of truth and giving tools or even a foundation for other sector sciences. The Laplace transform and the inverse Laplace transform together have a number of properties that make them useful for analyzing linear dynamic systems. Laplace transform is a powerful mathematical technique for solving ordinary and partial differential equations and also initial and boundary value problems of differential equation arising in mathematics. We have studied the applications of mechanics and analyzed its solution nature. The graphical representations in this paper are made by using Mathematica.

Keywords: Transform, Application, Force

I. Introduction

Laplace transform was mainly originated in attempts to justify rigorously certain operational rules used by Heaviside in the latter part of the 19th century for solving equations of electromagnetic theory. In the early part of 20th century through the efforts of Bromwich, van der Pol, Carson and other mathematicians were finally succeed in these attempts. The term 'Laplace' was named by PIERRE SIMON DE LAPLACE (23 March 1749-5 March 1827). He was a great mathematician and astronomer whose work was pivotal to the development of mathematical astronomy and statistics [01]. He summarized and extended the work of his predecessors in his five-volume Celestial Mechanics (1799–1825). This work translated the geometric study of classical mechanics to one based on calculus, opening up a broader range of problems. In statistics, the Bayesian interpretation of probability was developed mainly by Laplace [02].

Laplace transforms and Inverse Laplace transform is a very useful part in mathematics. Laplace and Inverse Laplace transform both are applicable to solve various types of problems. Laplace transform is highly useful in various types of problems. If we take Laplace transform of a mechanics, the ordinary differential equations with variable coefficients, and the partial differential equations, problems on electrical circuits and boundary value problems. Then it becomes an algebraic equation for the transform equation. Solving these equation and applying inverse Laplace transform. We get the required result [03].

A particle P of mass 2 grams moves on the X axis and is attracted toward origin O with force numerically equal to $8X$. If it is initially at rest at $X = 10$, find its position at any subsequent time assuming (a) no other forces act, (b) a damping force numerically equal to 12 times the instantaneous velocity acts. [04]

(a) Choose the positive direction to the right figure 01 When $X > 0$, the net force is to the left (i.e. is negative) and must be given by $-8X$. When $X < 0$ the net force is to the right (i.e. is positive) and must be given by $-8X$. [05]. Then by Newton's law,
(Mass).(Acceleration) = Net force

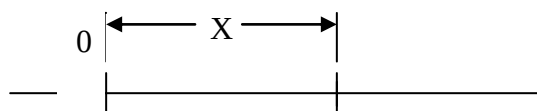


Fig. 01

$$2 \cdot \frac{d^2 X}{dt^2} = -8X$$

$$\frac{d^2 X}{dt^2} + 4X = 0 \quad (1) \text{ The initial conditions are: } (2) X(0) = 10, (3) X'(0) = 0.$$

Taking the Laplace transform of (1) and using condition (2) and (3), we have, if $x = \mathcal{L}\{X\}$,

$$s^2 x - 10s + 4x = 0$$

$$\text{Or, } x = \frac{10s}{s^2 + 4}$$

$$\text{Then } X = \mathcal{L}^{-1}\{x\}$$

$= 10 \cos 2t$ Then the graph of the motion is shown in figure 02 below. The amplitude [maximum displacement from 0] is 10, the period [time for a complete cycle] is π , the frequency [number of cycles per second] is $1/\pi$.

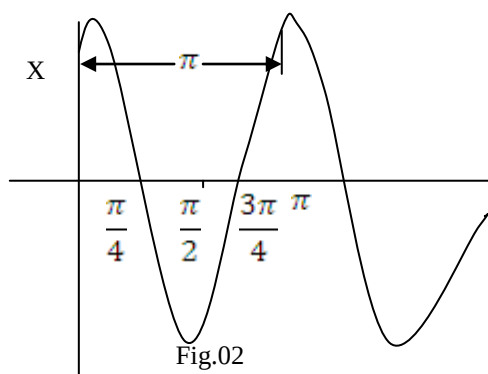


Fig.02

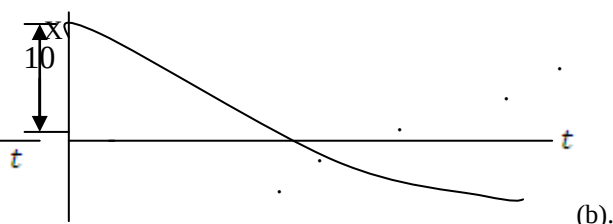


Fig.03

when $X > 0$ and $dX/dt > 0$, p is on the right of 0 and moving to the right. Then the damping force is to the left (i.e. is negative) and must be given by $-8 dX/dt$. similarly when $X < 0$ and $dX/dt < 0$, P is on the left moving to the left so the damping force is to the right (i.e. is positive) and must also be given by $-8 dX/dt$. the damping force is also $-8 dX/dt$. for the cases $X > 0, dX/dt < 0$ and $X < 0, dX/dt > 0$. [06] Then

(Mass)(Acceleration)=Net force

$$\text{Or, } 2 \frac{d^2 X}{dt^2} = -8X - 8 \frac{dX}{dt}$$

$$\text{i. e. } \frac{d^2 X}{dt^2} + 4 \frac{dX}{dt} + 4X = 0 \quad (03)$$

$$\text{with initial condition } X(0)=10 \quad (04)$$

$$X(0)=0 \quad (05)$$

Taking the Laplace transform of (03) and using conditions (04) and (05) we have

$$s^2 x - 10s + 4(sx - 10) + 4x = 0$$

$$x = \frac{10s + 40}{s^2 + 4s + 4}$$

$$\text{Then } X = \mathcal{L}^{-1}\{x\}$$

$$\begin{aligned} &= \mathcal{L}^{-1}\left\{\frac{21s + 40}{(s + 2)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{10(s + 2) + 20}{(s + 2)^2}\right\} = 10\mathcal{L}^{-1}\left\{\frac{1}{s + 2}\right\} + 20\mathcal{L}^{-1}\left\{\frac{1}{(s + 2)^2}\right\} \\ &= 10e^{-2t} + 20te^{-2t} \\ &= 10e^{-2t}(1 + 2t) \end{aligned}$$

The graph of X versus t is shown in above figure. We observe that the motion is non-oscillatory. The particle approaches 0 but never reaches it.

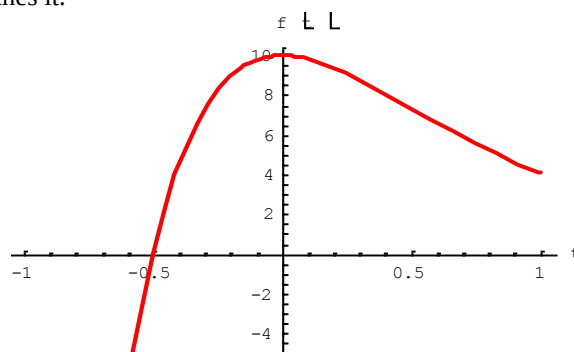


Fig.04 Graph of $10e^{-2t}(1 + 2t)$

Find the solution of Laplace Transform following boundary value problem

$$\frac{\partial^2 Y}{\partial t^2} = 9 \frac{\partial^2 Y}{\partial x^2}, Y(0, t) = 0, Y(2, t) = 0, Y(x, 0) = 20 \sin 2\pi x \text{ and with physical interpretation [04].}$$

Solution:

$$\text{Given that } \frac{\partial^2 Y}{\partial t^2} = 9 \frac{\partial^2 Y}{\partial x^2} \quad (06)$$

Taking the Laplace Transform of the partial differential equation (06) we have

$$\begin{aligned} \Lambda\left\{\frac{\partial^2 Y}{\partial t^2}\right\} &= 9 \Lambda\left\{\frac{\partial^2 Y}{\partial x^2}\right\} \\ \text{Or, } s^2 Y - sY(x, 0) - Y_t(x, 0) &= 9 \frac{d^2 y}{dx^2} \\ \text{Or, } \frac{d^2 y}{dx^2} - \left(\frac{s}{4}\right)^2 y &= -\frac{20}{9} \sin 2\pi x \\ \text{Or, } \frac{d^2 y}{dx^2} - \left(\frac{s}{4}\right)^2 y &= -\frac{20}{9} \sin 2\pi x \end{aligned} \quad (07)$$

$$\text{Now the Auxiliary equation is } m^2 - \left(\frac{s}{3}\right)^2 = 0$$

$$\therefore m = \pm \frac{s}{3}$$

$\therefore$ Complementary function (C.F) is

$$y_c = c_1 e^{\frac{s}{3}x} + c_2 e^{-\frac{s}{3}x} \quad (08)$$

$$\text{Particular integral (P.I) is, } u_p = y_p = \frac{1}{D^2 - \frac{s^2}{9}} \left(-\frac{20}{9} \sin 2\pi x\right) = \frac{20 \sin 2\pi x}{s^2 + 36\pi^2} \quad (09)$$

Hence the general solution of (07) is

$$\begin{aligned} y &= y_p + y_c \\ \frac{dy}{dx} &= c_1 e^{\frac{s}{3}x} + c_2 e^{-\frac{s}{3}x} + \frac{20 \sin 2\pi x}{s^2 + 36\pi^2} \end{aligned} \quad (10)$$

Taking the t Laplace transform of those boundary conditions which involve t, we have

$$\Lambda\{Y(0, t)\} = Y(0, s) = 0$$

$$\Lambda\{Y(2, t)\} = y(2, s) = 0$$

Using the first condition $y(0, s)$ in (31), we have

$$\text{Or, } c_1 + c_2 = 0 \quad (11)$$

Again using the condition $y(2, s) = 0$ in (31), we get

$$\begin{aligned} 0 &= c_1 e^{\frac{2s}{3}} + c_2 e^{-\frac{2s}{3}} \\ \text{Or, } c_1 e^{\frac{2s}{3}} + c_2 e^{-\frac{2s}{3}} &= 0 \end{aligned} \quad (12)$$

Solving the above two equations (11) and (12) for c_1 and c_2 , we have

$$c_1 = 0, c_2 = 0$$

Putting $c_1 = 0, c_2 = 0$ in (10) we get

$$y = \frac{20 \sin 2\pi x}{s^2 + 36\pi^2} \quad (13)$$

Now taking the inverse Laplace Transform of both sides of (13) we get

$$\mathcal{L}^{-1}\{y\} = 20 \sin 2\pi x \mathcal{L}^{-1}\left\{\frac{1}{s^2 + (6\pi)^2}\right\}$$

$$\text{Or, } Y(x, t) = 20 \sin 2\pi x \cos 6\pi t$$

This is the required solution.

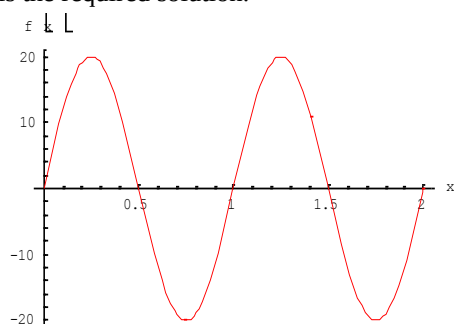


Fig.05 Graph of $20\sin 2\pi x$

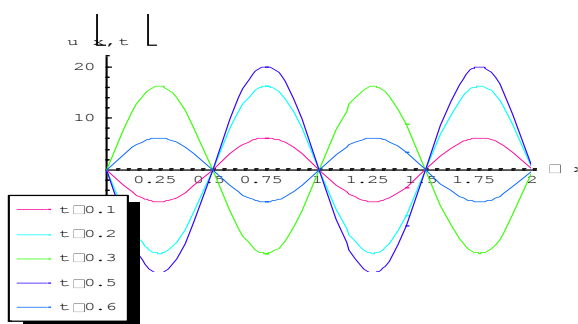


Fig.06 Graph of $20 \sin 2\pi x \cos 6\pi t$

Physical interpretation: The given boundary value problem can be interpreted physically in terms of the vibrations of string. This string has its ends fixed at $x = 0$ and $x = 2$ and is given an initial shape $f(x) = 20 \sin 2\pi x$. Which is shown in the figure 3.5. It is then released so that its initial velocity is zero. [07] The required result $Y(x, t) = 20 \sin 2\pi x \cos 6\pi t$ gives the displacement of any point x of the string at any time t . From figure we conclude that the displacement, $Y(x, t)$, decreases for the value of $t = 0.1, t = 0.2$ and $t = 0.5$ in the first portion while the reverse trend is found in the next portion. [08] Also we see that the particular value of $t (t = 0.3, t = 0.6)$ decreases for increasing the displacement, $Y(x, t)$, while the opposite effects is observed for the next section. [09]

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