



Solutal Convection in a Gravity Modulated Mushy Layer During the Solidification of Binary Alloys

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Abstract- The present paper deals with the analytical study of the solutal convection in a mushy layer during the solidification of binary alloys, subject to vertical stratification. This is a challenging problem in applied mathematics, and computational techniques are used for the analysis. The gravitational field in the present case is no longer a constant, but comprises a sinusoidally varying part also. The analysis is carried out for both the synchronous and subharmonic solutions. The results of the present investigation reveal that, up to the transition point, the increase in the frequency of vibrations enormously stabilizes the system while it has a destabilizing effect above the transition point. Another interesting point observed is, the destabilizing effect of Stefan-s.c ratio.

Key words- Mushy layer, Convective instability, gravity modulation, solidification, Rayleigh number

I. Introduction

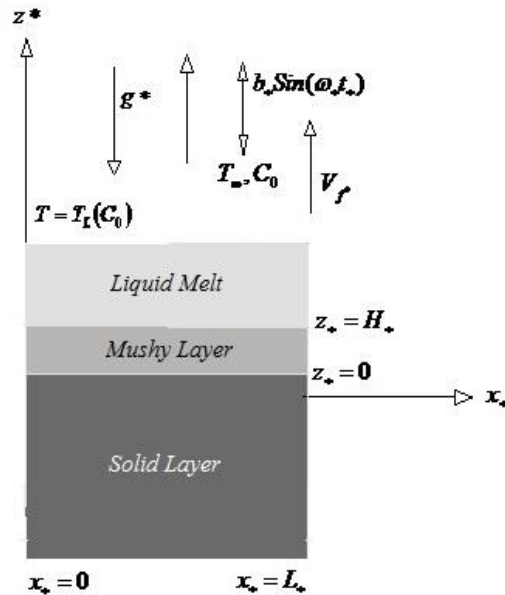
Alloyed components have extensive applications and a consistent internal structure is paramount to the performance as well as the integrity of the component. The internal structure is associated with its solidification from liquid melt phase to solid. During the solidification process of a molten alloy, a mixed phase occurs between the liquid melt and the frozen solid which advances into the liquid melt from the cooled surface. Such a mixed phase is called a mushy zone, in the literature which is found to be a loose matrix of crystalline solids, in the form of dendrites, through which the percolation of the liquid melt is possible. For all practical purposes, a mushy layer is considered as a porous medium and hence the study is also of great interest to the researchers in the porous media. In fact, the analogous behavior that exists between a solidifying metallic alloy and a freezing aqueous salt solution has led to numerous investigations. The origin of freckles was initially proposed by [5] and [6] proposed a mathematical model for the mushy layer for the limiting case where there is no coupling between convection and solidification processes. In [11] the author performed a stability analysis and predicted two modes of convection viz., the boundary layer mode and mushy layer mode. A weakly nonlinear analysis of convection in a mushy layer is conducted in [1], [2] and [3] extended the model of [1] by adopting large Stefan number scaling and observed oscillatory mode of convection different to the double diffusive mode observed by [4],[11],[12] comprises linear as well as non-linear analysis of convection in a mushy layer for constraints like modulation, rotation, etc. Recently, [9] have studied convection in a mushy layer in the presence of gravity modulation. The dynamics associated with a mushy layer is governed by the complex interaction between the compositional convection and the solidification process. The formulation of freckles is a direct result of compositional convection and they are in fact the non-uniformities that manifest themselves as vertical channels which are of a different composition than the surrounding solid. The objective of the present work is to analyze the stability of solutal convection in a mushy layer including the effects of rotation with a Stefan number of unit order of magnitude, during the solidification of binary alloys subject to a constant vertical stratification i.e., modulated mushy-layer convection under rotation.

The study is structured as follows: Section 2 deals with the formulation of the problem. In section 3, a linear stability analysis is carried out in detail and section 4, deals with the results and conclusion

II. Formulation of the Problem

In figure 1, a binary alloy cooled from below, subject to vibration parallel to the gravitational field in the vertical direction and uniform rotation is presented. The solidification process results in three distinct regions forming viz., The solid region, of a temperature below the eutectic temperature T_E a liquid melt region, with a temperature above the liquidus temperature $T_L(C_0)$ and a mushy layer is sandwiched between the solid layer and the liquid melt. The composition at the mush-liquid interface is C_0 and the composition at the mush-solid interfaces C_E . The mushy layer is of constant height H^* and width L^* , similar to the model of [1] and [7]

Figure 1 Schematic of the problem showing the imposed boundary conditions



This results in the mushy layer having rigid and isothermal upper and lower boundary conditions where the vertical components of velocity is zero are presented by physically isolating and dynamically decoupling the mushy layer from the solid region below and the liquid melt above. Subject to these conditions and performing a transformation for the translating frame of reference (for solidification), the following dimensional set of governing equations for continuity, energy, solute and Darcy, are proposed

$$\left. \begin{aligned} \nabla_* \cdot \mathbf{V}_* &= 0 \\ q \left(\frac{\partial}{\partial t_*} - \mathbf{V}_{f*} \cdot \frac{\partial}{\partial \mathbf{z}_*} \right) T_* + q_f \mathbf{V}_* \cdot \nabla_* T_* &= \nabla_* \cdot (k_m \nabla_* T_*) + h_{fs} \left(\frac{\partial}{\partial t_*} - \mathbf{V}_{f*} \cdot \frac{\partial}{\partial \mathbf{z}_*} \right) \phi \\ (1 - \phi) \left(\frac{\partial}{\partial t_*} - \mathbf{V}_{f*} \cdot \frac{\partial}{\partial \mathbf{z}_*} \right) C_* + \mathbf{V}_* \cdot \nabla_* T &= \nabla_* \cdot (D_m \nabla_* T_*) + (C_* - C_s) \left(\frac{\partial}{\partial t_*} - \mathbf{V}_{f*} \cdot \frac{\partial}{\partial \mathbf{z}_*} \right) \phi \\ \frac{1}{(1 - \phi)} \left(\frac{\partial}{\partial t_*} - \mathbf{V}_{f*} \cdot \frac{\partial}{\partial \mathbf{z}_*} \right) \mathbf{V}_* + \frac{\mu_*}{\prod_*} \frac{1}{\rho_*} \mathbf{V}_* &= \frac{-1}{\rho_*} \nabla_* p_* + \rho_* (g_* + b_* \omega_* \sin(\omega_* t_*)) \hat{e}_g - 2\Omega \hat{e}_w \times \vec{V} \end{aligned} \right\} (I)$$

Here b^* and ω^* refers to the amplitude and frequency of the imposed vibration. The specific heat per unit volume (q) and the mush thermal conductivity (k_m) of the mushy layer are given as

The dimensionless governing differential equations are:

$$\nabla \cdot \mathbf{V} = 0 \quad (1)$$

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial z} \right) (T - St\phi) + \mathbf{V} \cdot \nabla T = \nabla^2 T \quad (2)$$

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial z} \right) ((1 - \phi)T - \zeta\phi) + \mathbf{V} \cdot \nabla T = \frac{1}{Le} \nabla \cdot ((1 - \phi) \nabla T), \quad (3)$$

$$\frac{1}{(1 - \phi)\chi_0} \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial z} \right) \mathbf{V} + \prod(\phi) \cdot \mathbf{V} = -\nabla p - Ra_m [1 + \Lambda \sin(\Omega t)] T \cdot \hat{e}_z - T^{\frac{1}{2}} a \hat{e}_w \quad (4)$$

A number of dimensionless parameters emanate from dimensionless analysis. In Equation (2), St is the Stefan number, and represents the ratio of latent heat (h_{fs}) to heat content or internal energy, and is defined as $St = h_{fs} / (q_{eff} \Delta T)$. One also observes the Lewis number, Le , in Equation (2), which represents the ratio of thermal to solutal diffusivity and is defined as: $Le = \lambda_* / D_f$ and $[2\omega^* \pi_0 / \delta_* Q_0]^2$ is the Taylor number. The effective thermal diffusivity λ_* is defined as the ratio of the thermal conductivity and specific heat per unit volume: $\lambda_* = k_{eff} / q_{eff}$. The composition ratio ζ , in the Equation (2), relates the difference in the characteristic compositions of the liquid and solid phases with the varying composition of liquid within the mushy layer, and is defined as $\zeta = (C_s - C_0) / (C_0 - C_E)$. In the case of solidifying binary alloys, the Lewis number Le , usually assumes large values thereby resulting in the right hand side of the Equation (2) becoming negligible. Equation (2) may therefore be reduced to,

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial z} \right) ((1 - \phi)T - \zeta\phi) + V \cdot \nabla T = 0, \quad (5)$$

for large Lewis numbers. In the equation (4), the modified Darcy-Prandtl number (defined as $\chi_o = Pr \mathcal{G}_o$), relevant to the solidification-type problems, normally assumes small values for binary alloy mixtures [13] and is thereby resulting in the retention of the time derivative in the Darcy Equation. The mushy layer Rayleigh number Ra_m is defined as $Ra_m = \Pi_0 \beta_c \Delta C / \nu_* V_{f*}$. Note that μ_* refers to the dynamic viscosity, ρ_* is the density, and g_* is the gravitational acceleration.

In Equation (4), the retardability function is defined as $\Pi(\phi) = \Pi_0 / \Pi_*$ (where Π_0 is the characteristic permeability and Π_* is the permeability of mushy layer), and the dimensional amplitude Λ is defined as $\Lambda = (\kappa \cdot Fr_m) \cdot \Omega^2$ (where $\kappa = b_* / H_*$ and $\Omega = (\delta^2 \omega_* \lambda_*) / V_{f*}^2$).

The modified Froude Number, $Fr_m = \lambda_* / (H_*^3 g_*) = Fr_C \delta^2$, where Fr_C is the classic Froude Number, $Fr_C = V_{f*} / (H_* g_*)$ defined in terms of the front velocity, gravitational acceleration and mushy layer height.

The boundary conditions are: rigid and isothermal upper and lower boundary conditions together with vanishing of the vertical component of velocity. This physically isolates and dramatically decomposes the mushy layer from the solid rigid below and the liquid melt above.

III. Linear Stability Analysis

For the equations (1) to (4), a new scaling is introduced for the dependant variables in terms of the mushy layer depth δ are as follows:

$$\bar{X} = \delta \tilde{X}, t = \delta^2 \tilde{t}, R^2 = \delta Ra_m, p = R \tilde{p}, \bar{V} = \frac{R}{\delta} \tilde{V}$$

Basic State solution

Using an expansion in δ where the basic state has expansions:

$$[T_B, V_B, \phi_B, p_B] = [T_{B0}, V_{B0}, \phi_{B0}, p_{B0}] + \delta [T_{B1}, V_{B1}, \phi_{B1}, p_{B1}] + \delta^2 [T_{B2}, V_{B2}, \phi_{B2}, p_{B2}] \quad (6)$$

And a motionless state associated with basic flow implies that,

$$V_B = 0, \partial T / \partial \tilde{t} = 0, \partial T / \partial \tilde{X} = 0, \partial T / \partial \tilde{Y} = 0,$$

$$\partial \phi / \partial \tilde{t} = 0, \partial \phi / \partial \tilde{X} = 0 \quad \text{and} \quad \partial \phi / \partial \tilde{Y} = 0,$$

Substituting Equation (6) in Equation (1-4) yields the motionless basic state solution for the temperature and solid fraction to each order of δ subject to the boundary conditions:

$$T_B = T_{B0} + \delta T_{B1} = (\tilde{z} - 1) + \delta \left(-\frac{(1 + \tilde{S} / C_s)}{2} \tilde{z}^2 + \frac{2 + (1 + \tilde{S} / C_s)}{2} \tilde{z} - 1 \right) \quad (7)$$

$$\phi_B = \delta \phi_{B1} + \delta^2 \phi_{B2} = (\tilde{z} - 1) + \delta \left(\frac{1 - \tilde{z}}{C_s} \right) + \delta^2 \left(\frac{1}{C_s} \left(\frac{1}{2} \left(\frac{2}{C_s} + (1 + \tilde{S} / C_s) \right) \tilde{z}^2 \right) \right)$$

$$-\left(1 + \frac{(1 + \tilde{S}/C_s)}{2} + \frac{2}{C_s}\right) \tilde{z} + 1 + \frac{1}{C_s} \Bigg) \Bigg) \quad (8)$$

It can be observed from the Equation (8) that to $O(\delta^0)$, $\varphi_{B0} = 0$. This result clearly shows that for $\delta \ll 1$, a small amount of solid is formed for the near eutectic approximation. To analyze the stability of the basic state solution (7, 8) we apply small perturbations about of the form,

$$[T, V, \varphi, p] = [T_B, 0, \varphi_B, p_B] + \varepsilon [T_1, V_1, \varphi_1, p_1] + \varepsilon^2 [T_2, V_2, \varphi_2, p_2] \quad (9)$$

Where $\varepsilon \ll 1$, as required by the linear theory.

We now present the analysis for very small rotation rates. We now substitute (9) into Equations (1-4) and the $\theta(\varepsilon)$ equations are considered. This yields:

$$\left(\frac{\partial}{\partial t} - \frac{1}{\eta_0} \nabla^2 \right) T_1 = 1 - R\omega_1 \quad (10)$$

$$\left[\frac{1}{\chi} \frac{\partial}{\partial t} + 1 \right] V_1 = -\nabla p_1 - R[1 + \Lambda \sin(\Omega t)] T_1 \hat{e}_z \quad (11)$$

where $\eta_0 = 1 + \tilde{S}/C_s$, $\chi = \delta^2 \chi_0 = Da / p_r$

Applying the curl operator twice to equation (11) and eliminating the pressure term using equation (10), a single equation for the temperature perturbation is obtained:

$$\left[\frac{1}{\chi} \frac{\partial}{\partial t} + 1 \right] \nabla^2 \left[\frac{\partial}{\partial t} - \frac{1}{\eta_0} \nabla^2 \right] T_1 = R^2 [1 + \Lambda \sin(\Omega t)] \nabla_1^2 T = 0 \quad (12)$$

where ∇_1^2 is the 2D-Laplacian operator.

Assuming an expansion into the normal modes in the x- and y-directions, and a time dependant amplitude $\theta(t)$, we obtain,

$$T_1 = \theta(t) e^{i(S_x x + S_y y)} \sin(\pi z) + c.c. \quad (13)$$

where c.c. represents the complex conjugate terms $S^2 = S_x^2 + S_y^2$. Substituting the Equations (13) into (12) yields,

$$\frac{d^2 \theta}{dt^2} + 2p \frac{d\theta}{dt} - F(\alpha) \gamma [\hat{R} - \hat{R}_0] + \hat{R} \Lambda \sin(\Omega t) \theta = 0 \quad (12)$$

where $\alpha = S^2 / \pi^2$, $\gamma = \chi / \pi^2$, $\hat{R} = \tilde{R}^2$, $\tilde{R} = R / \pi^2$, $\hat{R}_0 = \tilde{R}_0^2$, $\tilde{R}_0 = R_0 / \pi^2$, R_0 is the un-modulated Rayleigh number defined as $\tilde{R}_0 = \pi^2 (\alpha + 1)^2 \eta_0 \alpha$, $2p = \pi^2 [(\alpha + 1) \eta_0 + \gamma]$ and $F(\alpha) = \pi^6 \alpha / (\alpha + 1)$.

Using the transformation $t = \left(\frac{\pi}{2} - 2\tau \right) / \Omega$, Equation (14) may be cast, as indicated by [8.], into the canonical form of the Mathieu equation:

$$\frac{d^2 X}{d\tau^2} + [a + 2q \cos(2\tau)] X = 0 \quad (15)$$

The solution to Equation(15) follows the form $G(\tau) = e^{-\sigma\tau} \chi(\tau)$ where $G(\tau)$ is a periodic function with a period of π or 2π and σ is characteristic exponent which is a complex number, and is function of a and q . In this paper the definitions of a , q and σ are obtained upon transforming equation (14) into the canonical form of Mathieu's equation, and are defined as:

$$\frac{2}{\sqrt{-a}} = \frac{\Omega}{\left[F(\alpha)\gamma(\hat{R}-E)\right]^{\frac{1}{2}}} \quad (16)$$

$$\frac{q}{2} = F(\alpha)\gamma\hat{R}\kappa Fr \quad (17)$$

$$\sigma = -\frac{2p}{\Omega} \quad (18)$$

$$E = -\frac{\hat{R}_0 E_0}{4\gamma(\alpha+1)} \left[\frac{\alpha+1}{E_0} - \gamma \right]^2 \quad (19)$$

When $\sigma = 0$, the solution to equation (15) is defined in terms of Mathieu function, c_e and d_f , (where $e = 1, 2, 3, 4, \dots$ and $f = 1, 2, 3, \dots$), such that for each Mathieu function, c_e and d_f , there exists a relation between a and q . We also propose the following definition for the modified characteristic exponent, $\xi = \frac{\sigma}{\sqrt{-a}}$. We now present the relation for the characteristic Rayleigh number in terms of newly defined parameter ξ , by substituting $\xi = \frac{\sigma}{\sqrt{-a}}$ in Equation (16), and rearranging to yield,

$$\hat{R} = \frac{\hat{R}_0 - E}{\xi^2} + E \quad (20)$$

IV . Results and Conclusion

The graphs of $\hat{R}$ Vs ξ are presented in figure 1 and 2, for $E_0=1, 1.5, 2.0$; $\gamma = 1.5$, $\kappa Fr = 10^{-3}$ and $\alpha = 1.5$ respectively. From the figures it is apparent that $\hat{R}$ increases with E for particular value of ξ and decreases as ξ increases. This indicates the stabilizing effect ξ on the solutal convection. For small values of ξ in the range $0.1 \leq \xi \leq 0.5$, $\hat{R}$ is always positive for all values of E , where as for the range, $10 \leq \xi \leq 20$, $\hat{R}$ is negative for $E_0=1$. The results are very encouraging.

Figure 2 $\hat{R}$ V/S ξ

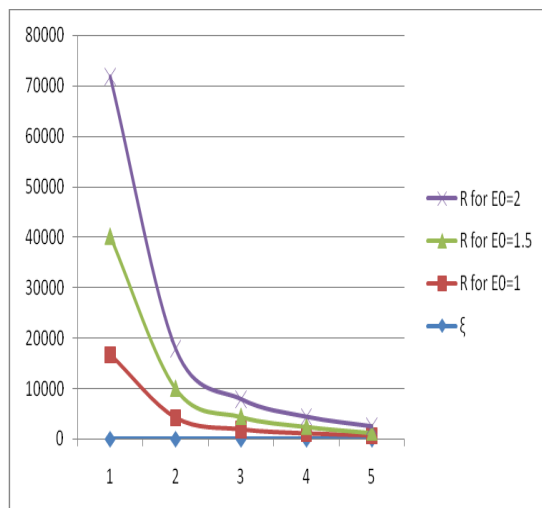
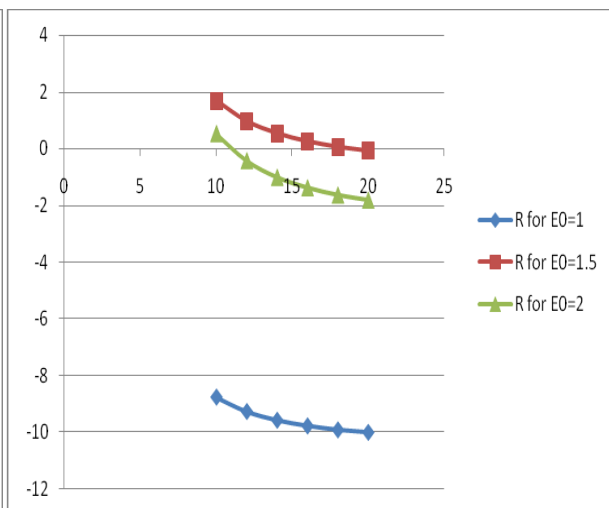


Figure 3 $\hat{R}$ V/S ξ



V. Acknowledgment

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