



STABILITY IN A DISCRETE NONLINEAR PREY - PREDATOR MODEL WITH FUNCTIONAL RESPONSE

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Abstract: This paper deals with discrete time predator – prey system. Equilibrium points are computed and local stability conditions are obtained. Numerical simulations are performed for different sets of parameter values to understand the behaviour of the system under various conditions. Also limit cycles and bifurcation diagram is provided for selected range of growth parameter.

Key words: difference equations, predator – prey system, equilibrium points, stability.

I. INTRODUCTION

Population dynamics has always been a core topic in theoretical ecology. Population modeling is also important for species management: managing fisheries for the highest possible sustainable yield, developing recovery plans for species threatened by extinction, or trying to contain or prevent the spread of invasive species. Hence to analyze these situations, we are often interested in knowing how interacting populations, responds to a change in system parameters. The first model to describe the size (density) dynamics of two populations interacting as a predator-prey system was suggested independently by A. Lotka (1925) and V. Volterra (1931):

$$\dot{x} = ax - bxy, \dot{y} = -cy + dxy,$$

The predator – prey interactions may be influenced by a behavioral response of the predator. The simplest model of functional response is obtained by assuming that in the time available for searching, the total change in the prey density is proportional to the prey density. Examples of functional response functions are

$$f(x) = \frac{ax}{b+x}, f(x) = \frac{ax^2}{b+cx^2}, f(x) = \frac{ax^2}{b+cx+x^2}$$

where a, b, c are positive constants. The most important and useful functional response is the Holling type II functional response given by $f(x) = \frac{ax}{b+x}$. This is also called a prey-dependent response function. Michael

L.Rosenzweig and Robert H.Mac Arthur studied predator – prey model with $f(x) = \frac{x}{\alpha+x}$ which gives a saturation effect in functional response when prey is abundant.

II. MODEL DESCRIPTION AND FIXED POINTS

The discrete time models governed by difference equations can also provide efficient computational models of continuous models for numerical simulations. The maps defined by simple difference equation can lead to rich complicated dynamics. This paper considers the following system of difference equations which describes interactions between two species with functional response.

$$\begin{aligned} x(n+1) &= rx(n)[1-x(n)] - \frac{ax(n)y(n)}{1+x(n)} \\ y(n+1) &= \frac{cx(n)y(n)}{1+x(n)} + (1-d)y(n) \end{aligned} \quad (1)$$

Where $a, c, d > 0$. We shall compute the equilibrium points for the above system (1). The system (1) has the fixed points $E_0 = (0, 0)$, $E_1 = \left(\frac{(r-1)}{r}, 0\right)$ and $E_2 = \left(\frac{d}{c-d}, \frac{c}{a(c-d)^2}((r-1)(c-d)-dr)\right)$. The fixed point E_0 is the trivial equilibrium point, whereas E_1 is the axial equilibrium point and E_2 corresponds to coexistence of both species. The interior point E_2 is a positive fixed point provided $r > 1 - \frac{d}{c}$.

III. STABILITY ANALYSIS

Non linear systems exhibit more complex dynamics and are much harder to analyze. One of the most useful techniques for analyzing nonlinear systems qualitatively is the analysis of the behavior of the solutions near fixed points using linearization technique. The local stability analysis of the model can be carried out by computing the Jacobian corresponding to each equilibrium point. The Jacobian matrix J for the system (1) is

$$J(x, y) = \begin{pmatrix} r - 2rx - \frac{ay}{(1+x)^2} & -\frac{ax}{1+x} \\ \frac{cy}{(1+x)^2} & \frac{cx}{1+x} - d + 1 \end{pmatrix} \quad (2)$$

The characteristic equation of Jacobian Matrix can be written as

$$\lambda^2 - \text{Tr } \lambda + \text{Det} = 0$$

where Trace and Determinant of the Jacobian Matrix $J(x, y)$ are given by

$$\text{Tr} = 1 + r - d - 2rx - \frac{ay}{(1+x)^2} + \frac{cx}{1+x}, \text{Det} = \left(\frac{cx}{1+x} + 1 - d \right) \left(r - 2rx - \frac{ay}{(1+x)^2} \right) + \frac{acxy}{(1+x)^3}.$$

We shall analyze the behaviour of the system near the equilibrium points.

1. When $E_0 = (0, 0)$, the Jacobian matrix of system (1) is given by

$$J(E_0) = \begin{pmatrix} r & 0 \\ 0 & 1 - d \end{pmatrix}$$

The eigenvalues of the matrix, $J(E_0)$ are $\lambda_1 = r$ and $\lambda_2 = 1 - d$.

Proposition 1. The fixed point E_0 is a

- a) sink if $r < 1$ and $0 < d < 2$.
 - b) source if $r > 1$ and $d > 2$.
 - c) saddle if $r < 1$ and $d > 2$.
 - d) non hyperbolic if either $r = 1$ or $d = 2$.
2. When $E_1 = \left(\frac{(r-1)}{r}, 0 \right)$, the Jacobian matrix of system (1) is given by

$$J(E_1) = \begin{pmatrix} 2 - r & \left(\frac{a(1-r)}{2r-1} \right) \\ 0 & \left(\frac{c(r-1)}{2r-1} \right) - d + 1 \end{pmatrix}$$

The eigenvalues of the matrix $J(E_1)$ are $\lambda_1 = 2 - r$ and $\lambda_2 = \left(\frac{c(r-1)}{2r-1} \right) - d + 1$.

Proposition 2. The fixed point E_1 is a

- a) sink if $1 < r < 3$ and $r < \frac{c-d}{(c-2d)}$.
 - b) source if $r > 3$ and $r > \frac{c-d}{(c-2d)}$.
 - c) saddle if $1 < r < 3$ and $r > \frac{c-d}{(c-2d)}$.
3. When $E_2 = \left(\frac{d}{c-d}, \frac{c}{a(c-d)^2} ((r-1)(c-d) - dr) \right)$, the Jacobian matrix of system (1) is given by

$$J(E_2) = \begin{pmatrix} 1 + \frac{d}{c}(2r-1) - \frac{2dr}{c-d} & -\frac{ad}{c} \\ \frac{c(r-1) - d(2r-1)}{a} & 1 \end{pmatrix}$$

Computation yields $\text{Tr} = 2 + \frac{d}{c}(2r-1) - \frac{2dr}{c-d}$ and $\text{Det} = 1 + d(r-1) + \frac{d(2r-1)(1-d)}{c} - \frac{2rd}{c-d}$.

Proposition 3. The fixed point E_2 is a

- a) sink if $B_1 < r < B_2$.
- b) source if $r > B_1$ and $r > B_2$.
- c) saddle if $r < B_1$.

$$\text{where, } B_1 = \frac{(c-d)[d(c-d+2)-4c]}{d[(c-2d)(c-d)-4d]}, B_2 = \frac{(c-d)(1+c-d)}{(c-d)(c+2-2d)-2c}$$

IV. NUMERICAL STUDY

In this section, we will examine the behaviour of the system (1) in a phase plane where the densities of predator and prey at time t are plotted and linked to their rate of change at those densities. A phase plane can be used to describe change in coupled difference equations. The dynamics resulting from the estimated parameter values can be found by simulation.

Example 1. We consider the values $r = 0.7, a = 0.037, c = 1.65, d = 0.95$. At equilibrium point E_0 , the Eigen values are $\lambda_1 = 0.7$ and $\lambda_2 = 0.05$ so that $|\lambda_{1,2}| < 1$. Hence the trivial equilibrium point is stable. The time plot and the phase diagram illustrate the result, see Figure - 1. Both prey and predator population will go to extinction.

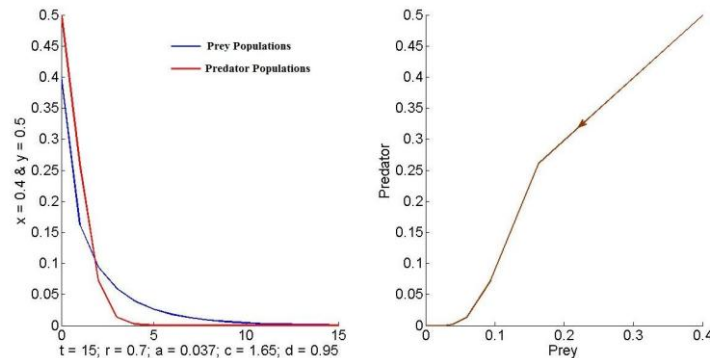


Figure 1.

Example 2. For the values $r = 1.75, a = 0.45, c = 2.54, d = 0.90$, we obtain $E_1 = (0.43, 0)$. This is an axial equilibrium point. Eigen values are $\lambda_1 = 0.25$ and $\lambda_2 = 0.86$ so that $|\lambda_{1,2}| < 1$. Hence the equilibrium point is stable. The time plot and the phase diagram can be seen in Figure - 2. In this case the prey population survives and the predator population goes to extinction.

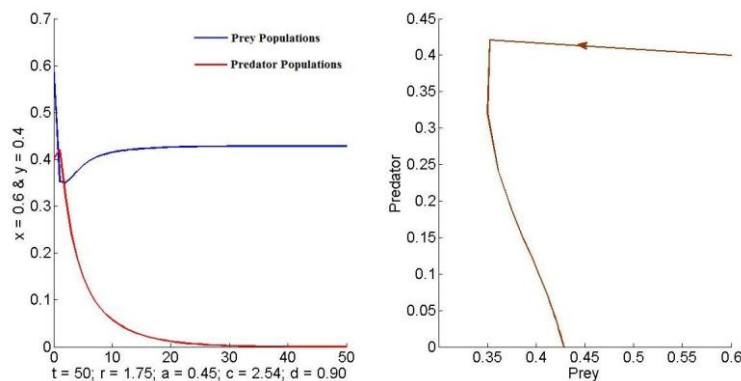


Figure 2.

Example 3. In this example, we take $r = 1.74, a = 1.5, c = 2.65$ and $d = 0.275$. Computations yield $E_2 = (0.11, 0.40)$ which is a point in the first quadrant of $\mathbb{R}^2$. The eigen values are $\lambda_{1,2} = 0.9272 \pm i 0.3570$ and $|\lambda_{1,2}| = 0.9935 < 1$. Hence the criteria for stability are satisfied. The phase portrait in Figure - 3 shows a sink and the trajectory spirals towards the equilibrium point E_2 . The species co-exist under these conditions.

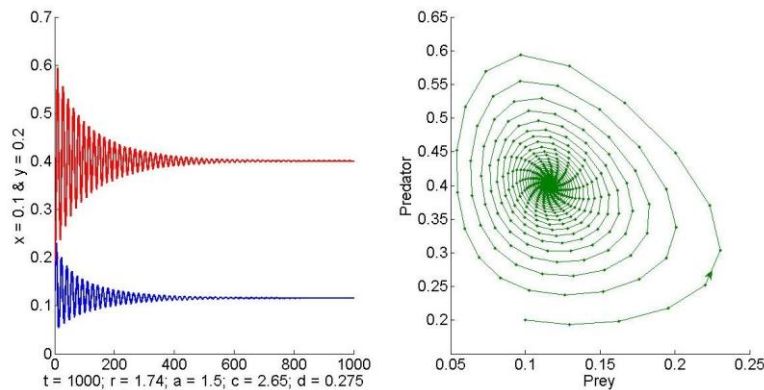


Figure 3.

V. BIFURCATION

Bifurcation has been used to describe significant and qualitative changes that occur in the solution of a dynamical system, as the system parameters are varied. It is used to describe the qualitative stability changes of the solution of a nonlinear dynamical system. Many dynamical systems depend on parameters. Normally a gradual variation of a parameter in the system corresponds to the gradual variation of the solutions of the problem. However, there exists a large number of problems for which the number of solutions changes abruptly and the structures of solution manifolds vary dramatically when a parameter passes through some critical values. This kind of phenomenon is called bifurcation and these parameter values are called bifurcation values.

Example 4. The parameters are assigned the values $a = 1.9, c = 2.6, d = 0.3$ and the bifurcation diagram is plotted for the growth parameter in the range $1.5 - 3.2$. Both prey and predator population undergoes chaos, see Figure-4.

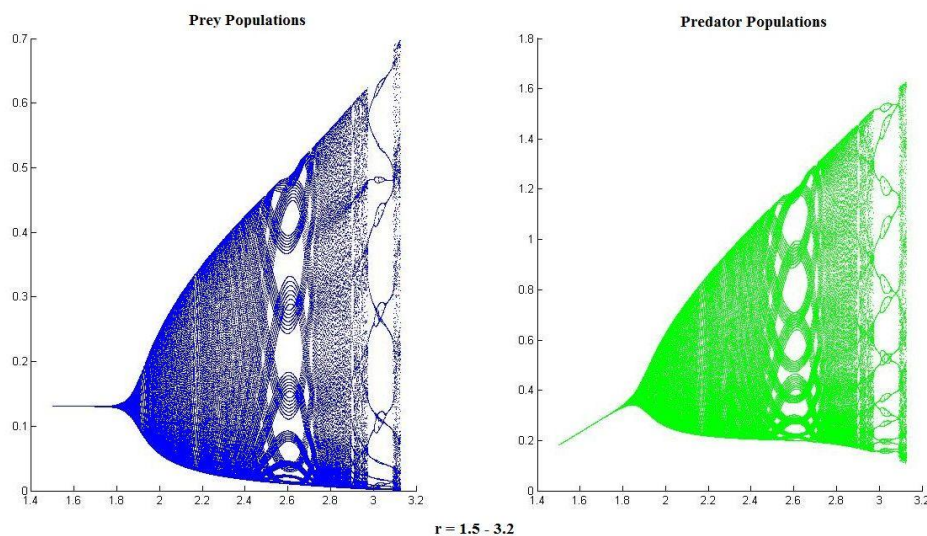


Figure 4.

VI. References

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