



Variational Homotopy Perturbation Method for Longitudinal Dispersion arising in Fluid Flow through Porous Media

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Abstract: In this paper, Variational Homotopy Perturbation Method is employed to solve dispersion equation arising to predict the contaminant concentration distribution of unsteady unidirectional seepage flows through semi-infinite, homogeneous, isotropic porous media. The results obtained are presented numerically and graphically. It shows remarkable high precision. Thus Variational Homotopy Perturbation Method can solve a large number of non-linear problems effectively and accurately having wide applications in engineering and physics.

Keywords: Variational Iteration Method (VIM), Homotopy Perturbation Method (HPM) and Variational Homotopy Perturbation Method (VHPM), Dispersion, Burger's equation.

I. Introduction

Dispersion is essentially macroscopic phenomenon caused by a combination of molecular diffusion and hydrodynamic mixing occurring with laminar flow through porous media. The pattern of point traces as its move downstream from its source tends to a normal distribution both longitudinal and transverse. Furthermore, the longitudinal component is larger than that of the transverse so that the major axis of mixing occurs in the direction of flow. Hydrodynamic dispersion is normally referred to as mixing of miscible fluid. It is a hydraulic mixing process by which waste concentration is attenuated while waste contaminants are being transported downstream. At the macro scale level, contaminant transport is defined by the average groundwater velocity. However, at the micro scale level the actual velocity of water may vary from point to point and can be either lower or higher than the average velocity. This difference in micro scale level water velocities arises due to pore size, path length and friction in pores. Because of these differences in velocities mixing occurs along the flow path. This mixing is known as mechanical dispersion or hydrodynamic dispersion [4]. The dispersion sources in uniform groundwater flow were reported by Hunt [8]. In general, the theory of dispersion in porous media was presented by Scheidegger [14]. Dispersion phenomena occur in many problems of ground water flow, in chemical engineering processes, in oil reservoir engineering, etc In this paper a theoretical model is developed for the dispersion problem in porous media in which the flow is one dimensional and the average flow is unsteady.

Analysis of flow against dispersion in porous media was presented by Al-Niami and Rushton [1]. The study of flow against dispersion in non-adsorbing porous media was presented by Marino. Hunt [8] applied the perturbation method to longitudinal and lateral dispersion in no uniform seepage flow through heterogeneous aquifers. Analytical solution of one dimensional time-dependent transport equation was also presented by Basha and Habel. An analytical solution for solute transport with depth dependent transformation or sorption coefficient was developed by Flury et al. [4]. Analytical solutions to transient, unsaturated transport of water and contaminants through horizontal porous media were discussed by Sander Braddock [13]. Analytical solution for conservative solute transport in one-dimensional homogeneous porous formations with time dependent velocity was also derived by Singh et al. [15]. R. Meher et al[9] solved dispersion problem in porous media in which the flow is one dimensional and the average flow is unsteady using Adomain decomposition method Mistry Piyush [10] solved by Variation Iteration Method.

The present paper discusses the analytical solution of the nonlinear differential equation for longitudinal dispersion phenomena which takes places when miscible fluids mix in the direction of flow by means of Variational Homotopy Perturbation Method [3]. Variational Homotopy Perturbation Method (VHPM) which is formulated by the coupling of variational iteration method and He's polynomials. The proposed VHPM provides the solution in a rapid convergent series which may lead the solution to a closed form. The proposed iterative scheme takes full advantage of variational iteration and the homotopy perturbation methods and absorbs all the positive features of the coupled techniques.

II. Mathematical formulation of the problem

The problem is to find the concentration as a function of time 't' and position 'x' as the two miscible fluid flow through porous media on either sides of the mixed region. The single fluid equation describes the motion of fluid. Here the mixing takes place longitudinally as well as transversely at t=0 and a dot of fluid having C_0 concentration is injected over the phase. The dot moves in the direction of flow as well as perpendicular to the flow. Finally it takes the shape of the ellipse with C_n .

According to Darcy's law the equation of continuity for the mixture in case of incompressible fluids is given by

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \bar{v}) = 0 \quad (1)$$

Where ρ is the density for mixture and $\bar{v}$ is the pore seepage velocity.

The equation of diffusion for a fluid flow through a homogeneous porous medium without increasing or decreasing the dispersing material is given by

$$\frac{\partial C}{\partial t} + \nabla \cdot (C \bar{v}) = \nabla \cdot \left[\rho \bar{D} \nabla \left(\frac{C}{\rho} \right) \right] \quad (2)$$

where 'C' is the concentration of a fluid in a porous media. $\bar{D}$ is the Coefficient of dispersion with nine components D_{ij} . In a laminar flow for an Incompressible fluid through homogeneous porous medium, density ρ is constant. Then equation (2) becomes,

$$\frac{\partial C}{\partial t} + \bar{v} \cdot \nabla C = \nabla \cdot (\bar{D} \nabla C) \quad (3)$$

Let us assume that the seepage velocity $\bar{v}$ is along the x- axis, then $\bar{v} = u(x, t)$ and the nonzero components will be $D_{11} \approx D_L = \gamma$ (Coefficient of longitudinal dispersion) and other Components will be zero .

Equation (3) becomes,

$$\frac{\partial C}{\partial t} + u \frac{\partial C}{\partial x} = \gamma \frac{\partial^2 C}{\partial x^2} \quad (4)$$

where u is the component velocity along x axis which is time dependent as well as concentration along x axis in $x \geq 0$ direction x axis and $D_L > 0$ and it is the cross sectional flow velocity in porous media.

$u = \frac{C(x, t)}{C_0}$, where $x > 0$ and for $C_0 \approx 1$ by Mehta (2006). Equation (4) becomes

$$\frac{\partial C}{\partial t} + C \frac{\partial C}{\partial x} - \gamma \frac{\partial^2 C}{\partial x^2} = 0 \quad (5)$$

This is the non linear Burger's equation for longitudinal dispersion of miscible fluid flow through porous media. The theory that follows is confined to dispersion in unidirectional seepage flow through semi-infinite homogeneous porous media. The seepage flow velocity is assumed unsteady. The dispersion systems to be considered are subject to an input concentration of contaminants C_0 . The governing partial differential equation (5) for longitudinal hydrodynamic dispersion with in a semi-infinite non adsorbing porous medium in a unidirectional flow field in which γ is the longitudinal dispersion coefficient, C is the average cross-sectional concentration, u is the unsteady seepage velocity, x is a coordinate parallel to flow and t is time.

The initial and boundary conditions are,

$$C(x, 0) = f(x) = e^{-x}, x \geq 0 \quad (6-a)$$

$$C(0, t) = \varepsilon \text{ (where } \varepsilon \approx 1), 0 < t \leq 0.001 \quad (6-b)$$

Since Concentration is decreasing with distance x. Therefore for the sake of convenience f(x) is considered as negative exponential function [9].

III. Variational iteration and homotopy perturbation method

To illustrate the basic concepts of the VIM and HPM, at first we consider the following nonlinear differential equation

$$Lu + Nu = g(x) \quad (2)$$

Where L = A linear operator, N = A nonlinear operator, $g(x)$ = An inhomogeneous term

According to the VIM [6,16] we can construct a correction functional as follows:

$$u_{n+1}(x) = u_n(x) + \int_0^t \lambda(\tau) (Lu_n(\tau) + Nu_n(\tau) - g(\tau)) d\tau \quad (3)$$

where $\lambda(\tau)$ is a general Lagrange multiplier, which can be identified optimally via variational theory. The second term on the right hand side is called the correction and is considered as a restricted variation, i.e., $\delta u_n = 0$. By this method, it is required first to determine the Lagrangian multiplier $\lambda(\tau)$ that will be identified optimally. The successive approximations $u_{n+1}(x, t)$, $n \geq 0$ of the solution $u(x, t)$ will be readily obtained upon using the determined Lagrangian multiplier and any selective function $u_0(x, t)$. Consequently, the solution is given by

$$u(x, t) = \lim_{n \rightarrow \infty} u_n(x, t) \quad (4)$$

To illustrate the basic concept of homotopy perturbation method, consider the following nonlinear system of differential equations

$$A(U) = f(r), \quad r \in \Omega \quad (5)$$

with boundary conditions

$$B(U, \frac{\partial u}{\partial n}) = 0, \quad r \in \Gamma \quad (6)$$

Where A = A differential operator, B = A boundary operator, $f(r)$ = A known analytic function, Γ = The boundary of the domain Ω .

The operator A can be divided into two parts L (linear) and N (nonlinear operator). Therefore (5) can be rewritten

$$L(U) + N(U) - f(r) = 0 \quad (7)$$

We construct a homotopy

$$v(r, p) : \Omega \times [0, 1] \rightarrow R \text{ which satisfies}$$

$$H(v, p) = (1-p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0 \quad (8)$$

$$p \in [0, 1], \quad r \in \Omega$$

Or

$$H(v, p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(r)] = 0 \quad (9)$$

where u_0 is initial approximation of equation (6). In this method, using the homotopy parameter p , we have the following power series presentation for

$$v = v_0 + pv_1 + p^2v_2 + \dots \quad (10)$$

Setting $p=1$ yields in the approximate solution of equation (10) to

$$u = \lim_{p \rightarrow 1} v = v_0 + v_1 + v_2 + \dots \quad (11)$$

The convergence of series (11) is discussed by J. Biazar, H. Ghazvini [17].

IV. Variational Homotopy Perturbation Method

All In the homotopy perturbation method [7], the basic assumption is that the solutions can be written as a power series in p

$$u = \sum_{i=0}^{\infty} p^i u_i = u_0 + p u_1 + p^2 u_2 + \dots \quad (12)$$

To illustrate the concept of the variational homotopy perturbation method [18] we consider the general differential equation (5). We construct the correction functional (6) and apply the homotopy perturbation method to obtain [7,17].

$$\sum_{i=0}^{\infty} p^i u_i = u_0(x, t) + p \int_0^t \lambda(\tau) \left(N \sum_{n=0}^{\infty} p^{(i)} u_i(x, \tau) - g(x, \tau) \right) d\tau \quad (13)$$

As we see, the procedure is formulated by the coupling of variational iteration method and homotopy perturbation method. A comparison of like powers of p gives solutions of various orders.

V. Implementation of VHPM to Solve Dispersion Equation

According to Variational Homotopy Perturbation method, we construct the correction functional for equation (5)

$$C_{n+1}(x, t) = C_n(x, t) + \int_0^t \lambda(\tau) \left(\frac{\partial C_n}{\partial t} + C_n \frac{\partial C_n}{\partial x} - \frac{\partial^2 C_n}{\partial x^2} \right) d\tau$$

Making the above functional stationary, the Lagrange multiplier can be determined as $\lambda = -1$, which yields the following iteration formula

$$C_{n+1}(x, t) = C_n(x, t) - \int_0^t \left(\frac{\partial C_n}{\partial t} + C_n \frac{\partial C_n}{\partial x} - \frac{\partial^2 C_n}{\partial x^2} \right) d\tau$$

Applying the variational homotopy perturbation method, we have

$$(C_0 + p C_1 + p^2 C_2 + \dots) = C_n(x, 0) - \int_0^t \left[\left(\frac{\partial}{\partial t} (C_0 + p C_1 + p^2 C_2 + \dots) + (C_0 + p C_1 + p^2 C_2 + \dots) \frac{\partial}{\partial x} (C_0 + p C_1 + p^2 C_2 + \dots) \right) - \frac{\partial^2}{\partial x^2} (C_0 + p C_1 + p^2 C_2 + \dots) \right] d\tau$$

Comparing the coefficient of like powers of p , we have

$$p^0 : C_0(x, t) = e^{-x}$$

$$p^1 : u_1(x, t) = e^{-2x} (1 + e^x) t$$

$$p^2 : C_2(x, t) = \frac{1}{2} e^{-3x} (3 + e^x (6 + e^x)) t^2$$

$$p^3 : C_3(x, t) = \frac{1}{6} e^{-4x} (16 + 51e^x + 28e^{2x} + e^{3x}) t^3$$

Similarly further approximations can be obtained up to desired accuracy. The solution becomes

$$C(x, t) = C_0 + C_1 + C_2 + C_3 + C_4 + \dots$$

VI. Graphical and Numerical Representation

Table I Numerical Values of Concentration for longitudinal dispersion of miscible fluid flow through porous media by VHPM

T	0.001	0.002	0.003	0.004	0.005
x=0.1	0.90656	0.90830	0.910045	0.91179	0.91355
x=0.2	0.820223	0.82172	0.82322	0.82473	0.82625
x=0.3	0.74211	0.74340	0.74471	0.74601	0.74733
x=0.4	0.64144	0.67256	0.67369	0.67483	0.67597
x=0.5	0.60750	0.60848	0.60947	0.61045	0.61144
x=0.6	0.54966	0.55051	0.55137	0.55223	0.55309
x=0.7	0.49733	0.49807	0.49882	0.49957	0.50033
x=0.8	0.44998	0.45063	0.45129	0.45194	0.452609
x=0.9	0.40714	0.40771	0.40829	0.40887	0.40944
x=1	0.36838	0.36888	0.36939	0.36990	0.37041

Figure 1 Curves refers Concentration values $0 < x < 1$ for fixed $t=0,0.001, \dots, 0.005$

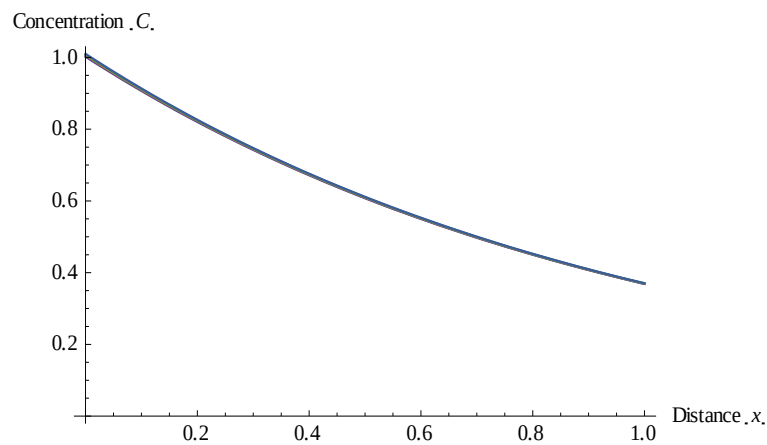


Figure 2 Curves refers Concentration values where $0 < t < 0.005$, for fixed $x=0,0.1, \dots, 1$

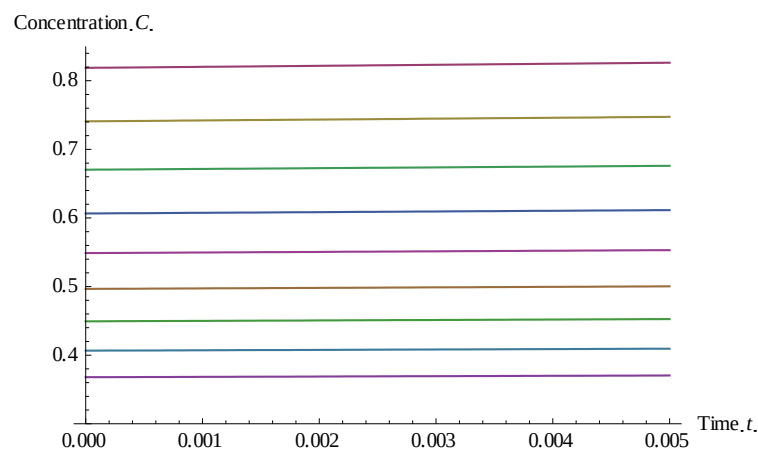
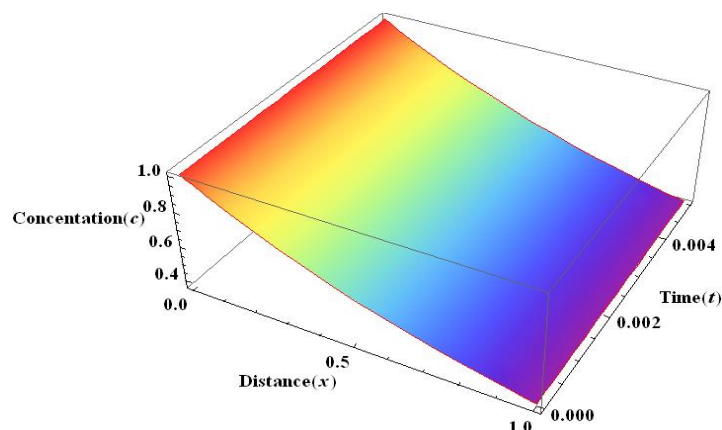


Figure 3 The 3D behaviour of Concentration versus x for different values of time by VHPM



VII. Conclusion

It can be easily concluded that the method is elegant and reduces computational work with high accuracy; the numerical values have been tabulated. The graphical representations have been developed for predicting the possible concentration of a given dissolved substance in unsteady unidirectional seepage flows through semi-infinite, homogeneous, isotropic porous media subject to the source concentrations that vary negative exponentially with distance which shows Concentration decreases with distance and slightly increases with time. In previous paper authors have already used Adomian polynomials to decompose the nonlinear terms in equations. The solution procedure is simple, but the calculation of Adomian polynomials is complex. To overcome this shortcoming, Variational Homotopy Perturbation Method is used.

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