



Acceleration of Predictor –Corrector Halley Method in Astrophysics Application

Mohammed S. Rasheed
Applied Sciences Department
University of Technology
University of Technology-P. O. Box 35317-Baghdad
IRAQ

E-mail: msr197575@yahoo.com

Abstract: In this paper, an iterative method named accelerated predictor-corrector Halley method for finding the true anomaly from mean anomaly and the eccentricity for a planet in an elliptical orbit around the sun (Kepler's equation). With the purpose of reducing the number of iteration, we find that the purpose method is the most efficient versus the predictor-corrector Halley method and Newton method.

Keywords: satellite orbit, mean anomaly, Kepler's equation, approximate solution, Astrophysics.

I. Introduction

Astronomy has always been a source of scientific inspiration. Probably the most significant revolution in science was spurred by astronomy. Since antiquity, civilizations have wondered about the laws governing the motion of Celestial bodies. The theory in force during the middle-ages was due to Ptolemy. In this model, the sun, the moon, and all the stars revolved about the earth, and their motion was determined by super-imposing circular motions. Some "stars" (which turned out to be the planets) were erratic, and each required many parameters to describe their motion. The theory gave very accurate predictions [3,6,2].

One of the problems frequently appearing in the celestial mechanics literature is the resolution of famous Kepler's equation. It gives the relation between the polar coordinates of a celestial body (such as a planet) and the time elapsed from a given initial point. Kepler's equation is of fundamental importance in celestial mechanics, but cannot be directly inverted in terms of simple functions in order to determine where the planet will be a given time [7,10]. Let M be the mean anomaly (a parameterization of time) and E the eccentric anomaly (a parameterization of polar angle) of a body orbiting on an ellipse with eccentricity e , then

$$M = E - e \sin(E)$$

For M not a multiple of π , Kepler's equation has a unique solution, but is a transcendental equation and so cannot be inverted and solved directly for E given an arbitrary M . However, many algorithms have been derived for solving the equation as a result of its importance in celestial mechanics [4,5,7,8,9,14]. One of the usual ways to solve algebraic or transcendental equation is by the mean of iterative algorithms. The functional iteration is the easiest and more used, its convergence is linear; nevertheless, when a very good approximation is required, one must consider methods of higher order of convergence. The Newton–Raphson is the most widely used, because of their simple implementation and their quadratic order of convergence [11]. With the purpose of reducing the number of iteration in the determination of the roots of a nonlinear equation, they have improved the iterative methods by increasing the order of convergence.

In this paper, we make a revision of Newton method and protector –corrector Halley method and proposed the accelerated protector –corrector Halley method in the determination of the roots of Kepler's equation to find which one is more efficient. Actually, we find that the accelerated protector –corrector Halley method is the most efficient and turns out to be highly competitive versus the other two methods.

II. Iterative Method

Consider the nonlinear equation of the type

$$f(x) = 0 \tag{1}$$

Assume that α is a simple root of equation (1) and γ is an initial guess sufficiently close to α . Using the Taylor's series expansion of the function $f(x)$, we have

$$f(\gamma) + (x - \gamma)f'(\gamma) + \frac{(x - \gamma)^2}{2}f''(\gamma) = 0 \text{ from which we have}$$

$$x = \gamma - \frac{2f(\gamma)f'(\gamma)}{2(f'(\gamma))^2 - f(\gamma)f''(\gamma)}$$

Then the following iterative method can be suggested for solving the nonlinear equation (1).

Algorithm (1): Newton method for a given x_0 , find the approximate solution by the iterative scheme

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n=0,1,2,\dots$$

Algorithm (2): Predictor-Corrector Halley method [13] for a given x_0 , find the approximate solution x_{n+1} by the iterative schemes

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = y_n - \frac{2f(y_n)f'(y_n)}{2(f'(y_n))^2 - f(y_n)f''(y_n)}, \quad n=0,1,2,\dots$$

Algorithm (3) _{new}: Accelerated Predictor-Corrector Halley method

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = y_n - \frac{2f(y_n)f'(y_n)}{2(f'(y_n))^2 - f(y_n)f''(y_n)}, \quad n=0,1,2,\dots$$

$$z_n = x_n - \frac{(x_{n+1} - x_n)^2}{x_{n+2} - 2x_{n+1} + x_n}, \quad n=0,1,2,\dots$$

and this is the main motivation of this paper.

III. Satellite Orbital motion

In this section we shall consider the behavior of the accelerated method versus some other methods used to determine the root of Kepler's equation in elliptic motion. Consider Earth's motion. The earth revolves in an elliptical orbit around the sun, which is at one focus of the ellipse. The eccentricity of this ellipse is 0.0167. In addition, the time it takes for the earth to make one complete orbit of the sun is approximately 365 days. We also known that the earth approaches closest to the sun approximately January 4 of each year; in other words, perihelion occurs approximately January 4. Now say we want to know the Earth's position February 1, or 27 days since perihelion. Finding the Earth's position at this time requires solving Kepler's equation of Elliptical motion. In this case, $e=0.0167$, $T=365$ days, $t=27$ days. Therefore the mean anomaly $M = \frac{2\pi t}{T}$, (it is the time since periapse and T is the period of the motion).

$M = 2 * 3.14159 * \frac{27}{365} = 0.46478$ radians. Finding the Earth's at February 1 requires solving Kepler's equation of Elliptical motion for E (the eccentric anomaly) [1,2,6]

$$E - e \sin(E) = M \quad (2)$$

with $0 \leq e < 1$ and $0 \leq M < \pi$.

The proposed new algorithm (algorithm 3) will be used to solve eq. (2) which can be summarized by the following steps:

- (1) Put $E = E_0$ (1st guess).
- (2) Find the value of $f = M - E_0 + e \sin E_0$
- (3) If $\text{abs}(f) < \epsilon$ goto step 7

Else

- (4) Find $f = e \cos E_0 - 1$

$$f_1 = \cos E - 1$$

$$f_2 = -e \sin E_0$$

$$E = E - \frac{f}{f_1}$$

$$M = E - E + e \sin E$$

$$f_1 = \cos E - 1$$

$$f_2 = -e \sin E$$

- (5) Take the new guess of E as

$$(6) E = E - \frac{(2 f f_1)}{2 f_1^2 - f f_2}$$

Find the new accelerated value of E as

$$E_n = E_n - \frac{(E_{n+1} - E_n)^2}{(E_{n+2} - 2E_{n+1} + E_n)}$$

Go to step 2

- (7) Present value of E is correct within ϵ

IV. Results and Discussion

To compare the different methods of iterations algorithms 1 and 2 against the proposed algorithm (algorithm 3) used the initial guess E_0 in the region $0 \leq M < \pi$, $0 \leq e < 1$.as follows:

$$\text{If } M < 0.1 \text{ then } E_0 = M + (6M)^2 - M e^2 \quad (3)$$

$$\text{If } M \geq 0.1 \text{ then } E_0 = M + 0.85 e \quad (4)$$

If $0 \leq M < \pi$ then $E_o = M + 0.85 e$ (5)
 Next, we shall take the following couple of values M and e as shown in table 1.

Table 1 shows different values of M and e

Cases	M	e
Case 1	0.9	0.09
Case 2	1.5	0.5
Case 3	3	0.09
Case 4	3	0.9
Case 4	0.2	0.2
Case 5	0.1	0.95
Case 6	0.05	0.09
Case 7	0.05	0.95
Case 8	0.099	0.95
Case 9	0.001	0.95
Case 10	0.05	0.5
Case 11	0.08	0.95
Case 12	0.08	0.08
Case 13	0.07	0.95
Case 14	0.07	0.09
Case 15	0.06	0.95
Case 16	0.06	0.09
Case 17	0.0005	0.09

In the table (2), the initial guess is $E_o = M + 0.85 e$ for cases (1-6) and $E_o = M + ((6M)^{1/3} - M)e^2$ for cases (7-18). Convergence is considered to be achieved if $|E - \text{esin}E - M| < 10^{-15}$. Then the number of iterations required using algorithms (1-3) are included in table (1)

Cases	Number of iteration			The root
	Algorithm (1)	Algorithm (2)	Algorithm (3)	
Case (1)	4	3	1	0.974466199670940
Case (2)	5	4	2	1.962189287578571
Case (3)	4	3	1	3.011660975704094
Case (4)	5	4	2	3.067037496680689
Case (5)	5	3	1	0.249355982228356
Case (6)	6	4	2	0.741917668105762
Case (7)	5	4	2	0.054942321531551
Case (8)	10	6	3	0.531420642466566
Case (9)	19	25	22	0.738568830061134
Case (10)	10	5	2	0.01997462949968
Case (11)	8	4	2	0.0998342433686
Case (12)	13	6	3	0.669752004546650
Case (13)	12	4	2	0.086996999248733
Case (14)	13	7	4	0.628653087236876
Case (15)	7	4	2	0.076915578596776
Case (16)	20	8	5	0.583020838445192
Case (17)	6	4	2	0.065929343219182
Case (18)	6	4	2	5.499505467163154e-4

Table 2 shows that the algorithms 1 and 2 require a large number of iterations to converge a good estimate with M small and e closed to 1 while Algorithm 3 provides correct values of a true anomaly and converges faster with small number of iterations for any values of M and e.

Table (3) illustrates the results of combining the initial guess $E_o = M + 0.85 e$ for $0 \ll M \ll \pi$ with the three algorithms for cases (7-18).

Cases	Number of iteration			The root
	Algorithm (1)	Algorithm (2)	Algorithm (3)	
Case (7)	5	3	1	0.054942321531551
Case (8)	7	4	2	0.531420642466566
Case (9)	8	5	2	0.738568830061134
Case (10)	9	5	2	0.019974762949968
Case (11)	7	4	2	0.99834243368842
Case (12)	8	5	2	0.669752004546650
Case (13)	5	3	1	0.086946999248733
Case (14)	7	5	2	0.628653087236876
Case (15)	8	4	2	4.336808689942018e-11
Case (16)	9	4	2	0.583020838445192
Case (17)	5	3	1	0.065929343219182
Case (18)	5	4	2	5.49450546713154e-4

It's noticed that only one or two iterations are to be used when using algorithm 3 together with the initial guess given in eq. (5).

V. Conclusions

For our comparison of our accelerated method with other methods Kepler equation was solved for different values of M and e such that $M \in [0, \pi)$ and $e \in [0, 1]$. A solution stopped when $|f| < 10^{-15}$, and the number of iteration was recorded. It's clear from the tables 2,3 that the other methods need a large number of iteration in order to reach the correct root whereas new method converts with less number of iterations. Converge with a case when M near to zero and e near to 1 needed large number of iterations. But for much more critical values of M and e can be treated successfully by our algorithm.

VI. References

- [1] Arovass D., "Lecture Notes on Classical Mechanics, (A Work in Progress), University of California, and San Diego, August 22, (2012), Available on: IVSL.org.
- [2] BATE R. R., MUELLER D.D., "Fundamentals of Astrodynamics, Dover Publications, Inc., New York, (1971).
- [3] Ben Mahmoud K. B., " Kepler's celestial two-body equation: a second attempt to establish a continuous and integrable solution via the BPES: Boubaker Polynomials Expansion Scheme", Astrophys Space Sci., 327: 77–81, DOI 10.1007/s10509-010-0284-3, (2010), Available on: IVSL.org.
- [4] Boyd J. P., "Chebyshev expansion on intervals with branch points with application to the root of Kepler's equation: A Chebyshev–Hermite–Padé method", Journal of Computational and Applied Mathematics, 223, 693–702, (2009), Available on: IVSL.org.
- [5] Carinena J. F., Lopez C., "CONFORMAL GEOMETRY OF THE KEPLER ORBIT SPACE", Celestial Mechanics and Dynamical Astronomy, 307-343, (1991).
- [6] COLWELL P. P., " Kepler's Equation and Newton's Method", Celestial Mechanics and Dynamical Astronomy, 203-204, 2 September, (1991). Available on: IVSL.org.
- [7] Eisinberg A., Fedele G., A. Ferrise, D. Frascino, " On an integral representation of a class of Kapteyn (Fourier Bessel) series: Kepler's equation, radiation problems and Meissel's expansion", Applied Mathematics Letters, 23, 1331-1335, (2010), Available on: IVSL.org.
- [8] Fukushima T., "A Fast Procedure Solving Kepler's Equation for Elliptic Case", Astronomical Journal, Vol. 112, No. 6, Dec. (1996), Available on: IVSL.org.
- [9] Fukushima T., "A Method Solving without Transcendental Function Evaluations" National Astronomical Observatory, 2-21-1, Ohsawa, Mitka, Tokyo 181, Japan, 7 Jan. (1997), Available on: IVSL.org.
- [10] M. Palacios, "Kepler equation and accelerated Newton method", Journal of Computational and Applied Mathematics, 138 (2002) 335–346, Available on: IVSL.org.
- [11] Nievergelt Y., "Computing the distance from a point to a helix and solving Kepler's equation", Nuclear Instruments and Methods in Physics Research A, 598, 788–794, (2009), Available on: IVSL.org.
- [12] Noor I. N., Norr A. M., "Predictor –corrector Hally method for nonlinear equations", Applied Mathematics and Computation 188 (2007) 1587-1591, Available on: IVSL.org.
- [13] Palacios M., "Kepler equation and accelerated Newton method", Journal of Computational and Applied Mathematics, 138, 335–346, (2002), Available on: IVSL.org.
- [14] Tang R., Yi D., " TAIC Algorithm for the Visibility of the Elliptical Orbits' Satellites", Changsha, Hunan Province, 410073, P. R. China, 1-4244-1212-9/07, IEEE, (2007), Available on: IVSL.org.