



An Efficient Algorithm for n^{th} Order Integro-Differential Equations Using New Haar Wavelets Matrix Designation

Suha N. Shihab⁽¹⁾, Amal A. Mohammed⁽²⁾

⁽¹⁾Applied Sciences Department, ⁽²⁾College of Engineering

⁽¹⁾University of Technology, ⁽²⁾Al-mustansiriyah University

⁽¹⁾University of Technology-P. O. Box 35317-Baghdad-Iraq
IRAQ

Email: alrawy1978@yahoo.com

Abstract: Haar wavelet method is applied to solve linear n^{th} order integro-differential equations (IDE). A simple way is proposed to perform the calculations for the matrix representation in order to avoid the tedious calculations. The procedure is taking the Haar series for the n^{th} order of the IDE and integrates the series, which is applicable for different kinds of IDE (Fredholm and Volterra). Some test problems for which the exact solution is known, are considered.

Keywords: Haar Wavelet; integro-differential equation; collocation method.

I. Introduction

Orthogonal functions and polynomial series have received considerable attention in dealing with various problems. The main characteristics of this technique are that it reduces these problems to those of solving a system of algebraic equations, thus greatly simplifying the problem. Wavelet theory is relatively new and it has been applied in various systems.

Wavelets permit the accurate representation of a variety of functions and operators. Special attention has been given to application of the Legendre wavelets [1,5,6,7] the CAD wavelets [2]. Chebyshev wavelets [3] and the sine-cosine wavelets [4]. In this paper, some operational matrices with the Haar wavelets of integration are derived and use them for obtaining approximate solution of the following n^{th} order IDE,

$$1. \text{ Fredholm IDE: } f^{(n)}(x) = g(x) + \int_0^1 k(x,t)f^{m_1}(t)dt, \quad n \geq m \quad (1)$$

$$\alpha_i = f^{(n-i)}(0), \quad i = 1, 2, 3, \dots, n$$

$$2. \text{ Volterra IDE: } f^{(n)}(x) = g(x) + \int_0^x k(x,t)f^{m_1}(t)dt, \quad n \geq m \quad (2)$$

$$\alpha_i = f^{(n-i)}(x), \quad i = 1, 2, 3, \dots, n$$

where $k(x,t)$ and $g(x)$ are known functions, and $f(x)$ is an unknown function.

II. Haar Wavelet

Haar Wavelet is the simplest wavelet. Haar transform or Haar wavelet transform has been used as an earliest example for orthogonal wavelet transform with compact support.

The Haar wavelet family for $t \in [0,1]$ is defined as follows [8,9]

$$h_i(t) = \begin{cases} 1 & \text{for } t \in \left[\frac{k}{m}, \frac{k+0.5}{m}\right) \\ -1 & \text{for } t \in \left[\frac{k+0.5}{m}, \frac{k+1}{m}\right) \\ 0 & \text{otherwise} \end{cases} \quad (3)$$

The integer $m = 2^j$ ($j = 0, 1, \dots, J$) indicates the level of the wavelet; $k = 0, 1, 2, \dots, m-1$ is the translation parameter. Maximum level of resolution is J . The index i is calculated according the formula $i = m + k + 1$; in the case of minimal values, $m = 1$, $k = 0$, we have $i = 2$, the maximal value of i is $i = 2M = 2^{J+1}$. It is assumed that the value $i = 1$ corresponds to the scaling function for which $h \equiv 1$ in $[0,1]$.

III. Haar Wavelet Method for Fredholm IDE with n^{th} Order

For solving Fredholm IDE with n^{th} order, eq. (1). A Function $f^{(n)}(x)$ which is defined in the interval $x \in [0,1]$ can be expanded into the Haar wavelet series

$$f^{(n)}(x) = \sum_{i=1}^{2M} a_i h_i(t) \quad (4)$$

where a_i are the wavelet coefficients.

Integrate eq. (4), n times, yields

$$f(x) = \sum_{i=1}^{2M} a_i \int_0^x \dots \int_0^x h_i(t) dt + \sum_{j=0}^{n-1} \frac{x^j}{j!} \alpha_{n-j} \quad (5)$$

Using the following formula $\int_0^x \dots \int_0^x h_i(t) dt = \frac{1}{(n-1)!} \int_0^x (x-t)^{n-1} h_i(t) dt$, therefore eq. (5) becomes

$$\sum_{i=1}^{2M} a_i \frac{1}{(n-1)!} \int_0^x (x-t)^{n-1} h_i(t) dt + \sum_{j=0}^{n-1} \frac{x^j}{j!} \alpha_{n-j} \quad (6)$$

Let $k_n(x, t) = \frac{(x-t)^{n-1}}{(n-1)!}$ and $s_i^n(x) = \int_0^x k_n(x, t) h_i(t) dt$, $i = 1, 2, \dots, 2M$, this leads to

$$f(x) = \sum_{i=1}^{2M} a_i s_i^n(x) + \sum_{j=0}^{n-1} \frac{x^j}{j!} \alpha_{n-j}$$

$$\text{In similar way, one can get } f^{(m)}(x) = \sum_{i=1}^{2M} a_i s_i^{n-m}(x) + \sum_{j=0}^{n-m-1} \frac{x^j}{j!} \alpha_{n-m-j}, n \geq m \quad (7)$$

Substituting equs (4) and (7) into eq. (8), yields

$$\sum_{i=1}^{2M} a_i h_i(x) = g(x) + \int_0^1 k(x, t) \left\{ \sum_{i=1}^{2M} a_i s_i^{n-m}(t) + \sum_{j=0}^{n-m-1} \frac{t^j}{j!} \alpha_{n-m-j} \right\} \quad (8)$$

$$\text{or } \sum_{i=1}^{2M} a_i [h_i(x) - R_i(x)] = g(x) + \sum_{j=0}^{n-m-1} \frac{\alpha_{n-m-j}}{j!} Q_j(x) \quad (9)$$

$$\text{where: } R_i(x) = \int_0^1 k(x, t) S_i^{n-m}(t) dt \quad i = 1, 2, \dots, 2M$$

$$Q_j(x) = \int_0^1 k(x, t) t^j dt \quad j = 0, 1, 2, \dots, n-m-1$$

Next, the interval $x \in [0, 1]$ is divided into $2M$ parts of equal length $\Delta x = \frac{1}{2M}$ and introduce the collocation points $X_L = \frac{(L-0.5)}{2M}$, $L = 1, 2, \dots, 2M$ (10)

Eq. (9) is satisfied only at the collocation points (10), we get a system of linear equations

$$\sum_{i=1}^{2M} a_i [h_i(x_L) - R_i(x_L)] = g(x_L) + \sum_{j=0}^{n-m-1} \frac{\alpha_{n-m-j}}{j!} Q_j(x_L), \quad L = 1, 2, \dots, 2M$$

$$\text{The matrix form of this system is } a(H - R) = G + \sum_{j=0}^{n-m-1} \frac{\alpha_{n-m-j}}{j!} Q_j \quad (11)$$

where $H_{iL} = h_i(x_L)$, $R_{iL} = R_i(x_L)$, $G_L = g(x_L)$ and $Q_{jL}(x_L) = Q_j(x_L)$

IV. Design of the matrix R

When Haar wavelets are integrated n times, then the following integral must be evaluated,

$$S_i^n(x) = \int_0^x \frac{(x-t)^{n-1}}{(n-1)!} h_i(t) dt, \quad i = 1, 2, \dots, 2M \quad (12)$$

$$\text{If } i = 1, \text{ that is } h_1(t) \equiv 1 \text{ so } S_1^n(x) = \int_0^x \frac{(x-t)^{n-1}}{(n-1)!} h_1(t) dt = \frac{x^n}{n!} \quad (13)$$

For $i = 2, 3, \dots, 2M$, we have

$$S_i^n(x) = \frac{1}{n!} \begin{cases} 0 & x < t_1 \\ (x - t_1)^n & t_1 \leq x \leq t_2 \\ (x - t_1)^n - 2(x - t_2)^n & t_2 \leq x \leq t_3 \\ (x - t_1)^n - 2(x - t_2)^n + (x - t_3)^n & t_3 \leq x \leq 1 \end{cases} \quad (14)$$

Therefore; the matrix $R_i(x)$ can be constructed as follows

$$\text{Since } R_i(x) = \int_0^1 k(x, t) S_i^{n-m}(t) dt, \quad i = 1, 2, \dots, 2M$$

$$R_i(x) = \frac{1}{(n-m)!} \begin{cases} \int_0^1 k(x, t) t^{n-m} dt & i = 1 \\ \int_{t_1}^1 k(x, t) (t - t_1)^{n-m} dt + \int_{t_2}^1 k(x, t) (t - t_2)^{n-m} dt & i > 1 \end{cases} \quad (15)$$

V. Haar Wavelet Method for Volterra IDE with nth Order

For solving volterra IDE with nth order, eq. 3 stages of section 4 will be followed to get

$$\sum_{i=1}^{2M} a_i [h_i(x_L) - R_i(x_L)] = g(x_L) + \sum_{j=0}^{n-m-1} \frac{\alpha_{n-m-j}}{j!} Q_j(x_L)$$

$$\text{But } R_i(x_L) = \int_0^{x_L} k(x, t) \int_i^{n-m}(t) dt \quad i = 1, \dots, 2M$$

$$Q_j(x_L) = \int_0^{x_L} k(x, t) t^j dt \quad j = 0, \dots, n-m-1$$

where $S_i^{n-m}(t)$ as in eq. 5, eq. 7 and eq. 8

$$\text{That is } R_i(x_L) = \frac{1}{(n-m)!} \int_0^{x_L} k(x_L, t) t^{n-m} dt = p_L^0 \text{ where } i = 2, \dots, 2M$$

and

$$R_i(x_L) = \frac{1}{(n-m)!} \begin{bmatrix} \int_{t_1}^{x_L} k(x_L, t) (t - t_1)^{n-m} dt = p_L^1, & x_L < t_1, & t_1 \leq x_L \leq t_2 \\ \int_{t_1}^{x_L} k(x_L, t) (t - t_1)^{n-m} dt - \int_{t_2}^{x_L} k(x_L, t) (t - t_2)^{n-m} dt = p_L^2, & x_L < t_1, & t_2 \leq x_L \leq t_3 \\ \int_{t_1}^{x_L} k(x_L, t) (t - t_1)^{n-m} dt - 2 \int_{t_2}^{x_L} k(x_L, t) (t - t_2)^{n-m} dt + \int_{t_3}^{x_L} k(x_L, t) (t - t_3)^{n-m} dt = p_L^3, & x_L < t_1, & t_3 \leq x_L \leq 1 \end{bmatrix}$$

Let $p_L^1 = p_1(x_L) = p_{1L} = \int_{t_s}^{x_L} k(x_L, t)(t - t_s)^{n-m} dt$ $L = 1, 2, \dots, 2M$, $s = 0, 1, 2, 3$, $t_0 = 0$

$$p_L^2 = p^2(x_L) = p_{1L} - 2p_{2L} = p_1(x_L) - 2p_2(x_L)$$

$$p_L^3 = p^3(x_L) = p_{1L} - 2p_{2L} + p_{3L} = p_L^2 + p_{3L} = p^2(x_L) + p_3(x_L)$$

If $M=4$ we have $L=1, \dots, 2M=8$; $i=1, \dots, 2M=8$; $x_L = \frac{L-0.5}{2M}$ then the matrix R is shown below

$$R = \begin{bmatrix} p_1^0 & p_2^0 & & & & & & p_8^0 \\ p_1^1 & p_1^1 & & p_4^1 & p_5^1 & & & p_8^1 \\ p_1^1 & p_2^1 & p_3^1 & p_4^1 & p_5^1 & & & p_8^1 \\ 0 & 0 & 0 & 0 & p_5^1 & p_6^1 & p_7^1 & p_8^1 \\ p_1^1 & p_2^1 & p_3^1 & p_4^1 & p_5^1 & & & p_8^1 \\ 0 & 0 & p_3^1 & p_4^1 & p_5^1 & & & p_8^1 \\ 0 & 0 & 0 & 0 & p_5^1 & p_6^1 & p_7^1 & p_8^1 \\ 0 & 0 & 0 & 0 & 0 & 0 & p_7^1 & p_8^1 \end{bmatrix}$$

VI. Numerical Examples

Example (1):

Consider the following Fredholm integro differential equation:

$$\left. \begin{aligned} f^{(3)}(x) &= \frac{55}{12} - 2x + \int_0^1 (x+t)f(t) dt \\ f^{(2)}(0) &= 2, \quad f^{(1)}(0) = f(0) = 0 \end{aligned} \right\} \quad (16)$$

The exact solution of this equation is $f(x) = x^3 + x^2$.

We solve equation (16) by using our method with differential value of J . The accuracy for the results was estimated by the error function

$$e_j = \max(|f(t_L) - f_{ex}(t_L)|) \quad 1 \leq L \leq 2M \text{ or } e_{rr} = \sum_{L=1}^{2M} [f(t_L) - f_{ex}(t_L)]^2$$

Where $f_{ex}(t_L)$ is the exact solution. Table 1 shows the errors e_j and e_{rr}

Table I Shows the results of example 1

J	2M	e_j	e_{rr}
2	8	0	0
3	16	2.2204 e-016	4.9304 e-032
4	32	2.2204 e-016	1.1093 e-031
5	64	0	0

Example (2): consider the following integro-differential equation

$$f^{(4)}(x) = e^x + (1 - e)x - 2 + \int_0^1 (x+2t)f^{(2)}(t) dt$$

$$\alpha_1 = f^{(3)}(0) = 1$$

$$\alpha_2 = f^{(2)}(0) = 1$$

$$\alpha_3 = f^{(1)}(0) = 1$$

$$\alpha_4 = f(0) = 1$$

The exact solution of this problem is $f(x) = e^x$. The errors e_j and e_{rr} are shown in table (2)

Table II Shows the results of example 2

J	2M	e_j	e_{rr}
2	8	2.6122e-004	1.0610e-007
3	16	7.2822e-005	1.3569e-008
4	32	1.9208e-005	1.7058e-009
5	64	4.9316e-006	2.1353e-010

Example (3):

Consider the following Volterra integro-differential equation

$$f^{(5)}(x) = -2\sin x + 2x\cos x - x + \int_0^x (x+t)f^{(3)}(t) dt$$

$$\alpha_5 = f(0) = 1$$

$$\alpha_4 = f^{(1)}(0) = 0$$

$$\alpha_3 = f^{(2)}(0) = -1$$

$$\alpha_2 = f^4(0) = 0$$

$$\alpha_1 = f^4(0) = 1$$

The exact solution of this equation is $f(x) = \cos x$. Table (3) shows The errors e_f and e_{rr} .

Table III Shows the results of example 3

J	$2M$	e_f	e_{rr}
2	8	4.1260 e-005	2.4214 e-009
3	16	1.1784 e-005	3.1535 e-010
4	32	3.1415 e-006	3.9820 e-011
5	64	8.1059 e-007	4.9902 e-012

VII. Conclusions

In this work, we solved IDE by using Haar wavelets in collocation method. Haar wavelets are well behaved basics functions that are orthogonal on $[0, 1)$. This allow us to transform any function square integral on the interval time $[0, 1)$ into Haar wavelets series. Applications of the wavelets allow the creation of more effective and faster algorithms than the ordinary ones. Illustrative examples are included to demonstrate the validity and applicability of the technique. The main advantage of this method is its simplicity and small computation costs. We can use the proposed method to solve the following n^{th} order Fredholm integro-differential equation

$$f^{(n)}(x) = g(x) + \int_a^b k(x, t) f^{(m)}(t) dt$$

Since the Haar wavelets are defined only for the integral $[0, 1]$, we must first normalize the above equation. This can be done by the change of the variables

$$x = \frac{(x-\alpha)}{(\beta-\alpha)}, t_\alpha = \frac{(t-\alpha)}{(\beta-\alpha)}$$

VIII. References

- [1] Mesgarani, H., Numerical Solution of nth Order Integro-Differential Equations with Trigonometric Wavelet, Applied Mathematical Sciences, Vol. 4, No. 45, PP. 2241-2248, (2010).
- [2] Kajani, M. and Dadkhah, M., Numerical Solution of Nonlinear Fredholm-Volterra Integro-Differential Equations Using Legendre Wavelets, Proceedings of the 6th IMT Conference on Mathematics, Statistics and its Applications (ICMSA), Malaysia, (2010).
- [3] Soleymanivarak M. and Jafari H., Legendre wavelet for solving fractional differential equations, Journal of applied mathematics, Vol. 7, No. 4 (27), pp.65-70, (2011).
- [4] Mahdavi, S., Nonlinear Integro-Differential Equations, Journal of Mathematical Extension, Vol. 4, No. 2, PP. 107-117, (2010).
- [5] Rawashdeh E. A., Legendre Wavelets Method for Fractional Integro-Differential Equations, Applied Mathematical Sciences, Vol. 5, No. 50, PP. 2467-2474, (2011).
- [6] Lepik, V. and Tamme E., Solution of Nonlinear Fredholm Integral Equations Via the Haar Wavelet Method Proc. Estonian Acad. Sci. phys. Math., PP. 2691, 17-27, (2007).
- [7] Ibraheem K. I. nd Al-Bayati, A. Y., A modified Wavelet Algorithm to Solve BVPS with an infinite Number of Boundary Conditions, Int. J. Open Problems Comp. Math., Vol. 4, No. 2, (2011).
- [8] Chang P. and Pian P., Haar Wavelets Matrices Designation in Numerical Solution of Ordinary Differential Equations, IAENG International Journal of Applied Mathematics, 38:3, IJAM, 38-3-11.
- [9] Tamme E. and Lepik V., Application of the Haar Wavelets for Solution of Linear Integral Equations, Dynamical Systems and Applications, Proceedings, PP. 494-507, 5-10, (2004).