



Some New Relationships Between the Derivatives of First and Second Chebyshev Wavelets

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Abstract: In this study, using the properties of first and second chebyshev wavelets $\psi_r'(t)$ and $\psi_r^2(t)$ respectively, we explicitly present the relationship between them. Then a new formula expressing the first derivative of first kind Chebyshev wavelets $\frac{d\psi_r'(t)}{dt}$ interms of $\psi_r^2(t)$, and a formula expressing the derivative of second kind Chebyshev wavelets $\frac{d\psi_r^2(t)}{dt}$ interms of $\psi_r'(t)$ and $\psi_{r-2}^2(t)$ is deduced. The relation between the first derivatives of $\psi_r^2(t)$ is also given in this paper. All the proposed results are of direct interest in many applications.

Keywords: first Chebyshev wavelets, $\psi_r'(t)$, second Chebyshev wavelets $\psi_r^2(t)$, operational matrix of derivative.

I. Introduction

Wavelets theory is a relatively new emerging in mathematical research [1-4]. It has been applied in a wide range of engineering disciplines, particularly, wavelets play an important role in establishing algebraic methods for the solution of integral differential equations [5,6], analysis of time-varying or time delay system and optimal control [7-9]. This paper is organized as follow: section 2 talks about the first and second shifted Chebyshev wavelets. Section 3, gives some important properties of first and second Chebyshev wavelets, the relation between $\psi_r'(t)$ and $\psi_r^2(t)$, the operational matrices of derivative of $\psi_r'(t)$ and $\psi_r^2(t)$. Section 4, discusses some important relationship between the operational matrices of derivative for $\psi_r'(t)$ and $\psi_r^2(t)$ interms of $\psi_r'(t)$ and $\psi_r^2(t)$ themselves.

II. Chebyshev Wavelets

Wavelets constitute a family of signal functions constructed from dilation called the mother wavelet when the dilation parameter a and translation parameter b vary continuously, we have the following family of wavelets [5].

$$\Psi_{a,b}(t) = |a|^{\frac{1}{2}} \Psi\left(\frac{t-b}{a}\right) \quad a, b \in \mathbb{R} \quad a \neq 0$$

where $\Psi(t) = [\Psi_0(t), \Psi_1(t), \dots, \Psi_{M-1}(t)]^T$

The elements $\Psi_0(t), \Psi_1(t), \dots, \Psi_{M-1}(t)$ are the basis functions, orthogonal on the $[0, 1]$. We should note in dealing with Chebyshev wavelets the weight function $w(t)$ have to dilated and translated as

$$w_n(t) = w(2^k t - 2n + 1)$$

Chebyshev polynomials are encountered in several areas of numerical analysis, and they hold particular importance in various subjects such as orthogonal polynomials, polynomials approximation, numerical integration and spectral methods [10-13]. Depending on the definition of Chebyshev polynomials we can present the definitions of Chebyshev wavelets for first and second kinds [3,6].

Definition (1) Shifted First Chebyshev Wavelets

first Chebyshev wavelets $\Psi_{n,m}^1 = \Psi_{k,n,m,t}^1$ have four arguments; n argument k can assume any positive integer and t is the normalized time. They are defined on the interval $[0, 1]$ by

$$\Psi_{n,m}^1(t) = \begin{cases} 2^{\frac{k+1}{2}} T_m^*(2^{k+1}t - 2n + 1) & \frac{n}{2^k} \leq t < \frac{n+1}{2^k} \\ 0 & \text{O.W} \end{cases}$$

$$\text{where } T_m^* = \begin{cases} \frac{1}{\sqrt{\pi}} T_m & m = 0 \\ \sqrt{\frac{2}{\pi}} T_m & m \geq 1 \end{cases}, \quad m = 0, 1, \dots, M \text{ and } n = 0, 1, 2, \dots, 2^k - 1 \quad (1)$$

Definition (2) Shifted Second Chebyshev Wavelets $\Psi_{n,m}^2$.

Second Chebyshev wavelets $\Psi_{n,m}^2 = \Psi_{k,n,m,t}^2$ have four arguments; n argument k can assume any positive integer and t is the normalized time. They are defined on the interval $[0,1]$ by

$$\Psi_{nm}^2(t) = \begin{cases} 2^{\frac{k}{2}} U_m^*(2^{k+1}t - 2n - 1) & \frac{n}{2^{k-1}} \leq t < \frac{n+1}{2^{k-1}} \\ 0 & \text{O.W} \end{cases} \quad (2)$$

$m = 0, 1, 2, \dots, M$, $n = 0, 1, 2, \dots, 2^{k-1} - 1$ where $U_m^*(t) = \sqrt{\frac{2}{\pi}} U_m(t)$

A function $f(t)$ defined over $L^2[0,1]$ can be expanded in terms of (Ψ_{nm}) either $\psi_r'(t)$ or $\psi_r^2(t)$

$$f(t) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} C_{nm} \Psi_{nm} \quad (3)$$

where $C_{nm} = (f(t), \Psi_{nm}(t))$

If infinite series in (3) is truncated then it can be written as

$$f(t) = \sum_{n=0}^{2^{k-1}-1} \sum_{m=0}^{M-1} C_{nm} \Psi_{nm} \quad (4)$$

$$C = [C_{00}, C_{01}, \dots, C_{0M}, \dots, C_{(2^{k-1}-1)M}, \dots, C_{(2^{k-1}-1)0}, C_{(2^{k-1}-1)1}, \dots, C_{(2^{k-1}-1)M}]^T$$

$$\Psi(t) [\Psi_{00}, \Psi_{01}, \dots, \Psi_{0M}, \dots, \Psi_{(2^{k-1}-1)M}, \dots, \Psi_{(2^{k-1}-1)0}, \Psi_{(2^{k-1}-1)1}, \dots, \Psi_{(2^{k-1}-1)M}]^T \quad (5)$$

then C and $\Psi(t)$ are $2^k(M+1) \times 1$ matrices.

III. New Relation between Ψ_{nm}^1 and Ψ_{nm}^2 .

$$\text{For } \Psi_{nm}^2 - \Psi_{nm-2}^2 = \frac{2\sqrt{2}}{c_m} \Psi_{nm}^1(t) \quad (6)$$

$$\text{where } c_m = \begin{cases} \sqrt{2} & m = 0 \\ 2 & m = 1, 2, \dots \end{cases} \quad (7)$$

From the following relation between T_m, U_m ;

$$U_m(x) - U_{m-2}(x) = 2 T_m(x), \quad m = 2, 3, \dots$$

we have

$$\begin{aligned} \Psi_{nm}^2 &= \sqrt{\frac{2}{\pi}} 2^{\frac{k}{2}} (2T_m + U_{m-2})(2^{k+1}t - 2n - 1) \\ &= \sqrt{\frac{2}{\pi}} 2^{\frac{k}{2}} 2T_m(2^{k+1}t - 2n - 1) + \sqrt{\frac{2}{\pi}} 2^{\frac{k}{2}} U_{m-2}(2^{k+1}t - 2n - 1) \\ \Psi_{n,m}^2 &= \Psi_{n,m-2}^2 + 2^{\frac{k}{2}} \sqrt{\frac{2}{\pi}} 2T_m(2^{k+1}t - 2n - 1) \end{aligned}$$

$$\text{or } \Psi_{n,m}^2 - \Psi_{n,m-2}^2 = \sqrt{2} \frac{2}{\sqrt{\pi}} 2^{\frac{k}{2}} T_m(2^{k+1}t - 2n - 1) \text{ then } \Psi_{n,m}^2 - \Psi_{n,m-2}^2 = \frac{2\sqrt{2}}{c_m} \Psi_{n,m}^1(t), \quad m = 2, 3, \dots$$

IV. Operational Matrices of Derivative for Ψ_{nm}^1 and Ψ_{nm}^2 .

Theorem (1):

Let $\Psi^1(t)$ be Chebyshev wavelets vector defined by

$$\Psi^1(t) [\Psi_{00}^1, \Psi_{01}^1, \dots, \Psi_{0M}^1, \dots, \Psi_{(2^{k-1}-1)M}^1, \dots, \Psi_{(2^{k-1}-1)0}^1, \Psi_{(2^{k-1}-1)1}^1, \dots, \Psi_{(2^{k-1}-1)M}^1]^T$$

The derivative of this vector $\Psi^1(t)$ can be expressed by

$$\frac{d\Psi^1}{dt} = D_{\Psi^1} \Psi^1(t) \quad (8)$$

where D is the $2^k(M+1)$ Operational matrix of derivative defined as follow

$$D_{\Psi^1} = \begin{bmatrix} F & 0 & \dots & 0 \\ 0 & F & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & F \end{bmatrix}$$

In which F is $(M+1) \times (M+1)$ matrix and its (r, s) the element is defined as follow

$$F_{r,s} = \begin{cases} 2^{k+1}m & r = 2, \dots, (M+1), s = 1, \dots, r-1 \\ 0 & (r+s) \text{ odd} \end{cases} \quad (9)$$

Theorem (2)

Let Ψ^2 be the second Chebyshev wavelets vector defined by

$$\Psi^2(t) [\Psi_{00}^2, \Psi_{01}^2, \dots, \Psi_{0M}^2, \dots, \Psi_{(2^{k-1}-1)M}^2, \dots, \Psi_{(2^{k-1}-1)0}^2, \Psi_{(2^{k-1}-1)1}^2, \dots, \Psi_{(2^{k-1}-1)M}^2]^T$$

then the derivative of the Ψ^2 can be expressed by.

$$\frac{d\Psi^2}{dt} = D_{\Psi^2} \Psi^2(t)$$

where D_{Ψ^2} is the $2^{k-1}(M+1)$ operational matrix of derivative defined as follow

$$D_{\Psi^2} = \begin{bmatrix} L & 0 & \cdots & 0 \\ 0 & L & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & L \end{bmatrix} \quad (10)$$

In which L is $(M + 1) \times (M + 1)$ matrix and its (r, s) the element is defined as follow

$$L_{r,s} = \begin{cases} 2^{k+1}s & r=2,3,\dots,(M+1) \quad s=1,2,\dots,(r-1) \\ 0 & \text{otherwise} \end{cases} \quad (11)$$

Proof:

By using shifted second Chebyshev polynomials in to $[0,1]$ the r th element of vector $\Psi^2(t)$ can be written as

$$\Psi_r^2(t) = \begin{cases} 2^{\frac{k}{2}} \sqrt{\frac{2}{\pi}} U_m^*(2^k t - n) & t \in \left[\frac{n}{2^{k-1}}, \frac{n-1}{2^{k-1}} \right] \\ 0 & \text{O.W} \end{cases} \quad (12)$$

$r = 1, 2, \dots, 2^{k-1}(M + 1)$

where $r = n(M + 1) + (m + 1)$ $m = 0, 1, \dots, M$ and $n = 0, 1, \dots, (2^{k-1} - 1)$

Differentiation equation (12) w.r.t t we have

$$\frac{d\Psi_r^2}{dt} = \begin{cases} 2^{\frac{k}{2}} \sqrt{\frac{2}{\pi}} 2^k U_m^*(2^{k-1} t - n) & \frac{n}{2^{k-1}} \leq t < \frac{n+1}{2^{k-1}} \\ 0 & \text{otherwise} \end{cases} \quad (13)$$

that is $\Psi_i^2(t)$, $i = n(M + 1) + 1, n(M + 1) + 2, \dots, (n + 1)(M + 1)$ so, its second. Chebyshev wavelets expansion has the following form

$$\frac{d\Psi_r^2}{dt} = \sum_{i=n(M+1)+1}^{(n+1)(M+1)} a_i \Psi_i^2$$

This implies that the operational matrix D_{Ψ^2} is ablok matrix as defined in (10), Moreover; we have $\frac{dU_0^*(t)}{dt} = 0$

This results that $\frac{d\Psi_r^2}{dt} = 0$ for $r = 1, (M + 1) + 1, \dots, (2^{k-1} - 1)(M + 1) + 1$.

Consequently the first row of matrix L defined in (11) is zero.

with the aid of the first derivative of shifted second kind Chebyshev polynomial given by the following formula

$U_m^{*'} = \sum_{k=0}^{m-1} 4(k + 1) U_k^*(t)$, $m = 1, 2, \dots, m + k$ odd

$$\frac{d\Psi_r^2}{dt} = 2^{\frac{k}{2}} \sqrt{\frac{2}{\pi}} 2^k \sum_{j=0}^{m-1} 4(j + 1) U_j^*(2^{k-1} t - n) \quad (j + m) \quad (14)$$

$$= 2^{k+1} \sum_{s=1}^{r-1} s \Psi_{n(M+1)+s}^2 t \quad (s + r) \text{ odd} \quad (15)$$

so if we choose $L_{r,s}$

$$L_{r,s} = \begin{cases} 2^{k+1}s & r = 2, \dots, (M + 1), s = 1, \dots, r - 1 \text{ and } (r + s) \text{ odd} \\ 0 & \text{O.W} \end{cases} \quad (16)$$

Equation $\frac{d\Psi_r^2}{dt} = D_{\Psi^2}$ is hold.

V. Some New Relationships between the Derivatives of Ψ_{nm}^1 and Ψ_{nm}^2 .

Some new relationships between $\Psi_{n,m}^1(t)$ and $\Psi_{n,m}^2(t)$ are derived and given throughout the following lemmas.

Lemma (1)

The first derivative of $\Psi_r^1(t)$ in terms of $\Psi_r^2(t)$ is formulated as

$$\frac{d\Psi_r^1(t)}{dt} = 2^{k+1} m \sum_{s=1}^{r-1} \Psi_{n(M+1)+s}^2 \quad (17)$$

Proof

By using shifted first Chebyshev polynomials in to $[0,1]$, $\Psi_{n,m}^1(t)$ can be written as

$$\Psi_{n,m}^1(t) = \begin{cases} 2^{\frac{k+1}{2}} T_m^*(2^{k+1} t - 2n - 1) & \frac{n}{2^k} \leq t < \frac{n+1}{2^k} \\ 0 & \text{O.W} \end{cases}$$

Then differentiation with respect to t yields,

$$\begin{aligned} \frac{d\Psi_r^1(t)}{dt} &= 2^k \sum_{s=1}^{r-1} 4m \sqrt{\frac{c_{r-1}}{c_{s-1}}} \Psi_{n(M+1)+s}^1(t) & r = 2, \dots, (M + 1), s = 1, \dots, r - 1 \\ & & (r + s) \text{ odd} \\ &= 2^{k+2} m \sqrt{\frac{c_{r-1}}{c_{s-1}}} \sum_{s=1}^{r-1} \Psi_{n(M+1)+s}^1(t) & (s + r) \text{ odd} \end{aligned}$$

or

$$\frac{d\Psi_r^1(t)}{dt} = (2^{k+1} m) 2 \sqrt{\frac{c_{r-1}}{c_{s-1}}} \sum_{s=1}^{r-1} \Psi_{n(M+1)+s}^1(t) \quad (18)$$

where $r = 2, \dots, M + 1$

$$c_{s-1} = \begin{cases} 2 & s - 1 = 0 \\ 1 & \text{O.W} \end{cases}$$

Since $\Psi_{r-1}^1(t) = \frac{2}{\sqrt{\pi}}$ then $\frac{d\Psi_r^1(t)}{dt} = 2^{k+1}m\sqrt{2}\Psi_{r-1}^1$

That is $\frac{d\Psi_r^1(t)}{dt} = 2^{k+1}m\Psi_{r-1}^2(t)$

Finally If $\frac{c_{r-1}}{c_{s-1}} = 1$ Then equation (18) becomes $\frac{d\Psi_r^1(t)}{dt} = 2^{k+1}m\sum_{s=1}^{r-1}\Psi_{n(M+1)+s}^2$ which is the required result.

Lemma 3:

The derivative of $\Psi_r^2(t)$ in terms of $\Psi_r^1(t)$ and $\Psi_{r-2}^2(t)$ is formulated as

$$\frac{d\Psi_r^2(t)}{dt} = (2^{k+1}s)[\sum_{s=1}^{r-1}\frac{2\sqrt{2}}{c_{s-1}}\Psi_{n(M+1)+s}^1(t) + \sum_{s=1}^{r-1}\Psi_{n(M+1)+s-2}^2(t)] \quad (19) \text{ where}$$

$$r = 2, \dots, (M+1), s = 1, 2, \dots, r-1 \quad \text{and} \quad c_{s-1} = \begin{cases} \sqrt{2} & s-1=0 \\ 2 & \text{other wise} \end{cases}$$

Proof:

From theorem (2)

$$\frac{d\Psi_r^2(t)}{dt} = 2^{k+1}s\sum_{s=1}^{r-1}\Psi_{n(M+1)+s}^2(t) \quad s+r=\text{odd} \quad (20)$$

and from relationships between Ψ_{nm}^1 and Ψ_{nm}^2 in chapter 2 and substitute it in(20) we obtained following equation

$$\frac{d\Psi_r^2(t)}{dt} = (2^{k+1}s)\sum_{s=1}^{r-1}(\frac{2\sqrt{2}}{c_{s-1}}\Psi_{n(M+1)+s}^1(t) + \Psi_{n(M+1)+s-2}^2(t))$$

or

$$\frac{d\Psi_r^2(t)}{dt} = (2^{k+1}s)[\sum_{s=1}^{r-1}\frac{2\sqrt{2}}{c_{s-1}}\Psi_{n(M+1)+s}^1(t) + \sum_{s=1}^{r-1}\Psi_{n(M+1)+s-2}^2(t)]$$

then equation (20) is hold.

Lemma (3)

The first derivative of $\Psi_r^2(t)$ in terms of the second derivative of $\Psi_r^1(t)$ is given by the formula:

$$\frac{d\Psi_r^2(t)}{dt} = \frac{1}{2^{k+1}m}\sum_{s=1}^r\Psi_{n(M+1)+s+1}^{1''}(t) \quad (r+s)\text{odd} \quad (21)$$

Proof:

From lemma (1)

$$\frac{d\Psi_r^1(t)}{dt} = (2^{k+1}m)\sum_{s=1}^{r-1}\Psi_{n(M+1)+s}^2(t)$$

Differentiation above equation to obtain

$$\frac{d^2\Psi_r^1(t)}{dt^2} = (2^{k+1}m)\sum_{s=1}^{r-1}\frac{d\Psi_{n(M+1)+s}^2(t)}{dt}$$

Divided both sides $(2^{k+1}m)$

$$\frac{1}{2^{k+1}m}\frac{d^2\Psi_r^1(t)}{dt^2} = \sum_{s=1}^{r-1}\frac{d\Psi_{n(M+1)+s}^2(t)}{dt}$$

$$r-1 = n(M+1) + s \quad r = n(M+1) + s + 1$$

$$\text{then} \quad \frac{d\Psi_r^2(t)}{dt} = \frac{1}{2^{k+1}m}\sum_{s=1}^r\Psi_{n(M+1)+s+1}^{1''}(t)$$

$$r = 3, \dots, (M+1), s = 1, \dots, (r-1)$$

which is the required result.

VI. Conclusion

In this work, at first, we demonstrate the relation between Chebyshev wavelets of the first and second kinds. Then we derived some important relationships between Chebyshev wavelets and their operational matrices of derivative, which plays an important role for algebraic methods.

VII. References

- [1] Chi. K. & Liu. M., "A Wavelet approach to fast approximation of Power Electronics circuits", International Journal of Circuit theory and Applications, 31(2003), 591-610.
- [2] W. Jie. , Liu. M., "Application of Wavelet Transform to steady-state Approximation of power electronics Waveforms", Department of Electronic, Hong Kong Polytechnic University, (2003).
- [3] Biazar.J., Ebrahimi.H., "A Strong Method for Solving Systems of Integro-Differential Equations", Applied Mathematical, Vol.2, (2011), 1105-1113, <http://WWW.ivsl.org/?language=ar>.
- [4] Sohrabi. S., "Comparison Chebyshev Wavelets Method with BPFs Method For Solving Abel's Integral equation", in Shams Engineering Journal, Vol.2, (2011).
- [5] Arsalani. M. & Vali. M. A., "Numerical Solution of Nonlinear Problems With Moving Boundary Conditions by Using Chebyshev Wavelets", Applied Mathematical Sciences, Vol.5, (2011), No. 20.

- [6] Ayyaswamy. S. K. & Venkatesh. S.G., Haar, "Wavelets for Solving Initial, Boundary Value Problems of Bratu-type", International Journal of Computational and Mathematical Sciences, (2010), 4:6.
- [7] Fatihi M. R. and Samavat M., "state Analysis and optimal control of Linear Time-Invariant Scaled Systems Using the Chebyshev wavelets, Contemporary Engineering Sciences, Vol. 5, (2012), No. 2, 91-105. Available on: IVSL.org.
- [8] Kumar R. and Kumar C., "A wavelet Approach for Identification of Linear Time Invariant System", International Journal of Computer Applications, Vol. 50, No. 20, (2002), 0975-8857. Available at: IVSL.org.
- [9] Elaydi H. and Abu Haya A., "Solving Optimal Control Problem for Linear Time Invariant Systems Via Chebyshev wavelet", International Journal of Electrical Engineering, Vol. 5, No. 5, (2012), pp 541-556. Available from: IVSL.org.
- [10] Eslahchi M. R. and Dehghan M., "The third and fourth kinds of Chebyshev polynomials and best uniform approximation", Mathematical and Computer Modelling, 55 (2012) 1746-1762. Available from: IVSL.org.
- [11] Belforte G. and Gay P., "Some new properties of Chebyshev polynomials", Journal of Computational and Applied Mathematics, 117 (2000), 175-181. Available at: IVSL.org.
- [12] Mason J. C. and Elliott G. H., "Near-minimax complex approximation by four kinds of Chebyshev polynomial expansion", Journal of Computational and Applied Mathematics, 46, (1993), 291-300. Available from: IVSL.org.