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A Fast Procedure for Solving Two-Body Problem in Celestial Mechanics

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Abstract: This paper discusses the solution of two body problem in a parabolic orbit, where the true anomaly v (as a function of the time) can be obtained by solving a cubic equation for $(\tan \frac{v}{2})$ named Bakers equation. We have suggested and analyzed an iterative method which works well for Barker's equation with suitable suggested initial guesses for the iterative. The convergence of the new method is discussed. It is established that the method has convergence of order five. In terms of computational cost, it requires evaluation of three functions and two first order derivatives per iteration.

Keywords: two-body problem, iterative method, convergence criteria, Barker's equation.

I. Introduction

The determination of the position and velocity in two-body orbits leads to the solution of transcendental equation commonly referred to as "Kepler's equation" which relates the dependence of position in orbit with time. In classical analysis, the shape of these two-body orbits is described through the use of conics and corresponding to each conic Kepler's equation has a different form. A useful quantity in classifying conics is a constant e called eccentricity [1], for conic sections we have the following classifications:

- (i) for $0 \leq e < 1$ the orbit is an ellipse,
- (ii) for $e = 1$ the orbit is parabola,
- (iii) for $e > 1$ the orbit is hyperbola.

It should be noted that the case $e = 1$ also includes the rectilinear ellipse, parabola and hyperbola. The case $e = 0$ is the special case of the ellipse of zero eccentricity (i.e a circle) [2]. Also obtainable from elementary considerations is the general polar equation for the conic which can be stated as

$$r = \frac{h^2/\mu}{1+e \cos f} = \frac{p}{1+e \cos f} \quad (1)$$

Where h is the massless angular momentum, μ is the product of the universal gravitational constant and the sum of the masses of the two bodies, p is the semi-latus rectum or parameters and f called the anomaly, is the angle between the radius vector and the direction of pericenter or point of closest approach of the two bodies [3]. In virtually every decade from 1650 to the present there have appeared papers devoted to Kepler's problem and its solution [4,5,6,7,8]. One of the usual ways to Kepler's equation is by the mean of iterative algorithms [9,10]. Several numerical methods have been suggested and analyzed under certain conditions. These numerical methods have been constructed using different technique such as Laguerre algorithm [11], Baoubaker Polynomials Expansion Scheme [12], and Richardson [13], others can be found in [14-18]

In this paper, we present a new three step method to solve two-body problem in a parabolic orbit, with the purpose of reducing the root of iteration in the determination of the root of a nonlinear equation, it has improved the iterative method by increasing the order of convergence.

II. Properties of Parabolic Orbits

In this type of two body motion (where $e = 1$) the orbit is open, the second body approaching the first from infinity until, at its nearest approach when the relative velocity is a maximum it begins to recede to infinity. The equation of the parabolic orbit is obtained by putting $e = 1$ in eq. (1) [2]

$$r = \frac{p}{1+\cos f} \quad (2)$$

where p and f are the semilatus rectum and true anomaly respectively. The integral of areas is

$$r^2 \dot{f} = h \quad \text{where} \quad p = \frac{h^2}{\mu} \quad (3)$$

Now, eq. (2) may be written as

$$r = \frac{p}{2} \sec^2 \frac{f}{2} = \frac{p}{2} (1 + \tan^2 \frac{f}{2}), \text{ therefore; eq. (3) gives}$$

$$\left(\frac{p}{2}\right)^2 \sec^4 \frac{f}{2} = (p\mu)^{\frac{1}{2}} \quad \text{or} \quad 4\left(\frac{\mu}{p^3}\right)^{\frac{1}{2}} dt = \left(\sec^2 \frac{f}{2} + \sec^2 \frac{f}{2} \tan^2 \frac{f}{2}\right)$$

$$\text{Integrating, yields } 2\left(\frac{\mu}{p^3}\right)^{\frac{1}{2}} (t - \tau) = \left(\tan \frac{f}{2} + \frac{1}{3} \tan^3 \frac{f}{2}\right) \quad (4)$$

Where τ is the time of perihelion passage.

Define $\bar{n}$ by the equation

$$\bar{n}^2 p^3 = \mu \quad (5)$$

Let $D = \tan \frac{f}{2}$ hence, eq. (4) may be written as

$$D + \frac{D^3}{3} = 2\bar{n} (t - \tau) \quad (6)$$

Equations (4) and (6) are versions of Barker's equation, which has been extensively used in studies of the orbits of comets and is now used in Astrodynamics [1].

III. Three step new iterative method

Newton's method has deservedly been the traditional favorite when successive approximations are used to solve the equation. It was simple to use in the days of logarithm, it's propensities or seeking out some unwanted solution are given little scope when applied when applied to Kepler's equation. Consequently, we have found the new method is stable and has a convergence of order five to determine the solution of eq. (6).

Eq. (6) can be written as follows

$$f(D) = D^3 + 3D - 6\bar{n}(t - \tau) \quad (7)$$

The suggested algorithm for solving eq. (7) is the following three step iterative method

$$y_n = D_n - \frac{f(D_n)}{\hat{f}(D_n)} \quad (8)$$

$$Z_n = y_n - \frac{f(D_n) - 2f(y_n)}{f(D_n) - 4f(y_n)} - \frac{f(y_n)}{\hat{f}(D_n)} \quad (9)$$

$$D_{n+1} = Z_n - \frac{f(Z_n)}{\hat{f}(Z_n)} \quad (10)$$

for $n = 0, 1, 2, \dots$

$$\text{where } D_0 \text{ in the initial guess, and we suggested to be } D_0 = \frac{6\bar{n}(t-\tau)}{4} \quad (11)$$

now, the algorithm described by (8-10).

If α is the root and e_n be the error at n th iteration, then $e_n = D_n - \alpha$, using Taylor's expansion, we have

$$f(D_n) = \hat{f}(\alpha)[e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + 0(e_n^5)] \quad (12)$$

$$\hat{f}(D_n) = \hat{f}(\alpha)[1 + 2c_2 e_n + 3c_3 e_n^2 + 4c_4 e_n^3 + 0(e_n^4)] \quad (13)$$

Where $c_k = \frac{f^{(k)}(\alpha)}{k! \hat{f}(\alpha)}$, $k=2, 3, \dots$

From equations (7) and (8), we have

$$\frac{f(D_n)}{\hat{f}(D_n)} = [e_n - c_2 e_n^2 - (2c_3 e_n^3 - 2c_2^2) e_n^3 - (3c_4 - 7c_2 c_3 + 4c_2^3) e_n^4 + 0(e_n^5)] \quad (14)$$

From eq. (14), we have

$$f(y_n) = \hat{f}(\alpha)[c_2 e_n^2 - (-2c_3 + 2c_2^2) e_n^3 + (3c_4 - 7c_2 c_3 + 4c_2^3) e_n^4 + 0(e_n^5)] \quad (15)$$

$$\text{From equations (12) and (15), one can obtain} \quad (16)$$

$$f(D_n) - 2f(y_n) = \hat{f}(\alpha)[e_n - c_2 e_n^2 + c - 3c_3 + 4c_2^2] e_n^3 + (14c_2 c_3 - 10c_2^3 - 5c_4) e_n^4 + 0(e_n^5)] \quad (17)$$

$$f(D_n) - 4f(y_n) = \hat{f}(\alpha)[e_n - 3c_2 e_n^2 + (-7c_3 + 8c_2^2) e_n^3 + (-11c_4 + 28c_2 c_3 - 20c_2^3) e_n^4 + 0(e_n^5)] \quad (18)$$

Hence,

$$\frac{f(D_n) - 2f(y_n)}{f(D_n) - 4f(y_n)} = 1 + 2c_2 e_n + (-10c_2^2 + 4c_3) e_n^2 + (-36c_2^3 + 6c_4 + 12c_2 c_3) e_n^3 + (-228c_2^4 - 110c_2^2 c_3 - 4c_2 c_4 - 28c_2^3 e_n^4 + 0e_n^5) \quad (19)$$

$$\text{and } \frac{f(D_n) - 2f(y_n)}{f(D_n) - 4f(y_n)} - \frac{f(y_n)}{\hat{f}(D_n)} = c_2 e_n^2 + c_2 c_3 - 2c_2^2 e_n^3 + (-10c_2 c_3 + 5c_2^3 + 3c_4) e_n^4 + (6c_2 c_4 + 26c_2^4 - 28c_2^2 c_3) e_n^5 + 0(e_n^6) \quad (20)$$

$$Z_n = \alpha - c_2 e_n^2 + (2c_2^2 - 2c_3) e_n^3 + (10c_2 c_3 - 5c_2^3 - 3c_4) e_n^4 + (-6c_2 c_4 - 26c_2^4 + 28c_2^2 c_3) e_n^5 + 0(e_n^6) \quad (21)$$

Using (14) and (20) in (9), we have

$$f(Z_n) = f(\alpha) + (Z_n - \alpha) \hat{f}(\alpha) + \frac{1}{2!} (Z_n - \alpha)^2 \hat{f}(\alpha) + \frac{1}{3!} (Z_n - \alpha)^3 \hat{f}(\alpha) + \quad (22)$$

Using (21) in (22) on simplifying yields

$$f(Z_n) = \hat{f}(\alpha)[-c_2 e_n^2 + (2c_2^2 - 2c_3) e_n^3 + (10c_2 c_3 - 4c_2^3 - 3c_4) e_n^4 + (20c_2^4 + 24c_2^2 c_3 - 6c_2 c_4) e_n^5 + 0(e_n^6)] \quad (23)$$

$$\text{And } \hat{f}(Z_n) = \hat{f}(\alpha)[1 - c_2 e_n^2 + (4c_2^2 - 4c_2 c_3) e_n^3 + (10c_2^2 c_3 - 3c_2^3 - 3c_4) e_n^4 + (28c_2^4 + 16c_2^2 c_3 - 6c_2 c_4) e_n^5 + 0(e_n^6)] \quad (24)$$

Using equations (21), (23) and (24) in eq. (10) to get

$$D_{n+1} = \alpha + (6c_2^2 c_3 - 10c_2^3 c_3 + 5c_2^5 + 3c_2^2 c_4) e_n^5 + 0(e_n^6) \text{ or } e_{n+1} = (6c_2^2 c_3 - 10c_2^3 c_3 + 5c_2^5 + 3c_2^2 c_4) e_n^5 + 0(e_n^6) \quad (25)$$

Thus, we observe that the proposed algorithm (8-10) has fifth order convergence.

IV. Application of the new method to solve parabolic orbit equation

To compare the different methods of iterations algorithms 1 and 2 against the proposed algorithm (algorithm 3) To demonstrate the performance of the new method, we solve eq. (7). We shall determine the consistency and stability of results by examining the convergence of the proposed method. The results are examined using three iterative methods:

$$\text{Method (1): Newton Method (NM)} \quad D_{n+1} = D_n - \frac{f(D_n)}{f'(D_n)} \quad (26)$$

$$\text{Method (2): Two -step method (TSM):} \quad y_n = D_n - \frac{f(D_n)}{f'(D_n)} \\ D_{n+1} = y_n - \frac{f(D_n) - 2f(y_n)}{f'(D_n) - 4f'(y_n)} \left(\frac{f'(y_n)}{f'(D_n)} \right) \quad (27)$$

Method (3): Three step Method (THSM), given in equations (8-10).

We take $\epsilon = 10^{-15}$ as tolerance. The following criteria is used for estimating the zero

$$\sigma = |D_{n+1} - D_n| < \epsilon, \quad |f(D_n)| < \epsilon$$

For convergence criteria, it was required that σ the distance between two consecutive iterates was less than 10^{-15} , n represents the number of iterations and $f(D_n)$, the absolute value of the function. Also the computational order of convergence (COC) can be approximated using the formula [10]

$$\text{COC} = \frac{\ln \left| \frac{D_{n+1} - D_n}{D_n - D_{n-1}} \right|}{\ln \left| \frac{D_n - D_{n-1}}{D_{n-1} - D_{n-2}} \right|}$$

Consider eq. (6), where $(t - \tau) = 1.2025$ TU (time unit) in a parabolic orbit where $p = 2$ AU (angular distance unit) and $\mu = 1$. Using eq. (5) to obtain $\bar{n}$, $\bar{n}^2 p^3 = \mu$ that is $\bar{n} = 0.353553390593274$, therefore $b = 6 \bar{n} (t - \tau) = 2.550887713130470$. When taking the initial guess $D_0 = \frac{6 \bar{n} (t - \tau)}{4}$, the numerical results for eq. (6) are listed in the table (1)

Table (1) shows the results when $D_0 = 0.637721928282618$

Methods	IT	D_n	σ	COC
NM	6	0.723865337018299	9.4920e-16	1.973
TSM	4	0.723865337018299	0	2.88
THSM	3	0.723865337018299	0	4.8

Taking some other initial values $D_0 = b = 2.550887713130470$, $D_0 = \frac{b}{2} = 1.275443856565235$, $D_0 = \frac{3b}{4} = 1.9131657848447853$, the obtained results are listed in table (2).

Table (2) shows the results for different initial guesses $b = 2.550887713130470$, $\frac{b}{2} = 1.275443856565235$, $\frac{3b}{4} = 1.9131657848447853$

Initial guess D_0	IT			D_n
	NM	TSM	THSM	
b	9	6	3	0.723865337018299
$\frac{b}{2}$	7	5	3	0.723865337018299
$\frac{3b}{4}$	8	4	2	0.723865337018299

V. Conclusions

In this paper, we have demonstrated the performance of new fifth iterative method, namely three step iterative method which works well for solving the two body problem in parabolic orbit (Barker's equation). Also different initial guesses for iterative are suggested depending on μ , p and $(t - \tau)$, we have shown numerically and verified that the new method converge of the order five. We have performed some tests for solving Barker's equation (eq. 6) for several initial values for the iteration $D_0 = b$, $D_0 = \frac{b}{4}$, $D_0 = \frac{b}{2}$ and $D_0 = \frac{3b}{4}$ where $b = 6 \bar{n} (t - \tau)$ and the results are obtained and appear in tables (1), (2). We observed that the efficiency of the new three step iterative method considerably improve that of Newton method and the given two step method. Remark that only 2 or 3 iterations are needed to reach the exact solution with small tolerance, while Newton's

method requires at least 6 iterations when the initial guess is $\frac{b}{4}$. And the proposed two step method requires 4 iterations or more to reach the exact value.

VI. References

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