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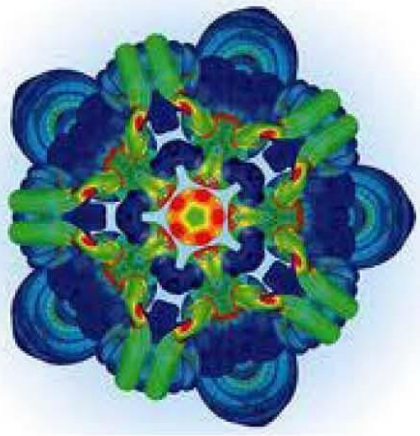
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A Discrete Fractional Order Lyapunov Type Inequality For Boundary Value Problem

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***Abstract**: In this present work, we consider a discrete nonlinear two point boundary value problem with fractional order. We study some useful inequalities satisfied by the Green function and also obtain Lyapunov type inequality for a discrete fractional order boundary value problem.*

***Keywords:** Discrete Fractional Calculus, Green's Function, Lyapunov type inequality, Two-point boundary value problem.*

I. Introduction

In recent years the theory of fractional calculus, in particular fractional difference equations have attracted great attention both in science and engineering as well. In many situations, the mathematical models based on fractional difference operators provide better analogous results than involving classical integer order depravities [4]. Taking advantage of some well-known properties of the Green's functions for some discrete boundary value problems, [2] we shall deduce a Lyapunov-type inequality [3].

Initially, Atici and Eloe [1, 2] published some important results on discrete fractional calculus with delta operators. Goodrich [4, 5] investigated the existence and uniqueness of solutions using tools from nonlinear functional analysis and proved some new results.Also, in [6] Chidouh and et al. proved generalization of the Lyapunov in equality of [7]. Inspired by the above cited works, in this paper, we discuss the Lyapunov type inequality for the following discrete fractional order boundary value problem of the form

$$\Delta^{\alpha}u(t)+q(t+\alpha-1)u(t+\alpha-1)=0 \quad (1)$$

for $t\in[0,b+1]_{\mathbb{N}_0}=\{0,1,...,b+1\}$, subject to the conditions

$$\text{(i) } u(\alpha-2)=0 \text{ and (ii) } \Delta u(\alpha-2)=\Delta u(\alpha+b) \quad (2)$$

where $q:[\alpha-1,\alpha+b]_{\mathbb{N}_{\alpha-1}}\to[0,\infty), 1<\alpha\le 2$ and $b\in\mathbb{N}_0$.

II. Preliminaries

In this section, let us first recollect some basic definitions and lemmas essential for us in the sequel [1] - [2].

Definition 2.1. We define

$$t^{\underline{\alpha}}=\frac{\Gamma(t+1)}{\Gamma(t+1-\alpha)}, \quad (3)$$

for any t and α for which the right hand side is defined. We also appeal to the conventionthat if $t+1-\alpha$ is a pole of the Gamma function and $t+1$ is not a pole, then $t^{\underline{\alpha}}=0$.

Definition 2.2. The αth fractional sum of a function f, for $\alpha>0$ is defined as

$$\Delta^{-\alpha}f(t)=\Delta^{-\alpha}f(t,a):=\frac{1}{\Gamma(\alpha)}\sum_{s=a}^{t-\alpha}(t-s-1)^{\underline{\alpha-1}}f(s), \quad (4)$$

for $t\in\{a+\alpha,a+\alpha+1,...\}=\mathbb{N}_{a+\alpha}$. We also define the αth fractional difference for $t^{\underline{\alpha}}=0$ by $\Delta^{\alpha}_{a+\alpha}f(t):=\Delta^{N}\Delta^{\alpha-N}f(t)$, where $t\in\mathbb{N}_{a+\alpha}$ and $N\in\mathbb{N}$ is chosen so that $0\le N-1<\alpha\le N$.

Now we give some important lemmas.

Lemma 2.1. Let t and α be any numbers for which $t^{\underline{\alpha}}$ and $t^{\underline{\alpha-1}}$ are defined. Then $\Delta t^{\underline{\alpha}}=\alpha t^{\underline{\alpha-1}}$.

Lemma 2.2. Assume $\mu,\alpha>0$ and $f:\mathbb{N}_a\to\mathbb{R}$ be a real valued function. Moreover, let $\mu,\alpha>0$. Then we find that

$$\Delta^{-\alpha}\left[\Delta^{-\mu}f(t)\right]=\Delta^{-(\mu+\alpha)}f(t)=\Delta^{-\mu}\left[\Delta^{-\alpha}f(t)\right], (5)$$

where $t\in\mathbb{N}_{\mu+\alpha+a}$.

Lemma 2.3. Let $0\le N-1<\alpha\le N$. Then

$$\Delta^{-\alpha}\Delta^{\alpha}u(t)=u(t)+C_1t^{\underline{\alpha-1}}+C_2t^{\underline{\alpha-2}}+...+C_Nt^{\underline{\alpha-N}}, (6)$$

for some $C_i\in\mathbb{R}$ with $1\le i\le N$.

III. Main Results

In this section, we will derive the Green's function and Lyapunov type inequality to the discrete fractional boundary value problem (1) - (2). Before stating the next suitable theorem, let us introduce the following notation, which will be vital in the sequel:

$$T_1:=\left\{(t,s)\in[\alpha-1,\alpha+b]_{\mathbb{N}_{\alpha-1}}\times[0,b+1]_{\mathbb{N}_0}:0\le s<t-\alpha+1\le b+1\right\}.$$

$$T_2:=\left\{(t,s)\in[\alpha-1,\alpha+b]_{\mathbb{N}_{\alpha-1}}\times[0,b+1]_{\mathbb{N}_0}:0\le t-\alpha+1\le s\le b+1\right\}.$$

Theorem 3.1. Let $1<\alpha\le 2$. The unique solution of the discrete fractional boundary value problem (1) - (2) is given by

$$u(t):=-\frac{1}{\Gamma(\alpha)}\sum_{s=0}^{b+1}G(t,s)q(s+\alpha-1)u(s+\alpha-1), (7)$$

where $G(t,s)$ is Green's function for the problem

$$-\Delta^{\alpha}u(t)=0,\ u(\alpha-2)=0 \text{ and } \Delta u(\alpha-2)=\Delta u(\alpha+b), (8)$$

which is given by

$$G(t,s):=\begin{cases}\dfrac{(\alpha+b-s-1)^{\underline{\alpha-2}}t^{\underline{\alpha-1}}}{\Gamma(\alpha-1)-(\alpha+b)^{\underline{\alpha-2}}}+(t-s-1)^{\underline{\alpha-1}}, & (t,s)\in T_1\\ \dfrac{(\alpha+b-s-1)^{\underline{\alpha-2}}t^{\underline{\alpha-1}}}{\Gamma(\alpha-1)-(\alpha+b)^{\underline{\alpha-2}}}, & (t,s)\in T_2\end{cases}$$

Proof: From Lemma 2.3, we obtain a general solution of the fractional difference equation(1) - (2) is

$$u(t)=-\Delta^{-\alpha}q(t+\alpha-1)u(t+\alpha-1)+C_1t^{\underline{\alpha-1}}+C_2t^{\underline{\alpha-2}},$$

where $C_1,C_2\in\mathbb{R}$. Hence, by definition 2.2, we get that

$$u(t)=-\frac{1}{\Gamma(\alpha)}\sum_{s=0}^{t-\alpha}(t-s-1)^{\underline{\alpha-1}}q(s+\alpha-1)u(s+\alpha-1)+C_1t^{\underline{\alpha-1}}+C_2t^{\underline{\alpha-2}}.$$

We find the values of C_1 and C_2 so that the boundary conditions in (8) hold. Now weapply boundary condition (i) $u(\alpha-2)=0$, we get

$$-\frac{1}{\Gamma(\alpha)}\sum_{s=0}^{-2}(\alpha-s-3)^{\underline{\alpha-1}}q(s+\alpha-1)u(s+\alpha-1)+C_1(\alpha-2)^{\underline{\alpha-1}}+C_2(\alpha-2)^{\underline{\alpha-2}}=0. (9)$$

Now, it is easy to show by definition 2.1 that $(\alpha-2)^{\underline{\alpha-1}}=\dfrac{\Gamma(\alpha-1)}{\Gamma(0)}=0$ and $(\alpha-2)^{\underline{\alpha-2}}=\Gamma(\alpha-1)$. Finally,

$$-\frac{1}{\Gamma(\alpha)}\sum_{s=0}^{-2}(\alpha-s-3)^{\underline{\alpha-1}}q(s+\alpha-1)u(s+\alpha-1)=0$$

by the standard convention on sums.

So, in summary, we conclude that (9) implies $C_2\Gamma(\alpha+1)=0$. Therefore, $C_2=0$.

On the other hand, using the right boundary condition $\Delta u(\alpha-2)=\Delta u(\alpha+b)$, we find that

$$\Delta u(t)=-\Delta^{-(\alpha-1)}q(t+\alpha-1)u(t+\alpha-1)+C_1\Delta t^{\underline{\alpha-1}} (10)$$

Note that

$$\Delta t^{\underline{\alpha-1}}=(\alpha-1)\frac{\Gamma(t+1)}{\Gamma(t-\alpha+3)} (11)$$

and

$$\Delta^{-(\alpha-1)}q(t+\alpha-1)u(t+\alpha-1)=\frac{1}{\Gamma(\alpha-1)}\sum_{s=0}^{t-\alpha+1}(t-s-1)^{\underline{\alpha-2}}q(s+\alpha-1)u(s+\alpha-1) (12)$$

Substituting the equations (11) and (12) in (10), we get

$$\Delta u(t) = -\frac{1}{\Gamma(\alpha-1)}\sum_{s=0}^{t-\alpha+1}(t-s-1)^{\underline{\alpha-2}}q(s+\alpha-1)u(s+\alpha-1)+C_1(\alpha-1)\frac{\Gamma(t+1)}{\Gamma(t-\alpha+3)} \quad (13)$$

In view of $\Delta u(\alpha-2)=\Delta u(\alpha+b)$, we have

$$C_1=\frac{1}{\Gamma(\alpha-1)\left[(\alpha-1)(\alpha+b)^{\underline{\alpha-2}}-(\alpha-1)\Gamma(\alpha-1)\right]}\sum_{s=0}^{b+1}(\alpha+b-s-1)^{\underline{\alpha-2}}q(s+\alpha-1)u(s+\alpha-1).$$

Using C_1 and C_2 in $u(t)$, we get

$$\begin{aligned}u(t) &= -\frac{1}{\Gamma(\alpha)}\sum_{s=0}^{t-\alpha}(t-s-1)^{\underline{\alpha-1}}q(s+\alpha-1)u(s+\alpha-1)\\ &\quad+\sum_{s=0}^{b+1}\frac{(\alpha+b-s-1)^{\underline{\alpha-2}}\,t^{\underline{\alpha-1}}}{\Gamma(\alpha-1)\left[(\alpha-1)(\alpha+b)^{\underline{\alpha-2}}-(\alpha-1)\Gamma(\alpha-1)\right]}q(s+\alpha-1)u(s+\alpha-1)\\ &= -\frac{1}{\Gamma(\alpha)}\left\{\sum_{s=0}^{t-\alpha}\left[\frac{(\alpha+b-s-1)^{\underline{\alpha-2}}\,t^{\underline{\alpha-1}}}{\left[\Gamma(\alpha-1)-(\alpha+b)^{\underline{\alpha-2}}\right]}+(t-s-1)^{\underline{\alpha-1}}\right]\right.\\ &\quad\left.+\sum_{s=t-\alpha+1}^{b+1}\left[\frac{(\alpha+b-s-1)^{\underline{\alpha-2}}\,t^{\underline{\alpha-1}}}{\left[\Gamma(\alpha-1)-(\alpha+b)^{\underline{\alpha-2}}\right]}\right]\right\}q(s+\alpha-1)u(s+\alpha-1)\end{aligned}$$

from which it is immediately clear that we may write

$$u(t)=-\frac{1}{\Gamma(\alpha)}\sum_{s=0}^{b+1}G(t,s)q(s+\alpha-1)u(s+\alpha-1)$$

where

$$G(t,s):=\begin{cases}\dfrac{(\alpha+b-s-1)^{\underline{\alpha-2}}\,t^{\underline{\alpha-1}}}{\Gamma(\alpha-1)-(\alpha+b)^{\underline{\alpha-2}}}+(t-s-1)^{\underline{\alpha-1}}, & (t,s)\in T_1\\ \dfrac{(\alpha+b-s-1)^{\underline{\alpha-2}}\,t^{\underline{\alpha-1}}}{\Gamma(\alpha-1)-(\alpha+b)^{\underline{\alpha-2}}}, & (t,s)\in T_2\end{cases}$$

is Green's function for (8). This completes the proof.

Theorem 3.2. The Green's function $G(t,s)$ defined in Theorem 3.1 satisfies the followingconditions:

(i) $G(t,s)\geq 0$ for all $t\in[\alpha-1,\alpha+b]_{\mathbb{N}_{\alpha-1}}$ and $s\in[0,b+1]_{\mathbb{N}_0}$;

(ii) $\max_{t\in[\alpha-1,\alpha+b]_{\mathbb{N}_{\alpha-1}}} G(t,s)=G(s+\alpha-1,s)$, for each $s\in[0,b+1]_{\mathbb{N}_0}$.

Proof: Similar to the one found in [7].

Theorem 3.3. The function $G(s+\alpha-1,s)$ has a unique maximum given by

$$\max_{s\in[0,b+1]_{\mathbb{N}_0}} G(s+\alpha-1,s)=\frac{\Gamma(\alpha+b+1)\Gamma(\alpha-1)(b+2)}{\Gamma(\alpha-1)\Gamma(b+3)-\Gamma(\alpha+b+1)}.$$

Proof: To find maximum of $G(s+\alpha-1,s)$ over s we first apply the difference operator to $G(s+\alpha-1,s)$. Specifically:

$$\begin{aligned}\Delta G(s+\alpha-1,s) &= \Delta\left[\frac{(s+\alpha-1)^{\underline{\alpha-1}}(\alpha+b-s-1)^{\underline{\alpha-2}}}{\Gamma(\alpha-1)-(\alpha+b)^{\underline{\alpha-2}}}\right]\\ &= \frac{1}{\Gamma(\alpha-1)-(\alpha+b)^{\underline{\alpha-2}}}\Delta\left[\frac{\Gamma(s+\alpha)}{\Gamma(s+1)}\frac{\Gamma(\alpha+b-s)}{\Gamma(b-s+2)}\right]\end{aligned}$$

$$\Delta G(s+\alpha-1,s)=\frac{1}{\Gamma(\alpha-1)-(\alpha+b)^{\underline{\alpha-2}}}$$
$$\times\left[\frac{\Gamma(s+\alpha+1)}{\Gamma(s+2)}\frac{\Gamma(\alpha+b-s-1)}{\Gamma(b-s+1)}-\frac{\Gamma(s+\alpha)}{\Gamma(s+1)}\frac{\Gamma(\alpha+b-s)}{\Gamma(b-s+2)}\right]$$
$$=\frac{1}{\Gamma(\alpha-1)-(\alpha+b)^{\underline{\alpha-2}}}$$
$$\times\left[\frac{(b-s+1)\Gamma(s+\alpha+1)\Gamma(\alpha+b-s-1)}{\Gamma(s+2)\Gamma(b-s+2)}-\frac{(s+1)\Gamma(s+\alpha)\Gamma(\alpha+b-s)}{\Gamma(s+2)\Gamma(b-s+2)}\right]$$
$$=\frac{b(\alpha-1)+s(3-2\alpha)+1}{\Gamma(\alpha-1)-(\alpha+b)^{\underline{\alpha-2}}}\frac{\Gamma(s+\alpha)\Gamma(\alpha+b-s-1)}{\Gamma(s+2)\Gamma(b-s+2)}$$
$$\Delta G(s+\alpha-1,s)=\frac{b(\alpha-1)+s(3-2\alpha)+1}{\Gamma(\alpha-1)\Gamma(b+3)-\Gamma(\alpha+b+1)}\frac{\Gamma(b+3)\Gamma(s+\alpha)\Gamma(\alpha+b-s-1)}{\Gamma(s+2)\Gamma(b-s+2)}$$

From which it follows that $\Delta G(s+\alpha-1,s)=\left[b(\alpha-1)+s(3-2\alpha)+1\right]f(s)$ with $f(s)>0$ for all $s\in[0,b+1]_{\mathbb{N}_0}$. Now, if $\alpha<3/2$ then $h(s)=b(\alpha-1)+s(3-2\alpha)+1$ is increasing and since $h(0)=b(\alpha-1)+1>0$, then we conclude that $G(s+\alpha-1,s)$ is increasing. On the other hand, if $\alpha\geq 3/2$ then $h(s)=b(\alpha-1)+s(3-2\alpha)+1$ is decreasing but nevertheless positive. Since $h(b+1)=b(2-\alpha)-2(\alpha-2)>0$. In conclusion, $G(s+\alpha-1,s)$ is increasing for all s. Therefore,

$$\max_{s\in[0,b+1]_{\mathbb{N}_0}} G(s+\alpha-1,s)=G(b+\alpha,b+1)=\frac{\Gamma(\alpha+b+1)\Gamma(\alpha-1)(b+2)}{\Gamma(\alpha-1)\Gamma(b+3)-\Gamma(\alpha+b+1)}.$$

Hence the theorem.

Theorem 3.4. Let $q:[\alpha-1,\alpha+b]_{\mathbb{N}_{\alpha-1}}\to[0,\infty)$ be a nontrivial function. If the discrete fractional boundary value problem (1) - (2) has a nontrivial solution given by (7), then

$$\sum_{s=0}^{b+1}\left|q(s+\alpha-1)\right|\geq\frac{\Gamma(\alpha)\left[\Gamma(\alpha-1)\Gamma(b+3)-\Gamma(\alpha+b+1)\right]}{(b+2)\Gamma(\alpha+b+1)\Gamma(\alpha-1)}.$$

Proof: Since the discrete fractional boundary value problem (1) - (2) has a nontrivial solution as

$$u(t)=-\frac{1}{\Gamma(\alpha)}\sum_{s=0}^{b+1}G(t,s)q(s+\alpha-1)u(s+\alpha-1)$$
$$\|u\|\leq\frac{1}{\Gamma(\alpha)}\sum_{s=0}^{b+1}\left|G(t,s)\right|\left|q(s+\alpha-1)\right|\left|u(s+\alpha-1)\right|$$
$$1\leq\frac{1}{\Gamma(\alpha)}\sum_{s=0}^{b+1}G(s+\alpha-1,s)\left|q(s+\alpha-1)\right|$$

from which it follows that

$$\sum_{s=0}^{b+1}\left|q(s+\alpha-1)\right|\geq\frac{\Gamma(\alpha)}{G(s+\alpha-1,s)}$$
$$\sum_{s=0}^{b+1}\left|q(s+\alpha-1)\right|\geq\frac{\Gamma(\alpha)\left[\Gamma(\alpha-1)\Gamma(b+3)-\Gamma(\alpha+b+1)\right]}{(b+2)\Gamma(\alpha+b+1)\Gamma(\alpha-1)}.$$

Hence the result.

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EXISTENCE OF IMPULSIVE INTEGRODIFFERENTIAL EQUATIONS WITH INTEGRAL BOUNDARY CONDITIONS

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***Abstract**: This paper deals with the existence of integrodifferential equations involving integral boundary conditions with impulses. Sufficient conditions are established by using Schaefer fixed point theorem. Examples has given to verify the results*

***Keywords**: Impulses, Integrodifferential equations, Schaefer fixed point theorem, Integral boundary conditions, Greens function*

I. Introduction

Differential condition can be utilized to demonstrate the elements of numerous certifiable phenomena. Many transformation processes are portrayed by the way that at specific snapshots of time they encounter a sudden difference in state. These processes are subject to short-term perturbations whose duration is negligible in compare with the duration of the process. Since the late 1990's the creators have delivered a broad arrangement of results on differential conditions and differential incorporations experiencing drive impacts. Many authors have been studied about existence of impulsive and integrodifferential equations[1-4,6-8], which are important for complementing the theory of impulses. As a consequence, applications of impulsive differential equations have been found in numerous framework including biology, ecology, spacecraft, etc. Many evolution processes are characterized by the fact that at certain moments of time they experience a change of state abruptly. It is normal to expect that the comparing hypothesis of impulsive differential and integral inequalities would be similarly valuable in the examination of impulsive differential condition. An integrodifferential condition is a functional equation which involves both integrals and derivatives of the functions.

Inspired by[5,9,10], we investigate the impulsive integrodifferential equation with integral boundary value problem of the form,

$$v''(t) + g\left(t, v(t), \int_0^t h(t, v(s))ds\right) = 0\,, t \in \Omega_0 = \Omega \setminus \{t_1, t_2, \dots, t_m\}$$

$$\Delta v(t_k) = J_k(v(t_k)) \text{ and } \Delta' v(t_k) = J_k^*(v(t_k)) \quad k = 1,2, \dots, m$$

$$\alpha v(0) - \beta v'(0) = \int_0^T v(t)dH_1(t) \text{ and } \gamma v(T) - \delta v'(T) = \int_0^T v(t)dH_2(t) \qquad \dots (1)$$

Where $\Omega = [0,T], 0 < t_1 < t_2 < \cdots < t_m < T$, $g : [0,T] \times R \times R \to R$ is continuous on $(t,x,y) \in \Omega_0 \times R \times R$, $g(t_k^+, x, y) = \lim_{t \to t_k^+} g(t,x,y)\,, g(t_k^-, x, y) = \lim_{t \to t_k^-} g(t,x,y)$ exist, $g(t_k^-, x, y) = g(t_k, x, y)$; $\Delta v(t_k) = v(t_k^+) - v(t_k^-), \Delta v'(t_k) = v'(t_k^+) - v'(t_k^-)$; $J_k, J_k^* \in C(R,R)$ and $\alpha, \beta, \gamma, \delta$ are non negative constants which verify some additional conditions and H_1, H_2 are functions of bounded variations.

II. Preliminaries

In this section, we introduce the space of functions and lemmas before starting our main results.

$$PC(\Omega) = \{v: \Omega \to R : v \text{ is continuous for any } t \in \Omega_0, v(t_k^+), v(t_k^-) \text{ exist and } v(t_k^-) = v(t_k), k = 1,2, \dots, m\}$$

$$PC^1(\Omega) = \{v: \Omega \to R : v \text{ is continuously differentiable for any } t \in \Omega_0, v'(t_k^+), v'(t_k^-) \text{ exist and } v'(t_k^-) = v'(t_k), k = 1,2, \dots, m\}$$

$PC(\Omega)$ and $PC^1(\Omega)$ are Banach space with the norms

$$\left||v|\right|_{PC} = \sup_{t\in\Omega} |v(t)| \text{ and } \left||v|\right|_{PC^1} = \max\{\left||v|\right|_{PC}, \left||v'|\right|_{PC}\} \quad \dots(2)$$

A solution to the impulsive BVP (1) is a function $v \in PC^1(\Omega) \cap C^2(\Omega_0)$ that satisfies (1) for each $t \in \Omega_0$. Consider the following impulsive BVP with $\lambda > 0$,

$$-v''(t) + \lambda^2 v(t) = \rho(t), t \in \Omega_0$$

$$\Delta v(t_k) = J_k(v(t_k)) \text{ and } \Delta' v(t_k) = J_k^*(v(t_k)) \quad k = 1,2, \dots, m$$
$$\alpha v(0) - \beta v'(0) = \int_0^T v(t) dH_1(t) \text{ and } \gamma v(T) - \delta v'(T) = \int_0^T v(t) dH_2(t) \quad \dots(3)$$

Where $\rho \in PC(\Omega)$. For convenience, we set $J_k = J_k(v(t_k))$, $J_k^* = J_k^*(v(t_k))$.

Lemma 2.1 $v \in PC^1(\Omega) \cap C^2(\Omega_0)$ is a solution of (3) if and only if $v \in PC^1(\Omega)$ is a solution of the impulsive integral equation

$$v(t) = \int_0^T G(t,s)\rho(s)ds + \sum_{k=1}^{m} [G(t,t_k)(J_k^*) + W(t,t_k)(J_k)] + \int_0^T W^*(t,s)v(t)dH_1(t) \dots(4)$$

Where

$$G(t,s) = \frac{1}{2\lambda}\begin{cases} 0 & 0 \le s < t \le T \\ e^{-\lambda(t-s)} - e^{\lambda(t-s)} & 0 \le t \le s \le T \end{cases}$$

$$W(t,s) = \frac{1}{2}\begin{cases} 0 & 0 \le s < t \le T \\ e^{\lambda(t-s)} - e^{-\lambda(t-s)} & 0 \le t \le s \le T \end{cases} \quad \dots(5)$$

$$W^*(t,s) = \frac{1}{2\lambda}\begin{cases} 0 & 0 \le s < t \le T \\ \dfrac{e^{-\lambda t}}{\frac{\alpha}{2\lambda} + \frac{\beta}{2}} + \dfrac{e^{\lambda t}}{\frac{\alpha}{2\lambda} - \frac{\beta}{2}} & 0 \le t \le s \le T \end{cases}$$

Proof: If $v \in PC^1(\Omega) \cap C^2(\Omega_0)$ is a solution of (3), setting

$$w(t) = v'(t) + \lambda v(t) \quad \dots(6)$$

Then by the first equation of (3), we have

$$w'(t) - \lambda w(t) = \rho(t) \quad , t \ne t_k \quad \dots(7)$$

Multiplying (7) by $e^{-\lambda t}$ and integrating on $[0, t_1)$ and $(t_1, t] (t_1 < t \le t_2)$, respectively, we get

$$e^{-\lambda t_1} w(t_1^-) - w(0) = -\int_0^{t_1} e^{-\lambda s}\rho(s)ds$$
$$e^{-\lambda t} w(t) - e^{-\lambda t_1} w(t_1^+) = -\int_{t_1}^{t} e^{-\lambda s}\rho(s)ds \quad \dots(8)$$

So,

$$w(t) = e^{\lambda t}\left[w(0) - \int_0^t e^{-\lambda s}\rho(s)ds + e^{-\lambda t_1}\Delta w(t_1)\right], \qquad t_1 < t \le t_2 \quad \dots(9)$$

In the same way, we can obtain that

$$w(t) = e^{\lambda t}\left[w(0) - \int_0^t e^{-\lambda s}\rho(s)ds + \sum_{0<t_k<t} e^{-\lambda t_k}[J_k^* + \lambda J_k]\right], \qquad t \in \Omega \quad \dots(10)$$

Where $w(0) = v'(0) + \lambda v(0)$. Integrating (6), we have

$$v(t) = e^{-\lambda t}\left[v(0) + \int_0^t e^{\lambda s}w(s)ds + \sum_{0<t_k<t} e^{\lambda t_k} J_k\right], \qquad t_1 < t \le t_2 \quad \dots(11)$$

By (10), we get

$$\int_0^t e^{\lambda s}\, w(s)ds = \frac{1}{2\lambda}\left[w(0)(e^{2\lambda t} - 1) - \int_0^t \left(e^{2\lambda t} - e^{2\lambda s}\right)\rho(s)e^{-\lambda s}ds + \sum_{0<t_k<t}\left(e^{2\lambda t} - e^{2\lambda t_k}\right)e^{-\lambda t_k}[J_k^* + \lambda J_k]\right], \quad \dots(12)$$

Substituting (12) into (11), we obtain

$$v(t) = \frac{1}{2\lambda}\left[\left(\lambda v(0) - v'(0)\right)e^{-\lambda t} + \left(v'(0) + \lambda v(0)\right)e^{\lambda t} + \int_0^t \left(e^{-\lambda(t-s)} - e^{\lambda(t-s)}\right)\rho(s)ds + \sum_{0<t_k<t} e^{\lambda(t-t_k)}[J_k^* + \lambda J_k] - \sum_{0<t_k<t} e^{-\lambda(t-t_k)}[J_k^* - \lambda J_k]\right], t \in \Omega \quad \dots(13)$$

$$v'(t) = \frac{1}{2}\left[-\left(\lambda v(0) - v'(0)\right)e^{-\lambda t} + \left(v'(0) + \lambda v(0)\right)e^{\lambda t} - \int_0^t \left(e^{-\lambda(t-s)} + e^{\lambda(t-s)}\right)\rho(s)ds + \sum_{0<t_k<t} e^{\lambda(t-t_k)}[J_k^* + \lambda J_k] + \sum_{0<t_k<t} e^{-\lambda(t-t_k)}[J_k^* - \lambda J_k]\right], t \in \Omega \quad \dots(14)$$

In view of $\alpha v(0) - \beta v'(0) = \int_0^T v(t)dH_1(t)$, we have

$$v'(0) + \lambda v(0) = \frac{\int_0^T v(t)dH_1(t)}{\frac{\alpha}{2\lambda} - \frac{\beta}{2}} \quad \text{and} \quad \lambda v(0) - v'(0) = \frac{\int_0^T v(t)dH_1(t)}{\frac{\alpha}{2\lambda} + \frac{\beta}{2}} \dots(15)$$

Substituting (15) into (14), by routine calculation, we can get (4).

Conversely, if v is a solution of (4), then direct differentiation of (4) gives $-v''(t) + \lambda^2 v(t) = \rho(t), t \ne t_k$. Moreover, we obtain $\Delta v(t_k) = J_k\left(v(t_k)\right)$ and $\Delta' v(t_k) = J_k^*\left(v(t_k)\right)$, $\alpha v(0) - \beta v'(0) = \int_0^T v(t)dH_1(t)$ and $\gamma v(T) - \delta v'(T) = \int_0^T v(t)dH_2(t)$. Hence $v \in PC^1(\Omega) \cap C^2(\Omega_0)$ is a solution of (3).

Remark 2.2 We call $G(t,s)$ above the Green functions for the following homogeneous BVP:

$$-v''(t) + \lambda^2 v(t) = 0\,, t \in \Omega$$

$$\alpha v(0) - \beta v'(0) = \int_0^T v(t)dH_1(t) \text{ and } \gamma v(T) - \delta v'(T) = \int_0^T v(t)dH_2(t) \dots(16)$$

Define a mapping $B: PC^1(\Omega) \to PC^1(\Omega)$ by

$$Bv(t) = \int_0^T G(t,s)[g\left(s, v(s), \int_0^t h\left(s, v(s)\right)ds\right) + \lambda^2 v(s)]ds + \sum_{k=1}^m [G(t,t_k)(J_k^*) + W(t,t_k)(J_k)] + \int_0^T W^*(t,s)v(t)dH_1(t)\,, t \in [0,T] \dots(17)$$

In view of lemma 2.1, we easily see that v is a fixed point of operator B if and only if v is a solution to the impulsive boundary value problem (1). It is easy to check that

$$|G(t,s)| \le \frac{e^{2\lambda s} + e^{2\lambda t}}{2\lambda(e^{\lambda(t+s)})}, \quad |W(t,s)| \le \frac{e^{2\lambda t} - e^{2\lambda s}}{2(e^{\lambda(t+s)})}$$

$$|W^*(t,s)| \le \frac{2\lambda^2\beta\left[e^{\lambda t} - e^{-\lambda t}\right] + 2\lambda\alpha[e^{\lambda t} + e^{-\lambda t}]}{\alpha^2 - \beta^2\lambda^2} \qquad \dots(18)$$

Lemma 2.3 If $v \in PC^1(\Omega)$and $\alpha v(0) - \beta v'(0) = \int_0^T v(t)dH_1(t)$, then

$$\left||v|\right|_{PC} \le \frac{1}{\alpha}\int_0^t |v(t)|dH_1(t) + \frac{\beta}{\alpha}\sum_{k=1}^m |\Delta v'(t_k)| + \sum_{k=1}^m |\Delta v(t_k)| + \int_0^T |v'(s)|ds \ \dots(19)$$

Proof : Since $v \in PC^1(\Omega)$, we have

$$v(t) = v(0) + \sum_{0<t_k<t} \Delta v(t_k) + \int_0^t v'(s)ds \qquad \dots(20)$$

Set t=T, we obtain from $\alpha v(0) - \beta v'(0) = \int_0^T v(t)dH_1(t)$ that

$$v(0) = \frac{1}{\alpha}\left[\int_0^T v(t)dH_1(t) + \beta\sum_{k=1}^m \Delta v'(t_k)\right] \dots(21)$$

Substituting (21) into (20), we get

$$|v(t)| = \left|\frac{1}{\alpha}\left[\int_0^T v(t)dH_1(t) + \beta\sum_{k=1}^m \Delta v'(t_k)\right] + \sum_{0<t_k<t} \Delta v(t_k) + \int_0^t v'(s)ds\right|$$

$$\le \frac{1}{\alpha}\int_0^t |v(t)|dH_1(t) + \frac{\beta}{\alpha}\sum_{k=1}^m |\Delta v'(t_k)| + \sum_{k=1}^m |\Delta v(t_k)| + \int_0^T |v'(s)|ds \ \dots(22)$$

Hence the proof is complete.

Lemma 2.4(Schaefer fixed point theorem[11]) Let E be a normed linear space with $H: E \to E$ a compact operator. If the set $S = \{x \in E \,|x = \lambda Hx, \text{for some } \lambda \in (0,1)\}$ is bounded, then H has at least one fixed point.

III. Main results

In this section, we study the existence of solutions for BVP (1). For this purpose we assume that there exist constants $0 < \eta < 1$, functions $a, b, c, e \in C(\Omega, [0, +\infty))$, and non negative constants $\alpha_k, \beta_k, \gamma_k, \delta_k$ $(k = 1,2, \dots, m)$ such that

$$(H_1) \quad g\left(t, v, \int_0^t h(t,v)ds\right) \le a(t)|v| + b(t)|v|^\eta + c(t)|v|^{1+\eta} + e(t)$$

$$(H_2) |J_k(v)| \le \alpha_k|v| + \beta_k \ \text{ and } \ |J_k^*(v)| \le \gamma_k|v| + \delta_k, \ k = 1,2, \dots, m$$

hold.

Remark 3.1 (H_1) means that the nonlinearity growths at most linearly in v, (H_2) implies that the impulses are (at most) linear. For convenience, let

$$p_1 = 2\left(\int_0^T a(t)dt + \lambda^2 T + \sum_{i=1}^m \gamma_i\right)$$

$$p_2 = 2\int_0^T b(t)dt \ \text{ and } \ p_3 = 2\int_0^T c(t)\ dt$$

$$p_4 = 2\left(\int_0^T e(t)dt + \sum_{i=1}^m \delta_i\right) \text{ and } p_5 = \frac{1}{\delta}\left[\int_0^T v(t)dH_2(t) + \gamma|v(T)|\right] \qquad \dots(23)$$

Theorem 3.2 Suppose that conditions (H_1) and (H_2) are satisfied. Further assume that

$$Tq_1 + \frac{\beta^2 m}{\alpha^2}\sum_{i=1}^{m}\gamma_i^2 + m\sum_{i=1}^{m}\alpha_i^2 + \frac{\beta}{\alpha}\sum_{i=1}^{m}\alpha_i\gamma_i + \frac{2\beta}{\alpha}\sqrt{Tq_1}\sum_{i=1}^{m}\gamma_i + 2\sqrt{Tq_1}\sum_{i=1}^{m}\alpha_i < 1 \ldots (24)$$

Holds, where $q_1 = \int_0^T a(t)dt + \sum_{i=1}^{m}(p_1\alpha_i + \gamma_i)$ and p_1 as in (23). Then, BVP (1) has atleast one solution.

Proof**:** It is easy to check by Arzela-Ascoli theorem that the operator B is completely continuous. Assume that v is a solution of the equation

$$v = \mu Bv, \quad \mu \in (0,1) \qquad \ldots (25)$$

Then,

$$v''(t) = \mu(Bv)''(t) = -\mu\left[g\left(t, v(t), \int_0^t h(t, v(s))ds\right)\right] - \lambda^2(\mu - 1)v(t) \qquad \ldots (26)$$

$$-v(t)v''(t) \le \mu v(t) g\left(t, v(t), \int_0^t h(t, v(s))ds\right) \qquad \ldots (27)$$

Integrating (26) from 0 to T, we get

$$v'(T) - v'(0) = -\mu\int_0^T g\left(t, v(t), \int_0^t h(t, v(s))ds\right)dt - \lambda^2(\mu - 1)\int_0^T v(t)dt + \sum_{i=1}^{m} J_i^* \qquad \ldots (28)$$

In view of $\gamma v(T) - \delta v'(T) = \int_0^T v(t)dH_2(t)$, we obtain by (28) that

$$|v'(0)| \le \left|\int_0^T g\left(t, v(t), \int_0^t h\big(t, v(s)\big)ds\right)dt\right| + \lambda^2\int_0^T |v(t)|dt + \sum_{i=1}^{m}|J_i^*| + \frac{1}{\delta}\left(\int_0^T |v(t)|dH_2(t) + \gamma|v(T)|\right) \qquad \ldots (29)$$

Integrating (26) from 0 to t, we get

$$v'(t) - v'(0) = -\mu\int_0^t g\left(t, v(t), \int_0^t h(t, v(s))ds\right)dt - \lambda^2(\mu - 1)\int_0^t v(t)dt + \sum_{i=1}^{m} J_i^* \qquad \ldots (30)$$

From (29) and (30), we have

$$\begin{aligned}|v'(t)| &\le \left|v'(0) + \int_0^T g\left(s, v(s), \int_0^t h\big(s, v(s)\big)ds\right)ds\right| + \lambda^2\int_0^T |v(s)|ds + \sum_{i=1}^{m}|J_i^*| \\ &\le 2\int_0^T \big(a(t)|v| + b(t)|v|^{\eta} + c(t)|v|^{1+\eta} + e(t)\big)dt \\ &\quad + 2\lambda^2\int_0^T |v(t)|dt + 2\sum_{i=1}^{m}|J_i^*| + \frac{1}{\delta}\left(\int_0^T |v(t)|dH_2(t) + \gamma|v(T)|\right) \\ &\le 2\left(||v||_{PC}\int_0^T a(t)dt + ||v||_{PC}^{\eta}\int_0^T b(t)dt + ||v||_{PC}^{1+\eta}\int_0^T c(t)dt + \int_0^T e(t)dt\right) + 2\lambda^2||v||_{PC}T \\ &\quad + 2\sum_{i=1}^{m}\left(\gamma_i||v||_{PC} + \delta_i\right) + \frac{1}{\delta}\left(\int_0^T |v(t)|dH_2(t) + \gamma|v(T)|\right) \ldots (31)\end{aligned}$$

That is,

$$\begin{aligned}||v'||_{PC} &\le 2\left(\int_0^T a(t)dt + \lambda^2 T + \sum_{i=1}^{m}\gamma_i\right)||v||_{PC} \\ &\quad + 2\int_0^T b(t)dt\,||v||_{PC}^{\eta} + 2\int_0^T c(t)dt\,||v||_{PC}^{1+\eta} + 2\left(\int_0^T e(t)dt + \sum_{i=1}^{m}\delta_i\right) \\ &\quad + \frac{1}{\delta}\left(\int_0^T |v(t)|dH_2(t) + \gamma|v(T)|\right) \ldots (32)\end{aligned}$$

Thus,

$$||v'||_{PC} \le p_1||v||_{PC} + p_2||v||_{PC}^{\eta} + p_3||v||_{PC}^{1+\eta} + p_4 + p_5 \ldots (33)$$

Where p_1, p_2, p_3, p_4, p_5 are in (23). Integrating (27) from 0 to T, we get

$$-\int_0^T v(t)v''(t)dt \le \mu\int_0^T v(t) g\left(t, v(t), \int_0^t h(t, v(s))ds\right)dt \qquad \ldots (34)$$

In view of $\alpha v(0) - \beta v'(0) = \int_0^T v(t)dH_1(t)$ and $\gamma v(T) - \delta v'(T) = \int_0^T v(t)dH_2(t)$, we have

$$\int_0^T v(t)v''(t)dt = \int_0^T v(t)d\big(v'(t)\big)$$

$$= \int_0^{t_1} v(t)d\big(v'(t)\big) + \int_{t_1}^{t_2} v(t)d\big(v'(t)\big) + \cdots + \int_{t_n}^{T} v(t)d\big(v'(t)\big)$$

$$= (t)v'(t)|_0^{t_1} - \int_0^t \big(v'(t)\big)^2 dt + v(t)v'(t)|_{t_1}^{t_2} - \int_{t_1}^{t_2} \big(v'(t)\big)^2 dt + \cdots + v(t)v'(t)|_{t_n}^{T}$$

$$- \int_{t_n}^{T} \big(v'(t)\big)^2 dt$$

$$= v(t_1 - 0)v'(t_1 - 0) - v(0)v'(0) - v(t_2 - 0)v'(t_2 - 0) - v(t_1 + 0)v'(t_1 + 0) + \cdots$$

$$+ v(T)v'(T) - v(t_n - 0)v'(t_n - 0) - \int_0^T \big(v'(t)\big)^2 dt$$

$$= v(t_1 - 0)v'(t_1 - 0) - v(t_1 + 0)v'(t_1 + 0) + \cdots + v(t_n - 0)v'(t_n - 0)$$

$$- v(t_n + 0)v'(t_n + 0) - \frac{1}{\alpha}\left[\beta v'(0) + \int_0^T v(t)dH_1(t)\right]\frac{1}{\beta}\left[\int_0^T v(t)dH_1(t) - \alpha v(0)\right]$$

$$+ \frac{1}{\gamma}\left[\int_0^T v(t)dH_2(t) - \delta v'(T)\right]\frac{1}{\delta}\left[\int_0^T v(t)dH_2(t) - \gamma v(T)\right] - \int_0^T \big(v'(t)\big)^2 dt$$

$$= -v(t_1 - 0)J_1^* - v'(t_1 + 0)J_1 - \ \ldots\ - v(t_n - 0)J_n^* - v'(t_n + 0)J_n$$

$$- \frac{1}{\alpha}\left[\beta v'(0) + \int_0^T v(t)dH_1(t)\right]\frac{1}{\beta}\left[\int_0^T v(t)dH_1(t) - \alpha v(0)\right]$$

$$+ \frac{1}{\gamma}\left[\int_0^T v(t)dH_2(t) - \delta v'(T)\right]\frac{1}{\delta}\left[\int_0^T v(t)dH_2(t) - \gamma v(T)\right] - \int_0^T \big(v'(t)\big)^2 dt \quad \ldots(35)$$

Substituting (36) into (35), we obtain by (H_1), (H_2) and (34) that

$$\int_0^T \big(v'(t)\big)^2 dt \le \mu \int_0^T v(t)g\left(t, v(t), \int_0^t h(t, v(s))ds\right) dt - v(t_1 - 0)J_1^* - v'(t_1 + 0)J_1 - \ \ldots\ - v(t_n - 0)J_n^*$$

$$- v'(t_n + 0)J_n - \frac{1}{\alpha}\left[\beta v'(0) + \int_0^T v(t)dH_1(t)\right]\frac{1}{\beta}\left[\int_0^T v(t)dH_1(t) - \alpha v(0)\right]$$

$$+ \frac{1}{\gamma}\left[\int_0^T v(t)dH_2(t) - \delta v'(T)\right]\frac{1}{\delta}\left[\int_0^T v(t)dH_2(t) - \gamma v(T)\right]$$

$$\le \int_0^T [(a(t)|v^2(t)| + b(t)|v(t)|^{1+\eta} + c(t)|v(t)|^{2+\eta} + e(t)|v(t)|)]\, dt + \big||v|\big|_{PC} \sum_{i=1}^{m} (\gamma_i \big||v|\big|_{PC} + \delta_i)$$

$$+ \big||v'|\big|_{PC} \sum_{i=1}^{m} (\alpha_i \big||v|\big|_{PC} + \beta_i) - \frac{1}{\alpha}\left[\beta v'(0) + \int_0^T v(t)dH_1(t)\right]\frac{1}{\beta}\left[\int_0^T v(t)dH_1(t) - \alpha v(0)\right]$$

$$+ \frac{1}{\gamma}\left[\int_0^T v(t)dH_2(t) - \delta v'(T)\right]\frac{1}{\delta}\left[\int_0^T v(t)dH_2(t) - \gamma v(T)\right]$$

$$\le \big||v|\big|_{PC}^2 \int_0^T a(t)dt + \big||v|\big|_{PC}^{1+\eta} \int_0^T b(t)dt + \big||v|\big|_{PC}^{2+\eta} \int_0^T c(t)dt + \big||v|\big|_{PC} \int_0^T e(t)dt$$

$$+ \sum_{i=1}^{m} \gamma_i \big||v|\big|_{PC}^2 + \sum_{i=1}^{m} \delta_i \big||v|\big|_{PC} + \frac{1}{\alpha}\left[\beta v'(0) + \int_0^T v(t)dH_1(t)\right]\frac{1}{\beta}\left[\int_0^T v(t)dH_1(t) - \alpha v(0)\right]$$

$$+ \frac{1}{\gamma}\left[\int_0^T v(t)dH_2(t) - \delta v'(T)\right]\frac{1}{\delta}\left[\int_0^T v(t)dH_2(t) - \gamma v(T)\right] + (p_1 \big||v|\big|_{PC} + p_2 \big||v|\big|_{PC}^{\eta}$$

$$+ p_3 \big||v|\big|_{PC}^{1+\eta} + p_4 + p_5) \sum_{i=1}^{m} (\alpha_i \big||v|\big|_{PC} + \beta_i) \ldots(36)$$

Thus,

$$\int_0^T \big(v'(t)\big)^2 dt \le \big||v|\big|_{PC}^2 q_1 + q_2 \big||v|\big|_{PC}^{1+\eta} + q_3 \big||v|\big|_{PC}^{2+\eta} + \big||v|\big|_{PC} q_4 + \big||v|\big|_{PC}^{\eta} q_5 + q_6 \quad \ldots(37)$$

Where,

$$q_1 = \int_0^T a(t)dt + \sum_{i=1}^{m} \gamma_i + p_1\alpha_i) \ , \qquad q_2 = \int_0^T b(t)dt + \ p_3 \sum_{i=1}^{m} \beta_i + p_2 \sum_{i=1}^{m} \alpha_i$$

$$q_3 = \int_0^T c(t)dt \ + p_3 \sum_{i=1}^{m} \alpha_i \quad , \qquad q_4 = \int_0^T e(t)dt + \sum_{i=1}^{m} \delta_i + p_1 \sum_{i=1}^{m} \beta_i + (p_4 + p_5) \sum_{i=1}^{m} \alpha_i$$

$$q_5 = (p_2 + p_4 + p_5) \sum_{i=1}^{m} \beta_i$$

$$q_6 = \frac{1}{\alpha}\left[\beta v'(0) + \int_0^T v(t)dH_1(t)\right]\frac{1}{\beta}\left[\int_0^T v(t)dH_1(t) - \alpha v(0)\right]$$
$$+ \frac{1}{\gamma}\left[\int_0^T v(t)dH_2(t) \ - \delta v'(T)\right]\frac{1}{\delta}\left[\int_0^T v(t)dH_2(t) - \gamma v(T)\right] \quad \dots (38)$$

By Lemma (2.3) and (37), we have

$$\begin{aligned}
||v||_{PC}^2 &\le \left[\frac{1}{\alpha}\int_0^t |v(t)|dH_1(t) + \frac{\beta}{\alpha}\sum_{k=1}^{m}|\Delta v'(t_k)| + \sum_{k=1}^{m}|\Delta v(t_k)| + \int_0^T |v'(s)|ds\right]^2 \\
&\le T\left(\int_0^T |v'(s)|ds\right)^2 + \frac{1}{\alpha^2}\int_0^t |v(t)|^2 dH_1(t) + \frac{\beta^2}{\alpha^2} m \sum_{k=1}^{m}|J_i^*|^2 + m\sum_{k=1}^{m}|J_i|^2 \\
&\quad + \frac{2\sqrt{T}}{\alpha}\left(\int_0^T |v'(t)|^2 dt\right)^{1/2}\int_0^t |v(t)|dH_1(t) + \frac{2\beta}{\alpha^2}\int_0^t |v(t)|dH_1(t)\sum_{k=1}^{m} J_i + \frac{2\beta}{\alpha}\left[\sum_{k=1}^{m} J_i^* J_i\right] \\
&\quad + \frac{2\beta\sqrt{T}}{\alpha}\left(\int_0^T |v'(t)|^2 dt\right)^{1/2}\sum_{k=1}^{m} J_i^* + 2\sqrt{T}\sum_{k=1}^{m} J_i\left(\int_0^T |v'(t)|^2 dt\right)^{1/2} + \frac{2}{\alpha}\int_0^t |v(t)|dH_1(t)\sum_{k=1}^{m} J_i \\
&\le \left(Tq_1 + \frac{\beta^2}{\alpha^2} m \sum_{k=1}^{m}{\gamma_i}^2 + m\sum_{k=1}^{m}{\alpha_i}^2 + \frac{\beta}{\alpha}\sum_{k=1}^{m}\alpha_i\gamma_i + \frac{2\beta}{\alpha}\sqrt{Tq_1}\sum_{k=1}^{m}\gamma_i + 2\sqrt{Tq_1}\sum_{k=1}^{m}\alpha_i\right)||v||_{PC}^2 + \cdots
\end{aligned}$$

… (39)

It follows from the above inequality and (24) that there exists $M_1 > 0$ such that $||v||_{PC} \le M_1$. Hence we get by (33)that

$$||v'||_{PC} \le a_1 M_1 + a_2 M_1^{\eta} + a_3 M_1^{1+\eta} + a_4 + a_5 = M_2 \dots (40)$$

Thus, $||v||_{PC^1} = \max\{M_1, M_2\}$. It follows from lemma 1.1 that BVP (1) has atleast one solution. The proof is complete.

Theorem 3.3 Assume that (H_2) holds. Suppose that there exists a continuous and non decreasing function $\phi; [0,\infty) \to [0,\infty)$ and a non negative function $c_1 \in C(\Omega)$ with

$$\left|g\left(t, v, \int_0^t h(t,v)dt\right) + \lambda^2 v\right| \le c_1(t)\phi(|v|), \qquad t \in \Omega, v \in R \quad \dots (41)$$

Moreover suppose that

$$\limsup_{v\to\infty} \frac{\phi(v)}{v} < L \qquad \dots (42)$$

Holds, where

$$L = \frac{1 - \left(\frac{e^{2\lambda s} + e^{2\lambda t}}{2\lambda(e^{\lambda(t+s)})}\sum_{i=1}^{m}\gamma_i + \frac{e^{2\lambda t} - e^{2\lambda s}}{2(e^{\lambda(t+s)})}\sum_{i=1}^{m}\alpha_i\right)}{\frac{e^{2\lambda s} + e^{2\lambda t}}{2\lambda(e^{\lambda(t+s)})}\int_0^T c_1(s)\,ds} \dots (43)$$

Then, BVP(1) has at least one solution.

Proof**:** From (42), there exist $0 < \epsilon < L \ and \ M > 0$ such that

$$\phi(w) \le (L - \epsilon)w, \quad w \ge M \ \dots (44)$$

Thus, there exists $K > 0$ such that

$$\phi(w) \le (L - \epsilon)w + K, \quad w \ge 0 \qquad \dots (45)$$

Assume that v is a solution of the equation

$$v = \mu B v, \ \ \mu \in (0,1) \dots (46)$$

Then we have by (18),(41) and (45) that

$$|v(t)| = \mu\left|\int_0^T G(t,s)\left[g\left(s,v(s),\int_0^t h(s,v(s))ds\right)+\lambda^2 v(s)\right]ds + \sum_{k=1}^{m}[G(t,t_k)(J_k^*)+W(t,t_k)(J_k)]+\int_0^T W^*(t,s)v(t)dH_1(t)\right|$$

$$\leq \frac{e^{2\lambda s}+e^{2\lambda t}}{2\lambda(e^{\lambda(t+s)})}\int_0^T c_1(s)\phi(|v|)\,ds+\frac{e^{2\lambda s}+e^{2\lambda t}}{2\lambda(e^{\lambda(t+s)})}\sum_{i=1}^{m}(\gamma_i||v||_{PC}+\delta_i)+\frac{e^{2\lambda t}-e^{2\lambda s}}{2(e^{\lambda(t+s)})}\sum_{i=1}^{m}(\alpha_i||v||_{PC}+\beta_i)$$
$$+\int_0^T \frac{2\lambda^2\beta[e^{\lambda t}-e^{-\lambda t}]+2\lambda\alpha[e^{\lambda t}+e^{-\lambda t}]}{\alpha^2-\beta^2\lambda^2}\,v(t)dH_1(t) \quad \ldots(47)$$

Thus, we have

$$||v||_{PC} \leq \frac{e^{2\lambda s}+e^{2\lambda t}}{2\lambda(e^{\lambda(t+s)})}\int_0^T c_1(s)\left[(L-\epsilon)||v||_{PC}+K\right]ds+\left(\frac{e^{2\lambda s}+e^{2\lambda t}}{2\lambda(e^{\lambda(t+s)})}\sum_{i=1}^{m}\gamma_i+\frac{e^{2\lambda t}-e^{2\lambda s}}{2(e^{\lambda(t+s)})}\sum_{i=1}^{m}\alpha_i\right)||v||_{PC}$$
$$+\left(\frac{e^{2\lambda s}+e^{2\lambda t}}{2\lambda(e^{\lambda(t+s)})}\sum_{i=1}^{m}\delta_i+\frac{e^{2\lambda t}-e^{2\lambda s}}{2(e^{\lambda(t+s)})}\sum_{i=1}^{m}\beta_i\right) \quad \ldots(48)$$

That is,

$$\epsilon\frac{e^{2\lambda s}+e^{2\lambda t}}{2\lambda(e^{\lambda(t+s)})}\int_0^T c_1(s)\,ds||v||_{PC} \leq \frac{e^{2\lambda s}+e^{2\lambda t}}{2\lambda(e^{\lambda(t+s)})}K\int_0^T c_1(s)\,ds+\left(\frac{e^{2\lambda s}+e^{2\lambda t}}{2\lambda(e^{\lambda(t+s)})}\sum_{i=1}^{m}\delta_i+\frac{e^{2\lambda t}-e^{2\lambda s}}{2(e^{\lambda(t+s)})}\sum_{i=1}^{m}\beta_i\right)\ldots(49)$$

Which implies that there exists $M_3 > 0$ such that $||v||_{PC} \leq M_3$, by (29), (30) and (45) we get

$$|v'(t)| \leq 2\int_0^T g\left(s,v(s),\int_0^t h(s,v(s))ds\right)ds+2\lambda^2\int_0^T|v(s)|ds+2\sum_{i=1}^{m}|J_i^*|+\frac{1}{\delta}\left(\int_0^T|v(t)|dH_2(t)+\gamma|v(T)|\right)$$
$$\leq 2\int_0^T c_1(s)ds\left(L||v||_{PC}+K\right)+2\lambda^2||v||_{PC}T$$
$$+2\sum_{i=1}^{m}\left(\gamma_i||v||_{PC}+\delta_i\right)+\frac{1}{\delta}\left(\int_0^T|v(t)|dH_2(t)+\gamma|v(T)|\right)\ldots(50)$$

Which implies that

$$||v'||_{PC} \leq (2L\int_0^T c_1(s)ds+2\lambda^2T+2\sum_{i=1}^{m}\gamma_i)\,||v||_{PC}+2K\int_0^T c_1(s)ds$$
$$+2\sum_{i=1}^{m}\delta_i+\frac{1}{\delta}\left(\int_0^T|v(t)|dH_2(t)+\gamma|v(T)|\right)$$
$$\leq 2(L\int_0^T c_1(s)ds+\lambda^2T+2\sum_{i=1}^{m}\gamma_i)\,M_3+2K\int_0^T c_1(s)ds$$
$$+2\sum_{i=1}^{m}\delta_i+\frac{1}{\delta}\left(\int_0^T|v(t)|dH_2(t)+\gamma|v(T)|\right)\ldots(51)$$

Hence, $||v||_{PC^1} \leq \max\{M_3, M_4\}$. It follows from Lemma 1.1 that BVP (1) has at least one solution. The proof is complete.

IV. Example

In this section, we give an example to illustrate the effectiveness of our results.

Example 4.1

Consider the impulsive integrodifferential equations with integral boundary conditions is of the form

$$v''(t)-\frac{1}{4}t^3[v(t)-2]-\frac{5}{18}\left[\int_0^t t^2s^4v(s)ds\right]^2-\frac{1}{8}\left[\int_0^t t^3s^2v(s)ds\right]^2=0, t\in[0,1]\setminus\frac{1}{2}$$
$$\Delta v(t_1)=\frac{1}{12}\sin(v(t_1))+\frac{1}{8}\ \ and\ \ \Delta v'(t_1)=\frac{1}{10}(v(t_1))+\frac{1}{6}\ \ and\ t_1=\frac{1}{2} \quad \ldots(52)$$

$$\alpha v(0) - \beta v'(0) = \int_0^T v(t)dH_1(t) \text{ and } \gamma v(T) - \delta v'(T) = \int_0^T v(t)dH_2(t)$$

From the above system of equations, we take

$$g(t,v,s) = -\frac{1}{4}t^3[v(t)-2] - \frac{5}{18}\left[\int_0^t t^2 s^4 v(s)ds\right]^2 - \frac{1}{8}\left[\int_0^t t^3 s^2 v(s)ds\right]^2$$

$$I_1(v) = \frac{1}{12}\sin\big(v(t_1)\big) + \frac{1}{8} \text{ and } I_1^*(v) = \frac{1}{10}\big(v(t_1)\big) + \frac{1}{6}, T = 1, J = [0,1]$$

It is easy to show that

$$|g(t,v,s)| \le a(t)|v| + b(t)|v|^2 + c(t)$$

where

$$a(t) = -\frac{1}{4}t^3, b(t) = -\frac{5}{18}\left[\int_0^t t^2 s^4 v(s)ds\right]^2 - \frac{1}{8}\left[\int_0^t t^3 s^2 v(s)ds\right]^2 \text{ and } c(t) = \frac{2t^3}{4}$$

And

$$|I_1(v)| = \frac{1}{12}|v| + \frac{1}{8} \text{ and } |I_1^*(v)| \le \frac{1}{10}|v| + \frac{1}{6}$$

Thus (H_1) and (H_2)holds. Obviously,

$$\alpha_1 = \frac{1}{12}, \beta_1 = \frac{1}{8}, \gamma_1 = \frac{1}{10}, \delta_1 = \frac{1}{6}, m = 1 \text{ and } \lambda = \frac{1}{100},$$

We have

$$p_1 = 2\left(\int_0^T a(t)dt + \lambda^2 T + \sum_{i=1}^m \gamma_i\right) = 0.2121$$

$$q_1 = \int_0^T a(t)dt + \sum_{i=1}^m \gamma_i + p_1\alpha_i = 0.1137$$

Therefore,

$$Tq_1 + \frac{\beta^2}{\alpha^2}m\sum_{k=1}^m {\gamma_i}^2 + m\sum_{k=1}^m {\alpha_i}^2 + \frac{\beta}{\alpha}\sum_{k=1}^m \alpha_i\gamma_i + \frac{2\beta}{\alpha}\sqrt{Tq_1}\sum_{k=1}^m \gamma_i + 2\sqrt{Tq_1}\sum_{k=1}^m \alpha_i = 0.3635 < 1.$$

Which implies that (24) holds. So, all the conditions of Theorem 3.2 are satisfied. By Theorem 3.2, BVP (52) has at least one solution.

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Stability of Second Order System of Differential Equation

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Abstract: *We investigate the Hyers-Ulam stability and Hyers-Ulam-Rassias stability of the system of second order differential equation* $x''(t) = f(t, x(t), x'(t))$ *with initial conditions* $x(a) = x_0$ *and* $x'(a) = x_1$ *in Banach space.*

Keywords: *Hyers-Ulam stability; Hyers-Ulam-Rassias stability; system of differential equations; Initial conditions.*

I. Introduction

In pure mathematics, differential equations are studied from several different perspectives, mostly concerned with their solutions. Only the simplest differential equations are solvable by explicit formulas. However, some properties of solutions of a given differential equation may be determined without finding their exact form. Equilibria are not always stable. Since stable and unstable equilibrium play quite different roles in the dynamics of a system, it is useful to be able to classify equilibrium points based on their stability.

The Hyers-Ulam stability property was introduced by S.M. Ulam [1] in 1940. Then in the next year, D.H. Hyers [2] was handled the issue of Ulam for Cauchy additive functional equation in Banach spaces. From that point forward, Aoki [3], Bourgin [4] and Rassias [5] are generalized the Hyers result. Starting there onwards, colossal number of authors are shown the Hyers-Ulam stability problem for various functional equation on different spaces (see [6, 7, 8]).

As of late, the Hyers-Ulam stability property was proposed by supplanting functional equations by differential equations. In 1998, Alsina et. al. [9] are the primary authors who contemplated the Ulam-Hyers stability of $x'(t) = x(t)$. Then Takashi et.al. [10] are generalized the result reported in [9] for Banach space valued function. The Hyers-Ulam stability of differential equations was developed in a series of papers [11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24]. Encouraged by the above ideas, the purpose of this paper is to refine the Hyers-Ulam stability and Generalized Hyers-Ulam stability of the system of second order differential equation

$$x''(t) = f(t, x(t), x'(t)) \quad (1.1)$$

in Banach space with initial conditions

$$x(a) = x_0 \quad \text{and} \quad x'(a) = x_1 \quad (1.2)$$

for all $t \in I$, $x(t) \in C^2(I, \mathbb{B})$ and where $I = [a, b)$, $a \in \mathbb{R}$, $b \in \overline{\mathbb{R}}$, $-\infty < a < b < +\infty$.

II. Preliminaries

Let $(\mathbb{B}, \|.\|)$ be a Banach Space (Real or Complex), and $I = [a, b)$, $a \in \mathbb{R}$, $b \in \overline{\mathbb{R}}, a < b \leq +\infty, \varepsilon$ be a positive real number, $f: I \times \mathbb{B} \to \mathbb{B}$ continuous operator and $\phi: I \to \mathbb{R}_+$ be a continuous function. Let us consider the system $x''(t) = f(t, x(t), x'(t))$ for all $t \in I$, where f is defined and continuous on $I \times \mathbb{B}$. Let G be a closed and bounded set of $\mathbb{B}$ containing the origin then there exists a real number $M > 0$ such that $G = \{x: \|x\| \leq M\}$ and let G_0 be a subset of G.

Definition 2.1 Let $x^*(t, x_0, x_1)$ be the solution of the system (1.1) satisfying $x^*(a) = x_0$ and $x^{*\prime}(a) = x_1$. If for every $\varepsilon > 0$, $x_0, x_1 \in G_0$ and each $x^*(t, x_0, x_1)$ is in G for all $t \in I$, then the origin is said to be (G_0, G, ε)-

practically stable. The solution which starts initially in G_0 remain thereafter in G.

Definition 2.2 We say that the differential equation (1.1) has the Hyers-Ulam stability with initial condition (1.2), if there exists a constant $S_f > 0$ such that for every $\varepsilon > 0$ and for each solution $x(t) \in C^2(I, \mathbb{B})$ satisfying the inequality $\|x''(t) - f(t, x(t), x'(t))\| \leq \varepsilon$, for all $t \in I$. Then there exists some $y \in C^2(I, \mathbb{B})$ satisfies the differential equation (1.1) with initial conditions (1.2) such that $\|x(t) - y(t)\| \leq S_f \varepsilon$, for all $t \in I$. We call such S as the Hyers-Ulam stability constant for (1.1) with (1.2).

Definition 2.3 We say that the differential equation (1.1) has the Hyers-Ulam-Rassias stability with initial condition (1.2) with respect to ϕ if there exists a constant $S_{f,\phi} > 0$ such that for every $\varepsilon > 0$ and for each solution $x(t) \in C^2(I, \mathbb{B})$ satisfies the inequality $\|x''(t) - f(t, x(t), x'(t))\| \leq \varepsilon\phi(t)$,for all $t \in I$. Then there exists some $y \in C^2(I, \mathbb{B})$ satisfies the differential equation (1.1) with initial conditions (1.2) such that $\|x(t) - y(t)\| \leq S_{f,\phi}\varepsilon\phi(t)$, for all $t \in I$. We call such S as the Hyers-Ulam-Rassias stability constant for (1.1) with initial condition (1.2).

Definition 2.4 The differential equation (1.1) is said to have the Generalized Hyers-Ulam stability with initial condition (1.2), if there exists $\theta_f \in C(\mathbb{R}_+, \mathbb{R}_+)$ with $\theta_f(0) = 0$ such that $x \in C^2(I, \mathbb{B})$ satisfying the inequality $\|x''(t) - f(t, x(t), x'(t))\| \leq \varepsilon$, for all $t \in I$. Then there exists some $y \in C^2(I, \mathbb{B})$ satisfies the differential equation (1.1) with initial conditions (1.2) such that $\|x(t) - y(t)\| \leq \theta_f(\varepsilon)$, for all $t \in I$.

Definition 2.5 The differential equation (1.1) is said to have the Generalized Hyers-Ulam-Rassias stability with initial condition (1.2) with respect to ϕ if there exists a constant $S_{f,\phi} > 0$ such that for every $\varepsilon > 0$ and for each solution $x(t) \in C^2(I, \mathbb{B})$ satisfies the inequality $\|x''(t) - f(t, x(t), x'(t))\| \leq \phi(t)$,for all $t \in I$. Then there exists some $y \in C^2(I, \mathbb{B})$ satisfies the differential equation (1.1) with initial conditions (1.2) such that $\|x(t) - y(t)\| \leq S_{f,\phi}\phi(t)$, for all $t \in I$. We call such S as the Generalized Hyers-Ulam-Rassias stability constant for (1.1) with initial condition (1.2).

Remark 2.6 Consider the system of differential equation (1.1) with initial condition

$$x(a) = x_0 \in G_0 \quad \text{and} \quad x'(a) = x_1 \in G_0, \tag{2.1}$$

where f is defined and continuous on $I \times \mathbb{B}$ and equilibrium state is at the origin: $f(t, 0, 0) = 0$, for all $t \in I$. The system (1.1) to be (G_0, G, ε)-practically stable it is sufficient that there exists a continuous non increasing on the system (1.1) solutions Lyapunov function $V(x, t)$ such that

$$\wp = \{x \in \mathbb{B}: V(t, x(t), x'(t)) \leq 1\} \subseteq G, \quad t \in I, \tag{2.2}$$

$$G_0 \subseteq \{x \in \mathbb{B}: V(t, x(t), x'(t)) \leq 1\}. \tag{2.3}$$

Proof. We prove this remark by contradiction. Suppose that the conditions (2.2) and (2.3) are satisfied but there are $\mu \in I$ and $x_0, x_1 \in G_0$ such that the solution $x(t) = x(a, x_0, x_1)$ of (1.1) leaves the set G. From (2.2) follows the inequality $V(\mu, x(\mu), x'(\mu)) > 1$ which contradicts the condition (2.3). Hence the equilibrium of the system (1.1) is (G_0, G, ε)-practically stable.

III. Main Results

In this section, we establish the Hyers-Ulam stability, Hyers-Ulam-Rassias stability, Generalized Hyers-Ulam stability and Generalized Hyers-Ulam-Rassias stability of the differential equation (1.1) with (1.2). Firstly, we prove the Hyers-Ulam stability of (1.1) with (1.2).

Theorem 3.1 Assume that there exists a constant $S_f > 0$ such that for every $\varepsilon > 0$ and for each solution $x(t) \in C^2(I, \mathbb{B})$ satisfying the inequality $\|x''(t) - f(t, x(t), x'(t))\| \leq \varepsilon$, with (1.2) for all $t \in I$. Then there exists some $y \in C^2(I, \mathbb{B})$ satisfies (1.1) with initial condition $y(a) = y_0$ and $y'(a) = y_1$ such that

$$\|x(t) - y(t)\| \leq S_f \varepsilon, \quad \forall\ t \in I.$$

Proof. Given that for every $\varepsilon > 0$, and for each solution $x(t) \in C^2(I, \mathbb{B})$ satisfying

$$\|x''(t) - f(t, x(t), x'(t))\| \leq \varepsilon, \tag{3.1}$$

for all $t \in I$. Now, we are going to prove that there exists a real number $S_f > 0$ and for some y in $C^2(I, \mathbb{B})$ satisfies the inequality (1.1) with $y(a) = y_0$ and $y'(a) = y_1$ such that

$$\|x(t) - y(t)\| \leq S_f \varepsilon,$$

for all $t \in I$. Let G be a closed and bounded set, then there exists a real number $M > 0$ such that

$$G = \{x: \|x\| \leq M\}.$$

Now, let $x^* = f(a, x_0, x_1)$ satisfies the inequality (3.1) for arbitrary ε, then x^* be the solution of the differential equation (1.1) with (1.2). Since the equilibrium of (1.1) is (G_0, G, ε)-practically stable, then $x^* \in G$.

Hence $\|x^*\| \leq M$, since $M > 0$, and $\varepsilon > 0$ then there exists $S_f > 0$ such that $M = S_f \varepsilon$. Then we have

$$\|x^*\| \leq S_f \varepsilon,$$

for all $t \in I$. Obviously, $y(t) = 0$ be the solution of (1.1) with (1.2) such that

$$\|x^*(t) - y(t)\| \leq S_f \varepsilon,$$

for all $t \in I$. Hence by the virtue of Definition 2.2, the system of second order differential equation has the Hyers-Ulam stability.

The following corollary shows the Generalized Hyers-Ulam stability of the system (1.1) with (1.2).

Corollary 3.2 Assume that there exists $\theta_f > 0$ such that for every $\varepsilon > 0$ and for each solution $x(t) \in C^2(I, \mathbb{B})$ satisfies $\|x''(t) - f(t, x(t), x'(t))\| \leq \varepsilon$, with initial condition $x(a) = x_0$ and $x'(a) = x_1$ for all $t \in I$. Then there exists some $y \in C^2(I, \mathbb{B})$ satisfies (1.1) with initial condition $y(a) = y_0$ and $y'(a) = y_1$ such that

$$\|x(t) - y(t)\| \leq \theta_f(\varepsilon), \quad \forall\ t \in I.$$

Proof. We can prove this corollary by the similar way as in the proof of Theorem 3.1.

Next, we prove the Hyers-Ulam-Rassias stability of the system (1.1) with initial condition (1.2).

Theorem 3.3 Assume that there exists a constant $S_{f,\phi} > 0$ such that for every $\varepsilon \in (0, \varepsilon_0]$ and for each solution $x(t) \in C^2(I, \mathbb{B})$ satisfying the inequality

$$\|x''(t) - f(t, x(t), x'(t))\| \leq \varepsilon\phi(t),$$

with (1.2) for all $t \in I$. Then there exists some $y \in C^2(I, \mathbb{B})$ satisfies (1.1) with initial condition $y(a) = y_0$ and $y'(a) = y_1$ such that $\|x(t) - y(t)\| \leq S_{f,\phi}\varepsilon\phi(t), \quad \forall\ t \in I.$

Proof. Given that for every $\varepsilon \in (0, \varepsilon_0]$, and for each solution $x(t) \in C^2(I, \mathbb{B})$ satisfying

$$\|x''(t) - f(t, x(t), x'(t))\| \leq \varepsilon\phi(t), \tag{3.2}$$

for all $t \in I$. Now, we prove that there exists real number $S_{f,\phi} > 0$ and for some $y \in C^2(I, \mathbb{B})$ satisfying the inequality (1.1) with $y(a) = y_0$ and $y'(a) = y_1$ such that

$$\|x(t) - y(t)\| \leq S_{f,\phi}\varepsilon\phi(t),$$

for all $t \in I$. Let G be a closed and bounded set, then there exists a real number $M > 0$ such that

$$G = \{x: \|x\| \leq M\}.$$

If the equilibrium is (G_0, G, ε_0)-practically stable and there exists $\varepsilon_1 > 0$ such that

$$\varepsilon_1 \leq \phi(t)\varepsilon \leq \varepsilon_0$$

for all $t \in I$. Now, let

$$x^* = f(a, x_0, x_1)$$

satisfies the inequality (3.2) for arbitrary ε_1, then x^* be the solution of the differential equation (1.1) with (1.2). Since the equilibrium of (1.1) is (G_0, G, ε_0)-practically stable, then $x^* \in G$.

Hence $\|x^*\| \leq M$, since $M > 0$, and $\varepsilon_1 > 0$ then there exists $S_{f,\phi} > 0$ such that $M = S_{f,\phi}\varepsilon_1$. Then we have $\|x^*\| \leq S_{f,\phi}\varepsilon_1$, for all $t \in I$, hence

$$\|x^*\| \leq S_{f,\phi}\varepsilon\phi(t).$$

Obviously, $y(t) = 0$ be the solution of (1.1) with (1.2) such that $\|x^*(t) - y(t)\| \leq S_{f,\phi}\varepsilon\phi(t)$,
for all $t \in I$. Hence by the virtue of Definition 2.3, the system of second order differential equation has the Hyers-Ulam stability.

The following corollary shows the Generalized Hyers-Ulam-Rassias stability of the system (1.1) with (1.2).

Corollary 3.4 Assume that there exists a constant $S_{f,\phi} > 0$ such that for every $\varepsilon \in (0, \varepsilon_0]$ and for each solution $x(t) \in C^2(I, \mathbb{B})$ satisfying the inequality $\|x''(t) - f(t, x(t), x'(t))\| \leq \phi(t)$,with (1.2) for all $t \in I$. Then there exists some $y \in C^2(I, \mathbb{B})$ satisfies (1.1) with initial condition $y(a) = y_0$ and $y'(a) = y_1$ such that

$$\|x(t) - y(t)\| \leq S_{f,\phi}\phi(t), \quad \forall\ t \in I.$$

Proof. We can prove this corollary by the similar way as in the proof of Theorem 3.3.

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Stabilization of Memristor based Commensurate Order Fractional Derivative System

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***Abstract**: Discovery of memory resistors, the fourth circuit element has widely attracted the attention of the researchers. A four dimensional hyperchaotic memristor based commensurate order fractional derivative system is considered and stabilization of the system is carried out in this paper with feedback gain matrix obtained by pole placement technique. The asymptotic stability of the system for non integer arbitrary order in $(0,1)$ and $(1,2)$ is analyzed. Numerical simulations are performed for the different parameter values. The chaotic behavior of the system is discussed with time plots and 2-Dimensional and 3-Dimensional phase plane diagrams.*

***2010Mathematics Subject Classification**: 34A08,37N05,93C10,93D15*

***Keywords and Phrases**: Fractional order system, Stabilization by feedback, Nonlinear system*

1 Introduction

Memristor, an fourth circuit element that links charge and flux was introduced by Chua. Memory resistors are widely used in Neuromorphic computer architecture, pattern recognition system, adaptive filter etc. The hyperchaotic systems of fractional order has attracted the interest of many young researchers[[2],[3]]. Non linearities in the system representing the memristor model denotes the dynamical change in internal state of memristor that is caused by the flow of charge through it. The history of the past current flow through a memristor determines the present electrical resistance . The slope of the function between charge and flux is the memristance of a memristor[4]. The important application is in experimental modeling of unicellular organisms adaptive behavior using a electrical circuit with LC and memristor.

Fractional Calculus, the branch of mathematics dealing with the arbitrary order integration and differentiation is widely used in the field of control theory, acoustics and thermal systems, reconfigurable hardware and so on. Control theory, the branch of mathematics had its origin in 19th century but became a important and interesting field of research during the world war II. Modern control theory overcame the complexity of modeling a physical system as time domain models in classical control theory by converting the equations into state equations which can be dealt with linear algebra techniques. Feedback properties to control the system is one among the many ways in controlling a system. Not all the systems with feedback have predictable behavior, chaotic behavior is also exhibited by the systems.

The memistor based system with fractional commensurate order is taken for discussion of stability properties in $(0,2)$ without consideration of integer order. Notations and the model is presented with some preliminary theorems in Section 2 and Section 3 respectively. The asymptotic stability of the feedback controlled system of fractional derivative order in $(0,1)$ and $(1,2)$ are illustrated in Section 4 with numerical examples and simulations.The chaotic behavior of the memristor based system with 2-D and 3-D phase portraits in Section 5 followed with the conclusion in Section 6.

2 Prerequisites

Some basic theorems for asymptotic stability for commensurate fractional order system are

$$\frac{d^q w(t)}{dt^q} = A(w(t)) = Hw(t) + C(w(t)) + Mu(t) \tag{1}$$

with q is the commensurate fractional order,$w(t) = (w_1(t), w_2(t), \dots, w_n(t))^T \in \Re^n$ denoting the state vector and non linear mapping $C: \Re^n \to \Re^n$, $H \in \Re^{n\times n}$, feedback controller $u(t)$ and $M \in \Re^n$. The sum of linear and nonlinear together comprise $A(w(t))$.

With $u(t) = Kw(t)$,the system (1) can be expressed as

$$\frac{d^q w(t)}{dt^q} = \tilde{H}w(t) + C(w(t)) \tag{2}$$

where $\tilde{H} = H - MK$.

Theorem 1 *[1] If* (H, M) *is controllable,there must exist a* K *such that*

$$|arg\{\lambda(H - MK)\}| > \frac{q\pi}{2}$$

, then the system (2) is locally asymptotically stable with order $0 < q < 1$.

Theorem 2 *[5] The controlled system (2) is locally asymptotically stable with order* $1 < q < 2$*, if the following conditions hold for chosen feedback gain* K*,*

1. $C(w(t))$ satisfies $C(0) = 0$ and $\lim_{w \to 0} \frac{\|C(w(t))\|}{\|w(t)\|} = 0$.
2. $Re\lambda(\tilde{H}) < 0$ and $\eta = -max\{Re\lambda(\tilde{H})\} > (\Gamma(q))^{\frac{1}{q}}$.

Theorem 3 *[5] The controlled system (3) is globally asymptotically stable with order* $1 < q < 2$*, if the following conditions hold for chosen feedback gain* K*,*

1. $C(w(t))$ satisfies $C(0) = 0$ and is global Lipschitz with Lipschitz constant L, i.e, $|C(w_1) - C(w_2)| \leq L|w_1 - w_2|$ for all $w_1, w_2 \in \Re$
2. $Re\lambda(\tilde{H}) < 0$ and $\eta = -max\{Re\lambda(\tilde{H})\} > (Nl\Gamma(q))^{\frac{1}{q}}$ where N satisfies $\left\|e^{\tilde{H}t}\right\| \leq Ne^{-\eta t}$

3 Model description and Stability

The Memristor based commensurate order fractional derivative system is

$$\begin{aligned}
\frac{d^q w_1(t)}{dt^q} &= -\alpha w_1(t) - \beta w_2(t) - w_1(t)w_2(t);\\
\frac{d^q w_2(t)}{dt^q} &= \sigma w_1(t) + 1.2\ w_1(t)w_3(t) + (0.1)(1 + 0.2\ w_4(t)^2\ - 0.0016 w_4(t)^4)\ w_2(t);\\
\frac{d^q w_3(t)}{dt^q} &= -\kappa w_3(t) - 1.2\ w_1(t)w_2(t);\\
\frac{d^q w_2(t)}{dt^q} &= -\vartheta w_2(t);
\end{aligned} \tag{3}$$

where $\alpha, \beta, \sigma, \kappa, \vartheta$ are constants.

3.1 Stability of System (3)

The system (3) is unstable for the values $\alpha = 1, \beta = 1.3, \sigma = 0.56, \kappa = 1.1, \vartheta = 1$. On Stabilization, system (3) becomes

$$\frac{d^q w(t)}{dt^q} = A(w(t)) = Hw(t) + C(w(t)) + Mu(t) \tag{4}$$

Here, $w(t) = [w_1(t) \quad w_2(t) \quad w_3(t) \quad w_4(t)]^T$, $H = \begin{bmatrix} -1 & -1.3 & 0 & 0 \\ -0.56 & 0.1 & 0 & 0 \\ 0 & 0 & -1.1 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$

$$C(v(t)) = \begin{bmatrix} -w_1(t)w_2(t) \\ 1.2\ w_1(t)w_3(t) + 0.1(0.2\ (w_4(t))^2 - 0.0016\ (w_4(t))^4)w_2(t) \\ -1.2\ w_1(t)w_2(t) \\ 0 \end{bmatrix}$$,initial condition $(0.001, 0.07, 0.01, 0.02)$, the control matrix is $M = [0 \quad 1 \quad 0 \quad 1]^T$. Let us consider the feedback gain matrix as $K = [k_1 \quad k_2 \quad k_3 \quad k_4]$.

Choosing the feedback gain matrix $K = [-3 \quad 12 \quad 3.5 \quad -9.5]$ by pole placement technique, the stability of the system (4) in $(0,1)$ and $(1,2)$ are as follows.

3.1.1 Fractional order $0 < q < 1$

For controllable matrices (H, M) there exists K such that $\tilde{H}$ of (4) has eigenvalues $\lambda_1 = -1.307049854, \lambda_{2,3} = -1.046475070 \pm 3.491604169\ i$ and $\lambda_4 = -1.1$. The local asymptotic stability of the system follows from Theorem (1) for fractional order $q = 0.97$. Figure 1 and 2 provides the time plots and phase plane diagrams explaining the stability of the system (4) are presented in Figure 3 and 4.

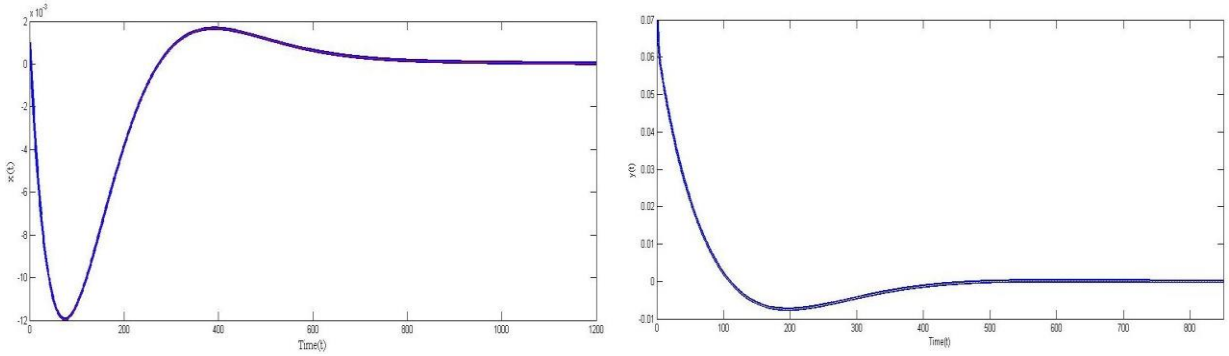


Figure 1: Time plots of the system (4) for 0<q<1

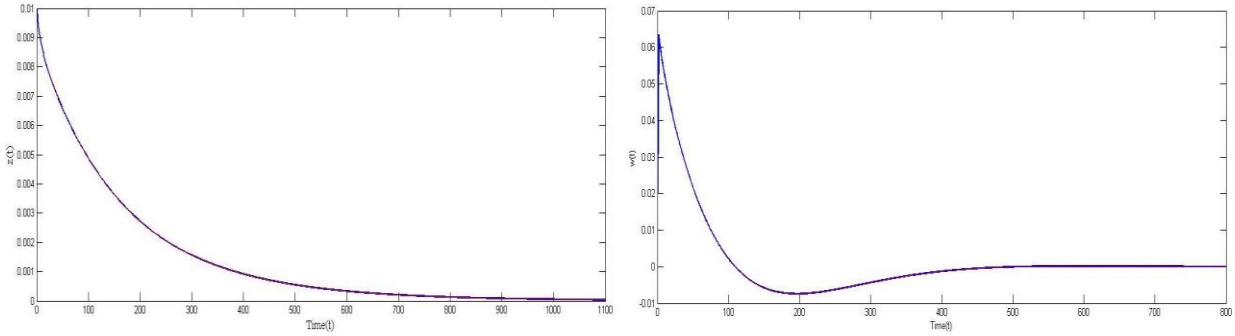


Figure 2: Time plots of the system (4) for 0<q<1

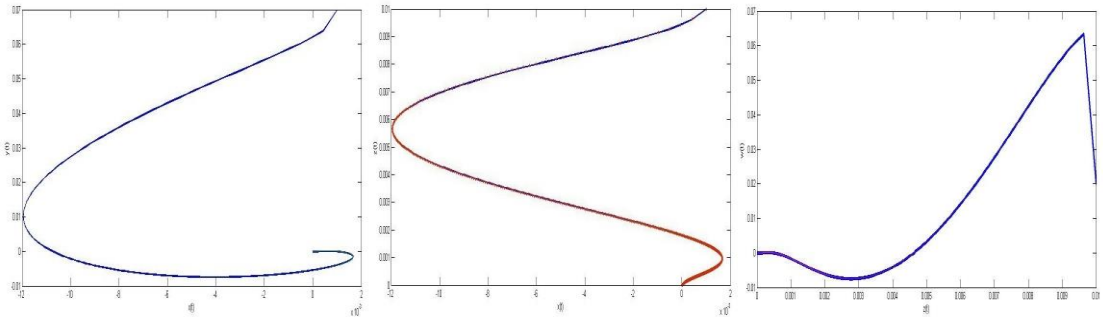


Figure 3: 2-D phase portraits of the system (4)for 0<q<1

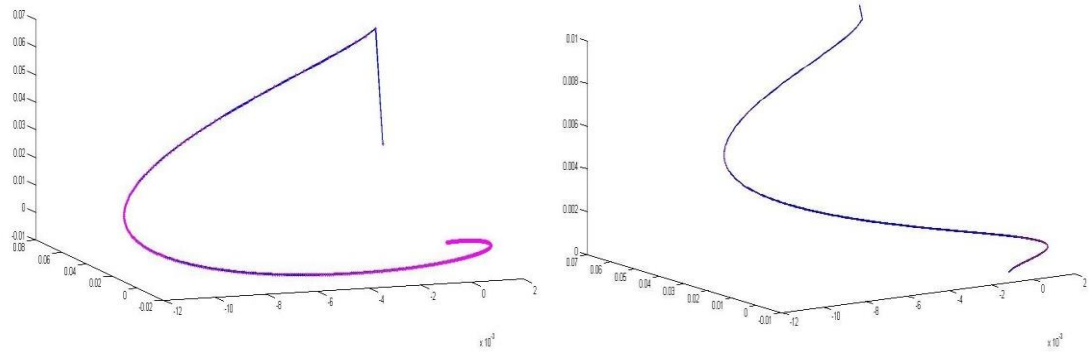


Figure 4: 3-D phase portraits of the system (4)for 0<q<1

3.2 Fractional order $1 < q < 2$

Local and global stability of the system (4) for the commensurate fractional order $1 < q < 2$ are as follows.

3.2.1 Local stability

Conditions in the Theorem (2) are to be satisfied for the system (4) to be locally asymptotically stable.

Clearly, $C(0) = 0$ and

$$\lim_{w\to 0} \frac{\|C(w(t))\|}{\|w(t)\|} \leq \lim_{w\to 0} |w_2| = 0.$$

$\lambda_1 = -1.307049854, \lambda_{2,3} = -1.046475070 \pm 3.491604169\ i$ and $\lambda_4 = -1.1$ are the eigen values of $\tilde{H}$ with negative real parts and

$$\eta = 1.0464750 > 0.9207002594.$$

Thus, the system (4) with feedback gain K with order 1.47 is locally asymptotically stable.

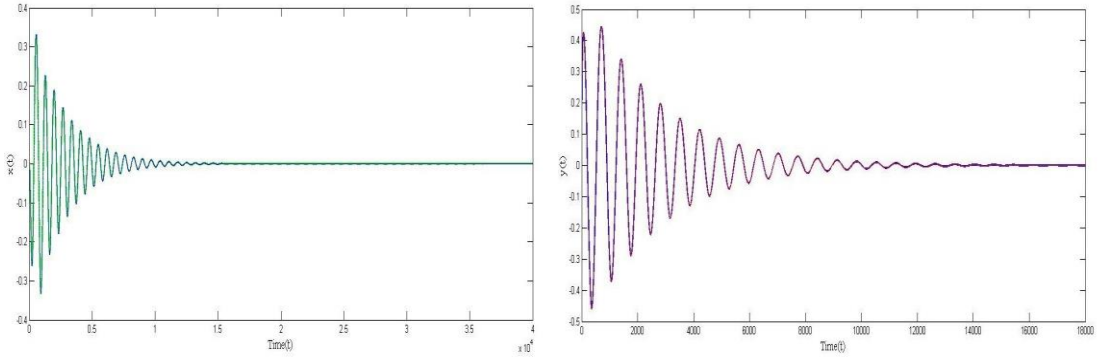


Figure 5: Time plots of the system (4) for 1<q<2

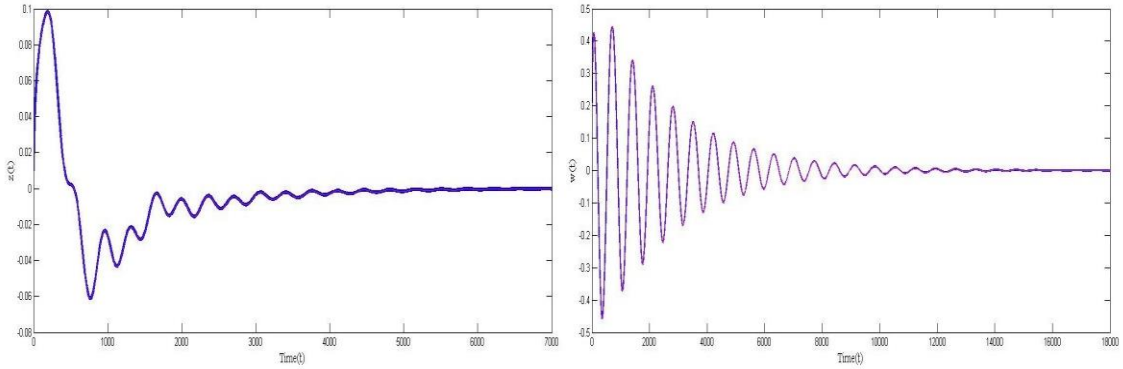


Figure 6: Time plots of the system (4) for 1<q<2

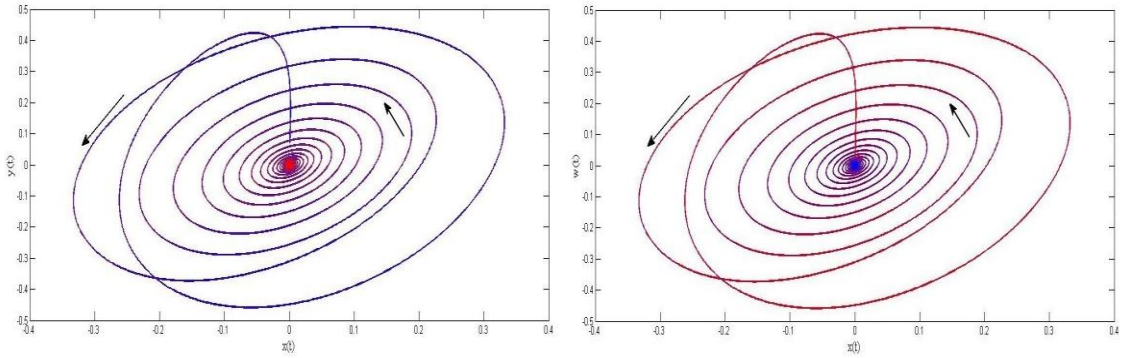


Figure 7: 2-D phase portraits of the system (4)for 1<q<2

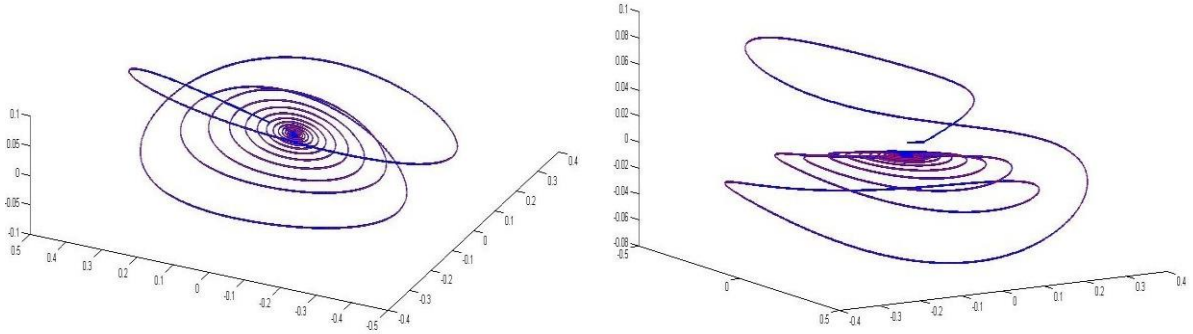


Figure 8: 3-D phase portraits of the system (4)for 1<q<2

3.2.2 Global Stability

The Global lipschitz condition of the nonlinear term with lipscitz constant L is

$$\|C(w)\| \leq L\|w\|.$$

The value of N can be chosen using the exponential of $\widetilde{H}$ and $e^{-\eta t}$ and the inequality in Theorem (3). Choosing $L = 0.5$ and $N = 2.3$ we have $\eta = 1.0464750 > 1.012533296$. The system (4) is Globally asymptotically stable.

The time plots and corresponding phase plane diagrams are provided in Figure 5,6,7and 8 respectively.

3.3 Chaotic Behavior of system (3)

Chaos is the unpredictable behavior of the complex system with random motion and great sensitivity to the slight variation in conditions [6]. Most of the natural systems like climate, weather and so on exhibits chaotic behavior. With initial development in study of chaos theory at the beginning of 20 th century, more intense research interest on this field arrived at mid 20 th century. Though chaos theory has its origin from weather prediction it is also used in geology, population dynamics etc. Chaos theory is very much useful in construction of predictive models of robots. In manufacturing of polymers the gas solubility is predicted by chaos theory. In this section, we explain the chaotic behavior of the system (3) with 2-D and 3-D diagrams. For the values of $\alpha = 7.05, \beta = 3.7, \sigma = 8.5, \kappa = 4.2, \vartheta = 11.8$ and initial conditions $(0.01,0.07,0.01,0.02)$, the behavior of the system for order $q = 1.41$ is explained by Figure 9, 10, 11 and 12.

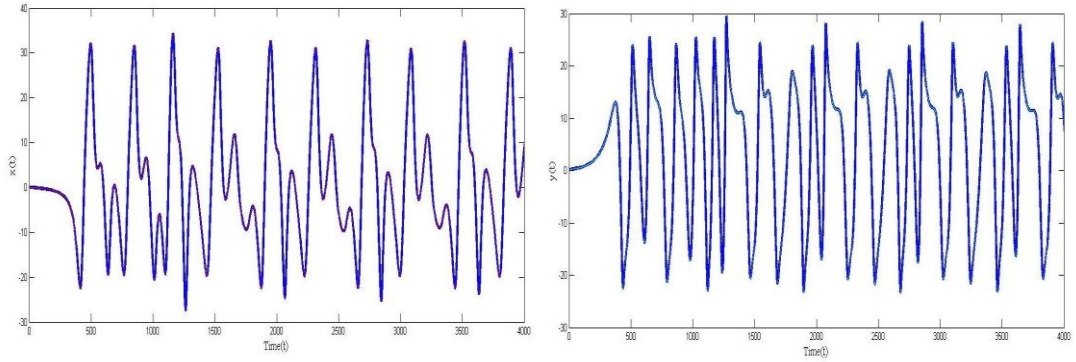


Figure 9: Chaotic behavior of the system (3)

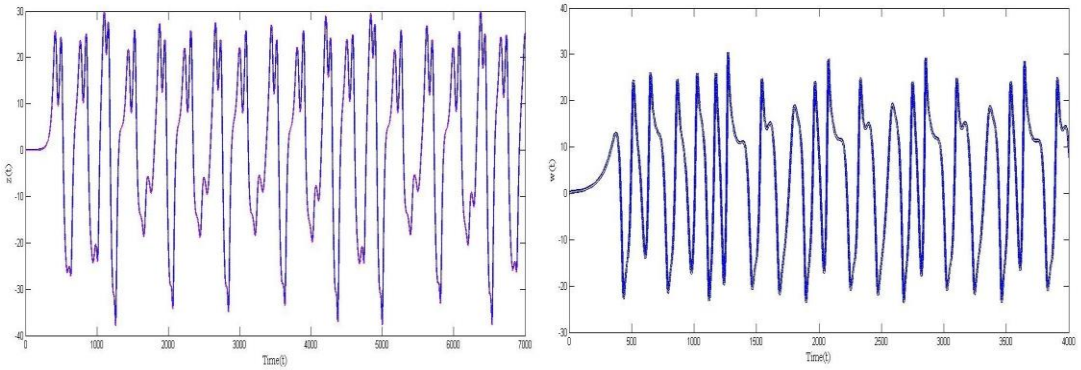


Figure 10: Chaotic behavior of the system (3)

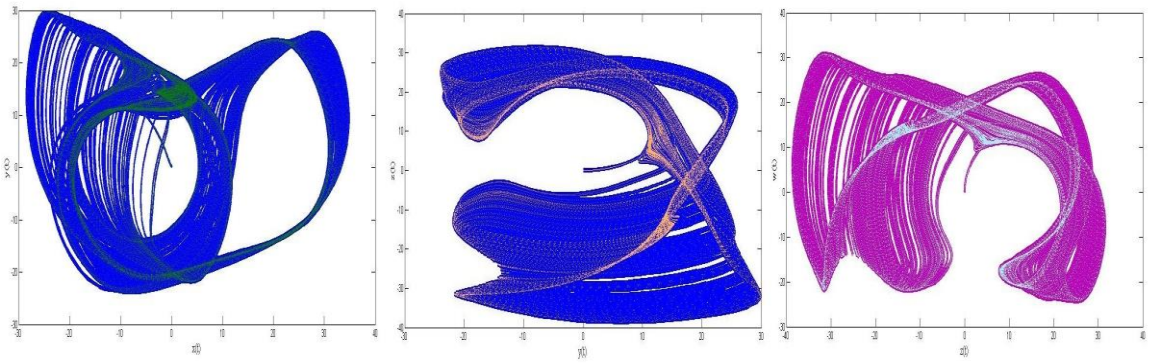


Figure 11: Chaotic behavior of the system (3)

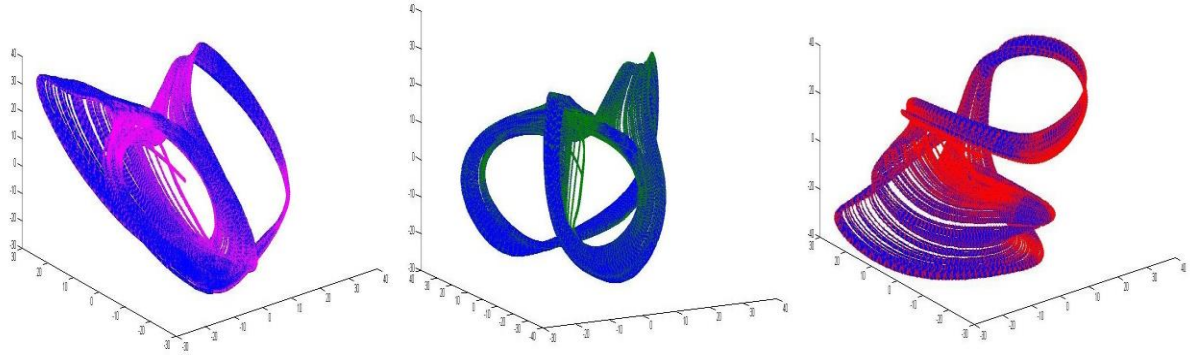


Figure 12: Chaotic behavior of the system (3)

4 Conclusion

The stability of feedback controlled hyper chaotic memristor based system is discussed in this paper. It is understood that the stability of these systems with fractional order $0 < q < 1$ take lesser time to attain equilibrium position rather than the system with order $1 < q < 2$ which take little longer time to approach the stable position. The Chaotic behavior of the system is illustrated with timeplots and phase plane diagrams. The dynamical behavior the system is thus discussed with numerical simulations for different set of values.

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Controllability results for fractional stochastic delay dynamical systems

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***Abstract**: This paper deals with the controllability of fractional stochastic delay dynamical systems, which involves Caputo derivatives. Sufficient conditions for controllability of the stochastic delay dynamical systems has established by using Banach fixed point theorem and the controllability Grammian matrix which is defined by Mittag-Leffler matrix function. Example has given to verify the results.*

***Keywords:** Controllability, Stochastic delay systems, Mittag-Leffler matrix function, Fractional differential equations.*

I. Introduction

Controllability is one of the fundamental concepts in mathematical control theory. It is a qualitative property of dynamical control systems and is of particular importance in control theory. Theory of controllability originates from the famous work of Kalmanin 1960, where the concept of controllability was defined for finite dimensional deterministic linear systems. The natural extension of the concept of controllability to infinite dimensional systems is studied by many authors [1]-[3].The notion of controllability has plays a central role throughout the history of modern control theory. Since, various researches have been carried out extensively in the context of finite-dimensional linear, nonlinear systems and infinite-dimensional systems using different kinds of approaches [3]-[5].

Recently there has been a great interest to differential equations with fractional order, that is fractional models are more accurate than integer models. Fractional calculus provides an excellent instrument for the description of systems with memory and hereditary properties. At the same time, the fluctuations in nature can be captured only by adding random elements into the differential equations, which are called stochastic differential equations (SDEs). It is used to model diverse phenomena such as fluctuating stock prices or physical systems subject to thermal fluctuations. In the literature, there are different definitions of controllability for SDEs, both for linear and nonlinear dynamical systems [5]. Fractional delay dynamical systems are an important kind of fractional order systems in real life. In these years, some authors pay attention to the study about the fractional delay dynamical systems [6], [7].

Controllability of both deterministic [8], [9] and stochastic [10], [11] linear systems with delays, and stochastic systems without delays [12]-[15] are studied. However, most literature in this field has been concerned with unconstrained controls. Some criteria for constrained controllability of linear dynamical systems with delays in control are presented in [16] and [17].

To the best our knowledge, no relevant report has been described in the control problem of fractional stochastic delay dynamical systems. In this regard, we make an attempt to investigate the controllability of the fractional stochastic delay dynamical systems. Numerical illustrations are included to clarify the theoretical results.

II. Preliminaries

Let (Ω, F, P) be a complete probability space with a filtration $\{F_t\}_{t\geq 0}$ satisfying the usual condit\ions (i.e. right continuous and F_0 containing all **P**-null sets). Let $\alpha, \beta > 0$, with $n-1 < \alpha < n$, $n-1 < \beta < n$and $n \in N$, $D$is the usual differential operator. Let R^mbe the m-dimensional Euclidean space, $R_+ = [0, \infty)$, and suppose $f \in L^1(R_+)$. The following definitions and properties are well known, for $\alpha, \beta > 0$ and $f$as a suitable function [18], [19].

(a) Riemann-Liouville fractional operators:

$$(I_{a+}^{n-\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{(\alpha-1)} f(t)dt,$$
$$(D_{0+}^{\alpha} f)(x) = D^n (I_{a+}^{n-\alpha} f)(x)$$

(b) Caputo fractional derivative:

$$\left({}^{C}\mathrm{D}_{0+}^{\alpha} f \right)(x) = (I_{a+}^{n-\alpha} D^n f)(x),$$

In particular $I_{0+}^{\alpha}\, {}^{c}D_{0+}^{\alpha} f(t) = f(t) - f(0), (0 < \alpha < 1)$.

The following is a well-known relation, for finite interval $[a, b] \in R_+$

$$(D_{0+}^{\alpha} f)(x) = \left({}^{C}\mathrm{D}_{0+}^{\alpha} f \right)(x) + \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{\Gamma(1+k+\alpha)} (x-a)^{k-\alpha}, n = R(\alpha) + 1$$

The Laplace transform of the Caputo fractional derivative is

$$\mathcal{L}\left\{ {}^{C}\mathrm{D}_{0+}^{\alpha} f(t) \right\} = s^{\alpha} F(s) - \sum_{k=0}^{n-1} f^{(k)}(0^+) s^{\alpha-1-k}.$$

From the above we notice that, the Riemann-Liouville fractional derivatives have singularity at zero and the fractional differential equations in the Riemann-Liouville sense require initial conditions of special form lacking physical interpretation. To overcome this difficulty Caputo introduced a new definition of fractional derivative but in general, both the Riemann-Liouville and the Caputo fractional operators possess neither semigroup nor commutative properties, which are inherent to the derivatives on integer order. Due to this fact, the concept of sequential fractional differential equations are discussed in Kilbas et al. [18] and Miller and Ross [19].

(c) Linear Sequential Derivative:

For $n \in N$ the sequential fractional derivative for suitable function f is defined by

$$f^{(k\alpha)} := (D^{k\alpha} f)(x) = (D^{\alpha} D^{(k-1)\alpha} f)(x),$$

where $k = 1, \ldots, n, (D^{\alpha} f)(x) = f(x)$, and D^{α} is any fractional differential operator, here we mention it as ${}^{C}\mathrm{D}_{0+}^{\alpha}$.

(d) Mittag-Leffler Function

$$E_{\alpha,\beta}(y) = \sum_{k=0}^{\infty} \frac{y^k}{\Gamma(ka+\beta)}, \qquad \alpha, \beta > 0.$$

The general Mittag-Leffler function satisfies

$$\int_0^{\infty} e^{-t} t^{\beta-1} E_{\alpha,\beta}(t^{\alpha} y) dt = \frac{1}{1-y}, \quad |y| < 1.$$

The Laplace transform of $E_{\alpha,\beta}(y)$ follows from the integral

$$\int_0^{\infty} e^{-st} t^{\beta-1} E_{\alpha,\beta}(\pm a t^{\alpha} y) dt = \frac{s^{\alpha-\beta}}{s \mp a}.$$

That is

$$\mathcal{L}\{t^{\beta-1} E_{\alpha,\beta}(\pm a t^{\alpha})\} = \frac{s^{\alpha-\beta}}{s \mp a},$$

For $\mathcal{R}(s) > |a|^{\frac{1}{\alpha}}$ and $R(\beta) > 0$. In particular for $\beta = 1$,

$$E_{\alpha,1}(\lambda y^{\alpha}) = E_{\alpha}(\lambda y^{\alpha}) = \sum_{k=0}^{\infty} \frac{\lambda^k y^{k\alpha}}{\Gamma(\alpha k + 1)}, \ \lambda, y \in \boldsymbol{C}$$

Have the interesting property ${}^{C}\mathrm{D}_{0+}^{\alpha} E_{\alpha}(\lambda y^{\alpha}) = \lambda E(\lambda y^{\alpha})$and

$$\mathcal{L}\{E_\alpha(\pm at^\alpha)\} = \frac{s^{\alpha-\beta}}{s \mp a}, \qquad for\ \beta = 1.$$

For brevity of notation let us take I_{0+}^q as I^q and ${}^{C}D_{0+}^{\alpha}$ as ${}^{C}D^{\alpha}$ and the fractional derivative is taken as Caputo sense.

III. Solution representation

Consider a fractional delay stochastic differential equation of the form

$$^{C}D^{\alpha}x(t) = Ax(t) + Bx(t-h) + \sigma(t)\frac{d\omega(t)}{dt}, 0 < \alpha \leq 1,$$

$$x(t) = \varphi(t), -h < t \leq 0. \tag{1}$$

where $x \in R^n$, A and B are $n \times n$ matrices $x \in R^n$, $\varphi(t)$ is a continuous function on $[-h, 0]$. By taking Laplace transform on both sides of (1) we get

$$s^{\alpha}X(s) - s^{\alpha-1}\varphi(0) = A\int_0^{\infty} x(t)e^{-st}dt + B\int_0^{\infty} e^{-st}x(t-h)dt + \int_0^{\infty}\sigma(t)\frac{d\omega(t)}{dt}dt.$$

And by simple calculation we have

$$s^{\alpha}X(s) - s^{\alpha-1}\varphi(0) = AX(s) + Be^{-hs}\int_{-h}^{0} e^{-s\tau}x(\tau)dt + Be^{-hs}X(s) + \sum(s)\frac{d\omega(s)}{ds},$$

Where, $X(s) = \int_0^{\infty} e^{-st}x(t)dt$, then

$$X(s) = \left[\frac{s^{\alpha-1}}{s^{\alpha}I - A - Be^{-hs}}\right]\varphi(0) + B\left[\frac{e^{-hs}}{s^{\alpha}I - A - Be^{-hs}}\right]\int_{-h}^{0} e^{-s\tau}\varphi(\tau)d\tau + \left[\frac{\sum(s)\frac{d\omega(s)}{ds}}{s^{\alpha}I - A - Be^{-hs}}\right].$$

Using inverse Laplace transform and convolution of Laplace we get

$$x(t) = L^{-1}\left[s^{\alpha-1}\left(s^{\alpha}I - A - Be^{-hs}\right)^{-1}\right](t)\varphi(0) + BL^{-1}\left[\frac{s^{\alpha-1}}{s^{\alpha}I - A - Be^{-hs}}s^{1-\alpha}\right](t)$$
$$* L^{-1}e^{-hs}\left[\int_{-h}^{0} e^{-s\tau}\varphi(\tau)d\tau\right](t) + L^{-1}\left[\sum(s)\frac{d\omega(s)}{ds}\right](t) * L^{-1}\left[\frac{s^{\alpha-1}}{s^{\alpha}I - A - Be^{-hs}}s^{1-\alpha}\right](t).$$

Let

$$\Phi_{\alpha}(t) = L^{-1}\left[s^{\alpha-1}\left(s^{\alpha}I - A - Be^{-hs}\right)^{-1}\right](t).$$

And

$$L^{-1}\left[\frac{s^{\alpha-1}}{s^{\alpha}I - A - Be^{-hs}}s^{1-\alpha}\right](t) = \Phi_{\alpha}(t) * \frac{t^{\alpha-2}}{\Gamma(\alpha-1)} = \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}\Phi_{\alpha}(s)ds,$$
$$= t^{\alpha-1}\Phi_{\alpha,\alpha}(t),$$

where

$$\Phi_{\alpha,\alpha}(t) = t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}\Phi_{\alpha}(s)ds$$

Define $\omega(t): [-h, \infty) \to [0,1]$ by $\omega(t) = 0$ for $t \geq 0$ and $\omega(t) = 1$ for $t < 0$. The function $\varphi(t)$ is extended to $(-h, \infty)$ by defining $\varphi(t) = \varphi(0)$ for $t \geq 0$, then

$$e^{-hs}\int_{-h}^{0} e^{-st}\varphi(t)dt = e^{-hs}\int_0^h e^{-s(-h+\lambda)}\varphi(-h+\lambda)d\lambda,$$
$$= \int_0^{\infty} e^{-s\lambda}\varphi(-h+\lambda)\omega(-h+\lambda)d\lambda,$$
$$= L[\varphi(-h+\cdot)\omega(-h+\cdot)](t).$$

Thus,

$$x(t) = \Phi_\alpha(t)\varphi(0) + B\int_0^t (t-s)^{\alpha-1}\Phi_{\alpha,\alpha}(t-s)\varphi(s-h)\omega(s-h)ds + \int_0^t (t-s)^{\alpha-1}\Phi_{\alpha,\alpha}(t-s)\sum(s)\frac{d\omega(s)}{ds}ds,$$

And so

$$x(t) = \Phi_\alpha(t)\varphi(0) + B\int_0^t (t-s)^{\alpha-1}\Phi_{\alpha,\alpha}(t-s)\varphi(s-h)\omega(s-h)ds + \int_0^t (t-s)^{\alpha-1}\Phi_{\alpha,\alpha}(t-s)\int_0^\tau \sigma(\theta)d\omega(\theta)ds,$$

Hence

$$x(t) = \Phi_\alpha(t)\varphi(0) + B\int_{-h}^0 (t-s+h)^{\alpha-1}\Phi_{\alpha,\alpha}(t-s+h)\varphi(s)ds + \int_0^t (t-s)^{\alpha-1}\Phi_{\alpha,\alpha}(t-s)\int_0^\tau \sigma(\theta)d\omega(\theta)ds.$$

IV. Linear Stochastic delay system

Consider the linear fractional delay stochastic dynamical system represented by the fractional delay stochastic differential equation of the form

$$\begin{aligned} {}^{C}D^{\alpha}x(t) &= Ax(t) + Bx(t-h) + Cu(t) + \sigma(t)\frac{d\omega(t)}{dt}, t \in J := [0,T] \\ x(t) &= \varphi(t), \end{aligned} \qquad (2)$$

where $0 < \alpha < 1, x(t) \in R^n$, $u \in R^m, A$ and B an $n \times n$ matrix, $\omega(t)$ isa givenl-dimensional Wienerprocesswiththefiltration$\mathcal{F}_t$generatedby$w(s), 0 \le s \le t$and$\sigma: [0,T] \to R^{n\times l}$is appropriatefunction.

Definition 4.1. System (2) is said to be controllable on *J* if for every complete state $\varphi(0)$ and every vector $x_1 \in R^n$ there exists a control $u(t)$ defined on J. Such that the corresponding trajectory of the system (2) satisfies $x(T) = x_1$.

Note that the solution of system (2) can be expressed in the following form

$$x(t) = x(t;\varphi) + \int_0^t (t-s)^{\alpha-1}\Phi_{\alpha,\alpha}(t-s)\left(\int_0^\tau \sigma(\theta)\, d\omega(\theta)\right)ds + \int_0^t (t-s)^{\alpha-1}\Phi_{\alpha,\alpha}(t-s)Cu(s)ds, \qquad (3)$$

where

$$x(t;\varphi) = \Phi_\alpha(t)\varphi(0) + B\int_{-h}^0 (t-s+h)^{\alpha-1}\Phi_{\alpha,\alpha}(t-s+h)\varphi(s)ds$$

Define the controllability Grammian matrix

$$W = \int_0^T [(T-s)^{\alpha-1}\Phi_{\alpha,\alpha}(T-s)C][(T-s)^{\alpha-1}\Phi_{\alpha,\alpha}(T-s)C)]^{\top} dt \qquad (4)$$

where the complete state $\varphi(0)$ and the vector $x_1 \in R^n$ are chosen arbitrarily and the $\top$ denotes the matrix transpose.

Theorem 4.1

The linear system (2) is controllable on $J = [0, T]$ if and only if controllability Grammian W is nonsingular.

Proof

Assume that W is nonsingular and so its inverse is well defined. Let $\varphi(t)$ be continuous on $[-h, 0]$ and let $x_1 \in R^n$. Let u be the control function given by

$$u(t) = [(T-t)^{\alpha-1} Z_{\alpha,\alpha}(T-t)C)]^{\top} W^{-1} \left[x_1 - x(T;\varphi) - \int_0^T (T-s)^{\alpha-1} Z_{\alpha,\alpha}(T-s) \int_0^{\tau} \sigma(\theta)\, d\omega(\theta) ds \right]$$

Substituting $t = T$ in the solution (3) we get

$$x(T) = x(T;\varphi) + \int_0^T (T-s)^{\alpha-1} \Phi_{\alpha,\alpha}(T-s) \left(\int_0^{\tau} \sigma(\theta)\, d\omega(\theta) \right) ds + \int_0^T (T-s)^{\alpha-1} \Phi_{\alpha,\alpha}(T-s) C u(s) ds$$

$$x(T) = x(T;\varphi) + \int_0^T (T-s)^{\alpha-1} \Phi_{\alpha,\alpha}(T-s) \left(\int_0^{\tau} \sigma(\theta)\, d\omega(\theta) \right) ds + \int_0^T (T-s)^{\alpha-1} \Phi_{\alpha,\alpha}(T-s) C \left[(T-t)^{\alpha-1} Z_{\alpha,\alpha}(T-t)C \right]^{\top} W^{-1} \left[x_1 - x(T;\varphi) - \int_0^T (T-s)^{\alpha-1} \Phi_{\alpha,\alpha}(T-s) \int_0^{\tau} \sigma(\theta)\, d\omega(\theta) ds \right]$$

$$= x(T;\varphi) + x_1 - x(T;\varphi)$$

$$x(T) = x_1.$$

Now, assume that W is singular. Then, there exists a vector $y \neq 0$. Such that $y^{\top} W y = 0$. It follows that

$$\int_0^T y^{\top} [(T-s)^{\alpha-1} \Phi_{\alpha,\alpha}(T-s)C][y^{\top}(T-s)^{\alpha-1} \Phi_{\alpha,\alpha}(T-s)C)]^{\top} ds = 0.$$

Therefore, $y^{\top}[(T-s)^{\alpha-1} \Phi_{\alpha,\alpha}(T-s)C] = 0$ for $s \in J$, consider the $\varphi = 0$ and the final point $x_1 = y$. Since the system is controllable there exists a control $u(t)$ on J that steers the response to $x_1 = y$ at $t = T$, that is $x(T) = y$; from $\varphi = 0$; $x(T;\varphi) = 0$ and $y^{\top} y \neq 0$ for $y \neq 0$. In other hand

$$y = x(T) = 0 + \int_0^T (T-s)^{\alpha-1} \Phi_{\alpha,\alpha}(T-s) \int_0^{\tau} \sigma(\theta)\, d(\theta) ds + \int_0^T (T-s)^{\alpha-1} \Phi_{\alpha,\alpha}(T-s) C u(s) ds$$

$$y^{\top} y = \int_0^T y^{\top} (T-s)^{\alpha-1} \Phi_{\alpha,\alpha}(T-s) \int_0^{\tau} \sigma(\theta)\, d(\theta) ds + \int_0^T (T-s)^{\alpha-1} \Phi_{\alpha,\alpha}(T-s) C u(s) ds$$

$$y^{\top} y = 0,$$

This is a contradiction for $y \neq 0$. Hence W is nonsingular. This means that control $u(t)$ steers the initial $\varphi(0)$ to the desired vector $z_1 \in R^n$ at time T.Thus system (2) is controllable on $[0, T]$.

V. Nonlinear Stochastic Delay Dynamical Systems

Consider the nonlinear fractional stochastic delay dynamical system represented by the following fractional differential equation

$$^{c}D^{\alpha} x(t) = Ax(t) + Bx(t-h) + Cu(t) + \sigma(t, x(t), x(t-h)) \frac{dw(t)}{dt} + f(t, x(t), x(t-h)),$$

$$x(t) = \varphi(t), -h < t \leq 0, \; t \in J := [0, T] \qquad (5)$$

where $0 < \alpha < 1, x(t) \in R^n$, $u \in R^m$, A and B an $n \times n$ matrix, $w(t)$ isa givenl-dimensionalWienerprocesswiththefiltration $\mathcal{F}_t$ generatedby $w(s), 0 \leq s \leq t$ and $\sigma: [0, T] \to R^{n \times l}$ is appropriatefunction and the nonlinear continuous function $f: J \times R^n \times R^n \times R^m \to R^n$.

The solution of (5) is given as

$$x(t) = x(t;\varphi) + \int_0^t (t-s)^{\alpha-1} \Phi_{\alpha,\alpha}(t-s) \left(\int_0^{\tau} \sigma(\theta, x(\theta), x(\theta-h))\, d\omega(\theta) \right) ds + \int_0^t (t-s)^{\alpha-1} \Phi_{\alpha,\alpha}(t-s) C u(s) ds + \int_0^t (t-s)^{\alpha-1} \Phi_{\alpha,\alpha}(t-s) f(s, x(s), x(s-h)) ds. \qquad (6)$$

Let $L^2_{\mathcal{F}_t}(J\times\Omega,R^n)$ be the Banach space of all $\mathcal{F}_t$- measurable square integrable processes $x(t)$with norm $\|x\|^2_{L^2}=\sup_{t\in J}E\|x(t)\|^2$, where $E(\cdot)$ denotes the expectation with respect to the measure P. Let $C=C([0,T];L^2_{\mathcal{F}_t})$ be the Banach space of continuous maps from $[0,T]$ into $L^2_{\mathcal{F}_t}(J\times\Omega,R^n)$ satisfying $\sup_{t\in J}E\|x(t)\|^2<\infty$.

(H1)f and σ satisfy Libschitz and linear growth conditions. That is, there exists some constants $L_1,L_2M_1,M_2>0$ such that

$$\|f(t,x,y)\|^2\le L_1(1+\|x\|^2+\|y\|^2)$$

$$\|\sigma(t,x,y)\|^2\le L_2(1+\|x\|^2+\|y\|^2)$$

$$\|f(t,x_1,y_1)-f(t,x_2,y_2)\|^2\le M_1(\|x_1-y_1\|^2+\|x_2-y_2\|^2)$$

$$\|\sigma(t,x_1,y_1)-\sigma(t,x_2,y_2)\|^2\le M_2(\|x_1-y_1\|^2+\|x_2-y_2\|^2).$$

(H2)$x(s)=\max\{x_1(s),x_2(s)\},\ y(s)=\max\{y_1(s),y_2(s)\}$.

For our convenience, let us introduce the following notations.

$$a_1=\max\{\|x(T;\varphi)\|^2,t\in J\},\mathrm{a}_2=\max\{\|(T-t)\|^{\alpha-1}\Phi_{\alpha,\alpha}\|x(T-t)\|^2,t,s\in J\}.$$

Theorem 5.1Assume that the hypotheses **(H1)** is satisfied and suppose that the linear fractional stochastic delay dynamical system (2) is controllable. Then the nonlinear system (5) is controllable on J.

Proof:

Firstly, we can write the control function u as

$$\mathrm{u(t)}=[(\mathrm{T-t})^{\alpha-1}\Phi_{\alpha,\alpha}(\mathrm{T-t)C})]^{\mathrm{T}}\mathrm{W}^{-1}\{\mathrm{x_1-x(T;\varphi)}-\int_0^{\mathrm{T}}(\mathrm{T-s})^{\alpha-1}\Phi_{\alpha,\alpha}(\mathrm{T-s})\int_0^{\tau}\sigma(\theta)\,\mathrm{d\omega(\theta)ds}$$
$$-\int_0^{\mathrm{T}}(\mathrm{T-s})^{\alpha-1}\Phi_{\alpha,\alpha}(\mathrm{T-s})\mathrm{f(s,x(s),x(s-h))ds}\}$$

Secondly, we define the operator $\mathcal{P}:C\to C$ by

$$\begin{aligned}\mathcal{P}(x)(T)=x(T;\varphi)&+\int_0^T(T-s)^{\alpha-1}\Phi_{\alpha,\alpha}(T-s)\left(\int_0^{\tau}\sigma(\theta)\,d\omega(\theta)\right)ds\\&+\int_0^T(T-s)^{\alpha-1}\Phi_{\alpha,\alpha}(T-s)Cds\Big[\big[(T-t)^{\alpha-1}\Phi_{\alpha,\alpha}(T-t)C\big]^{\mathrm{T}}W^{-1}\Big\{x_1-x(T;\varphi)\\&-\int_0^T(T-s)^{\alpha-1}\Phi_{\alpha,\alpha}(T-s)\int_0^{\tau}\sigma(\theta)\,d\omega(\theta)ds\\&-\int_0^T(T-s)^{\alpha-1}\Phi_{\alpha,\alpha}(T-s)f\big(s,x(s),x(s-h)\big)ds\Big]\\&+\int_0^T(T-s)^{\alpha-1}\Phi_{\alpha,\alpha}(T-s)f\big(s,x(s),x(s-h)\big)ds\Big\}.\end{aligned}$$

In order to prove thecontrollability to the system (5) it is enough to show that $\mathcal{P}$ has a fixed point in C.To do this, we can empty the contraction mapping principle. To apply the principle, first we show that $\mathcal{P}$ maps C into itself. We have

$$\begin{aligned}E\|\mathcal{P}(x)(T)\|^2\le 6a_1+6a_2E\|\int_0^T\int_0^{\tau}\sigma\big(\theta,x(\theta),x(\theta-h)\big)d\omega(\theta)ds\|^2+6E\|\int_0^T a_2C[a_2C]^{\mathrm{T}}ds\|^2\,\|W^{-1}\|\,E\|6x_1\\-6\int_0^T a_2\int_0^{\tau}\sigma(\theta)d\omega(\theta)ds\|^2-6E\|\int_0^T a_2f\big(s,x(s),x(s-h)\big)ds\|^2\\+6E\|6\int_0^T a_2f\big(s,x(s),x(s-h)\big)ds\|^2\end{aligned}$$

$$\leq 6a_1 + 6a_2E\| \int_0^T \int_0^\tau \sigma\big(\theta, x(\theta), x(\theta-h)\big)d\omega(\theta)ds\|^2 + 6E\| \int_0^T a_2C[a_2C]^\top\|^2 \|W^{-1}\| \Big\{E\|6x_1\|^2 + 6a_1 + 6a_2E\| \int_0^T \int_0^\tau \sigma\big(\theta, x(\theta), x(\theta-h)\big)d\omega(\theta)d\xi\|^2 - 6a_2E\| \int_0^T f\big(\xi, x(\xi), x(\xi-h)\big)d\xi\|^2\Big\}ds + 6a_2E\| \int_0^T f\big(s, x(s), x(s-h)\big)ds\|^2$$

$$\leq 6a_1 + 6a_2L_\sigma\| \int_0^T (\int_0^\tau E\|\sigma\big(\theta, x(\theta), x(\theta-h)\big)\|^2 d\theta)ds + 6a_2 \int_0^T E\|f\big(s, x(s), x(s-h)\big)ds\|^2 + 6a_2^2 \int_0^T E\|C\|^2\|C^\top\|^2 \|W^{-1}\| \Big\{6E\|x_1\|^2 + 6a_1 + 6a_2L_\sigma \int_0^T (\int_0^\tau E\|\sigma\big(\theta, x(\theta), x(\theta-h)\big)d\theta)\|^2 d\xi - 6a_2 \int_0^T E\|f\big(\xi, x(\xi), x(\xi-h)\big)d\xi\|^2\Big\}ds$$

Thus we have

$$E\|\mathcal{P}(x)(T)\|^2 \leq 6a_1 + 6a_2L_\sigma M_2 \int_0^T \Big[\int_0^\tau (1 + E\|x(\theta)\|^2 + E\|x(\theta-h)\|^2)d\theta\Big]ds + 6a_2L_1 \int_0^T (1 + E\|x(s)\|^2 + E\|x(s-h)\|^2)d\theta + 6a_2^2 \int_0^T E\|C\|^2\|C^\top\|^2 \|W^{-1}\| \Big\{6E\|x_1\|^2 + 6a_1 + 6a_2L_\sigma \int_0^T \int_0^\tau E\|\sigma\big(\theta, x(\theta), x(\theta-h)\big)\|^2 d\theta d\xi - 6a_2L_2 \int_0^T (1 + E\|x(\xi)\|^2 + E\|x(\xi - h)\|^2)d\xi\Big\}ds$$

Hence

$$E\|\mathcal{P}(x)(T)\|^2 \leq 6a_1 + 6a_2M_2L_\sigma\big(1 + T\|x\|_{L_2}^2 + T\|y\|_{L_2}^2\big) + 6a_2M_1\big(1 + T\|x\|_{L_2}^2 + T\|y\|_{L_2}^2\big) + 6a^2T\|C\|^2\|C^\top\|^2\|W^{-1}\|\big\{6\|x_1\|^2 + 6a_1 + 6a_2M_2L_\sigma\big(1 + T\|x\|_{L_2}^2 + T\|y\|_{L_2}^2\big) + 6a_2M_2\big(1 + T\|x\|_{L_2}^2 + T\|y\|_{L_2}^2\big)\big\}.$$

It follows from the above inequality and the condition (5) that there exists $c > 0$ such that

$$E\|\mathcal{P}(x)(T)\|^2 \leq c\big(1 + \|x\|_{L_2}^2 + \|y\|_{L_2}^2\big).$$

Therefore $\mathcal{P}$maps C into itself.

Secondly, we claim that $\mathcal{P}$ is a contraction mapping on C.For$x, y \in C$,

$$E\|\mathcal{P}(x)(T) - \mathcal{P}(y)(T)\|^2 \leq 6a_2E\| \int_0^T \int_0^\tau \big[\sigma\big(\theta, x(\theta), x(\theta-h)\big) - \sigma\big(\theta, y(\theta), y(\theta-h)\big)\big]d\omega(\theta)ds\|^2 + 6E\| \int_0^T a_2C[a_2C]^\top\|^2 \|W^{-1}\| \Big\{6a_2E\| \int_0^T \int_0^\tau [\sigma\big(\theta, x(\theta), x(\theta-h)\big) - \sigma\big(\theta, y(\theta), y(\theta-h)\big)]d\omega(\theta)d\xi\|^2 - 6a_2E\| \int_0^T [f\big(\xi, x(\xi), x(\xi-h)\big) - f\big(\xi, y(\xi), y(\xi-h)\big)]d\xi\|^2\Big\}ds + 6a_2E\| \int_0^T [f\big(s, x(s), x(s-h)\big) - f\big(s, y(s), y(s-h)\big)]ds\|^2$$

$$\leq 6a_2L_\sigma\int_0^T\Big(\int_0^\tau E\|\sigma\big(\theta,x(\theta),x(\theta-h)\big)-\sigma\big(\theta,y(\theta),y(\theta-h)\big)\|^2d\theta\Big)ds$$
$$+6a_2\int_0^T E\|f\big(s,x(s),x(s-h)\big)-f\big(s,y(s),y(s-h)\big)\|^2ds$$
$$+6a_2^2\int_0^T\|C\|^2\|C^\top\|^2\,\|W^{-1}\|\left\{6a_2L_2\int_0^T\Big(\int_0^\tau E\|\sigma\big(\theta,x(\theta),x(\theta-h)\big)\right.$$
$$\left.-\sigma\big(\theta,y(\theta),y(\theta-h)\big)\|^2d\theta\Big)d\xi\right.$$
$$\left.+6a_2\int_0^T E\|f\big(\xi,x(\xi),x(\xi-h)\big)-f\big(\xi,y(\xi),y(\xi-h)\big)\|^2d\xi\right\}ds$$

$$\leq 6a_2L_\sigma\int_0^T\left[\int_0^\tau E(\|x_1(s)-y_1(s)\|^2+\|x_2(s)-y_2(s)\|^2)ds\right]d\xi$$
$$+6a_2\int_0^T E(\|x_1(s)-y_1(s)\|^2+\|x_2(s)-y_2(s)\|^2)ds$$
$$+6a_2^2\int_0^T\|C\|^2\|C^\top\|^2\,\|W^{-1}\|\left\{6a_2L_2\int_0^T\left[\int_0^\tau E(\|x_1(s)-y_1(s)\|^2+\|x_2(s)-y_2(s)\|^2)ds\right]d\xi\right.$$
$$\left.+6a_2\int_0^T E(\|x_1(s)-y_1(s)\|^2+\|x_2(s)-y_2(s)\|^2)ds\right\}d\xi$$
$$\leq 6a_2M_2L_\sigma TE(\|x_1(s)-y_1(s)\|^2+\|x_2(s)-y_2(s)\|^2)+6a_2M_1TE(\|x_1(s)-y_1(s)\|^2+\|x_2(s)-y_2(s)\|^2)$$
$$+6\mathrm{a}^2\mathrm{T}\|\mathrm{C}\|^2\|\mathrm{C}^\top\|^2\|\mathrm{W}^{-1}\|\{6\mathrm{a}_2\mathrm{M}_2\mathrm{L}_\sigma TE(\|x_1(s)-y_1(s)\|^2+\|x_2(s)-y_2(s)\|^2)$$
$$+6\mathrm{a}_2\mathrm{M}_2TE(\|x_1(s)-y_1(s)\|^2+\|x_2(s)-y_2(s)\|^2)\}$$

It results that by **(H2)**

$$\leq[6a_2M_2L_\sigma+6a_2M_1+36a^2a_2M_2T\|C\|^2\|C^\top\|^2\|W^{-1}\|(1+L_\sigma)]T\sup_{t\in[0,T]}E(\|x(s)-y(s)\|^2)$$

Therefore, we conclude that if $[6a_2M_2L_\sigma+6a_2M_1+36a^2a_2M_2T\|C\|^2\|C^\top\|^2\|W^{-1}\|(1+L_\sigma)]T$ which $<1,\mathcal{P}$ is contradiction on mapping on $C\to C$ that $\mathcal{P}$
has unique fixed point. Hence we have

$$x(T)=x(T;\varphi)+\int_0^T(T-s)^{\alpha-1}\Phi_{\alpha,\alpha}(T-s)\left(\int_0^\tau\sigma(\theta)\,d\omega(\theta)\right)ds$$
$$+\int_0^T(T-s)^{\alpha-1}\Phi_{\alpha,\alpha}(T-s)Cds\left[\left[(T-t)^{\alpha-1}\Phi_{\alpha,\alpha}(T-t)C\right]^\top W^{-1}\left\{x_1-x(T;\varphi)\right.\right.$$
$$-\int_0^T(T-s)^{\alpha-1}\Phi_{\alpha,\alpha}(T-s)\int_0^\tau\sigma(\theta)\,d\omega(\theta)ds$$
$$\left.-\int_0^T(T-s)^{\alpha-1}\Phi_{\alpha,\alpha}(T-s)f\big(s,x(s),x(s-h)\big)ds\right]$$
$$\left.+\int_0^T(T-s)^{\alpha-1}\Phi_{\alpha,\alpha}(T-s)f\big(s,x(s),x(s-h)\big)ds\right\}.$$

Thus $x(t)$ is the solution of the system (5), and it is easy to verify that $x(T)=x_1$. Further the control function $u(t)$ steers (5) from the initial complete state $\varphi(0)$ to x_1 on J. Hence the system (5) is controllable on J.

VI. Example

In this section, example has provided to demonstrate the controllability results for proposed criteria.

Example 6.1

The problem of nonlinear fractional stochastic delay dynamical systems as follows

$${}^{C}D^{\frac{1}{2}}x(t)=\begin{pmatrix}0&1\\0&0\end{pmatrix}x(t)+\begin{pmatrix}3&1\\1&2\end{pmatrix}x(t-1)+\begin{pmatrix}2\\1\end{pmatrix}u(t)+\begin{pmatrix}(2t^2+1)x_1(t)e^{-t}&0\\0&x_2(t-1)e^{-t}\end{pmatrix}\frac{dw(t)}{dt}$$
$$+\begin{pmatrix}\dfrac{x_1(t-1)}{x_1^2(t)+x_1^2(t-1)}\\[2ex]\dfrac{x_2(t-1)}{x_2^2(t)+x_2^2(t-1)}\end{pmatrix}$$

$$x(0) = \varphi(0), \qquad (7)$$

where $\alpha = \frac{1}{2}, A = \begin{pmatrix} 0 & -2 \\ 3 & 0 \end{pmatrix}, B = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}, C = \begin{pmatrix} 2 \\ 1 \end{pmatrix},$

$$\sigma\big(t, x(t), x(t-1)\big) = \begin{pmatrix} (2t^2+1)x_1(t)e^{-t} & 0 \\ 0 & x_2(t-1)e^{-t} \end{pmatrix},$$

$$f\big(t, x(t), x(t-h)\big) = \begin{pmatrix} \dfrac{x_1(t-1)}{x_1^2(t) + x_1^2(t-1)} \\ \dfrac{x_2(t-1)}{x_2^2(t) + x_2^2(t-1)} \end{pmatrix}.$$

Now, we have to apply Theorems 4.1 and 5.1 to prove that system (7) is controllable on [0,3].
For t ∈ [0,3], solution (7),by utilizing Laplace and inverse Laplace transform technique we have,

$$\begin{aligned} x(t) = \Phi_{\frac{1}{2}}\left(At^{\frac{1}{2}}\right) &+ \int_{-1}^{0} (t-s-1)^{-\frac{1}{2}} \Phi_{\frac{1}{2},\frac{1}{2}}\left(A(t-s-1)^{\frac{1}{2}}\right) ds \\ &+ \int_0^t (t-s)^{-\frac{1}{2}} \Phi_{\frac{1}{2},\frac{1}{2}}\left(A(t-s)^{\frac{1}{2}}\right) \left(\int_0^{\tau} \sigma(\theta, x(\theta), x(\theta-1))\, d\omega(\theta) \right) ds \\ &+ \int_0^t (t-s)^{-\frac{1}{2}} \Phi_{\frac{1}{2},\frac{1}{2}}\left(A(t-s)^{\frac{1}{2}}\right) Cu(s) ds \\ &+ \int_0^t (t-s)^{-\frac{1}{2}} \Phi_{\frac{1}{2},\frac{1}{2}}\left(A(t-s)^{\frac{1}{2}}\right) f\big(s, x(s), x(s-1)\big) ds, \end{aligned}$$

and further simplification

$$\begin{aligned} x(t) = \Phi_{\frac{1}{2}}\left(At^{\frac{1}{2}}\right) &+ Bt^{\frac{1}{2}} \Phi_{\frac{1}{2},\frac{3}{2}}\left(At^{\frac{1}{2}}\right) + \int_0^t (t-s)^{-\frac{1}{2}} \Phi_{\frac{1}{2},\frac{1}{2}}\left(A(t-s)^{\frac{1}{2}}\right) \left(\int_0^{\tau} \sigma(\theta, x(\theta), x(\theta-1))\, d\omega(\theta) \right) ds \\ &+ \int_0^t (t-s)^{-\frac{1}{2}} \Phi_{\frac{1}{2},\frac{1}{2}}\left(A(t-s)^{\frac{1}{2}}\right) Cu(s) ds \\ &+ \int_0^t (t-s)^{-\frac{1}{2}} \Phi_{\frac{1}{2},\frac{1}{2}}\left(A(t-s)^{\frac{1}{2}}\right) f\big(s, x(s), x(s-1)\big) ds. \end{aligned}$$

The controllability Grammian matrix is defined by

$$W = \int_0^3 (3-s)^{-\frac{1}{2}} \Phi_{\frac{1}{2},\frac{1}{2}}\left(A(3-s)^{\frac{1}{2}}\right) CC^{\top} (3-s)^{-\frac{1}{2}} \Phi_{\frac{1}{2},\frac{1}{2}}\left(A^{\top}(3-s)^{\frac{1}{2}}\right) ds.$$

The Mittag-LefflerFunctions of the matrices are given by

$$\Phi_{\frac{1}{2},\frac{1}{2}}\left(A(3-s)^{\frac{1}{2}}\right) = \sum_{k=0}^{1} \frac{A^k (3-s)^{\frac{k}{2}}}{\Gamma\left(\frac{1}{2}(k+1)\right)} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \frac{(3-s)}{\sqrt{\pi}} + \begin{bmatrix} 0 & -2 \\ 3 & 0 \end{bmatrix} \frac{(3-s)}{1}$$

$$= \begin{bmatrix} \dfrac{3-s}{\sqrt{\pi}} & -2(3-s) \\ 3(3-s) & \dfrac{3-s}{\sqrt{\pi}} \end{bmatrix},$$

and

$$\Phi_{\frac{1}{2},\frac{1}{2}}\left(A^{\top}(3-s)^{\frac{1}{2}}\right) = \sum_{k=0}^{1} \frac{(A^{\top})^k (3-s)^{\frac{k}{2}}}{\Gamma\left(\frac{1}{2}(k+1)\right)} = \begin{bmatrix} \dfrac{3-s}{\sqrt{\pi}} & 3(3-s) \\ -2(3-s) & \dfrac{3-s}{\sqrt{\pi}} \end{bmatrix}, \quad CC^{\top} = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix},$$

$$W = \int_0^3 (3-s)^{-\frac{1}{2}} \begin{bmatrix} \dfrac{3-s}{\sqrt{\pi}} & -2(3-s) \\ 3(3-s) & \dfrac{3-s}{\sqrt{\pi}} \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} (3-s)^{-\frac{1}{2}} \begin{bmatrix} \dfrac{3-s}{\sqrt{\pi}} & 3(3-s) \\ -2(3-s) & \dfrac{3-s}{\sqrt{\pi}} \end{bmatrix} ds,$$

and evaluating, we get

$$W = \begin{pmatrix} 3.4188 & -25.7467 \\ -25.7467 & 193.8986 \end{pmatrix},$$

which is positive definite. Further the function $f(t, x(t), x(t-h))$ and $\sigma(t, x(t), x(t-h))$ satisfies the hypothesis mentioned in Theorem 5.1 and hence fractional system (7) is controllable on $[0,3]$.

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Oscillatory Behavior of Fractional Order Difference Equations with Damping

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Abstract: *In this paper, we investigates oscillatory behavior of fractional order difference equations with damping term of the form*

$$\Delta\left[r(t)\psi(u(t))\Delta^{\alpha}u(t)\right]+p(t)\psi(u(t))\Delta^{\alpha}u(t)+F[t,G(t)]=0, t\geq t_0>0$$

where $G(t)=\sum_{s=t_0}^{t-1+\alpha}(t-s-1)^{-\alpha}u(s)$ *and* Δ^{α} *denotes the Riemann-Liouville difference operator of order* $0<\alpha\leq 1$. *We establish some sufficient conditions of oscillation criteria for fractional order difference equations with damping term based on a generalized riccati transformation technique and some inequalities. Also apply the double sequence technique to illustrate our established results.*

Keywords: *Oscillation, Fractional Order Difference Equations, Riccati technique, Damping.*

I. Introduction

Recent years have witnessed the study of qualitative properties, especially oscillation of solutions, of fractional difference equations, [3, 4]. In this paper, we establish oscillatory behavior of fractional order difference equations with damping term of the form [2]

$$\Delta\left[r(t)\psi(u(t))\Delta^{\alpha}u(t)\right]+p(t)\psi(u(t))\Delta^{\alpha}u(t)+F[t,G(t)]=0, t\geq t_0>0 \quad (1)$$

where $G(t)=\sum_{s=t_0}^{t-1+\alpha}(t-s-1)^{-\alpha}u(s)$ and Δ^{α} denotes the Riemann-Liouville difference operator of order $0<\alpha\leq 1$. The following assumption to be hold throughout this paper:

[A1] $p(t)$ is positive sequences on $t_0>0$ and $r(t)\in C([t_0,\infty),R)$.

[A2] $r(t)\in C'([t_0,\infty),R_+)$ such that $r(t)\leq k$ for some $k>0$.

[A3] $0<\psi(u(t))\leq m$ for some positive constant m and for all $u\neq 0$.

[A4] $F[t,G(t)]\geq 0$ such that $\frac{F[t,G(t)]}{G(t)}\geq q(t)$, where $q(t)\geq 0$ and $G(t)\neq 0, t\geq t_0$.

A solution $u(t)$ of (1) is said to be oscillatory if it is neither eventually positive nor eventually negative; otherwise, it is nonoscillatory. Equation (1) is said to be oscillatory if all its solutions are oscillatory.

II. Preliminaries And Basic Lemmas

In this section, we introduce some preliminary results of discrete fractional calculus, which will be used throughout this paper.

Definition 1 [1] Let $\nu>0$. The ν-th fractional sum f is defined by

$$\Delta^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \sum_{s=0}^{t-\nu} (t-s-1)^{\nu-1} f(s)$$

Where f is defined for $s \equiv a \bmod(1)$ and $\Delta^{-\nu} f$ is defined for $t \equiv (a+\nu) \bmod(1)$ and $t^{(\nu)} = \frac{\Gamma(t+1)}{\Gamma(t-\nu+1)}$. The fractional sum $\Delta^{-\nu} f : N_a \to N_{a+\nu}$.

Definition 2 [1] Let $\mu > 0$ and $m-1 < \mu < m$ where m denotes a positive integer $m = \lceil \mu \rceil$. Set $\nu = m - \mu$. The μ -th fractional difference is defined as

$$\Delta^{\mu} f(t) = \Delta^{m-\nu} f(t) = \Delta^{m} \Delta^{-\nu} f(t)$$

Lemma 3 [5] Let $u(t)$ be a solution of (1) and $G(t) = \sum_{s=t_0}^{t-1+\alpha} (t-s-1)^{-\alpha} u(s)$, then

$$\Delta G(t) = \Gamma(1-\alpha) \Delta^{\alpha} u(t)$$

Proof.

$$\begin{aligned} G(t) &= \sum_{s=t_0}^{t-1+\alpha} (t-s-1)^{-\alpha} u(s) = \sum_{s=t_0}^{t-(1-\alpha)} (t-s-1)^{(1-\alpha)-1} u(s) \\ &= \Gamma(1-\alpha) \Delta^{-(1-\alpha)} u(t) \\ \Delta G(t) &= \Gamma(1-\alpha) \Delta \Delta^{-1} \Delta^{\alpha} u(t) \\ \Delta G(t) &= \Gamma(1-\alpha) \Delta^{\alpha} u(t) \end{aligned}$$

III. Main Results

In this section, we establish some sufficient conditions of oscillation criteria results and some inequalities.

Theorem 1 Assume that $(A_1)-(A_4)$ hold and there exists a positive function $g(t), t \geq t_0$, such that

$$\lim_{t\to\infty} \left(\frac{\Gamma(1-\alpha)}{mk} \right)^{1/2} \sum_{s=t_2}^{t-1} \frac{1}{g(s)} = \infty \tag{2}$$

and

$$\lim_{t\to\infty} \left\{ \left(\frac{m}{16k\Gamma(1-\alpha)} \right)^{1/2} \sum_{s=t_1}^{t-1} \left[\frac{p^2(s)g(s)}{k} + \frac{k\Delta^2 g(s)}{g(s)} - 2p(s)\Delta g(s) - \frac{4\Gamma(1-\alpha)}{m} g(s)q(s) \right] + \left(\frac{mk}{4\Gamma(1-\alpha)} \right)^{1/2} \Delta g(t) \right\} = \infty. \tag{3}$$

Then every solution of (1) is Oscillatory.

Proof. Suppose that equation (1) is Non oscillatory solution $u(t)$ on $t \geq t_0$. Without loss of generality, we assume that $u(t)$ is an eventually positive of equation (1), then $u(t) > 0$ and $G(t) > 0$ for $t > t_1 > t_0$. Let us define the Recatti substitution

$$\omega(t) = -g(t) \frac{r(t)\psi(u(t))\Delta^{\alpha} u(t)}{G(t)} \quad \text{fort} \geq t_1 \tag{4}$$

$$\begin{aligned} \Delta\omega(t) &= -\Delta \left[g(t) \frac{r(t)\psi(u(t))\Delta^{\alpha} u(t)}{G(t)} \right] \\ &= -\left[g(t+1)\Delta \left[\frac{r(t)\psi(u(t))\Delta^{\alpha} u(t)}{G(t)} \right] + \frac{r(t)\psi(u(t))\Delta^{\alpha} u(t)}{G(t)} \Delta g(t) \right] \\ &= -\Delta g(t) \frac{r(t)\psi(u(t))\Delta^{\alpha} u(t)}{G(t)} - g(t+1) \left[\frac{G(t)\Delta[r(t)\psi(u(t))\Delta^{\alpha} u(t)]}{G(t)G(t+1)} - \frac{r(t)\psi(u(t))\Delta^{\alpha} u(t)}{G(t)G(t+1)} \Delta G(t) \right] \end{aligned}$$

Now applying (1), (4) and Lemma(3), we obtain

$$\Delta\omega(t) = \Delta g(t)\frac{\omega(t)}{g(t)} - \frac{g(t+1)}{G(t+1)}\left[-p(t)\psi(u(t))\Delta^{\alpha}u(t) - F[t,G(t)]\right] + \frac{g(t+1)}{G(t)G(t+1)}\frac{\Gamma(1-\alpha)\left[r(t)\psi(u(t))\Delta^{\alpha}u(t)\right]^2}{r(t)\psi(u(t))}$$

Since $g(t) > 0$ and $\Delta G(t)$ is Non increasing function, we have

$$\Delta\omega(t) > \Delta g(t)\frac{\omega(t)}{g(t)} + \frac{g(t)}{G(t)}p(t)\psi(u(t))\Delta^{\alpha}u(t) + \frac{g(t)}{G(t)}F[t,G(t)] + \frac{g(t)}{\left(G(t)\right)^2}\frac{\Gamma(1-\alpha)\left[r(t)\psi(u(t))\Delta^{\alpha}u(t)\right]^2}{r(t)\psi(u(t))}$$

$$\Delta\omega(t) > \Delta g(t)\frac{\omega(t)}{g(t)} + \frac{g(t)p(t)\psi(u(t))\Delta^{\alpha}u(t)}{G(t)} + \frac{g(t)F[t,G(t)]}{G(t)} + \frac{\Gamma(1-\alpha)}{g(t)r(t)\psi(u(t))}\frac{\left[g(t)r(t)\psi(u(t))\Delta^{\alpha}u(t)\right]^2}{\left(G(t)\right)^2}$$

$$\Delta\omega(t) > \Delta g(t)\frac{\omega(t)}{g(t)} - p(t)\frac{\omega(t)}{r(t)} + g(t)q(t) + \frac{\Gamma(1-\alpha)}{g(t)r(t)\psi(u(t))}\omega^2(t) \quad (5)$$

Using (A_3), we get

$$\Delta\omega(t) > \frac{1}{g(t)}\left[\frac{\Gamma(1-\alpha)}{mr(t)}\omega^2(t) - \frac{g(t)p(t)\omega(t)}{r(t)} + \Delta g(t)\omega(t)\right] + g(t)q(t)$$

$$> \frac{1}{g(t)}\left\{\left[\left(\frac{\Gamma(1-\alpha)}{mr(t)}\right)^{1/2}\omega(t) + \frac{1}{2}\left(\frac{mr(t)}{\Gamma(1-\alpha)}\right)^{1/2}\Delta g(t)\right]^2 - \frac{g(t)p(t)\omega(t)}{r(t)}\right\}$$

$$-\frac{1}{g(t)}\frac{mr(t)\Delta^2 g(t)}{4\Gamma(1-\alpha)} + g(t)q(t)$$

Now taking $L_2(t) = \left(\frac{\Gamma(1-\alpha)}{mr(t)}\right)^{1/2}\omega(t) + \frac{1}{2}\left(\frac{mr(t)}{\Gamma(1-\alpha)}\right)^{1/2}\Delta g(t)$, we get

$$\Delta\omega(t) > \frac{1}{g(t)}\left[L_2^2(t) - \frac{1}{4}\frac{mr(t)}{\Gamma(1-\alpha)}\Delta^2 g(t) - \frac{g(t)p(t)\omega(t)}{r(t)}\right] + g(t)q(t) \quad (6)$$

Now,

$$L_2(t) - \frac{1}{2}\left(\frac{mr(t)}{\Gamma(1-\alpha)}\right)^{1/2}\Delta g(t) = \left(\frac{\Gamma(1-\alpha)}{mr(t)}\right)^{1/2}\omega(t)$$

$$\left(\frac{mr(t)}{\Gamma(1-\alpha)}\right)^{1/2}L_2(t) - \frac{1}{2}\frac{mr(t)}{\Gamma(1-\alpha)}\Delta g(t) = \omega(t)$$

$$\frac{g(t)p(t)}{r(t)}\left(\frac{mr(t)}{\Gamma(1-\alpha)}\right)^{1/2}L_2(t) - \frac{g(t)p(t)}{r(t)}\frac{1}{2}\frac{mr(t)}{\Gamma(1-\alpha)}\Delta g(t) = \frac{g(t)p(t)}{r(t)}\omega(t) \quad (7)$$

Using (7) in (6), we obtain

$$\Delta\omega(t) \geq \frac{1}{g(t)}\left[L_2^2(t) - \frac{g(t)p(t)}{r(t)}\left(\frac{mr(t)}{\Gamma(1-\alpha)}\right)^{1/2} L_2(t) + \frac{g(t)p(t)mr(t)}{2r(t)\Gamma(1-\alpha)}\Delta g(t)\right]$$
$$-\frac{1}{g(t)}\frac{mr(t)}{4\Gamma(1-\alpha)}\Delta^2 g(t) + g(t)q(t)$$

$$\Delta\omega(t) > \frac{1}{g(t)}\left[L_2^2(t) - 2\frac{1}{2}\frac{g(t)p(t)}{r(t)}\left(\frac{mr(t)}{\Gamma(1-\alpha)}\right)^{1/2} L_2(t) + \frac{g(t)p(t)m}{2\Gamma(1-\alpha)}\Delta g(t)\right] + \frac{1}{g(t)}$$
$$\left[\left[\frac{1}{2}\left(\frac{m}{r(t)\Gamma(1-\alpha)}\right)^{1/2} g(t)p(t)\right]^2 - \frac{mg^2(t)p^2(t)}{4r(t)\Gamma(1-\alpha)} - \frac{mr(t)\Delta^2 g(t)}{4\Gamma(1-\alpha)}\right] + g(t)q(t)$$

$$= \frac{1}{g(t)}\left[L_2(t) - \frac{1}{2}g(t)p(t)\left(\frac{m}{r(t)\Gamma(1-\alpha)}\right)^{1/2}\right]^2 - \frac{1}{g(t)}\frac{mg^2(t)p^2(t)}{4r(t)\Gamma(1-\alpha)}$$
$$-\frac{1}{g(t)}\frac{mr(t)\Delta^2 g(t)}{4\Gamma(1-\alpha)} + \frac{1}{g(t)}\frac{g(t)p(t)m}{2\Gamma(1-\alpha)}\Delta g(t) + g(t)q(t)$$

$$> \frac{1}{g(t)}\left[L_2(t) - \frac{1}{2}g(t)p(t)\left(\frac{m}{r(t)\Gamma(1-\alpha)}\right)^{1/2}\right]^2$$
$$-\frac{m}{4\Gamma(1-\alpha)}\left[\frac{p^2(t)g(t)}{r(t)} + \frac{r(t)\Delta^2 g(t)}{g(t)} - 2p(t)\Delta g(t) - \frac{4\Gamma(1-\alpha)}{m}g(t)q(t)\right]$$

Let us taking summation from t_1 to $t-1$, we have

$$\sum_{s=t_1}^{t-1}\Delta\omega(s) > \sum_{s=t_1}^{t-1}\frac{1}{g(s)}\left[L_2(s) - \frac{1}{2}g(s)p(s)\left(\frac{m}{r(s)\Gamma(1-\alpha)}\right)^{1/2}\right]^2 - \frac{m}{4\Gamma(1-\alpha)}$$
$$\sum_{s=t_1}^{t-1}\left[\frac{p^2(s)g(s)}{r(s)} + \frac{r(s)\Delta^2 g(s)}{g(s)} - 2p(s)\Delta g(s) - \frac{4\Gamma(1-\alpha)}{m}g(s)q(s)\right]$$

Now apply (A_2), which implies that

$$\omega(t) - \omega(t_1) > \sum_{s=t_1}^{t-1}\frac{1}{g(s)}\left[L_2(s) - \frac{1}{2}g(s)p(s)\left(\frac{m}{k\Gamma(1-\alpha)}\right)^{1/2}\right]^2 - \frac{m}{4\Gamma(1-\alpha)}$$
$$\sum_{s=t_1}^{t-1}\left[\frac{p^2(s)g(s)}{k} + \frac{k\Delta^2 g(s)}{g(s)} - 2p(s)\Delta g(s) - \frac{4\Gamma(1-\alpha)}{m}g(s)q(s)\right]$$

$$\omega(t) > \omega(t_1) + \sum_{s=t_1}^{t-1}\frac{1}{g(s)}\left[L_2(s) - \frac{1}{2}g(s)p(s)\left(\frac{m}{k\Gamma(1-\alpha)}\right)^{1/2}\right]^2 - \frac{m}{4\Gamma(1-\alpha)}$$
$$\sum_{s=t_1}^{t-1}\left[\frac{p^2(s)g(s)}{k} + \frac{k\Delta^2 g(s)}{g(s)} - 2p(s)\Delta g(s) - \frac{4\Gamma(1-\alpha)}{m}g(s)q(s)\right]$$

Applying (A_2) and Again consider $L_2(t) = \left(\frac{\Gamma(1-\alpha)}{mr(t)}\right)^{1/2}\omega(t) + \frac{1}{2}\left(\frac{mr(t)}{\Gamma(1-\alpha)}\right)^{1/2}\Delta g(t)$ and using the above equation, we have

$$L_2(t) > -\left(\frac{\Gamma(1-\alpha)}{mk}\right)^{1/2} \frac{m}{4\Gamma(1-\alpha)} \sum_{s=t_1}^{t-1}\left[\frac{p^2(s)g(s)}{k} + \frac{k\Delta^2 g(s)}{g(s)} - 2p(s)\Delta g(s) - \frac{4\Gamma(1-\alpha)}{m} g(s)q(s)\right]$$

$$\left(\frac{\Gamma(1-\alpha)}{mk}\right)^{1/2} \omega(t_1) + \left(\frac{\Gamma(1-\alpha)}{mk}\right)^{1/2} \sum_{s=t_1}^{t-1} \frac{1}{g(s)}\left[L_2(s) - \frac{1}{2} g(s)p(s)\left(\frac{m}{k\Gamma(1-\alpha)}\right)^{1/2}\right]^2 + \frac{1}{2}\left(\frac{mk}{\Gamma(1-\alpha)}\right)^{1/2} \Delta g(t)$$

$$> \left(\frac{\Gamma(1-\alpha)}{mk)}\right)^{1/2} \omega(t_1) + \left(\frac{\Gamma(1-\alpha)}{mk}\right)^{1/2} \sum_{s=t_1}^{t-1} \frac{1}{g(s)}\left[L_2(s) - \frac{1}{2} g(s)p(s)\left(\frac{m}{k\Gamma(1-\alpha)}\right)^{1/2}\right]^2$$

$$-\frac{1}{4}\left(\frac{m}{k\Gamma(1-\alpha)}\right)^{1/2} \sum_{s=t_1}^{t-1}\left[\frac{p^2(s)g(s)}{k} + \frac{k\Delta^2 g(s)}{g(s)} - 2p(s)\Delta g(s) - \frac{4\Gamma(1-\alpha)}{m} g(s)q(s)\right] + \frac{1}{2}\left(\frac{mk}{\Gamma(1-\alpha)}\right)^{1/2} \Delta g(t)$$

Using (3) and there exists $t_2 \geq t_1$, we obtain

$$L_2(t) > \left(\frac{\Gamma(1-\alpha)}{mk}\right)^{1/2} \sum_{s=t_1}^{t-1} \frac{1}{g(s)}\left[L_2(s) - \frac{1}{2} g(s)p(s)\left(\frac{m}{k\Gamma(1-\alpha)}\right)^{1/2}\right]^2$$

Let $Q(t) = \left(\frac{\Gamma(1-\alpha)}{mk}\right)^{1/2} \sum_{s=t_1}^{t-1} \frac{1}{g(s)}\left[L_2(s) - \frac{1}{2} g(s)p(s)\left(\frac{m}{k\Gamma(1-\alpha)}\right)^{1/2}\right]^2$, which implies that $L_2(t) > Q(t) > 0$. Now

$$\Delta Q(t) \geq \left(\frac{\Gamma(1-\alpha)}{mk}\right)^{1/2} \frac{1}{g(t)}\left[L_2(t) - \frac{1}{2} g(t)p(t)\left(\frac{m}{k\Gamma(1-\alpha)}\right)^{1/2}\right]^2$$

$$> \left(\frac{\Gamma(1-\alpha)}{mk}\right)^{1/2} \frac{1}{g(t)} Q^2(t) \quad \text{since,p(t)<0}$$

$$\frac{\Delta Q(t)}{Q^2(t)} > \left(\frac{\Gamma(1-\alpha)}{mk}\right)^{1/2} \frac{1}{g(t)}$$

$$\left(\frac{\Gamma(1-\alpha)}{mk}\right)^{1/2} \frac{1}{g(t)} < \frac{\Delta Q(t)}{Q^2(t)} < \frac{Q(t+1)}{Q^2(t)} - \frac{Q(t)}{Q^2(t)}$$

Since, $\Delta Q(t)$ is non decreasing sequence, we have

$$\left(\frac{\Gamma(1-\alpha)}{mk}\right)^{1/2} \frac{1}{g(t)} < \frac{Q(t+1)}{Q^2(t+1)} - \frac{Q(t)}{Q^2(t)} < \frac{1}{Q(t+1)} - \frac{1}{Q(t)}$$

Let us taking summation from t_2 to $t-1$, we get

$$\left(\frac{\Gamma(1-\alpha)}{mk}\right)^{1/2} \sum_{s=t_2}^{t-1} \frac{1}{g(s)} < \sum_{s=t_2}^{t-1}\left[\frac{1}{Q(s+1)} - \frac{1}{Q(s)}\right]$$

$$< \frac{1}{Q(t)} - \frac{1}{Q(t_2)} < \frac{1}{Q(t_2)}$$

Now taking limit as $t \to \infty$, we obtain

$$\lim_{t\to\infty}\left(\frac{\Gamma(1-\alpha)}{mk}\right)^{1/2} \sum_{s=t_2}^{t-1} \frac{1}{g(s)} < \frac{1}{Q(t_2)}$$

Which contradicted to (2), hence complete the proof.

Theorem 2 Suppose that (A_1) - (A_4) holds; there exists a positive function $g(t)$ and a double positive sequence $H(t,s)$ such that,

$$H(t,t)=0 \text{ for } t>t_0 \ \& \ H(t,s)>0 \text{ for } t>s>t_0$$
$$\Delta_s H(t,s)=H(t,s+1)-H(t,s)\le 0 \text{ for } t>s>t_0$$

if

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\sum_{s=t_0}^{t-1}\left[q(s)g(s)H(t,s)-\frac{h_+^2(t,s)mkg(s)}{4\Gamma(1-\alpha)H(t,s)}\right]=\infty \qquad (8)$$

where $h_+(t,s)=\Delta_2 H(t,s)+\frac{\Delta g(s)}{g(s)}H(t,s)-\frac{p(s)}{k}H(t,s)$, then every solution of (1) is Oscillatory.

Proof. Suppose to the contrary that $u(t)$ is a non-oscillatory solution of (1). Without loss of generality, we can assume that $u(t)$ is an eventually positive solution of (1). Proceeding as theorem – 1 and apply assumptions $(A_2)\&(A_3)$, we arrive at the equation (5),

$$\begin{aligned}\Delta\omega(t) &> \Delta g(t)\frac{\omega(t)}{g(t)}-p(t)\frac{\omega(t)}{r(t)}+g(t)q(t)+\frac{\Gamma(1-\alpha)}{g(t)r(t)\psi(u(t))}\omega^2(t)\\ &> \frac{\Gamma(1-\alpha)}{g(t)mk}\omega^2(t)+\Delta g(t)\frac{\omega(t)}{g(t)}-p(t)\frac{\omega(t)}{k}+g(t)q(t)\end{aligned}$$

Let us multiply by $H(t,s)$ and taking summation from t_1 to $t-1$, we get

$$\sum_{s=t_1}^{t-1}\Delta\omega(s)H(t,s)>\sum_{s=t_1}^{t-1}\left[\frac{\Gamma(1-\alpha)}{g(s)mk}\omega^2(s)+\Delta g(s)\frac{\omega(s)}{g(s)}-p(s)\frac{\omega(s)}{k}+g(s)q(s)\right]H(t,s)$$

$$\begin{aligned}-\sum_{s=t_1}^{t-1} g(s)q(s)H(t,s) &> -\sum_{s=t_1}^{t-1}\Delta\omega(s)H(t,s)\\ &+\sum_{s=t_1}^{t-1}\left[\frac{\Gamma(1-\alpha)}{g(s)mk}H(t,s)\omega^2(s)+\Delta g(s)\frac{\omega(s)}{g(s)}H(t,s)-p(s)\frac{\omega(s)}{k}H(t,s)\right]\end{aligned} \qquad (9)$$

Now using summation by parts formula, we obtain

$$-\sum_{s=t_1}^{t-1}\Delta\omega(s)H(t,s)=-\left[H(t,s)\omega(s)\right]_{s=t_1}^{t}+\sum_{s=t_1}^{t-1}\omega(s+1)\Delta_2 H(t,s)$$

$$-\sum_{s=t_1}^{t-1}\Delta\omega(s)H(t,s)=H(t,t_1)\omega(t_1)+\sum_{s=t_1}^{t-1}\omega(s+1)\Delta_2 H(t,s) \qquad (10)$$

Using (10) in (9), we get

$$\begin{aligned}-\sum_{s=t_1}^{t-1} g(s)q(s)H(t,s) &> H(t,t_1)\omega(t_1)+\sum_{s=t_1}^{t-1}\omega(s+1)\Delta_2 H(t,s)\\ &+\sum_{s=t_1}^{t-1}\left[\frac{\Gamma(1-\alpha)}{g(s)mk}H(t,s)\omega^2(s)+\Delta g(s)\frac{\omega(s)}{g(s)}H(t,s)-p(s)\frac{\omega(s)}{k}H(t,s)\right]\end{aligned}$$

Since, $\Delta\omega(t)>0$, which implies that $\omega(t+1)>\omega(t)$, we obtain

$$-\sum_{s=t_1}^{t-1} g(s)q(s)H(t,s) > H(t,t_1)\omega(t_1) + \sum_{s=t_1}^{t-1} \frac{\Gamma(1-\alpha)}{g(s)mk} H(t,s)\omega^2(s)$$
$$+ \sum_{s=t_1}^{t-1} \left[\Delta_2 H(t,s) + \frac{\Delta g(s)}{g(s)} H(t,s) - \frac{p(s)}{k} H(t,s) \right] \omega(s)$$

Let us consider $h_+(t,s) = \Delta_2 H(t,s) + \frac{\Delta g(s)}{g(s)} H(t,s) - \frac{p(s)}{k} H(t,s)$

$$-\sum_{s=t_1}^{t-1} g(s)q(s)H(t,s) > H(t,t_1)\omega(t_1) + \sum_{s=t_1}^{t-1} \left[h_+(t,s)\omega(s) + \frac{\Gamma(1-\alpha)}{g(s)mk} H(t,s)\omega^2(s) \right] \quad (11)$$

Now,

$$h_+(t,s)\omega(s) + \frac{\Gamma(1-\alpha)}{g(s)mk} H(t,s)\omega^2(s) = -\frac{h_+^2(t,s)mkg(s)}{4\Gamma(1-\alpha)H(t,s)}$$
$$+ \left[\sqrt{\frac{\Gamma(1-\alpha)H(t,s)}{mkg(s)}}\omega(s) + \frac{h_+(t,s)}{2}\sqrt{\frac{mkg(s)}{\Gamma(1-\alpha)H(t,s)}} \right]^2$$
$$> -\frac{h_+^2(t,s)mkg(s)}{4\Gamma(1-\alpha)H(t,s)}$$

Since, $\Delta_2 H(t,s) \le 0$ for $t > s \le t_0$ and $0 < H(t,t_1) \le H(t,t_0)$ for $t > s \le t_0$. Now equation (11) becomes,

$$-\sum_{s=t_1}^{t-1} g(s)q(s)H(t,s) > H(t,t_1)\omega(t_1) - \sum_{s=t_1}^{t-1} \frac{h_+^2(t,s)mkg(s)}{4\Gamma(1-\alpha)H(t,s)}$$
$$\sum_{s=t_1}^{t-1} g(s)q(s)H(t,s) < -H(t,t_1)\omega(t_1) + \sum_{s=t_1}^{t-1} \frac{h_+^2(t,s)mkg(s)}{4\Gamma(1-\alpha)H(t,s)}$$
$$\sum_{s=t_1}^{t-1} \left[g(s)q(s)H(t,s) - \frac{h_+^2(t,s)mkg(s)}{4\Gamma(1-\alpha)H(t,s)} \right] < -H(t,t_1)\omega(t_1)$$
$$\sum_{s=t_1}^{t-1} \left[g(s)q(s)H(t,s) - \frac{h_+^2(t,s)mkg(s)}{4\Gamma(1-\alpha)H(t,s)} \right] \le H(t,t_0)\omega(t_1) \quad (12)$$

Since, $0 < H(t,s) \le H(t,t_0)$ for $t > s \ge t_0$ and $0 < \frac{H(t,s)}{H(t,t_0)} \le 1$ for $t > s \ge t_0$. Now

$$\frac{1}{H(t,t_0)} \sum_{s=t_0}^{t-1} \left[g(s)q(s)H(t,s) - \frac{h_+^2(t,s)mkg(s)}{4\Gamma(1-\alpha)H(t,s)} \right] \le \frac{1}{H(t,t_0)} \sum_{s=t_0}^{t_1-1} \left[g(s)q(s)H(t,s) - \frac{h_+^2(t,s)mkg(s)}{4\Gamma(1-\alpha)H(t,s)} \right]$$
$$+ \frac{1}{H(t,t_0)} \sum_{s=t_1}^{t-1} \left[g(s)q(s)H(t,s) - \frac{h_+^2(t,s)mkg(s)}{4\Gamma(1-\alpha)H(t,s)} \right]$$

$$\frac{1}{H(t,t_0)} \sum_{s=t_0}^{t-1} \left[g(s)q(s)H(t,s) - \frac{h_+^2(t,s)mkg(s)}{4\Gamma(1-\alpha)H(t,s)} \right] \le \frac{1}{H(t,t_0)} \sum_{s=t_0}^{t_1-1} \left[g(s)q(s)H(t,s) - \frac{h_+^2(t,s)mkg(s)}{4\Gamma(1-\alpha)H(t,s)} \right]$$
$$+ \frac{1}{H(t,t_0)} H(t,t_0)\omega(t_1) \quad \text{by using (12)}$$

$$\frac{1}{H(t,t_0)}\sum_{s=t_0}^{t-1}\left[g(s)q(s)H(t,s)-\frac{h_+^2(t,s)mkg(s)}{4\Gamma(1-\alpha)H(t,s)}\right] \le \omega(t_1)+\frac{1}{H(t,t_0)}\sum_{s=t_0}^{t_1-1}g(s)q(s)H(t,s)$$

$$\le \omega(t_1)+\sum_{s=t_0}^{t_1-1}g(s)q(s)$$

Now letting $t\to\infty$, we get

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\sum_{s=t_0}^{t-1}\left[g(s)q(s)H(t,s)-\frac{h_+^2(t,s)mkg(s)}{4\Gamma(1-\alpha)H(t,s)}\right] < \omega(t_1)+\sum_{s=t_0}^{t_1-1}g(s)q(s)$$

which contradicts (8). Hence complete the proof.

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Controllability of higher order fractional damped delay dynamical systems with distributed delays in control

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***Abstract**: This paper is concerned with the controllability of higher order fractional damped delay dynamical systems with distributed delays in control, which involves Caputo derivatives of any different orders. A necessary and sufficient conditions for the controllability of linear fractional damped delay dynamical systems are obtained by using the Grammian matrix and are based on the Mittag-Leffler function. Sufficient conditions for controllability of the corresponding nonlinear damped delay dynamical system has established by the successive approximation technique. Two examples are given to verify the theory.*

***Keywords**: Relatively Controllability, Damped delay dynamical systems, Laplace transform, Mittag-Leffler Matrix function, Distributed delays.*

I. Introduction

The fractional derivative has recognized as a suitable tool in the modelling of many real-world problems in numerous fields of science and engineering. These formulations are used to represent many practical systems more accurate than integer order ones and gained significance in the fields of bioengineering, signal processing, electrochemistry, electromagnetism, thermal systems, filter design, circuit theory, nano materials, nonlinear oscillation of earthquakes and robotics [1]-[7].

Controllability shows a vital part in the expansion of recent mathematical control theory and it is used to influence an object's behaviour of a dynamical system to accomplish a desired goal [8],[9]. The study of control problems demonstrated by fractional differential systems is significant in various problems of an applied nature. Nowadays, the controllability has applied in the fields of reactor control, electric bulk power control systems, industrial and chemical process control, aerospace engineering and theory of quantum systems. Several authors studied the linear and nonlinear dynamical systems in finite dimensional spaces with the help of control theory [10]-[13].

A distinguished feature for delay dynamical system is the future development of system depends not only on the present state but also in the entire prehistory. The main significance of imposing fractional order derivatives into delay dynamical system is the new effective model involving both delayed and nonlocal terms, especially hereditary properties and more degrees of freedom [14],[15].

Yonggang and Xiu'e [16] introduced a fractional oscillator system in which the restoring force is denoted by a term having fractional derivative and the properties of oscillation is reserved. In the fractional oscillator model, numerous specific forcing functions and their resonance were analysed by Achar *et al.* [17]. Tofighi [18] has described and attained the expression of the inherent damping force in the fractional oscillator system. Some authors extended the explanation to the fractional oscillator and reported that fractional oscillations have finite numbers of zeros. Solution of the fractional oscillator system is obtained by Al-rabth *et al.* [19] using differential transform method. Recently, Balachandran *et al.* [11] and Junpeng Liu *et al.* [20] has discussed the controllability of fractional damped dynamical systems.

To the best our knowledge, no relevant report has been described in the control problem of higher order fractional damped delay dynamical systems with distributed delays in control. This article, we make an attempt to investigate the controllability of the higher order fractional damped delay dynamical systems with distributed delays in control. Numerical illustrations are included to clarify the theoretical results.

II. Preliminaries

In this section, let us recall some notations, basic definitions and preliminary facts [6],[7].

Definition 2.1

The Caputo fractional derivative of order $\alpha > 0, n-1 < \alpha \leq n$, is defined as

$$ {}^{C}D_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} \; f^{(n)}(s) ds. $$

where $f^{(n)}(s) = \frac{d^n f}{ds^n}$ and the function $f(t)$ has absolutely continuous derivative up to order n-1. For the brevity, Caputo fractional derivative ${}^{C}D_{0+}^{\alpha}$ is taken as ${}^{C}D^{\alpha}$.

The Laplace transform of Caputo derivative is

$$ L[\, {}^{C}D^{\alpha} \; x(t)](s) = s^{\alpha} L[x(t)](s) - \sum_{k=0}^{n-1} x^{k}(0) s^{\alpha-k-1}, n-1 < \alpha \leq n. $$

Definition 2.2

The Mittag-Leffler function of three parameters is defined as

$$ E_{\alpha,\beta}^{\gamma}(-\lambda\, t^{\alpha}) = \sum_{k=0}^{\infty} \frac{(\gamma)_k (-\lambda)^k}{k!\, \Gamma(\alpha\, k + \beta)} t^{\alpha k} $$

where $(\gamma)_n$ is a Pochhamer symbol which is defined as $\gamma(\gamma+1)\cdots(\gamma+n-1)$ and $(\gamma)_n = \frac{\Gamma(\gamma + n)}{\Gamma(\gamma)}$.

Definition 2.3

The Laplace transform of Mittag-Leffler function of three parameters is defined as

$$ L[t^{\beta-1} E_{\alpha,\beta}^{\gamma}(\pm\lambda t^{\alpha})](s) = \frac{s^{\alpha\gamma-\beta}}{(s^{\alpha} \pm \lambda)^{\gamma}}, Re(s) > 0, Re(\beta) > 0, |\lambda s^{-\alpha}| < 1. $$

Lemma 2.1

The Mittag-Leffler matrix function derivative of order $p (p \in N)$ is defined as

$$ \left(\frac{d}{dt}\right)^{(p)} \left(t^{\alpha-1} E_{\alpha-\beta,\alpha}(A t^{\alpha-\beta}) \right) = t^{\alpha-p-1} E_{\alpha-\beta,\alpha-p}(A t^{\alpha-\beta}). $$

III. Linear Damped Delay Systems

Consider the linear higher order fractional damped delay dynamical system with distributed delays in control of the form

$$ \begin{aligned} {}^{C}D^{\alpha}\, x(t) - A\, {}^{C}D^{\beta}\, z(t) &= Bz(t-\tau) + \int_{-\tau}^{0} d_{\eta}\, C(t,\eta) v(t+\eta), \;\; t \; in \; J: [0,T] \\ z(t) &= \varphi(t), z'(t) = \varphi'(t), \cdots, z^{(p)}(t) = \varphi^{(p)}(t), \\ v(t) &= \psi(t), \;\; -\tau \leq t \leq 0, \end{aligned} \qquad (1) $$

where $p-1 < \alpha < p, q-1 < \beta < q, q \leq p-1, z \in \mathbb{R}^n, v \in \mathbb{R}^m, A, B$ are $n \times n$ matrices and $C(t,\eta)$ is $n \times m$ matrix, continuous in t for fixed η and of bounded variations in η on $[-\tau, 0]$ for each t in $[0,T]$. The function $v: [-\tau, T] \to \mathbb{R}^m$ and t in $[0,T]$, we use symbol v_{η} to denote the function on $[-\tau, 0]$ defined by $v_t(\eta) = v(t+\eta)$ for η in $[-\tau, 0)$. The integral term is in the Lebesgue Stieltjes sense that is denoted by the symbol d_{η}. In order to get the solution of system of (1) by taking Laplace and inverse Laplace transform on both sides of the equation (1), and is using convolution of Laplace transforms, we have the solution of the following expression [18],

$$ z(t) = z(t;\varphi) + \int_0^t (t-s)^{\alpha-1} \Phi_{\alpha-\beta,\alpha}\left(A(t-s)^{\alpha-\beta}\right) \left[\int_{-\tau}^{0} d_{\eta} C(s,\eta) v(s+\eta) \right] ds $$

where

$$ \begin{aligned} z(t;\varphi) = \sum_{k=0}^{p-1} z^{k}(0) t^{k} \Phi_{\alpha-\beta,1+k}\left(A t^{\alpha-\beta}\right) - \sum_{k=0}^{q-1} z^{k}(0) t^{k} \Phi_{\alpha-\beta,\alpha-\beta+1+k}\left(A t^{\alpha-\beta}\right) \\ + B \int_{-\tau}^{0} (t-s-\tau)^{\alpha-1} \Phi_{\alpha-\beta,\alpha}\left(A(t-s-\tau)^{\alpha-\beta}\right) \varphi(s) ds \end{aligned} \qquad (2) $$

$$ \Phi_{\alpha-\beta,1+k}\left(A t^{\alpha-\beta}\right) = L^{-1}\left[\frac{s^{\alpha-\beta-1-k}}{s^{\alpha-\beta} I - A - B s^{-\beta} e^{-s\tau}} \right](t), $$

$$ \Phi_{\alpha-\beta,\alpha-\beta+1+k}\left(A t^{\alpha-\beta}\right) = L^{-1}\left[\frac{s^{\alpha-\beta-(\alpha-\beta+1+k)}}{s^{\alpha-\beta} I - A - B s^{-\beta} e^{-s\tau}} \right](t), $$

$$\Phi_{\alpha-\beta,\alpha}(\mathrm{At}^{\alpha-\beta}) = \mathrm{L}^{-1}\left[\frac{s^{\alpha-\beta-\alpha}}{s^{\alpha-\beta}I - A - Bs^{-\beta}e^{-s\tau}}\right](t).$$

Now, using the well know result of unsymmetric Fubini theorem [21] and change of order of integration in the last term, we have

$$z(t) = z(t;\varphi) + \int_{-\tau}^{0} dC_\eta \left[\int_0^t (t-s)^{\alpha-1}\Phi_{\alpha-\beta,\alpha}\big(A(t-s)^{\alpha-\beta}\big)C(s,\eta)v(s+\eta)\,ds\right]$$

$$z(t) = z(t;\varphi) + \int_{-\tau}^{0} dC_\eta \left[\int_\eta^0 \big(t-(s-\eta)\big)^{\alpha-1}\Phi_{\alpha-\beta,\alpha}\left(A\big(t-(s-\eta)\big)^{\alpha-\beta}\right)C(s-\eta,\eta)\psi(s)\,ds\right]$$
$$+ \int_{-\tau}^{0} dC_\eta \left[\int_0^{t+\eta} \big(t-(s-\eta)\big)^{\alpha-1}\Phi_{\alpha-\beta,\alpha}\left(A\big(t-(s-\eta)\big)^{\alpha-\beta}\right)C(s-\eta,\eta)v(s)\,ds\right]$$

$$z(t) = z(t;\varphi) + \int_{-\tau}^{0} dC_\eta \left[\int_\eta^0 \big(t-(s-\eta)\big)^{\alpha-1}\Phi_{\alpha-\beta,\alpha}\left(A\big(t-(s-\eta)\big)^{\alpha-\beta}\right)C(s-\eta,\eta)\psi(s)\,ds\right]$$
$$+ \int_0^t \left[\int_{-\tau}^{0} \big(t-(s-\eta)\big)^{\alpha-1}\Phi_{\alpha-\beta,\alpha}\left(A\big(t-(s-\eta)\big)^{\alpha-\beta}\right)d_\eta C_t(s-\eta,\eta)\right]v(s)ds \quad (3)$$

where

$$C_t(s,\eta) = \begin{cases} C(s,\eta), & s \le t \\ 0, & s > t \end{cases}$$

and dC_η denotes the integration of Lebesgue Stieltjes sense with respect to the variable η in the function $C(t,\eta)$.

For our convenience, let us present the notation

$$\mathbb{G}(t,s) = \int_{-\tau}^{0} \big(t-(s-\eta)\big)^{\alpha-1}\Phi_{\alpha-\beta,\alpha}\left(A\big(t-(s-\eta)\big)^{\alpha-\beta}\right)d_\eta C_t(s-\eta,\eta). \quad (4)$$

The controllability Grammian matrix ω is defined as follows

$$\omega = \int_0^T \mathbb{G}(T,s)\mathbb{G}^\top(T,s)\,ds, \quad (5)$$

where the $\top$ denotes the matrix transpose.

Definition 3.1

The set $y(t) = \{z(t), z'(t), \cdots, z^{(p)}(t), v_t\}$ is the complete state of the systems (1) at time t.

Definition 3.2

Systems (1) is called relatively controllable on [0,T], if for each complete state $y(t)$ and for every vectors $z_1 \in \mathbb{R}^n$, there exists a control $v(t)$ defined on $[0,T]$ such that the solution $z(t)$ of the corresponding system (1) satisfies the condition $z(T) = z_1$.

Theorem 3.1

The linear system (1) is relatively controllable on $[0,T]$, if and only if the controllability Grammian matrix ω is nonsingular for some $T > 0$.

Proof

We first show that necessity. Assume that ω is singular and so its inverse is does not exist, and there exists a nonzero vector y such that

$$y^\top \omega y = y^\top \int_0^T \mathbb{G}(T,s)\mathbb{G}^\top(T,s)y\,ds = 0,$$

and hence, for $s \in [0,T]$. Consider the zero initial function $\varphi = 0, \varphi' = 0, \cdots, \varphi^{(p)} = 0$ and $\psi = 0$ on $[-\tau, 0]$ and the final point $z_1 = y$. Since the system is relatively controllable, there exists a control $v(t)$ on J that steers the response to $z_1 = y$ at $t = T$ that is

$$z(T;\varphi) + \int_{-\tau}^{0} dC_\eta \left[\int_\eta^0 \big(t-(s-\eta)\big)^{\alpha-1}\Phi_{\alpha-\beta,\alpha}\left(A\big(t-(s-\eta)\big)^{\alpha-\beta}\right)C(s-\eta,\eta)\psi(s)\,ds\right] + \int_0^T \mathbb{G}(T,s)v(s)\,ds = y.$$

Form $\varphi = 0, \varphi' = 0, \cdots, \varphi^{(p)} = 0, z(T;\varphi = 0)$ and $y^\top y \neq 0$ for $y \neq 0$. On the other side

$$y = (T;\varphi) + \int_{-\tau}^{0} dC_\eta \left[\int_\eta^0 \big(t-(s-\eta)\big)^{\alpha-1}\Phi_{\alpha-\beta,\alpha}\left(A\big(t-(s-\eta)\big)^{\alpha-\beta}\right)C(s-\eta,\eta)\psi(s)\,ds\right]$$
$$+ \int_0^T \mathbb{G}(T,s)v(s)\,ds = \int_0^T \mathbb{G}(T,s)v(s)\,ds,$$

$$y^{\top}y = \int_0^T y^{\top}\mathbb{G}(T,s)v(s)\,ds = 0.$$

This is a contradiction to $y \neq 0$. Thus ω is nonsingular.

Next, we show that the sufficiency. Assume that ω is nonsingular, its inverse is well defined. Let control function v is defined by

$$\begin{aligned}\mathrm{v(t)} = \mathbb{G}^{\top}(T,t)\omega^{-1}\Bigg(&\mathrm{z}_1 - \mathrm{z(T;\varphi)} \\ &- \int_{-\tau}^{0} dC_\eta \left[\int_{\eta}^{0} \big(T-(s-\eta)\big)^{\alpha-1}\Phi_{\alpha-\beta,\alpha}\left(A\big(T-(s-\eta)\big)^{\alpha-\beta}\right)C(s \\ &- \eta,\eta)\psi(s)\,ds\right]\Bigg) \qquad (6)\end{aligned}$$

Substituting $t = T$ in (5) and inserting (6) we have,

$$\begin{aligned}z(T) = z(T;\varphi) &+ \int_{-\tau}^{0} dC_\eta \left[\int_{\eta}^{0} \big(T-(s-\eta)\big)^{\alpha-1}\Phi_{\alpha-\beta,\alpha}\left(A\big(T-(s-\eta)\big)^{\alpha-\beta}\right)C(s-\eta,\eta)\psi(s)\,ds\right] \\ &+ \int_0^T \left[\int_{-\tau}^{0} \big(T-(s-\eta)\big)^{\alpha-1}\Phi_{\alpha-\beta,\alpha}\left(A\big(T-(s-\eta)\big)^{\alpha-\beta}\right)d_\eta C_T(s \right. \\ &\left. - \eta,\eta)\right]\left[\int_{-\tau}^{0} \big(T-(s-\eta)\big)^{\alpha-1}\Phi_{\alpha-\beta,\alpha}\left(A\big(T-(s-\eta)\big)^{\alpha-\beta}\right)d_\eta C_T(s-\eta,\eta)\right]^{\top}\omega^{-1}\Bigg(\mathrm{z}_1 \\ &- \mathrm{z(T;\varphi)} \\ &- \int_{-\tau}^{0} dC_\eta \left[\int_{\eta}^{0} \big(T-(s-\eta)\big)^{\alpha-1}\Phi_{\alpha-\beta,\alpha}\left(A\big(T-(s-\eta)\big)^{\alpha-\beta}\right)C(s-\eta,\eta)\psi(s)\,ds\right]\Bigg)d\eta \\ &= z_1.\end{aligned}$$

This means that control $v(t)$ steers the initial $y(0)$ to the desired vector $z_1 \in \mathbb{R}^n$ at time T. Thus system (1) is relatively controllable on $[0,T]$.

IV. Nonlinear Damped Delay Dynamical Systems

Consider the nonlinear higher order fractional damped delay dynamical system with distributed delays in control represented by the following fractional differential equation

$$\begin{aligned}{}^{C}D^{\alpha}\,\mathrm{z(t)} - \mathrm{A}\,{}^{C}D^{\beta}\,z(t) &= Bz(t-\tau) + \int_{-\tau}^{0} d_\eta\, C(t,\eta)v(t+\eta) + \mathrm{f}\Big(\mathrm{t},\mathrm{z(t)},\mathrm{z(t-\tau)},\ {}^{C}D^{\alpha}\,\mathrm{z(t)},\ {}^{C}D^{\beta}\,\mathrm{z(t)},\mathrm{v(t)}\Big) \\ z(t) = \varphi(t), z'(t) = \varphi'(t), \cdots, z^{(p)}(t) &= \varphi^{(p)}(t), \qquad (7) \\ v(t) = \psi(t),\ -\tau \le t \le 0,&\end{aligned}$$

where the nonlinear function $\mathrm{f}: \mathrm{J}\times\mathbb{R}^{\mathrm{n}}\times\mathbb{R}^{\mathrm{n}}\times\mathbb{R}^{\mathrm{n}}\times\mathbb{R}^{\mathrm{n}}\times\mathbb{R}^{\mathrm{m}} \to \mathbb{R}^{\mathrm{n}}$ being continuous. The solution of the system (7) can be expressed as follows

$$\begin{aligned}z(t) = z(t;\varphi) &+ \int_0^t (t-s)^{\alpha-1}\Phi_{\alpha-\beta,\alpha}\big(A(t-s)^{\alpha-\beta}\big)\mathrm{f}\Big(s,z(s),z(t-\tau),\ {}^{C}D^{\alpha}\,z(s),\ {}^{C}D^{\beta}\,z(s),v(s)\Big)\,ds \\ &+ \int_0^t (t-s)^{\alpha-1}\,\Phi_{\alpha-\beta,\alpha}\big(A(t-s)^{\alpha-\beta}\big)\left[\int_{-\tau}^{0} d_\eta C(s,\eta)v(s+\eta)\right]ds,\end{aligned}$$

where $z(t;\varphi)$ is defined as in (2). Now, using the well know result of unsymmetric Fubini theorem [21] and change of order of integration in the last term, we have

$$\begin{aligned}z(t) = z(t;\varphi) &+ \int_0^t (t-s)^{\alpha-1}\Phi_{\alpha-\beta,\alpha}\big(A(t-s)^{\alpha-\beta}\big)\mathrm{f}\Big(\mathrm{s},\mathrm{z(s)},\mathrm{z(t}-\tau),\ {}^{C}D^{\alpha}\,\mathrm{z(s)},\ {}^{C}D^{\beta}\,\mathrm{z(s)},\mathrm{v(s)}\Big)\,ds \\ &+ \int_{-\tau}^{0} dC_\eta \left[\int_{\eta}^{0} \big(t-(s-\eta)\big)^{\alpha-1}\Phi_{\alpha-\beta,\alpha}\left(A\big(t-(s-\eta)\big)^{\alpha-\beta}\right)C(s-\eta,\eta)\psi(s)\,ds\right] \\ &+ \int_0^t \left[\int_{-\tau}^{0} \big(t-(s-\eta)\big)^{\alpha-1}\Phi_{\alpha-\beta,\alpha}\left(A\big(t-(s-\eta)\big)^{\alpha-\beta}\right)d_\eta C_t(s-\eta,\eta)\right]v(s)ds \qquad (8)\end{aligned}$$

where

$$C_t(s,\eta) = \begin{cases} C(s,\eta), & s \le t \\ 0, & s > t \end{cases}$$

and dC_η denotes the integration of Lebesgue Stieltjes sense with respect to the variable η in the function $C(t,\eta)$. For our convenience, let us present the notation

$$\delta(t,\psi(s)) = \int_{-\tau}^{0} dC_\eta \left[\int_{\eta}^{0} \left(t-(s-\eta)\right)^{\alpha-1} \Phi_{\alpha-\beta,\alpha}\left(A\left(t-(s-\eta)\right)^{\alpha-\beta}\right) C(s-\eta,\eta)\psi(s)\, ds \right],$$

and $\mathbb{G}(t,s)$ is already defined as in (4).

Define the control function

$$\mathrm{v(t)} = \mathbb{G}^{\top}(T,t)\omega^{-1}\Bigg(\mathrm{z_1} - \mathrm{z(T;\varphi)} - \delta\big(T,\psi(s)\big) - \int_0^T (T-s)^{\alpha-1}\Phi_{\alpha-\beta,\alpha}\big(A(T-s)^{\alpha-\beta}\big)\mathrm{f}\Big(\mathrm{s,z(s),z(t-\tau),\ {}^{C}D^{\alpha}\,z(s),\ {}^{C}D^{\beta}\,z(s),v(s)}\Big)\, ds\Bigg).$$

Consider the space $ZB = \left\{ z : z^{(p)}(t),\ {}^{C}D^{\alpha}\ \mathrm{z(t)}\ {}^{C}D^{\beta}\ \mathrm{z(t)} \in \mathrm{C(J,}\mathbb{R}^n\mathrm{)} \text{ and } \mathrm{v}(t) \in \mathbb{R}^m \right\}$ be a Banach space endowed with the norm $\|z\|_z = max_{t\in J}\left\{ \|z(t)\|, \left\| {}^{C}D^{\alpha}\, z(t) \right\|, \left\| {}^{C}D^{\beta}\, z(t) \right\|, \|v(t)\| \right\}$.

Further we assume the following hypothesis:

(E1) $\mathrm{f: J \times R^n \times R^n \times R^n \times R^n \times R^m \to R^n}$ is continuous and there exist positive constants K and L such that

$$\left\| \mathrm{f}\left(\mathrm{t,z(t),z(t-\tau),\ {}^{C}D^{\alpha}\ z(t),\ {}^{C}D^{\beta}\ z(t),v(t)}\right) \right\| \le K,$$

$$\begin{aligned} &\left\| \mathrm{f}\big(\mathrm{t,z_1,z_{\tau 1},z_{\alpha 1},z_{\beta 1},v_1}\big) - \mathrm{f}\big(\mathrm{t,z_2,z_{\tau 2},z_{\alpha 2},z_{\beta 2},v_2}\big) \right\| \\ &\quad \le \mathrm{L}\big[\|\mathrm{z_1 - z_2}\| + \|\mathrm{z_{\tau 1} - z_{\tau 2}}\| + \|\mathrm{z_{\alpha 1} - z_{\alpha 2}}\| + \left\|\mathrm{z_{\beta 1} - z_{\beta 2}}\right\| \\ &\quad + \|\mathrm{v_1 - v_2}\|\big],\ \mathrm{z_1, z_2, z_{\tau 1}, z_{\tau 2}, z_{\alpha 1}, z_{\alpha 2}, z_{\beta 1}, z_{\beta 2} \in R^n},\ \mathrm{v_1, v_2 \in R^m}. \end{aligned}$$

For brevity, let us define

$$m_1 = sup\{\|\mathrm{z(t;\varphi)}\|, t \in J\}, m_2 = sup\left\{\left\|\mathrm{(T-s)^{\alpha-1}\Phi_{\alpha-\beta,\alpha}(A(T-s)^{\alpha-\beta})}\right\|, T, s \in J\right\},$$

$$m_3 = \sup\left\{\left\|\mathrm{(t-(s-\eta))^{\alpha-1}\Phi_{\alpha-\beta,\alpha}(A(T-(s-\eta))^{\alpha-\beta})}\right\|\right\},$$

$$m_4 = \left\| \int_{\eta}^{0} \left(T-(s-\eta)\right)^{\alpha-1} \Phi_{\alpha-\beta,\alpha}\left(A\left(T-(s-\eta)\right)^{\alpha-\beta}\right) C(s-\eta,\eta)\psi(s)\, ds \right\|,$$

$$m_5 = \sup\|\mathbb{G}^{\top}(T,s)\|,\ \ m_6 = \|z_1\|,\ \ m_7 = m_5\|\omega^{-1}\|,$$

$$m_8 = sup\left\| \int_{-\tau}^{0} \left(t-(s-\eta)\right)^{\alpha-1} \Phi_{\alpha-\beta,\alpha}\left(A\left(t-(s-\eta)\right)^{\alpha-\beta}\right) d_\eta C_t(s-\eta,\eta) \right\|,\ m_9 = \|\varphi(0)\|,$$

$$m_{10} = \|\delta(t,\psi(s))\|, \qquad m_{11} = \sup\left\{ \left\| \left(t-(s-\eta)\right)^{-1} \Phi_{\alpha-\beta}\left(A(t-(s-\eta))^{\alpha-\beta}\right) \right\|, t,s \in J \right\},$$

$$m_{12} = \left\| \int_{-\tau}^{0} \left(s-(\xi-\eta)\right)^{\alpha-p-1} \Phi_{\alpha-\beta,\alpha-p}\left(A\left(s-(\xi-\eta)\right)^{\alpha-\beta}\right) d_\eta C_t(\xi-\eta,\eta) \right\|,$$

$$m_{13} = \sup\left\{ \left\| (t-s)^{-1} \Phi_{\alpha-\beta}\left(A(t-s)^{\alpha-\beta}\right) \right\|, t,s \in J \right\},$$

$$m_{14} = \sup\left\{ \left\| \left(t-(s-\eta)\right)^{\alpha-\beta-p+q-1} \Phi_{\alpha-\beta,\alpha-\beta-p+q}\left(A(t-(\eta-s))^{\alpha-\beta}\right) \right\|, t,s \in J \right\},$$

$$m_{15} = \sup\left\{ \left\| (t-s)^{\alpha-\beta-p+q-1} \Phi_{\alpha-\beta,\alpha-\beta-p+q}\left(A(t-s)^{\alpha-\beta}\right) \right\|, t,s \in J \right\}.$$

Theorem 4.1

Assume that the function f satisfy the condition **(E1).** Suppose that the linear system (1) is relatively controllable then the nonlinear system (7) is relatively controllable on $[0,T]$.

Proof: To prove the relatively controllability result we apply the successive approximation technique. For that, we define

$$z_0(t) = \varphi(t)$$

$$z_{n+1}(t) = z(t;\varphi) + \int_0^t (t-s)^{\alpha-1}\Phi_{\alpha-\beta,\alpha}(A(t-s)^{\alpha-\beta})\mathrm{f}\Big(\mathrm{s},\mathrm{z_n(s)},\mathrm{z_n(t-\tau)},\ {}^{C}D^{\alpha}\ \mathrm{z_n(s)},\ {}^{C}D^{\beta}\ \mathrm{z_n(s)},\mathrm{v_n(s)}\Big)\,ds + \delta\big(t,\psi(s)\big) + \int_0^t \left[\int_{-\tau}^{0}\big(t-(s-\eta)\big)^{\alpha-1}\Phi_{\alpha-\beta,\alpha}\Big(A\big(t-(s-\eta)\big)^{\alpha-\beta}\Big)\,d_\eta C_t(s-\eta,\eta)\right] v_n(s)ds \quad (9)$$

where

$$\mathrm{v_n(T)} = \mathbb{G}^{\top}(T,t)\omega^{-1}\Big(\mathrm{z_1} - \mathrm{z(T;\varphi)} - \delta\big(T,\psi(s)\big) - \int_0^T (T-s)^{\alpha-1}\Phi_{\alpha-\beta,\alpha}(A(T-s)^{\alpha-\beta})\mathrm{f}\Big(\mathrm{s},\mathrm{z_n(s)},\mathrm{z_n(t-\tau)},\ {}^{C}D^{\alpha}\ \mathrm{z_n(s)},\ {}^{C}D^{\beta}\ \mathrm{z_n(s)},\mathrm{v_n(s)}\Big)\,ds\Big) \quad (10)$$

and $n = 0, 1, 2$ ....

Since $\varphi(t)$ is a given vector and note that $\{z_n(t)\}$ are the known sequence of functions. Now we have to show that $\{z_n(t)\}$ is a Cauchy sequence in ZB. Noting that $z_{n+1}(t) = \varphi(t) + \sum_{i=0}^{n}\big(z_{i+1}(t) - z_i(t)\big)$, and it is necessary to prove that the series $\sum_{i=0}^{n}\big(z_{i+1}(t) - z_i(t)\big)$ converges uniformly with respect to $t \in J$. It is clear that

$$\|\mathrm{v_n(t)}\| \le \|\mathbb{G}^{\top}(T,t)\|\|\omega^{-1}\|\Big(\|\mathrm{z_1}\| + \|\mathrm{z(T;\varphi)}\| + \left\|\int_{-\tau}^{0} dC_\eta\left[\int_{\eta}^{0}\big(T-(s-\eta)\big)^{\alpha-1}\Phi_{\alpha-\beta,\alpha}\Big(A\big(T-(s-\eta)\big)^{\alpha-\beta}\Big)C(s-\eta,\eta)\psi(s)\,ds\right]\right\| + \int_0^T \left\|(T-s)^{\alpha-1}\Phi_{\alpha-\beta,\alpha}(A(T-s)^{\alpha-\beta})\right\|\left\|\mathrm{f}\Big(\mathrm{s},\mathrm{z_n(s)},\mathrm{z_n(t-\tau)},\ {}^{C}D^{\alpha}\ \mathrm{z_n(s)},\ {}^{C}D^{\beta}\ \mathrm{z_n(s)},\mathrm{v_n(s)}\Big)\right\| ds\Big)$$

$$\|v_n(t)\| \le m_5\|\omega^{-1}\|\,(\,\|\mathrm{z_1}\| + \mathrm{m_1} + m_4 + m_2 K\,T) \le m_7(m_6 + m_1 + m_4 + m_2 KT) = p$$

$$\|v_n(t) - v_{n-1}(t)\| \le \|\mathbb{G}^{\top}(T,t)\|\|\omega^{-1}\|\left[\int_0^T \left\|(T-s)^{\alpha-1}\Phi_{\alpha-\beta,\alpha}(A(T-s)^{\alpha-\beta})\right\|\left\|\mathrm{f}\Big(s,z_{n-1}(s),z_{n-1}(s-\tau),\ {}^{C}D^{\alpha}\ \mathrm{z_{n-1}(s)},\ {}^{C}D^{\beta}\ \mathrm{z_{n-1}(s)},v_{n-1}(s)\Big) - \mathrm{f}\Big(s,z_n(s),z_n(s-\tau),\ {}^{C}D^{\alpha}\ \mathrm{z_n(s)},\ {}^{C}D^{\beta}\ \mathrm{z_n(s)},v_n(s)\Big)\right\| ds\right]$$

$$\le m_7 LT\big[\|z_{n-1}(s) - z_n(s)\| + \|z_{n-1}(s-\tau) - z_n(s-\tau)\| + \|\ {}^{C}D^{\alpha}\,\mathrm{z_{n-1}(s)} - {}^{C}D^{\alpha}\,\mathrm{z_n(s)}\| + \|\ {}^{C}D^{\beta}\,\mathrm{z_{n-1}(s)} - {}^{C}D^{\beta}\,\mathrm{z_n(s)}\| + \|v_{n-1}(s) - v_n(s)\|\big].$$

Then

$$\|\mathrm{z_{n+1}(t) - z_n(t)}\| \le \int_0^t \left[\left\|\int_{-\tau}^{0}\big(t-(s-\eta)\big)^{\alpha-1}\Phi_{\alpha-\beta,\alpha}\Big(A\big(t-(s-\eta)\big)^{\alpha-\beta}\Big)\,d_\eta C_t(s-\eta,\eta)\right\|\right]\|v_n(s) - v_{n-1}(s)\|ds + \int_0^t \left\|(t-s)^{\alpha-1}\Phi_{\alpha-\beta,\alpha}(A(t-s)^{\alpha-\beta})\right\|\left\|\mathrm{f}\Big(s,z_n(s),z_n(s-\tau),\ {}^{C}D^{\alpha}\ \mathrm{z_n(s)},\ {}^{C}D^{\beta}\ \mathrm{z_n(s)},v_n(s)\Big) - \mathrm{f}\Big(s,z_{n-1}(s),z_{n-1}(s-\tau),\ {}^{C}D^{\alpha}\ \mathrm{z_{n-1}(s)},\ {}^{C}D^{\beta}\ \mathrm{z_{n-1}(s)},v_{n-1}(s)\Big)\right\| ds$$

$$\leq m_7 m_8 LT \int_0^t \left[\|z_{n-1}(s) - z_n(s)\| + \|z_{n-1}(s-\tau) - z_n(s-\tau)\| + \left\| {}^{C}D^{\alpha}\, \mathrm{z_{n-1}(s)} - {}^{C}D^{\alpha}\, \mathrm{z_n(s)} \right\| \right.$$
$$\left. + \left\| {}^{C}D^{\beta}\, \mathrm{z_{n-1}(s)} - {}^{C}D^{\beta}\, \mathrm{z_n(s)} \right\| + \|v_{n-1}(s) - v_n(s)\| \right] ds$$
$$+ m_2 L \int_0^t \left[\|z_{n-1}(s) - z_n(s)\| + \|z_{n-1}(s-\tau) - z_n(s-\tau)\| + \left\| {}^{C}D^{\alpha}\, \mathrm{z_{n-1}(s)} - {}^{C}D^{\alpha}\, \mathrm{z_n(s)} \right\| \right.$$
$$\left. + \left\| {}^{C}D^{\beta}\, \mathrm{z_{n-1}(s)} - {}^{C}D^{\beta}\, \mathrm{z_n(s)} \right\| + \|v_{n-1}(s) - v_n(s)\| \right] ds$$

$$\leq (m_7 m_8 LT + m_2 L) \int_0^t \left[\|z_{n-1}(s) - z_n(s)\| + \|z_{n-1}(s-\tau) - z_n(s-\tau)\| + \left\| {}^{C}D^{\alpha}\, z_{n-1}(s) - {}^{C}D^{\alpha}\, z_n(s) \right\| \right.$$
$$\left. + \left\| {}^{C}D^{\beta}\, z_{n-1}(s) - {}^{C}D^{\beta}\, z_n(s) \right\| + \|v_{n-1}(s) - v_n(s)\| \right] ds.$$

Further it is seen that

$$\|\mathrm{z_1(t) - z_0(t)}\| \leq m_1 + m_9 + m_{10}$$
$$+ \int_0^t \left\| \int_{-\tau}^{0} \left(t - (s-\eta)\right)^{\alpha-1} \Phi_{\alpha-\beta,\alpha}\left(A\left(t-(s-\eta)\right)^{\alpha-\beta}\right) d_\eta C_t(s-\eta,\eta) \right\| \|v_0(s)\| ds$$
$$+ \int_0^t \left\| (\mathrm{t-s})^{\alpha-1} \Phi_{\alpha-\beta,\alpha}(\mathrm{A(t-s)}^{\alpha-\beta}) \right\| \left\| f\left(s, z_0(s), z_0(s\right.\right.$$
$$\left.\left. - \tau),\ {}^{C}D^{\alpha}\, z_0(s),\ {}^{C}D^{\beta}\, z_0(s), v_0(s)\right) \right\| ds$$
$$\leq m_{11} + (m_8 p + m_2 K)t$$
$$\leq \mathrm{P}T,$$

assuming that $\mathrm{P} > 0$. The method of induction and using the above inequality we have the estimate

$$\|\mathrm{z_{n+1}\,(t) - z_n(t)}\| \leq \mathrm{P}\left(m_7 m_8 LT + m_2 L\right) \frac{T^{n+1}}{(n+1)!}.$$

By choosing sufficiently large value of n, then the right-hand side in the above estimate can be made arbitrarily small. This implies that $\{z_n(t)\}$ is a Cauchy sequence in ZB. Since ZB is complete, the sequence $\{z_n(t)\}$ converges uniformly to a continuous function $z(t)$ on J. Thus, we have

$$z(t) = z(t;\varphi) + \int_{-\tau}^{0} dC_\eta \left[\int_{\eta}^{0} \left(t-(s-\eta)\right)^{\alpha-1} \Phi_{\alpha-\beta,\alpha}\left(A\left(t-(s-\eta)\right)^{\alpha-\beta}\right) C(s-\eta,\eta)\psi(s)\, ds \right]$$
$$+ \int_0^t \left[\int_{-\tau}^{0} \left(t-(s-\eta)\right)^{\alpha-1} \Phi_{\alpha-\beta,\alpha}\left(A\left(t-(s-\eta)\right)^{\alpha-\beta}\right) d_\eta C_t(s-\eta,\eta) \right] v(s) ds$$
$$+ \int_0^t (t-s)^{\alpha-1} \Phi_{\alpha-\beta,\alpha}\left(A(t-s)^{\alpha-\beta}\right) \mathrm{f}\left(\mathrm{s, z(s), z(t-\tau),\ {}^{C}D^{\alpha}\, z(s),\ {}^{C}D^{\beta}\, z(s), v(s)}\right) ds$$

where

$$\mathrm{v(t)} = \mathbb{G}^{\top}(T,t)\omega^{-1}\left(\mathrm{z_1 - z(T;\varphi)} - \delta\left(T, \psi(s)\right) \right.$$
$$\left. - \int_0^T (T-s)^{\alpha-1} \Phi_{\alpha-\beta,\alpha}\left(A(T-s)^{\alpha-\beta}\right) \mathrm{f}\left(\mathrm{s, z(s), z(t-\tau),\ {}^{C}D^{\alpha}\, z(s),\ {}^{C}D^{\beta}\, z(s), v(s)}\right) ds \right)$$

which follows by taking limit as $n \to \infty$ on both sides of (9) and (10). Then

$$z_{n+1}^{(p)}(t) = z^{(p)}(t;\varphi) + \int_{-\tau}^{0} dC_\eta \left[\int_{\eta}^{0} \left(t-(s-\eta)\right)^{\alpha-p-1} \Phi_{\alpha-\beta,\alpha-p}\left(A\left(t-(s-\eta)\right)^{\alpha-\beta}\right) C(s-\eta,\eta)\psi(s)\, ds \right]$$
$$+ \int_0^t \left[\int_{-\tau}^{0} \left(t-(s-\eta)\right)^{\alpha-p-1} \Phi_{\alpha-\beta,\alpha-p}\left(A\left(t-(s-\eta)\right)^{\alpha-\beta}\right) d_\eta C_t(s-\eta,\eta) \right] v_n(s) ds$$
$$+ \int_0^t (t-s)^{\alpha-p-1} \Phi_{\alpha-\beta,\alpha-p}\left(A(t-s)^{\alpha-\beta}\right) \mathrm{f}\left(\mathrm{s, z_n(s), z_n(t}\right.$$
$$\left. - \tau),\ {}^{C}D^{\alpha}\, \mathrm{z_n(s)},\ {}^{C}D^{\beta}\, \mathrm{z_n(s), v_n(s)}\right) ds$$

$$\lim_{n\to\infty} z_{n+1}^{(p)}(t) = \lim_{n\to\infty} \Bigg(z^{(p)}(t;\varphi) + \int_{-\tau}^{0} dC_\eta \left[\int_{\eta}^{0} \big(t-(s-\eta)\big)^{\alpha-p-1} \Phi_{\alpha-\beta,\alpha-p}\left(A\big(t-(s-\eta)\big)^{\alpha-\beta}\right) C(s-\eta,\eta)\psi(s)\, ds \right] + \int_{0}^{t} \left[\int_{-\tau}^{0} \big(t-(s-\eta)\big)^{\alpha-p-1} \Phi_{\alpha-\beta,\alpha-p}\left(A\big(t-(s-\eta)\big)^{\alpha-\beta}\right) d_\eta C_t(s-\eta,\eta) \right] v_n(s) ds + \int_{0}^{t} (t-s)^{\alpha-p-1} \Phi_{\alpha-\beta,\alpha-p}(A(t-s)^{\alpha-\beta}) \mathrm{f}\Big(\mathrm{s}, \mathrm{z_n(s)}, \mathrm{z_n(t} - \tau), {}^{C}D^{\alpha}\, \mathrm{z_n(s)}, {}^{C}D^{\beta}\, \mathrm{z_n(s)}, \mathrm{v_n(s)}\Big) ds \Bigg)$$

$$= \mathrm{z^{(p)}(t)}.$$

Moreover, we have

$$\begin{aligned} \left\| {}^{C}D^{\alpha}\, z(t) - {}^{C}D^{\alpha}\, z_{n+1}(t) \right\| &= \left\| \frac{1}{\Gamma(\mathrm{p}-\alpha)} \int_0^t \big(t - (s-\eta)\big)^{p-\alpha-1} \left(\int_0^s \int_{-\tau}^{0} \big(s-(\xi-\eta)\big)^{\alpha-p-1} \Phi_{\alpha-\beta,\alpha-p}\left(A\big(s-(\xi-\eta)\big)^{\alpha-\beta}\right) d_\eta C_t(\xi - \eta,\eta)\big(v(\xi)-v_n(\xi)\big) d\xi \right) ds \right. \\ &\quad + \frac{1}{\Gamma(\mathrm{p}-\alpha)} \int_0^t (\mathrm{t} - \mathrm{s})^{\mathrm{p}-\alpha-1} \left(\int_0^s (\mathrm{s}-\xi)^{\alpha-\mathrm{p}-1} \Phi_{\alpha-\beta,\alpha-\mathrm{p}}(\mathrm{A}(\mathrm{s}-\xi)^{\alpha-\beta}) \mathrm{f}\Big(\mathrm{s}, z(\xi), \mathrm{z}(\xi - \tau), {}^{C}D^{\alpha}\, \mathrm{z}(\xi), {}^{C}D^{\beta}\, \mathrm{z}(\xi), \mathrm{v}(\xi)\Big) \right. \\ &\quad \left. \left. - \mathrm{f}\Big(\mathrm{s}, \mathrm{z_n}(\xi), \mathrm{z_n}(\xi - \tau), {}^{C}D^{\alpha}\, \mathrm{z_n}(\xi), {}^{C}D^{\beta}\, \mathrm{z_n}(\xi), \mathrm{v_n}(\xi)\Big) d\xi \right) ds \right\| \\ &= \left\| \int_0^t \big(t-(s-\eta)\big)^{-1} \Phi_{\alpha-\beta}\big(A(t-s)^{\alpha-\beta}\big) \int_{-\tau}^{0} \big(t-(s-\eta)\big)^{\alpha-p-1} \Phi_{\alpha-\beta,\alpha-p}\left(A\big(s-(s-\eta)\big)^{\alpha-\beta}\right) d_\eta C_t(s - \eta,\eta)\big(v(s)-v_n(s)\big) ds \right. \\ &\quad + \int_0^t (t-s)^{-1} \Phi_{\alpha-\beta}\big(A(t-s)^{\alpha-\beta}\big) \mathrm{f}\Big(\mathrm{s}, z(s), \mathrm{z}(s - \tau), {}^{C}D^{\alpha}\, \mathrm{z}(s), {}^{C}D^{\beta}\, \mathrm{z}(s), \mathrm{v}(s)\Big) \\ &\quad \left. - \mathrm{f}\Big(\mathrm{s}, \mathrm{z_n}(s), \mathrm{z_n}(s - \tau), {}^{C}D^{\alpha}\, \mathrm{z_n}(s), {}^{C}D^{\beta}\, \mathrm{z_n}(s), \mathrm{v_n}(s)\Big) ds \right\| \\ &\le m_{11} m_{12} T \|v(t) - v_n(t)\| \\ &\quad + m_{13} T \big[\|z(t) - z_n(t)\| + \|z(t-\tau) - z_n(t-\tau)\| + \left\| {}^{C}D^{\alpha}\, z(t) - {}^{C}D^{\alpha}\, \mathrm{z_n}(t) \right\| \\ &\quad + \left\| {}^{C}D^{\beta}\, z(t) - {}^{C}D^{\beta}\, \mathrm{z_n}(t) \right\| + \|v(t) - v_n(t)\| \big] \end{aligned}$$

$$\left\| {}^{C}D^{\beta} z(t) - {}^{C}D^{\beta} z_{n+1}(t) \right\|$$
$$= \left\| \frac{1}{\Gamma(q-\beta)} \int_0^t \Big(t - (s-\eta) \Big)^{q-\beta-1} \left(\int_0^s \int_{-\tau}^0 \big(s - (\xi-\eta) \big)^{\alpha-p-1} \Phi_{\alpha-\beta,\alpha-p} \left(A \big(s - (\xi-\eta) \big)^{\alpha-\beta} \right) d_\eta C_t(\xi - \eta, \eta) \big(v(\xi) - v_n(\xi) \big) d\xi \right) ds \right.$$
$$+ \frac{1}{\Gamma(q-\beta)} \int_0^t (t - s)^{q-\beta-1} \left(\int_0^s (s-\xi)^{\alpha-p-1} \Phi_{\alpha-\beta,\alpha-p}(A(s-\xi)^{\alpha-\beta}) f\Big(s, z(\xi), z(\xi - \tau), {}^{C}D^{\alpha} z(\xi), {}^{C}D^{\beta} z(\xi), v(\xi) \Big) \right.$$
$$\left. \left. - f\Big(s, z_n(\xi), z_n(\xi - \tau), {}^{C}D^{\alpha} z_n(\xi), {}^{C}D^{\beta} z_n(\xi), v_n(\xi) \Big) d\xi \right) ds \right\|$$

$$= \left\| \int_0^t \big(t - (s-\eta) \big)^{\alpha-\beta-p+q-1} \Phi_{\alpha-\beta,\alpha-\beta-p+q} \big(A(t-(\eta-s))^{\alpha-\beta} \big) \int_{-\tau}^0 \Big(t - (s-\eta) \Big)^{\alpha-p-1} \Phi_{\alpha-\beta,\alpha-p} \left(A \big(s - (s-\eta) \big)^{\alpha-\beta} \right) d_\eta C_t(s - \eta, \eta) \big(v(s) - v_n(s) \big) ds \right.$$
$$+ \int_0^t (t-s)^{\alpha-\beta-p+q-1} \Phi_{\alpha-\beta,\alpha-\beta-p+q} \big(A(t-s)^{\alpha-\beta} \big) f\Big(s, z(s), z(s - \tau), {}^{C}D^{\alpha} z(s), {}^{C}D^{\beta} z(s), v(s) \Big) - f\Big(s, z_n(s), z_n(s - \tau), {}^{C}D^{\alpha} z_n(s), {}^{C}D^{\beta} z_n(s), v_n(s) \Big) ds \Big\|$$
$$\leq m_{14} m_{12} T \| v(t) - v_n(t) \|$$
$$+ m_{15} T \Big[\| z(t) - z_n(t) \| + \| z(t-\tau) - z_n(t-\tau) \| + \left\| {}^{C}D^{\alpha} z(t) - {}^{C}D^{\alpha} z_n(t) \right\|$$
$$+ \left\| {}^{C}D^{\beta} z(t) - {}^{C}D^{\beta} z_n(t) \right\| + \| v(t) - v_n(t) \| \Big]$$

As $n \to \infty$, ${}^{C}D^{\alpha} z_{n+1}(t) \to {}^{C}D^{\alpha} z(t)$ and ${}^{C}D^{\beta} z_{n+1}(t) \to {}^{C}D^{\beta} z(t)$. Clearly $z(T) = z_1$ which means that the control $v(t)$ steers the system from the initial state $\varphi(t), \varphi'(t), \cdots, \varphi^{(p)}(t)$ to z_1 in time T. Hence the system (8) is relatively controllable on $[0, T]$.

V. Examples

In this section, we have provided two examples for our proposed criteria to illustrate the controllability results.

Example 5.1

Consider the higher order nonlinear fractional delay dynamical system with distributed delays in control of the form

$$ {}^{C}D^{\frac{5}{2}} z(t) - {}^{C}D^{\frac{3}{2}} z(t) = z(t-2) + \int_{-2}^{0} d_\eta e^{\eta} \cos t \, v(t+\eta) + f\left(\sin {}^{C}D^{\alpha} z_1(t) + \cos {}^{C}D^{\beta} z_2(t) + \frac{\sqrt{z_3(t) + v^2(t)}}{\log z_4(t)} \right)$$
$$z(t) = \varphi(t), z'(t) = \varphi'(t), z''(t) = \varphi''(t), \quad v(t) = \psi(t), t \in -2 \leq t \leq 0, \tag{11}$$

where $\alpha = \frac{5}{2}, \beta = \frac{3}{2}, A = 1, B = 1, C(t, \eta) = e^{\eta} \cos t$,

$f\Big(t, z(t), z(t-\tau), {}^{C}D^{\alpha} z(t), {}^{C}D^{\beta} z(t), v(t) \Big) = \sin {}^{C}D^{\alpha} z_1(t) + \cos {}^{C}D^{\beta} z_2(t) + \frac{\sqrt{z_3(t) + v^2(t)}}{\log z_4(t)}$.

Now, we have to apply Theorem 3.1 and 4.1 to prove that system (11) is relatively controllable on [0,2].

For $t \in [0,2]$, then by using the above technique, the solution of (11) is of the form

$$z(t) = \left[\Phi_{1,1}(t) - t\Phi_{1,2}(t)\right]\varphi(t) + \left[t\Phi_{1,2}(t) - t^2\,\Phi_{1,3}(t)\right]\varphi'(t) + t^2\,\Phi_{1,3}(t)\varphi''(t)$$
$$+ \int_{-2}^{0} dC_\eta \int_{\eta}^{0} \big(t-(s-\eta)\big)^{\frac{3}{2}} \Phi_{1,\frac{5}{2}}\big(t-(s-\eta)\big) C(s-\eta,\eta)\psi(s)ds$$
$$+ \int_{0}^{t}\int_{-2}^{0} \big(t-(s-\eta)\big)^{\frac{3}{2}} \Phi_{1,\frac{5}{2}}\big(t-(s-\eta)\big) d_\eta C_t(s-\eta)v(s)ds$$
$$+ \int_{0}^{t} (t-s)^{\alpha-1}\Phi_{\alpha-\beta,\alpha}\big(A(t-s)^{\alpha-\beta}\big) f\left(s, z(s), z(t-\tau),\ {}^{C}D^{\alpha} z(s),\ {}^{C}D^{\beta} z(s), v(s)\right) ds.$$

Define

$$\mathbb{G}(2,s) = \int_{-2}^{0} \big(2-(s-\eta)\big)^{\frac{3}{2}} \Phi_{1,\frac{5}{2}}\big(2-(s-\eta)\big)\, d_\eta C_2(s-\eta,\eta)v(s)ds,$$

since $C(t,\eta) = e^{\eta}cost$

$$\mathbb{G}(2,s) = \int_{-2}^{0} \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-1)^l \big(2-(s-\eta)\big)^{\frac{5}{2}(k+1)-1} s^{2(l+1)-1}\eta^m}{\Gamma\left(\frac{5}{2}(k+1)\right)\Gamma(2n+1)\Gamma(l+1)}\, d\eta,$$

then the Grammian matrix

$$\omega = \int_0^2 \mathbb{G}(2,s)\mathbb{G}^{\top}(2,s)ds = \int_0^2 a^2(s)ds,$$

where,

$$a(s) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-1)^{l+m}(2-s)^{\frac{5}{2}(k+1)+m} s^{2(l+1)-1} B_{\frac{1}{2-s}}\big(m+1, \alpha(k+1)\big)}{\Gamma\left(\frac{5}{2}(k+1)\right)\Gamma(2n+1)\Gamma(l+1)}$$

is positive definite. Further the nonlinear function f is bounded and all conditions stated in Theorem 4.1 are satisfied. Hence the fractional delay dynamical system (11) is relatively controllable on $[0,2]$.

Example 5.1 defines the conditions when A and B are constant. The next Example 5.2 demonstrate the conditions when A and B are matrices.

Example 5.2

Consider the problem of nonlinear fractional damped delay dynamical systems with distributed delay in control as follows

$$ {}^{C}D^{\frac{5}{2}} z_1(t) - {}^{C}D^{\frac{3}{2}} z_2(t) = z_1(t-1) + \int_{-1}^{0} [e^{\eta+t} v_1(t+\eta) + e^{\eta} sint\, v_2(t+\eta)]d\eta + \frac{z_1(t)}{1+z_1^2(t-1)+v_1(t)}$$

$$ {}^{C}D^{\frac{5}{2}} z_2(t) + {}^{C}D^{\frac{3}{2}} z_1(t)$$
$$= z_2(t-1) + \int_{-1}^{0} [e^{\eta+t} v_2(t+\eta) - e^{\eta} sint\, v_1(t+\eta)]d\eta + \frac{z_2(t)}{1+z_2^2(t)}$$
$$+ \left| {}^{C}D^{\frac{5}{2}} z_2(t) - cos\ {}^{C}D^{\frac{3}{2}} z_2(t)\right| \qquad (12)$$

for $t \in J$ and $\alpha = \frac{5}{2}, \beta = \frac{3}{2}$. In the matrix form, we have $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$,

$C(t,\eta) = \begin{pmatrix} e^{t+\eta} & e^{\eta}sint \\ -e^{\eta}sint & e^{t+\eta} \end{pmatrix}$ and the nonlinear function

$$\mathrm{f}\left(t, z(t), z(t-\tau),\ {}^{C}D^{\alpha} z(t),\ {}^{C}D^{\beta} z(t), v(t)\right) = \begin{pmatrix} \frac{z_1(t)}{1+z_1^2(t-1)+v_1(t)} \\ \frac{z_2(t)}{1+z_2^2(t)} + \left| {}^{C}D^{\frac{5}{2}} z_2(t) - cos\ {}^{C}D^{\frac{3}{2}} z_2(t)\right| \end{pmatrix}.$$

The Mittag-Leffler matrix of the given system is

$$\Phi_{\alpha}(At^{\alpha}) = \begin{pmatrix} 0 & \Phi_{\alpha}(t^{\alpha}) \\ -\Phi_{\alpha}(t^{\alpha}) & 0 \end{pmatrix}.$$

Further

$$\Phi_{\alpha-\beta,\alpha}\left(A\left(T-(s-\eta)\right)\right)=\begin{pmatrix}0 & \Phi_{\alpha-\beta,\alpha}\left(A\left(T-(s-\eta)\right)^{\alpha}\right)\\ -\Phi_{\alpha-\beta,\alpha}\left(A\left(T-(s-\eta)\right)^{\alpha}\right) & 0\end{pmatrix}$$

Also,

$$\mathbb{G}(T,s)=\int_{-1}^{0}\left(T-(s-\eta)\right)\Phi_{\alpha-\beta,\alpha}\left(A\left(T-(s-\eta)\right)^{\alpha}\right)d_{\eta}C_{T}(s-\eta,\eta)=\begin{pmatrix}L(s) & K(s)\\ -K(s) & L(s)\end{pmatrix},$$

where

$$L(s)=-\int_{-1}^{0}e_{\alpha}^{(T-(s-\eta))}e^{\eta}\sin(s-\eta)\,d\eta,$$

$$K(s)=\int_{-1}^{0}e_{\alpha}^{(T-(s-\eta))}e^{t+\eta}d\eta$$

where e_{α}^{t} is the α-exponential function defined by $e_{\alpha}^{t}=t^{\alpha-1}\Phi_{\alpha-\beta,\alpha}(t^{\alpha})$ [6]. Now, evaluating the Gramian matrix, we have

$$\omega(o,T)=\int_{0}^{T}\mathbb{G}(T,s)\mathbb{G}^{\top}(T,s)ds=\int_{0}^{T}[L^{2}(s)+K^{2}(s)]\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}ds$$

is positive definite for any $T>0$. Further, if the nonlinear function f is bounded, continuous and conditions stated in Theorem 4.1 are satisfied. Hence the fractional delay dynamical system (12) is relatively controllable on $[0,T]$.

VI. References

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Stability of n-Dimensional Mixed Type Additive and Quadratic Functional Equation

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***Abstract**: In this paper, the authors introduced n-dimensional mixed type additive and quadratic functional equation and established generalized Ulam-Hyers stability in Banach space using Hyers' direct method.*

***Keywords**: Additive functional equation, quadratic functional equation, Generalized Ulam-Hyers stability, JMRassias stability*

I. Introduction

The stability of functional equations had been first raised by S.M. Ulam [26]. In 1941, D. H. Hyers [12] gave a positive answer to the question of Ulam for Banach spaces. In 1950, T. Aoki [2] was the second author to treat this problem for additive mappings. Th.M. Rassias [20] succeeded in extending the result of Hyers' Theorem by weakening the condition for the Cauchy difference controlled by $\left(\|x\|^p+\|y\|^p\right)$, $p\in[0,1)$ to be unbounded. Taking into consideration a lot of influence of Ulam, Hyers and Rassias on the development of stability problems of functional equations, the stability phenomenon that was proved by Th.M. Rassias is called Hyers-Ulam-Rassias stability one can refer [1, 6, 13, 15].

In 1982, J.M. Rassias [18] followed the innovative approach of the Th.M. Rassias theorem [20] in which he replaced the factor $\|x\|^p+\|y\|^p$ by $\|x\|^p\|y\|^q$ for $p,q\in\mathbb{R}$ with $p+q\neq 1$. A generalization of all the above results was obtained by P. Gavruta [8] in 1994 by replacing the unbounded Cauchy difference by a general control function $\phi(x,y)$ in the spirit of J.M. Rassias approach.

In 2008, a special case of Gavruta's theorem for the unbounded Cauchy difference was obtained by Ravi et al., [24] by considering the summation of both the sum and the product of two p- norms in the sprit of Rassias approach. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [4, 5, 16, 19, 24]) and reference cited there in.

The functional equation

$$f(x+y)=f(x)+f(y) \tag{1.1}$$

is called the additive functional equation and it is the most famous functional equation. Since $f(x)=kx$ is the solution of the functional equation (1.1), every solution of the additive equation is called an additive function.

The functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y) \tag{1.2}$$

is said to be quadratic functional equation because the quadratic function $f(x)=ax^2$ is a solution of the functional equation (1.2). K.W. Jun and H.M. Kim [11] introduced the following generalized quadratic and additive type functional equation

$$f\left(\sum_{i=1}^{n}x_i\right)+(n-2)\sum_{i=1}^{n}f(x_i)=\sum_{1\le i<j\le n}f(x_i+x_j) \tag{1.3}$$

in the class of function between real vector spaces. For $n=3$, Pl. Kannappan proved that a function f satisfies the functional equation (1.3) if and only if there exists a symmetric bi-additive function A and additive function B such that $f(x)=B(x,x)+A(x)$ for all x (see [16]). The Ulam-Hyers stability for the equation (1.3) when

$n=3$ was proved by S.M. Jung [14]. The Hyers-Ulam-Rassias stability for the equation (1.3) when $n=4$ was also investigated by I.S. Chang et al., [7].

The general solution and the generalized Hyers-Ulam stability for the quadratic and additive type functional equation

$$f(x+ay)+af(x-y)=f(x-ay)+af(x+y) \tag{1.4}$$

for any positive integer a with $a\neq -1,0,1$ was discussed by K.W. Jun and H.M. Kim [10]. A. Najati and M.B. Moghimi [17] investigated the generalized Hyers-Ulam-Rassias stability for the quadratic and additive type functional equation of the form

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+2f(2x)-4f(x). \tag{1.5}$$

Recently, M. Arunkumar and S. Karthikeyan [3] introduced and proved the general solution and generalized Ulam-Hyers stability of n-dimensional mixed type additive and quadratic functional equation of the form

$$f(-x_1)+f\left(2x_1-\sum_{i=2}^{n}x_i\right)+f\left(2\sum_{i=2}^{n}x_i\right)+f\left(x_1+\sum_{i=2}^{n}x_i\right)-f\left(-x_1-\sum_{i=2}^{n}x_i\right)$$
$$-f\left(x_1-\sum_{i=2}^{n}x_i\right)-f\left(-x_1+\sum_{i=2}^{n}x_i\right)=3f(x_1)+3f\left(\sum_{i=2}^{n}x_i\right) \tag{1.6}$$

in Banach spaces.

In this paper, the authors introduced and established generalized Ulam-Hyers stability of n-dimensional mixed type additive and quadratic functional equation of the form

$$\sum_{i=1}^{n}f\left(\sum_{j=1}^{n}x_{ij}\right)=\left(\frac{-n^2+7n-6}{2}\right)\sum_{i=1}^{n}f\left(x_i\right)+\left(\frac{-n^2+5n-2}{2}\right)\sum_{i=1}^{n}f\left(-x_i\right)$$
$$+\left(\frac{n-4}{2}\right)\sum_{1\le i<j\le n}\left(f\left(x_i+x_j\right)+f\left(-x_i-x_j\right)\right) \tag{1.7}$$

where $x_{ij}=\begin{cases}-x_j & if \quad i=j,\\ x_j & if \quad i\neq j\end{cases}$ in Banach space using Hyers' direct method.

In Section 2, the authors discussed general solution of the functional equation (1.7). The generalized Ulam - Hyers stability of the functional equation (1.7) is presented in Section 3.

II. General Solution of the Functional Equation (1.7)

In this section, the authors investigate the general solution of the mixed type functional equation (1.7). Throughout this section let us consider X and Y be real vector spaces.

Theorem 2.1. Let $f:X\to Y$ be an odd function satisfies the functional equation (1.7) for all $x_1,x_2,\ldots,x_n\in X$ if and only if $f:X\to Y$ satisfies the functional equation

$$f(x+y)=f(x)+f(y) \tag{2.1}$$

for all $x,y\in X$.

Proof. Since f is an odd function, one can deduce from (1.7), we get

$$\sum_{i=1}^{n}f\left(\sum_{j=1}^{n}x_{ij}\right)=(n-2)\sum_{j=1}^{n}f(x_j) \tag{2.2}$$

for all $x_1,x_2,\ldots,x_n\in X$. The rest of the proof for this theorem can be derived follows from Theorem 2.1 in [4].

Theorem 2.2. Let $f:X\to Y$ be an even function satisfies the functional equation (1.7) for all $x_1,x_2,\ldots,x_n\in X$ if and only if $f:X\to Y$ satisfies the functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y) \tag{2.3}$$

for all $x,y\in X$.

Proof. Since f is an even function, one can deduce from (1.7), we arrive

$$\sum_{i=1}^{n}f\left(\sum_{j=1}^{n}x_{ij}\right)=\left(-n^2+6n-4\right)\sum_{i=1}^{n}f(x_i)+(n-4)\sum_{1\le i<j\le n}f(x_i+y_j) \tag{2.4}$$

for all $x_1,x_2,\ldots,x_n\in X$. The rest of the proof for this theorem can be derived follows from Theorem 2.1 in [5].

III. Stability Results: Hyers' Direct Method

In this section, the authors presented the generalized Ulam - Hyers stability of the functional equation (1.7) using Hyers' direct method.

Throughout this section, let X be a normed space and Y be a Banach space respectively. Define a mapping $Df : X \to Y$ by

$$Df\left(x_1, x_2, \ldots, x_n\right) = \sum_{i=1}^{n} f\left(\sum_{j=1}^{n} x_{ij}\right) - \left(\frac{-n^2+7n-6}{2}\right)\sum_{i=1}^{n} f\left(x_i\right) - \left(\frac{-n^2+5n-2}{2}\right)\sum_{i=1}^{n} f\left(-x_i\right) - \left(\frac{n-4}{2}\right)\sum_{1\le i<j\le n}\left(f\left(x_i+x_j\right)+f\left(-x_i-x_j\right)\right)$$

for all $x_1, x_2, \ldots, x_n \in X$.

Theorem 3.1 Let $j \in \{-1,1\}$ and $\alpha : X^n \to [0,\infty)$ be a function such that

$$\sum_{k=0}^{\infty} \frac{\alpha\left((n-2)^{kj}x_1, \ldots, (n-2)^{kj}x_n\right)}{(n-2)^{2kj}} \text{ converges to } \mathbb{R} \text{ and } \lim_{k\to\infty} \frac{\alpha\left((n-2)^{kj}x_1, \ldots, (n-2)^{kj}x_n\right)}{(n-2)^{2kj}} < \infty \tag{3.1}$$

for all $x_1, \cdots, x_n \in X$. Let $f_e : X \to Y$ be an even function satisfies the inequality

$$\left\|Df_e(x_1, \cdots, x_n)\right\| \le \alpha\left((n-2)^{kj}x_1, \ldots, (n-2)^{kj}x_n\right) \tag{3.2}$$

for all $x_1, \cdots, x_n \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ which satisfying (1.7) such that

$$\left\|f_e(x) - Q(x)\right\| \le \frac{1}{(n-2)^2} \sum_{i=\frac{1-j}{2}}^{\infty} \frac{\beta((n-2)^{ij}x)}{(n-2)^{2ij}} \tag{3.3}$$

for all $x \in X$, where $\beta((n-2)^{ij}x)$ and $Q(x)$ are defined by

$$\beta((n-2)^{ij}x) = \frac{1}{n}\alpha\left((n-2)^{ij}x, \ldots, (n-2)^{ij}x\right) \tag{3.4}$$

$$Q(x) = \lim_{k\to\infty} \frac{f_e((n-2)^{kj}x)}{(n-2)^{2kj}} \tag{3.5}$$

for all $x \in X$.

Proof. Assume $j = 1$. Since f is an even function. Replacing $(x_1, x_2, \ldots, x_n)$ by $(x, x, \ldots, x)$ and divided by $(n-2)^2$ in (3.2), we get

$$\left\|\frac{f_e((n-2)x)}{(n-2)^2} - f_e(x)\right\| \le \frac{1}{n(n-2)^2}\alpha\left(x, x, \ldots x\right) \tag{3.6}$$

for all $x \in X$. Letting $\beta(x) = \frac{1}{n}\alpha\left(x, x, \ldots, x\right)$ in (3.6), we arrive

$$\left\|\frac{f_e((n-2)x)}{(n-2)^2} - f_e(x)\right\| \le \frac{1}{(n-2)^2}\beta\left(x\right) \tag{3.7}$$

for all $x \in X$. Now replacing x by $(n-2)x$ and dividing by $(n-2)^2$ in (3.7), we get

$$\left\|\frac{f_e((n-2)^2x)}{(n-2)^4} - \frac{f_e((n-2)x)}{(n-2)^2}\right\| \le \frac{1}{(n-2)^4}\beta((n-2)x) \tag{3.8}$$

for all $x \in X$. Combining (3.7) and (3.8), we obtain

$$\left\|\frac{f_e((n-2)^2x)}{(n-2)^4} - f_e(x)\right\| \le \frac{1}{(n-2)^2}\left[\beta\left(x\right) + \frac{1}{(n-2)^2}\beta\left(\left(n-2\right)x\right)\right] \tag{3.9}$$

for all $x \in X$. In general for any positive integer n, we have

$$\begin{aligned}\left\|\frac{f_e((n-2)^2x)}{(n-2)^4} - f_e(x)\right\| &\le \frac{1}{(n-2)^2}\sum_{i=0}^{k-1}\frac{\beta\left(\left(n-2\right)^i x\right)}{(n-2)^{2i}} \\ &\le \frac{1}{(n-2)^2}\sum_{i=0}^{\infty}\frac{\beta\left(\left(n-2\right)^i x\right)}{(n-2)^{2i}}\end{aligned} \tag{3.10}$$

for all $x \in X$. In order to prove the convergence of the sequence $\left\{\frac{f_e((n-2)^k x)}{(n-2)^{2k}}\right\}$, replace x by $(n-2)^m x$ and dividing by $(n-2)^{2m} x$ in (3.10), for any $m, k > 0$, we arrive

$$\begin{aligned}\left\|\frac{f_e\left((n-2)^k (n-2)^m x\right)}{(n-2)^{2k+2m}} - \frac{f_e\left((n-2)^m x\right)}{(n-2)^{2m}}\right\| &= \frac{1}{(n-2)^{2m}}\left\|\frac{f_e\left((n-2)^k (n-2)^m x\right)}{(n-2)^{2k+2m}} - f_e\left((n-2)^m x\right)\right\| \\ &\le \frac{1}{(n-2)^2}\sum_{i=0}^{k-1}\frac{\beta\left((n-2)^i (n-2)^m x\right)}{(n-2)^{2i+2m}} \le \frac{1}{(n-2)^2}\sum_{i=0}^{\infty}\frac{\beta\left((n-2)^{i+m} x\right)}{(n-2)^{2i+2m}}\end{aligned} \tag{3.11}$$

for all $x \in X$. Since the right hand side of the inequality (3.11) tends to 0 as $m \to \infty$, the sequence $\left\{\frac{f_e((n-2)^k x)}{(n-2)^{2k}}\right\}$ is a Cauchy sequence. Since Y is complete, there exists a mapping $Q: X \to Y$ such that $Q(x) = \lim_{k\to\infty}\frac{f_e((n-2)^k x)}{(n-2)^{2k}}$, $\forall\, x \in X$. Letting $k \to \infty$ in (3.10), we see that (3.3) holds for all $x \in X$. To prove Q satisfies (1.7), replacing $(x_1, x_2, \ldots, x_n)$ by $\left((n-2)^k x_1, (n-2)^k x_2, \ldots, (n-2)^k x_n\right)$ and divided by $(n-2)^{2k}$ in (3.2), we arrive

$$\frac{1}{(n-2)^{2k}}\left\|Df_e\left((n-2)^k x_1, \cdots, (n-2)^k x_n\right)\right\| \le \frac{1}{(n-2)^{2k}}\alpha\left((n-2)^k x_1, \ldots, (n-2)^k x_n\right)$$

for all $x_1, x_2, \ldots, x_n \in X$. Letting $n \to \infty$ and in the above inequality, we see that

$$\left\|DQ\left((n-2)^k x_1, \cdots, (n-2)^k x_n\right)\right\| = 0$$

for all $x_1, x_2, \ldots, x_n \in X$. Hence Q satisfies (1.7) for all $x_1, x_2, \ldots, x_n \in X$. To prove Q is unique, let $Q'(x)$ be another quadratic mapping satisfying (1.7) and (3.3). Then

$$\begin{aligned}\|Q(x) - Q'(x)\| &= \frac{1}{(n-2)^{2k}}\left\{\left\|Q\left((n-2)^k x\right) - f_e\left((n-2)^k x\right)\right\| + \left\|f_e\left((n-2)^k x\right) - Q'\left((n-2)^k x\right)\right\|\right\} \\ &\le \frac{2}{(n-2)}\sum_{i=0}^{\infty}\frac{\beta\left((n-2)^{k+i} x\right)}{(n-2)^{2(k+i)}} \quad \to 0 \ \text{as} \ n \to \infty\end{aligned}$$

for all $x \in X$. Hence Q is unique. For $j = -1$, we can prove the similar type of stability result. Hence completes the proof.

The following corollary is an immediate consequence of Theorem 3.1 concerning the stability of (1.7).

Corollary 3.2. Let λ and s be nonnegative real numbers. Let an even function $f_e: X \to Y$ satisfies the inequality

$$\left\|Df(x_1, x_2, \ldots, x_n)\right\| \le \begin{cases} \lambda, \\ \lambda\sum_{i=1}^{n}\|x_i\|^s, & s \ne 2; \\ \lambda\prod_{i=1}^{n}\|x_i\|^s, & s \ne \frac{2}{n}; \\ \lambda\left\{\sum_{i=1}^{n}\|x_i\|^{ns} + \prod_{i=1}^{n}\|x_i\|^s\right\}, & s \ne \frac{2}{n} \end{cases} \tag{3.12}$$

for all $x_1, \ldots, x_n \in X$. Then there exists a unique quadratic function $Q: X \to Y$ such that

$$\left\| f_e(x) - Q(x) \right\| \le \begin{cases} \dfrac{\lambda}{n((n-2)^2-1)}, \\ \dfrac{\lambda \| x \|^s}{|(n-2)^2-(n-2)^s|}, \\ \dfrac{\lambda \| x \|^{ns}}{n|(n-2)^2-(n-2)^s|}, \\ \dfrac{\lambda(n+1)\| x \|^{ns}}{n|(n-2)^2-(n-2)^{ns}|} \end{cases} \tag{3.13}$$

for all $x \in X$.

Theorem 3.3. Let $j \in \{-1,1\}$ and $\alpha : X^n \to [0,\infty)$ be a function such that

$$\sum_{k=0}^{\infty} \frac{\alpha\left((n-2)^{kj}x_1,\ldots,(n-2)^{kj}x_n\right)}{(n-2)^{kj}} \text{ converges to } \mathbb{R} \lim_{k\to\infty} \frac{\alpha\left((n-2)^{kj}x_1,\ldots,(n-2)^{kj}x_n\right)}{(n-2)^{kj}} < \infty \tag{3.14}$$

for all $x_1,\cdots,x_n \in X$. Let $f_o : X \to Y$ be an odd function satisfying the inequality

$$\left\| Df_o(x_1,\cdots,x_n) \right\| \le \alpha\left((n-2)^{kj}x_1,\ldots,(n-2)^{kj}x_n\right) \tag{3.15}$$

for all $x_1,\cdots,x_n \in X$. Then there exists a unique additive mapping $A: X \to Y$ which satisfying (1.7) such that

$$\left\| f_o(x) - Q(x) \right\| \le \frac{1}{(n-2)} \sum_{i=\frac{1-j}{2}}^{\infty} \frac{\beta((n-2)^{ij}x)}{(n-2)^{ij}} \tag{3.16}$$

for all $x \in X$. The mapping $A(x)$ is defined by

$$A(x) = \lim_{k\to\infty} \frac{f_o((n-2)^{kj}x)}{(n-2)^{kj}} \tag{3.17}$$

for all $x \in X$.

Proof. Assume $j = 1$. Since f is an odd function. Replacing $(x_1, x_2, \ldots, x_n)$ by $(x, x, \ldots, x)$ and divided by $(n-2)$ in (3.2), we get

$$\left\| \frac{f_o((n-2)x)}{(n-2)} - f_o(x) \right\| \le \frac{1}{n(n-2)} \alpha\left(x, x, \ldots x\right) \tag{3.18}$$

for all $x \in X$. Letting $\beta(x) = \frac{1}{n}\alpha\left(x, x, \ldots, x\right)$ in (3.6), we arrive

$$\left\| \frac{f_o((n-2)x)}{(n-2)} - f_o(x) \right\| \le \frac{1}{(n-2)} \beta\left(x\right) \tag{3.19}$$

for all $x \in X$. Now replacing x by $(n-2)x$ and dividing by $(n-2)$ in (3.19), we get

$$\left\| \frac{f_o((n-2)^2x)}{(n-2)^2} - \frac{f_o((n-2)x)}{(n-2)} \right\| \le \frac{1}{(n-2)^2} \beta((n-2)x) \tag{3.20}$$

for all $x \in X$. Combining (3.19) and (3.20), we obtain

$$\left\| \frac{f_o((n-2)^2x)}{(n-2)^2} - f_o(x) \right\| \le \frac{1}{(n-2)} \left[\beta\left(x\right) + \frac{\beta\left((n-2)x\right)}{(n-2)} \right] \tag{3.21}$$

for all $x \in X$. In general for any positive integer n , we have

$$\begin{aligned} \left\| \frac{f_o((n-2)^2x)}{(n-2)^2} - f_o(x) \right\| &\le \frac{1}{(n-2)} \sum_{i=0}^{k-1} \frac{\beta\left((n-2)^i x\right)}{(n-2)^i} \\ &\le \frac{1}{(n-2)} \sum_{i=0}^{\infty} \frac{\beta\left((n-2)^i x\right)}{(n-2)^i} \end{aligned} \tag{3.22}$$

for all $x \in X$. In order to prove the convergence of the sequence $\left\{ \dfrac{f_o((n-2)^k x)}{(n-2)^k} \right\}$, replace x by $(n-2)^m x$ and dividing by $(n-2)^m x$ in (3.21), for any $m, k > 0$, we arrive

$$\left\|\frac{f_o\left((n-2)^k(n-2)^m x\right)}{(n-2)^{k+m}}-\frac{f_o\left((n-2)^m x\right)}{(n-2)^m}\right\|=\frac{1}{(n-2)^m}\left\|\frac{f_o\left((n-2)^k(n-2)^m x\right)}{(n-2)^{k+m}}-f_o\left((n-2)^m x\right)\right\|$$

$$\leq\frac{1}{(n-2)}\sum_{i=0}^{k-1}\frac{\beta\left((n-2)^i(n-2)^m x\right)}{(n-2)^{i+m}} \tag{3.23}$$

$$\leq\frac{1}{(n-2)}\sum_{i=0}^{\infty}\frac{\beta\left((n-2)^{i+m} x\right)}{(n-2)^{i+m}}$$

for all $x\in X$. Since the right hand side of the inequality (3.23) tends to 0 as $m\to\infty$, the sequence $\left\{\frac{f_o((n-2)^k x)}{(n-2)^k}\right\}$ is a Cauchy sequence. Since Y is complete, there exists a mapping $A: X\to Y$ such that $A(x)=\lim_{k\to\infty}\frac{f_o((n-2)^k x)}{(n-2)^k}$, $\forall\, x\in X$. Letting $k\to\infty$ in (3.22), we see that (3.16) holds for all $x\in X$. In order to prove A satisfies (1.7) and it is unique the proof is similar to that of Theorem 3.1.

The following corollary is an immediate consequence of Theorem 3.3 concerning the stability of (1.7).

Corollary 3.4 Let λ and s be nonnegative real numbers. If a odd function $f_o: X\to Y$ satisfies the inequality

$$\left\|Df(x_1,x_2,\ldots,x_n)\right\|\leq\begin{cases}\lambda, & \\ \lambda\sum_{i=1}^{n}\left\|x_i\right\|^s, & s\neq 1;\\ \lambda\prod_{i=1}^{n}\left\|x_i\right\|^s, & s\neq\frac{1}{n}\\ \lambda\left\{\sum_{i=1}^{n}\left\|x_i\right\|^{ns}+\prod_{i=1}^{n}\left\|x_i\right\|^s\right\}, & s\neq\frac{1}{n}\end{cases} \tag{3.24}$$

for all $x_1,\ldots,x_n\in X$. Then there exists a unique additive function $A: X\to Y$ such that

$$\left\|f_o(x)-A(x)\right\|\leq\begin{cases}\frac{\lambda}{n(n-3)},\\ \frac{\lambda\|x\|^s}{|(n-2)-(n-2)^s|},\\ \frac{\lambda\|x\|^{ns}}{n|(n-2)-(n-2)^s|},\\ \frac{\lambda(n+1)\|x\|^{ns}}{n|(n-2)-(n-2)^{ns}|}\end{cases} \tag{3.25}$$

for all $x\in X$.

Now, we are ready to prove our main theorem.

Theorem 3.5 Let $j\in\{-1,1\}$ and $\alpha: X^n\to[0,\infty)$ be a function with conditions (3.1) and (3.14) for all $x_1,\cdots,x_n\in X$. Let $f: X\to Y$ be a function satisfying the inequality

$$\left\|Df(x_1,\cdots,x_n)\right\|\leq\alpha(x_1,\cdots,x_n) \tag{3.26}$$

for all $x_1,\cdots,x_n\in X$. Then there exists a unique additive mapping $A: X\to Y$ and a unique quadratic mapping $Q: X\to Y$ which satisfying (1.7) such that

$$\|f(x)-A(x)-Q(x)\| \le \frac{1}{2}\left[\frac{1}{(n-2)^2}\sum_{i=\frac{1-j}{2}}^{\infty}\left(\frac{\beta((n-2)^{ij}x)}{(n-2)^{2ij}}+\frac{\beta(-(n-2)^{ij}x)}{(n-2)^{2ij}}\right)\right.$$
$$\left.+\frac{1}{(n-2)}\sum_{i=\frac{1-j}{2}}^{\infty}\left(\frac{\beta((n-2)^{ij}x)}{(n-2)^{ij}}+\frac{\beta(-(n-2)^{ij}x)}{(n-2)^{ij}}\right)\right] \tag{3.27}$$

for all $x \in X$. The mapping $A(x)$ and $Q(x)$ are defined in (3.5) and (3.17) respectively for all $x \in X$.

Proof. Let $f_o(x) = \frac{f(x)-f(-x)}{2}$ for all $x \in X$. Then $f_o(0) = 0$ and $f_o(-x) = -f_o(x)$ for all $x \in X$. Hence

$$\|Df_o(x_1,x_2,\ldots,x_n)\| \le \frac{1}{2}\{\alpha(x_1,x_2,\ldots,x_n)+\alpha(-x_1,-x_2,\ldots,-x_n)\} \tag{3.28}$$

for all $x_1,x_2,\ldots,x_n \in X$. By Theorem 3.3, we have

$$\|f_o(x)-A(x)\| \le \frac{1}{2(n-2)}\sum_{i=\frac{1-j}{2}}^{\infty}\left(\frac{\beta((n-2)^{ij}x)}{(n-2)^{ij}}+\frac{\beta(-(n-2)^{ij}x)}{(n-2)^{ij}}\right)\Bigg]$$

for all $x \in X$. Also, let $f_e(x) = \frac{f(x)+f(-x)}{2}$ for all $x \in X$. Then $f_e(0) = 0$ and $f_e(-x) = f_e(x)$ for all $x \in X$. Hence

$$\|Df_e(x_1,x_2,\ldots,x_n)\| \le \frac{1}{2}\{\alpha(x_1,x_2,\ldots,x_n)+\alpha(-x_1,-x_2,\ldots,-x_n)\} \tag{3.29}$$

for all $x_1,x_2,\ldots,x_n \in X$. By Theorem 3.1, we have

$$\|f_e(x)-Q(x)\| \le \frac{1}{2(n-2)^2}\sum_{i=\frac{1-j}{2}}^{\infty}\left(\frac{\beta((n-2)^{ij}x)}{(n-2)^{2ij}}+\frac{\beta(-(n-2)^{ij}x)}{(n-2)^{2ij}}\right) \tag{3.30}$$

for all $x \in X$. Define

$$f(x) = f_e(x) + f_o(x) \tag{3.31}$$

for all $x \in x$. From (3.28) and (3.29), we arrive

$$\|f(x)-A(x)-Q(x)\| = \|f_e(x)+f_o(x)-A(x)-Q(x)\| \le \|f_o(x)-A(x)\|+\|f_e(x)-Q(x)\|$$

$$\le \frac{1}{2}\left[\frac{1}{(n-2)^2}\sum_{i=\frac{1-j}{2}}^{\infty}\left(\frac{\beta((n-2)^{ij}x)}{(n-2)^{2ij}}+\frac{\beta(-(n-2)^{ij}x)}{(n-2)^{2ij}}\right)+\frac{1}{(n-2)}\sum_{i=\frac{1-j}{2}}^{\infty}\left(\frac{\beta((n-2)^{ij}x)}{(n-2)^{ij}}+\frac{\beta(-(n-2)^{ij}x)}{(n-2)^{ij}}\right)\right]$$

for all $x \in X$. Hence the theorem is proved.

Using Corollaries 3.2 and 3.4, we have the following corollary concerning the stability of (1.7).

Corollary 3.6 Let λ and s be nonnegative real numbers. If a function $f: X \to Y$ satisfies the inequality

$$\|Df(x_1,x_2,\ldots,x_n)\| \le \begin{cases} \lambda, & \\ \lambda\sum_{i=1}^{n}\|x_i\|^s, & s \ne 1,2; \\ \lambda\prod_{i=1}^{n}\|x_i\|^s, & s \ne \frac{1}{n},\frac{2}{n}; \\ \lambda\left\{\sum_{i=1}^{n}\|x_i\|^{ns}+\prod_{i=1}^{n}\|x_i\|^s\right\}, & s \ne \frac{1}{n},\frac{2}{n} \end{cases} \tag{3.32}$$

for all $x_1,\ldots,x_n \in X$. Then there exists a unique additive function $A: X \to Y$ and a unique quadratic function $Q: X \to Y$ such that

$$\|f(x)-A(x)-Q(x)\| \leq \begin{cases} \frac{\lambda}{n}\left(\frac{1}{(n-3)}+\frac{1}{((n-2)^2-1)}\right), \\ \lambda\left(\frac{1}{|(n-2)-(n-2)^s|}+\frac{1}{|(n-2)^2-(n-2)^s|}\right)\|x\|^s, \\ \frac{\lambda}{n}\left(\frac{1}{|(n-2)-(n-2)^s|}+\frac{1}{|(n-2)^2-(n-2)^s|}\right)\|x\|^{ns}, \\ \frac{\lambda(n+1)}{n}\left(\frac{1}{|(n-2)-(n-2)^{ns}|}+\frac{1}{|(n-2)^2-(n-2)^{ns}|}\right)\|x\|^{ns} \end{cases} \tag{3.33}$$

for all $x \in X$.

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Computation of Four Degree Based Topological Invariants of Some Nanostructures

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Abstract*: The topological indices provide complete information about the molecular graph. In particular, degree or valency based topological indices are very much useful in structure property/activity relationships. Carbon nanotubes are being used majorly for its excellent properties. In this article, we have computed four valency based indices such as Reciprocal Randic index, Reduced Reciprocal Randic index, Reduced second Zagreb index and Forgotten topological indices of Linear [n]- anthracene, V-anthracene – 2D Lattice, Nanotube and Nanotori.*

Keywords*: Molecular Descriptors; Degree based invariants; Molecular Structures; Nanostructures; Anthracene.*

I. Introduction

A molecular network is a simple and undirected network represents hydrogen suppressed structure of an organic molecule. The nodes of a molecular network are considered as a carbon atoms and its undirected links are considered as carbon-carbon bonds. Let H=[V,E] be a connected molecular network with vertex (atom) set V(H) and link (bond) set E(H). The degree or valency d_v of a vertex is the number of nodes adjacent to v. A network can be identified by a numerical value or a polynomial or a matrix. A numerical quantity that associated with a network that symbolises the topology of the graph. Several topological indices are widely used in QSAR and QSPR studies. The studies have great importance in modern chemistry and biochemistry. Moreover, the applications of topological indices are found in pharmaceutical drug design, chemical documentation and isomer discrimination. The notion of topological indices was introduced by well known chemist Herold Wiener in 1947 [16]. He defined a topological index called Wiener index, based on the path number associated with a chemical graph. Likewise, the notion topological indices based on the vertex degree was introduced by famous chemist Randic under the name branching index also called as connectivity index or Randic index.

Randic index [15] of any network H is denoted by R(H) defined as

$$R(H) = \sum_{uv \in E(H)} (d_u d_v)^{-1/2} \tag{1}$$

Later, this index was generalized for any real α instead of -1/2, referred as general Randic index. Recent past, Gutman et.al. [10-12] initiated the study on four node-valency based topological indices. The new/old node-valency based topological invariants are Reciprocal Randic index RR(H), Reduced Reciprocal Randic index RRR(H), Reduced second Zagreb index $RM_2(H)$ and Forgotten topological index F(H).

The Reciprocal Randic index of a network H is defined as,

$$\mathrm{RR(H)} = \sum_{uv \in E(H)} (d_u d_v)^{1/2} \tag{2}$$

The Reduced Reciprocal Randic index of a network H is defined as,

$$RRR(H) = \sum_{uv \in E(H)} \sqrt{(d_u - 1)(d_v - 1)} \tag{3}$$

The Reduced Second Zagerb index of a network H is defined as,

$$RM_2(H) = \sum_{uv \in E(H)} (d_u - 1)(d_v - 1) \tag{4}$$

The forgotten index or F-index of any network H is defined as

$$F(H) = \sum_{uv \in E(H)} d_u^2 + d_v^2 \tag{5}$$

A solid polycyclic aromatic hydrocarbon consisting of three fused benzene rings with the chemical formula $C_{14}H_{10}$ called anthracene or paranapthalene. It is used in manufacture of red dye alizarin. It is also used as an intermediate in the production of smoke screens, dyes, organic semi conductors research, scintillation counter crystals and as a coal tar component.

In 2017, Aslam et al. [2] found the results of three new/old vertex degree based topological indices of some dendrimers, Gaoa et al [8] calculated the vertex degree based topological indices of line network of subdivision network of some nanostructures and Jamil et al [13] calculated the same vertex degree based topological indices of $VC_5C_7[p;q]$ Nanotubes. In 2018, Xu et al [14] gave explicit expression for the new/old vertex degree based topological indices of certain nanostructures.

For more results on topological indices of several nanostructures and para-line network of nanostructures one can refer [1-9].

II. Degree Based Topological Invariants of Anthracene Nanostructures

In this section, we compute Reciprocal Randic index, Reduced Reciprocal Randic index, Reduced second Zagreb index and Forgotten topological indices of Linear [n]- anthracene, V-anthracene – 2D Lattice, Nanotube and Nanotori.

Theorem 1: Let G_1 be the molecular network of linear [n] – anthracene. Then

$$RR(G_1) = 6n\left[2\sqrt{6}+3\right] - 4\sqrt{6}$$
$$RRR(G_1) = 2\left[6\sqrt{2}n - 2\sqrt{2} + 6n - 1\right]$$
$$RM_2(G_1) = 48n - 18$$
$$F(G_1) = 2[132n - 38]$$

Proof: In G_1, there are 14n nodes and 18n-2 links. Based on the degree of end nodes the link partition of G_1 consists of three separations such as E_1, E_2 and E_3, where $E_1 = \{uv \in E(G_1) \ni d_u = 2 \text{ and } d_v = 2\}$, $E_2 = \{ uv \in E(G_1) \ni d_u = 2 \text{ and } d_v = 3\}$ and $E_3 = \{ uv \in E(G_1) \ni d_u = 3 \text{ and } d_v = 3\}$. Therefore, number of links in the link separations E_1, E_2 and E_3 are 6, 12n-4 and 6n-4 respectively. Hence,

$$RR(G_1) = \sum_{uv \in E(G_1)} \sqrt{d_u d_v}$$
$$= \sum_{uv \in E_1(G_1)} \sqrt{d_u d_v} + \sum_{uv \in E_2(G_1)} \sqrt{d_u d_v} + \sum_{uv \in E_3(G_1)} \sqrt{d_u d_v}$$
$$= 6\sqrt{2 \times 2} + (12n\text{-}4)\sqrt{2 \times 3} + (6n\text{-}4)\sqrt{3 \times 3}.$$
$$= 6n\left[2\sqrt{6}+3\right] - 4\sqrt{6}.$$

$$RRR(G_1) = \sum_{uv\in E(G_1)} \sqrt{(d_u-1)(d_v-1)}$$

$$= \sum_{uv\in E_1(G_1)} \sqrt{(d_u-1)(d_v-1)} + \sum_{uv\in E_2(G_1)} \sqrt{(d_u-1)(d_v-1)} + \sum_{uv\in E_3(G_1)} \sqrt{(d_u-1)(d_v-1)}$$

$$= 6\sqrt{1\times 1} + (12n-4)\sqrt{1\times 2} + (6n-4)\sqrt{2\times 2}.$$

$$= 2\left[6\sqrt{2}n - 2\sqrt{2} + 6n - 1\right].$$

$$RM_2(G_1) = \sum_{uv\in E(G_1)} (d_u-1)(d_v-1)$$

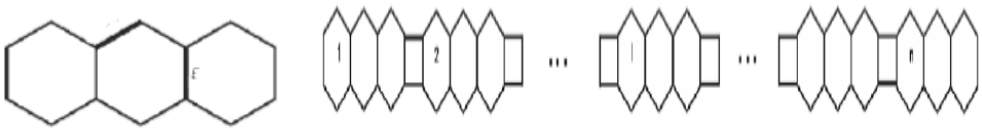
$$= \sum_{uv\in E_1(G_1)} (d_u-1)(d_v-1) + \sum_{uv\in E_2(G_1)} (d_u-1)(d_v-1) + \sum_{uv\in E_3(G_1)} (d_u-1)(d_v-1)$$

$= 6(1) + (12n-4)2 + (6n-4)4 = 48n - 18.$

$$F(G_1) = \sum_{uv\in E(G_1)} d_u^2 + d_v^2$$

$$= \sum_{uv\in E_1(G_1)} d_u^2 + d_v^2 + \sum_{uv\in E_2(G_1)} d_u^2 + d_v^2 + \sum_{uv\in E_3(G_1)} d_u^2 + d_v^2$$

$= 6\,(2^2+2^2) + (12n-4)(\,2^2 + 3^2) + (6n-4)(3^2+3^2) = 2[132n - 38].$



(a) (b)

Figure 1: (a) Linear [1] – Anthracene (b) Linear [n] – Anthracene

Theorem 2: Let $G_2[t,s]=G_2$ be the molecular network of 2D-lattice V- anthracene. Then

$$RR(G_2) = 63ts - 45t - 20s + 20 + \sqrt{6}[12t + 4s - 8]$$

$$RRR(G_2) = 42ts - 14s - 30t + 12 + 2\sqrt{2}[6t + 2s - 4]$$

$$RM_2(G_2) = 84ts - 36t - 22s + 4$$

$$F(G_2) = 378ts - 76s - 114t$$

Proof: In G_2, there are 14ts nodes and 21ts-3t-2s links. Based on the degree of end vertices, the link partition of G_2 consists of three separations such as E_1, E_2 and E_3, where $E_1 = \{uv\in E(G_2) \ni d_u = 2$ and $d_v = 2\}$, $E_2 = \{ uv \in E(G_2) \ni d_u = 2$ and $d_v = 3\}$ and $E_3 = \{ uv\in E(G_2) \ni d_u = 3$ and $d_v = 3\}$.Therefore, number of links in the link separations E_1, E_2 and E_3 are 2s+4, 12t+4s-8 and 21ts-15t-8s+4 respectively. Hence,

$$RR(G_2) = \sum_{uv \in E(G_2)} \sqrt{d_u d_v}$$

$$= \sum_{uv \in E_1(G_2)} \sqrt{d_u d_v} + \sum_{uv \in E_2(G_2)} \sqrt{d_u d_v} + \sum_{uv \in E_3(G_2)} \sqrt{d_u d_v}$$

= (2s+4)$\sqrt{2 \times 2}$ + (12t+4s-8) $\sqrt{2 \times 3}$ + (21ts-15t-8s+4) $\sqrt{3 \times 3}$.

= 63ts-45t-20s+20+ $\sqrt{6}$[12t+4s-8].

$$RRR(G_2) = \sum_{uv \in E(G_2)} \sqrt{(d_u - 1)(d_v - 1)}$$

$$= \sum_{uv \in E_1(G_2)} \sqrt{(d_u - 1)(d_v - 1)} + \sum_{uv \in E_2(G_2)} \sqrt{(d_u - 1)(d_v - 1)} + \sum_{uv \in E_3(G_2)} \sqrt{(d_u - 1)(d_v - 1)}$$

=(2s+4) (1) + $(12t + 4s - 8)$ 2 + $(21ts - 15t - 8s + 4)4$

= $42ts - 14s - 30t + 12 + 2\sqrt{2}[6t + 2s - 4]$.

$$RM_2(G_2) = \sum_{uv \in E(G_2)} (d_u - 1)(d_v - 1)$$

= $(2s + 4)$ (1) + $(12t + 4s - 8)$ 2 + $(21ts - 15t - 8s + 4)4$

= $84ts - 36t - 22s + 4$.

$$F(G_1) = \sum_{uv \in E(G_1)} d_u^2 + d_v^2$$

$$= \sum_{uv \in E_1(G_1)} d_u^2 + d_v^2 + \sum_{uv \in E_2(G_1)} d_u^2 + d_v^2 + \sum_{uv \in E_3(G_1)} d_u^2 + d_v^2$$

= $(2s + 4)(2^2+2^2)$ + $(12t + 4s - 8)(2^2+3^2)$ + $(21ts - 15t - 8s + 4)(3^2+3^2)$

= 378ts-76s-114t .

Theorem 3: Let $G_3[t,s]=G_3$ be the molecular network of V- anthracene-nanotube.

Then

$$RR(G_3) = 3t(4\sqrt{6} + 21s - 15)$$

$$RRR(G_3) = 6t(2\sqrt{2} + 7s - 5)$$

$$RM_2(G_3) = 12t(7s - 3)$$

$$F(G_3) = 378ts - 114t$$

Proof: In G_3, there are 14ts nodes and 21ts-3t links. Based on the degree of end vertices, the link partition of G_3 consists of two separations such as E_1 and E_2, where E_1 = { $uv \in E(G_3) \ni d_u = 2$ and $d_v = 3$} and E_2 = { $uv \in E(G_3) \ni d_u = 3$ and $d_v = 3$}. Therefore, number of links in the link separations E_1 and E_2 are 12t and 21ts-15t respectively. Hence,

$$RR(G_3) = \sum_{uv \in E(G_3)} \sqrt{d_u d_v}$$

$$= \sum_{uv \in E_1(G_3)} \sqrt{d_u d_v} + \sum_{uv \in E_2(G_3)} \sqrt{d_u d_v}$$

= (12t) $\sqrt{2 \times 3}$ + (21ts-15t) $\sqrt{3 \times 3}$.

= $3t(4\sqrt{6} + 21s - 15)$.

$$RRR(G_3) = \sum_{uv \in E(G_3)} \sqrt{(d_u - 1)(d_v - 1)}$$

$$= \sum_{uv \in E_1(G_3)} \sqrt{(d_u - 1)(d_v - 1)} + \sum_{uv \in E_2(G_3)} \sqrt{(d_u - 1)(d_v - 1)}$$

$$= (12t)\sqrt{1 \times 2} + (21ts - 15t)\sqrt{2 \times 2}.$$

$$= 6t(2\sqrt{2} + 7s - 5).$$

$$RM_2(G_3) = \sum_{uv \in E(G_3)} (d_u - 1)(d_v - 1)$$

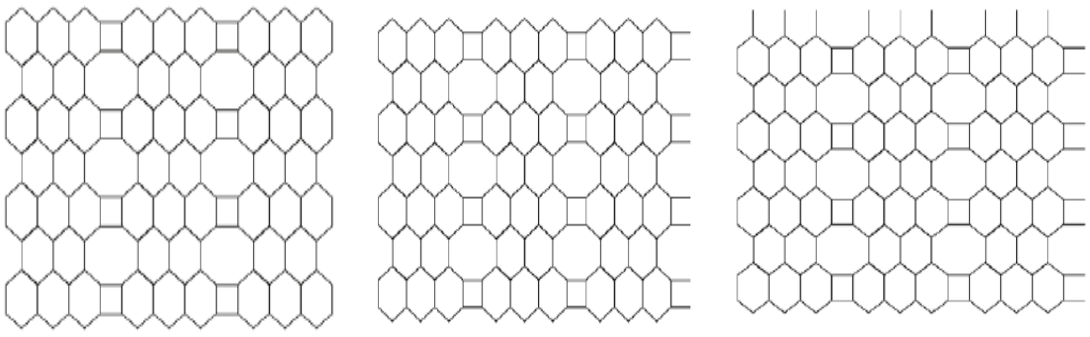
$$= \sum_{uv \in E_1(G_3)} (d_u - 1)(d_v - 1) + \sum_{uv \in E_2(G_3)} (d_u - 1)(d_v - 1)$$

$$= (12t)\,2 + (21ts - 15t)4 = 12t(7s - 3).$$

$$F(G_3) = \sum_{uv \in E(G_3)} d_u^2 + d_v^2$$

$$= \sum_{uv \in E_1(G_3)} d_u^2 + d_v^2 + \sum_{uv \in E_2(G_3)} d_u^2 + d_v^2$$

$= (12t)\,(2^2+3^2) + (21ts - 15t)\,(3^2+3^2) = 378ts-114t.$



(a) (b) (c)

Figure 2: (a) 2D-lattice V- anthracene $G_2[t,s]$, (b) V- anthracene-nanotube $G_3[t,s]$ (b) V- anthracene-nanotori $G_4[t,s]$, where t=3 and s=4.

Theorem 4: Let $G_4[t,s]=G_4$ be the molecular network of V- anthracene-nanotori.

Then

$$RR(G_4) = 63ts$$
$$RRR(G_4) = 42ts$$
$$RM_2(G_4) = 84ts$$
$$F(G_4) = 378ts$$

Proof: In G_4, there are 14ts nodes and 21ts links. Based on the degree of end vertices, the link partition of G_4 is E_1, where E1 = { $uv \in E(G_3) \ni d_u = 3$ and $d_v = 3$}. Therefore, number of links in the link partition E_1 is 21ts respectively. Hence,

$$RR(G_3) = \sum_{uv \in E(G_3)} \sqrt{d_u d_v}$$
$$= \sum_{uv \in E_1(G_3)} \sqrt{d_u d_v}$$
$$= (21ts)\sqrt{3 \times 3}. = 63ts.$$

$$RRR(G_3) = \sum_{uv \in E(G_3)} \sqrt{(d_u - 1)(d_v - 1)}$$
$$= \sum_{uv \in E_1(G_3)} \sqrt{(d_u - 1)(d_v - 1)}$$
$$= (21ts)\sqrt{2 \times 2}. = 42ts.$$

$$RM_2(G_3) = \sum_{uv \in E(G_3)} (d_u - 1)(d_v - 1)$$
$$= \sum_{uv \in E_1(G_3)} (d_u - 1)(d_v - 1)$$
$$= (21ts(4) = 84ts.$$

$$F(G_3) = \sum_{uv \in E(G_3)} d_u^2 + d_v^2$$
$$= \sum_{uv \in E_1(G_3)} d_u^2 + d_v^2$$
$$= (21ts)\ (3^2+3^2) = 378ts.$$

III. Conclusion

In this article, we have computed four valency based indices such as Reciprocal Randic index, Reduced Reciprocal Randic index, Reduced second Zagreb index and Forgotten topological indices of Linear [n]-anthracene, V-anthracene – 2D Lattice, Nanotube and Nanotori. Recent past, Gutman et al (re) introduced above four new/old degrees based topological indices and explained its significance in the study of chemical graph theory.

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EXTENDED RESULTS IN PENDANT DOMINATION POLYNOMIAL OF A GRAPH

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***Abstract**: Let G= (V, E) be a graph. A subset S of vertices in G is called a pendant dominating set if the sub graph induced by S contains a pendant vertex. The pendant domination polynomial of G is defined as $D_{pe}(G,x) = \sum_{i=1}^{n} d_{pe}(G,i)x^i$, where $d_{pe}(G,i)$ is the number of pendant dominating sets of size i. In this paper, we extend the study of pendant domination polynomial of a graph and recurrence relations are obtained for polynomials of path and cycle graphs*

***Keywords**: Domination, Pendant Domination, Domination Polynomial*

I. Introduction

Let $G = (V, E)$ be any graph of order n and size m. A vertex of degree zero is called an isolated vertex and a vertex of a degree one is called a pendant vertex. An edge incident to a pendant vertex is called a pendant edge. For a detailed treatment of the domination polynomial of a graph, the reader is referred to[1]. We have introduced and studied the concept of pendant domination polynomial of G, and obtained pendant domination polynomial for some standard graphs in [4]. The join of G and H denoted by $G \vee H$ is the graph such that $G \vee H = V(G) \cup V(H) and\ E(G \vee H) = E(G) \cup E(H) \cup \{uv: u \in V(G), v \in V(H)\backslash\}$. Let G_1 and G_2 be sub graphs of G. We say that G_1 and G_2 are disjoint if they have no vertex in common and edge-disjoint if they have no edge in common. The union $G_1 \cup G_2$ of G_1 and G_2 is the subgraph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$; if G_1 and G_2 are disjoint, we sometimes denote their union by $G_1 \vee G_2$. A graph G is called a bi-star if it can be constructed from K_2 by touching m edges in one vertex and n in the another vertex, denoted by $B(m,n)$. Graph polynomial is good representation for the graph. There are many polynomials which play an important role in Combinatorics. One of the recent graph polynomial is domination polynomial which was introduced in [1]. The dominating polynomial motivated us to introduce and study the pendant domination polynomial in graph.

Dominating set: A subset S of V (G) is a dominating set of G if each vertex $u \in V - S$ is adjacent to a vertex in S. The least cardinality of a domination set is called the domination number of G, denoted by $\gamma(G)$.

Isolated Dominating set: A subset S of V(G) is a called an isolated dominating set of G if $< S >$ contains at least one isolated vertex. The minimum cardinality of an isolated dominating set in G is called the isolated domination number of G, denoted by $\gamma_0\ (G)$.

Pendant Domination Number: A dominating set S in G is called a pendant dominating set if $< S >$ contains at least one pendant vertex. The minimum cardinality of a pendant dominating set is called the pendant domination number denoted by $\gamma_{pe}\ (G)$.The pendant domination polynomial is defined for all non-trivial connected graphs of order at least two. Hence, throughout the paper we assume that by a graph G, we mean a connected graph of order at least two.

II. Pendant Domination Polynomial of a graph:

Let $G = (V, E)$ be any graph of order $n \geq 2$. The pendant domination polynomial of G is denoted by $D_{pe}(G, x)$ and is defined as $D_{pe}(G, x) = \sum_{i=1}^{n} d_{pe}(G, i)x^i$, where $d_{pe}(G, i)$ is the number of pendant dominating sets of size i.

The pendant domination polynomial is defined for all non-trivial connected graphs of order at least two. Hence, throughout the paper we assume that by a graph G, we mean a connected graph of order at least two.

Observations 1.1: Let G be any connected graph of order $n \geq 2$, Then

1. $d_{pe}(G, n) = \begin{cases} 1, & if\ \delta(G) = 1. \\ 0, & otherwise \end{cases}$
2. $d_{pe}(G, i) = 0$ if and only if $i < \gamma_{pe}(G) or\ i > n$.
3. $D_{pe}(G, x)$ is a monotonically increasing function.
4. $D_{pe}(G, x)$ has no constant.
5. Zero is a root of $D_{pe}(G, x)$ of multiplicity $\gamma_{pe}(G)$.

Theorem 1.1: If a graph G consists of m components G_1 , $G_2, \dots, G_m$, then $D(G, x) = D(G_1, x)D(G_2, x) \cdots D(G_m, x)$.

Proof: Let G be a disconnected graph having components. Then, the pendant domination polynomial of each connected component will be a factor of pendant domination polynomial of G. Therefore, pendant domination polynomial of G is obtained by considering the product of pendant domination polynomial of each component.

Proposition 1.1[4]: Let G be a connected graph of order $n \geq 2$. Then $D_{pe}(G, x) = \binom{n}{2} x^2$ if and only if G is a complete graph of order n.

Theorem 1.2[4]: Let G_1, G_2 be connected graphs of order $n, m \geq 2$ respectively. Then $D_{pe}(G_1 \vee G_2, x) = nmx^2 + D_{pe}(G_1, x) + D_{pe}(G_2, x)$.

Theorem 1.3: Let G_1, G_2 be connected graphs of order $n, m \geq 2$ respectively. Then $\$D_{pe}(G_1 \cup G_2, x) = D_{pe}(G_1, x) + D_{pe}(G\_2, x)$.

Proposition 1.2. The following properties hold for coefficients of $D_{pe}(C_n, x)$.

1. $d_{pe}(C_n, n) = 1, if\ n \geq 2.$
2. $d_{pe}(C_n, n-1) = n, if\ n \geq 3.$
3. $d_{pe}(C_{3n}, n) = 0.$
4. $d_{pe}(C_{3n}, n+1) = n(n+1).$
5. $d_{pe}(C_{3n+1}, n+1) = n.$
6. $d_{pe}(C_{4n}, 2n) = 1.$

Proposition 1.3. The following properties hold for coefficients of $D_{pe}(P_n, x)$.

1. $d_{pe}(P_n, n) = 1, if\ n \geq 2.$
2. $d_{pe}(P_n, n-1) = n, if\ n \geq 3.$
3. $d_{pe}(P_{3n}, n) = 0.$
4. $d_{pe}(P_{3n}, n+1) = n(n+1).$
5. $d_{pe}(P_{3n+1}, n+1) = n.$
6. $d_{pe}(P_{4n}, 2n) = 1.$

Theorem 1.4 [4]: Let $G \cong K_{1,n}$ be a star of order n. Then $D_{pe}(G, x) = x((1+x)^{n-1} - 1))$.

Converse of the above result is also true. We state above theorem as follows:

Theorem 1.5: Let G be any connected graph n. Then $D_{pe}(G, x) = x((1+x)^{n-1} - 1))$ if and only if G is a star on n vertices.

Proof: Let G be a connected graph of order n with the pendant domination polynomial $D_{pe}(G, x) = x((1+x)^{n-1} - 1))$. It is clear from the polynomial that, there are exactly n-1 pendant dominating sets of

cardinality 2. Since, pendant dominating set of size two leads to an edge in G, it follows that G contains at least n-1 edges. For any integer $k, 2 \leq k \leq n$, the number of pendant dominating set of cardinality k is given by$\binom{n-1}{k-1}$. Thus a pendant dominating set of size k is obtained by choosing k-1 vertices out of n-1 vertices. Therefore, G is isomorphic to the sum of K_1 and a graph of order $n-1$ say H. That is, $G \cong K_1 \vee H$. Clearly H must be totally disconnected. On contrary, suppose H is connected. It is possible to choose a pendant dominating S of H of cardinality say k. Then S is also a pendant dominating set of G of cardinality k. This implies that, $d_{pe}(G,k)$ is at least $\binom{n}{k}+1$, which is a contradiction. This contradiction shows that $H \cong \overline{K}_{n-1}$ and hence$G \cong K_1 \vee \overline{K}_{n-1}$, a star on n vertices. Converse follows from Theorem 1.4.

Theorem 1.6: Let G be any connected graph m+n+2 vertices. Then $D_{pe}(G,x) = x^2(1+x)^{m+n}$ if and only if G is a bi-star on n vertices.

Proof: Let G be a connected graph of order m+n+2 with the pendant domination polynomial $D_{pe}(G,x) = x^2(1+x)^{m+n}$. It is clear from the polynomial that, there is only one pendant dominating set of cardinality 2. Therefore, G contains only one edge $e = uv$ of degree at least m+n. That is, every vertex in G is adjacent to u or v. For if all the vertices in G are adjacent to u only, then for any vertex w different from v, the edge $e' = uw$ will be a pendant dominating set in G, which is not possible. Thus, both u and v are of degree at least two. Suppose S_1, S_2 be any two subsets of $V(G)$ of cardinality m, n respectively, containing neither u nor v.

From the polynomial, it is clear that, there are $\binom{m+n}{k-2}$ ways to select pendant dominating sets of cardinality k by choosing vertices from S_1 and S_2 together with u, v. Now, we show that no vertex in S_1 is adjacent to a vertex in S_2. On contrary, if there is an edge between S_1 and S_2, then we any pendant dominating set of $< V(G) - \{u, v\} >$ of cardinality k will be a pendant dominating set of G. Therefore, there at least $\binom{m+n}{k-2}+1$ possibilities for a pendant dominating sets of cardinality k, a contradiction. This contradiction proves that, there are no edges between S_1 and S_2. Further, it is easy to observe that the sub graphs induced by S_1 and S_2 are totally disconnected. Therefore, G obtained by attaching m, n number of vertices to u, v respectively. That is, $G \cong B(m,n)$.

Theorem 1.7: Let G be a graph of connected graph of order $n \geq 3$ with r isolated vertices. Then

1. $r = n - d_{pe}(G, n-1)$.
2. $d(G,2) = |\{uv \in E(G) | deg(e) \geq 2(n-2) and\ N(v) \cap N(u) = \emptyset\}|$.

Proof:

1. Let G be a connected graph and let A be a collection of isolated vertices in G. Therefore by our assumption, $|A| = r$. Now, for any vertex $v \in V(G) - A$, the set $V(G) - \{v\}$ is a pendant dominating set of G. Therefore, $d_{pe}(G, n-1) = |V(G-A)| = n - r$, and $r = n - d_{pe}(G, n-1)$.
2. A pendant dominating set of cardinality is an edge $e = uv$ in G. Therefore $d_{pe}(G,2)$$ counts number of edges in G dominating G. That is, number of edges $e = uv$ with $deg(e) = deg(u) + deg(v)$ is at least $2(n-2)$ such that $N(v) \cap N(u) = \emptyset$.

A finite sequence of real numbers $\{d_0, d_1, \ldots, d_m\}$ is said to be unimodal if there exists an index $0 \leq j \leq m$ such that $d_0 \leq d_1 \leq \cdots \leq d_j\ and\ d_j \geq d_{j+1} \geq \cdots \geq d_m$$. A polynomial is said to be unimodal if its sequence of coefficients is unimodal.

Conjecture: The domination polynomial of any graph is unimodal.

When counting the number of dominating sets with cardinality close to n, it may simplify things to count the number of subsets which are not dominating. A subset $S \subseteq V(G)$ is not dominating if there exists a vertex v in G such that none of its neighbors, nor itself, is in S. That is, $N[v] \cap S = \emptyset$. The next proposition will help us identify which subsets are not dominating by looking at the subset's complement (with respect to V). If there exists a vertex v and subset $S \subseteq V$ such that $N[v] \subseteq S$ then we say S encompasses v or v is encompassed by S. We shall state the following proposition without proof.

Proposition 1.4: For a graph G and $S \subseteq V(G)$, S is not dominating if and only if there exists a vertex $v \in S$ which is encompassed by S.

III. Conclusion

In recent days, the study of domination and its related parameters has a major significance in graph theory and polynomial representation of graphs leads to study of graphs through graphs. In this article, we are interested in studying behavior of roots of pendant domination polynomial. We also study the unimodality of pendant domination polynomial and try to obtain expressions for $d_{pe}(G, n-2)\ and\ d_{pe}(G, n-3)$.

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VERTEX ODD MEAN LABELING OF SOME CYCLES WITH PARALLEL CHORDS

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Abstract*: During the mid-sixties of twentieth century, Graph labelings were first introduced and in the subsequent years many researchers have studied various graph labeling problems and found solutions. Applications of labeled graphs are found in coding theory, missile guidance codes, X-ray crystallography and communication network. The main aim of this paper is to identify some vertex odd mean graphs among cycles C_n with parallel chords. In this paper we prove that cycles with parallel chords, joining two copies of even cycles with parallel chords by a path, path union of cycles with parallel chords, Butterfly graphs with parallel chords and edge connected cycles with parallel chords are vertex odd mean graphs with suitable illustrations.*

Keywords*: Mean labeling, Vertex odd mean labeling, cycles with parallel chords,Butterfly graphs, Edge*

I. Introduction

The assignment of integers to the vertices (or) edges (or) both subject to certain conditions is defined as graph labeling. In this paper we consider a graph as $G = (V, E)$ where $|V(G)| = p$ and $|E(G)| = q$ which means a finite simple and undirected graph. Most of the references on graph labeling trace their origin to β-valuation introduced by Rosa [11] in 1967 which was later renamed as graceful labeling by Golomb [5]. A detailed survey on graph labeling can also be seen in Gallian [3]. A cycle on n vertices is denoted by C_n and by connecting two otherwise non-adjacent vertices of a cycle, achord (path) of a cycle is obtained.S.Somasundaram and R. Ponraj [13]calls a graph G with p vertices and q edges as a mean graph if there is an injection$f:V(G) \rightarrow \{0,1,2,...,q\}$ such that when each edge xy is labeled with $[f(x)+f(y)] / 2$ if $f(x)+f(y)$ is even, and $[f(x)+f(y)+1] / 2$ if $f(x)+f(y)$ is odd, then the resulting edge labels are distinct. Several results of mean labeling can be found in [4], [13], [16], [17]. K. Manickam and Marudai [8] introduced the concept of odd mean labeling. A graph G with p vertices and q edges is said to be odd mean if there exists an injective function $f : V(G) \rightarrow \{0,1,2,3,...,2q-1\}$ such that the induced map $f^*: E(G) \rightarrow \{1,3,5,...,2q-1\}$defined by $f^*(xy) = (f(x)+f(y)) / 2$ if $f(x) + f(y)$ is even, $(f(x)+f(y)+1) / 2$ if $f(x) + f(y)$ is odd is a bijection. New families of odd mean graphs can be found in [7], [8]. N. Revathi introduced vertex odd mean labeling and proved that Umbrella graph, $K_1 + C_n$ are vertex odd mean graphs [10]. Vertex odd mean labeling of Mangolian Tent, fan graph are discussed in [1]. Various labelings on cycles with zigzag chords, cycles with parallel chords are seen in [2], [6], [14], [15]. This paper is focussed on vertex odd mean labeling of cycles with parallel chords, and it is proved that new families of graphs obtained from cycles with parallel chords using graph operations admit vertex odd mean labeling. The basic definitions relevant to this paper are given below.

Definition 1: [10]

A Graph $G=(V, E)$ with $|V(G)|=p$, and $|E(G)|=q$ is said to be vertex odd mean graph if there exist an injective function $f : V(G) \rightarrow \{1, 3, 5, \ldots 2q-1\}$ such that the induced mapping $f^*: E(G) \rightarrow I\{\text{set of positive integers}\}$ defined by $f^*(uv) = \frac{f(u)+f(v)}{2}$ are distinct for each edge uv. Such a function f is called a vertex odd mean labeling.

Definition 2 [12]

A cycle with parallel chords is defined as a graph G obtained from a cycle C_n $(n \geq 6)$ with consecutive vertices $v_0, v_1, \ldots v_{n-1}, v_0$ by adding the chords $v_1 v_{n-1}$, $v_2 v_{n-2}$,$\ldots v_\alpha v_\beta$where $\alpha = \left\lfloor \frac{n}{2} \right\rfloor - 1$, $\beta = \left\lfloor \frac{n}{2} \right\rfloor + 1$if n is even and $\beta = \left\lfloor \frac{n}{2} \right\rfloor + 2$ if n is odd. Then G has n vertices and M edges where $M = (3n-3) / 2$ if n is odd and $M = (3n-2) / 2$ if n is even.

Definition 3 [9]
Butterfly is a graph obtained from two copies of even cycle sharing a common vertex with an arbitrary number of pendant edges attached at the common vertex.

Definition 4
Consider s copies of even cycle C_n ($n \geq 6$) with parallel chords. We define Edge connected cycles with parallel chords as a graphobtained by connecting a vertex of degree 2 in j^{th} copy of cycle $C_n$with a vertex of degree 2 in $(j+1)^{th}$ copy of cycle $C_n$by an edge for $j = 1, 2, ..., s - 1$.

II. Main Results

Theorem1: Every cycle C_n ($n \geq 6$) with parallel chords is a vertex odd mean graph
Proof:Let G be the cycle C_n with parallel chords. Then by definition 2,G has n vertices and M edges where $M = (3n-3) / 2$ if n is odd and $M = (3n-2) / 2$ if n is even. The n vertices of G are $v_0, v_1,\ldots,v_{n-1}$.Define the vertex labeling $f : V(G) \rightarrow \{1,3,5,\ldots 2M-1\}$ as follows. Two cases are considered here depending on n.
Case 1: Let n be even
Define
$f(v_i) = 4i - 1, 1 \leq i \leq n/2$
$f(v_{n-i}) = 4i + 1, 0 \leq i \leq n/2 - 1$
From the above it is seen that all the vertices are labeled and are distinct.
The edge set $E(G)$ is given by $E(G) = E_1 \cup E_2 \cup E_3 \cup E_4$ where
$E_1 = \{v_i v_{i+1}, 1 \leq i \leq n/2 - 1\}$
$E_2 = \{v_i v_{i+1}, n/2 + 1 \leq i \leq n - 1\}$
$E_3 = \{v_i v_{n-i}, 1 \leq i \leq n/2 - 1\}$ and
$E_4 = \{v_0 v_1, v_{n/2} v_{n/2+1}\}$.
Define the induced function $f^* : E(G) \rightarrow I$ (Set of Positive integers) as follows.
$f^*(v_i v_{i+1}) = 4i + 1, 1 \leq i \leq n/2 - 1$
$f^*(v_i v_{i+1}) = 4n - 4i - 1, n/2 + 1 \leq i \leq n - 1$
$f^*(v_i v_{n-i}) = 4i, 1 \leq i \leq n/2 - 1$
$f^*(v_0 v_1) = 2$
$f^*(v_{n/2} v_{n/2+1}) = 2n - 2$
It is clear from the above labeling that the edge labels of E_1, E_2 are odd and that of E_3 is even. To show that the edge labels of E_1, E_2 are distinct we will assume on the contrary that they are same.
For the edges of E_1 and E_2 :
if $i \neq j$, $1 \leq i \leq n/2 - 1$ and, $n/2 + 1 \leq j \leq n - 1$
let$f^*(v_i v_{i+1}) = f^*(v_j v_{j+1})$ Then
$4i + 1 = 4n - 4j - 1$
$2(i+j) = 2n - 1$
Here the left hand side is an even integer whereas the right hand side is an odd integer, a contradiction. Hence the edge labels of E_1 and E_2 are distinct. Thus all the edges of G have distinct labels. Hence G is a vertex odd mean graph. An illustration is given in Fig. 1a.
Case 2: Let n be odd
Define
$f(v_i) = 4i - 1, 1 \leq i \leq (n-1) / 2$
$f(v_{n-i}) = 4i + 1, 0 \leq i \leq (n-1) / 2$
From the above it is seen that all the vertices are labeled and are distinct.
The edge set $E(G)$ is given by $E(G) = E_1 \cup E_2 \cup E_3 \cup E_4$ where
$E_1 = \{v_i v_{i+1}, 1 \leq i \leq n-3/2\}$
$E_2 = \{v_{n-i} v_{n-(i+1)}, 0 \leq i \leq (n-3)/2\}$
$E_3 = \{v_i v_{n-i}, 1 \leq i \leq (n-3)/2\}$ and
$E_4 = \{v_0 v_1, v_{(n-1)/2}\ v_{(n+1)/2}\}$.
Define the induced function $f^* : E(G) \rightarrow I$ (Set of Positive integers) as follows.
$f^*(v_i v_{i+1}) = 4i + 1, \ 1 \leq i \leq (n-3)/2$
$f^*(v_{n-i} v_{n-(i+1)}) = 4i + 3, 0 \leq i \leq (n-3) / 2$
$f^*(v_i v_{n-i}) = 4i , 1 \leq i \leq (n-3)/2$
$f^*(v_0 v_1) = 2$
$f^*(v_{(n-1)/2} v_{(n+1)/2}) = 2n - 2$
It is clear from the above labeling that the edge labels of E_1, E_2 are odd and that of E_3 is even. As proved in case 1, it can be shown that the edge labels of E_1 and E_2 are distinct. Thus all the edges of G have distinct labels. Hence G is a vertex odd mean graph. An illustration is given in Fig. 1b

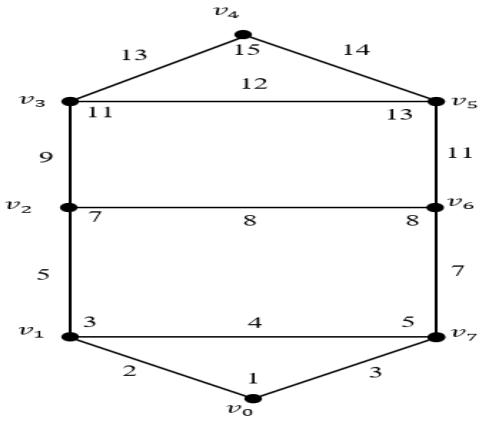


Fig 1a: Vertex odd mean labeling of C_8 with parallel chords

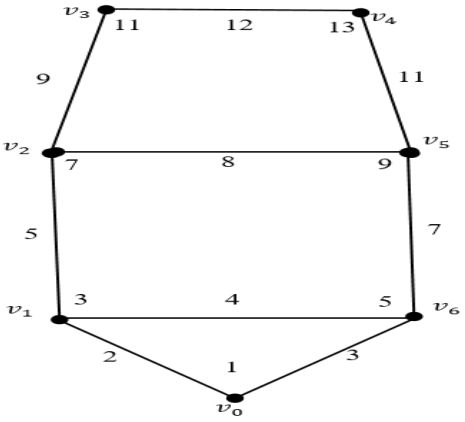


Fig 1b: Vertex odd mean labeling of $C_7$with parallel chords

Theorem 2: Joining two copies of even cycle $C_n(n \geq 6)$ with parallel chords by a path H_g of even order g is a vertex odd mean graph

Proof: Consider two copies of even cycle $C_n$with parallel chords. Let $v_0, v_1,\ldots v_{n-1}$ be the vertices of first copy of cycle C_n and let $w_0, w_1, \ldots w_{n-1}$ be the vertices of second copy of cycle C_n. Joining two cycles with a path H_g of even order g yield the resultant graph G. Let $h_1, h_2,...h_g$ be the vertices of the path H_g with $h_1 = v_{\frac{n}{2}}$ and $h_g = w_{\frac{n}{2}}$. Then $|V(G)| = 2n + g - 2$, $|E(G)| = 2M + g - 1$. Define the vertex labeling $f : V(G) \rightarrow \{1,3,5\ldots. 4M + 2g - 3\}$as follows.

$f(v_0) = 1$

$f(v_i) = 4i - 1,\ 1 \leq i \leq n/2$

$f(v_{n-i}) = 4i + 1,\ 1 \leq i \leq n/2 - 1$

For the path H_g

$f(h_i) = 2n + 2i - 3,\ 2 \leq i \leq g-1$

$f(w_0) = 4n + 2g - 5$

$f(w_i) = 4n + 2g - 4i - 3,\ 1 \leq i \leq (n-2)/2$

$f(w_{n-i}) = 4n + 2g - 4i - 5,\ 1 \leq i \leq (n-2)/2$

From the above it is seen that all the vertices are labeled and are distinct.

The edge set $E(G)$ is given by $E(G) = E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_1' \cup E_2' \cup E_3' \cup E_4' \cup E_5$ where

$E_1 = \{v_i v_{i+1}, 1 \leq i \leq (n-2)/2\}$

$E_2 = \{v_i v_{i+1}, (n+2)/2 \leq i \leq n-1\}$

$E_3 = \{v_i v_{n-i},\ 1 \leq i \leq (n-2)/2\}$

$E_4 = \{v_0v_1, v_{\frac{n}{2}}v_{\frac{n+2}{2}}\}$

$E_1' = \{ w_iw_{i+1},\ 1 \leq i \leq (n-2)/2\}$

$E_2' = \{w_{n-i}w_{n-(i+1)},\ 0 \leq i \leq (n-4)/2\}$

$E_3' = \{w_iw_{n-i},\ 1 \leq i \leq (n-2)/2\}$

$E_4' = \{w_0w_1, w_{\frac{n}{2}}w_{\frac{n+2}{2}}\}$

$E_5 = \{h_1h_2, h_2h_3,\ldots h_{g-1}h_g\}$.

Define the induced function $f^*: E(G) \rightarrow$ I (Set of positive integers) as follows.

$f^*(v_iv_{i+1}) = 4i + 1,\ 1 \leq i \leq (n-2)/2$

$f^*(v_iv_{i+1}) = 4n - 4i - 1,\ (n+2)/2 \leq i \leq n-1$

$f^*(v_iv_{n-i}) = 4i,\ 1 \leq i \leq (n-2)/2$

$f^*(v_0v_1) = 2$

$f^*(v_{n/2}v_{(n+2)/2}) = 2n - 2$

$f^*(w_iw_{i+1}) = 4n + 2g - 4i - 5,\ 1 \leq i \leq (n-2)/2$

$f^*(w_{n-i}w_{n-(i+1)}) = 4n + 2g - 4i - 7,\ 0 \leq i \leq (n-4)/2$

$f^*(w_iw_{n-i}) = 4n + 2g - 4i - 4,\ 1 \leq i \leq (n-2)/2$

$f^*(w_0w_1) = 4n + 2g - 6$

$f^*(w_{\frac{n}{2}}w_{\frac{n+2}{2}}) = 2n + 2g - 2$

$f^*(h_ih_{i+1}) = 2n + 2(i - 1),\ 1 \leq i \leq g-1$

From the above it is clear that all the edge labels are distinct. Hence the graph is a vertex odd mean graph. An illustration is given in Fig 2.

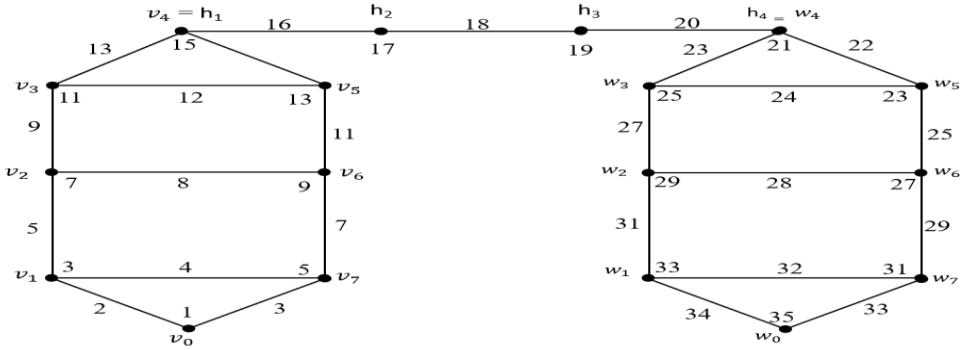


Fig 2: Vertex odd mean labeling of joining two copies of C_8 with parallel chords by a path H_4

Theorem 3: Path union of cycle C_n ($n \geq 6$) with parallel chords is a vertex odd mean graph

Proof: Consider s copies of cycle C_n with parallel chords. For j=1,2,3,……s let $v_{0,j}$, $v_{1,j}$, … $v_{n-1,j}$ be the n vertices of jth copy of cycle C_n with parallel chords. Let G be their path union obtained by joining $v_{0,\,j}$ with $v_{0,\,j+1}$ for j=1,2,3,…s-1. Then $|V(G)|= sn$ and $|E(G)|= sM + s - 1$ where $M = (3n-2) / 2$ if n is even and $M = (3n - 3) / 2$ if n is odd.

Define the vertex labeling $f : V(G) \rightarrow \{1, 3, 5, \ldots 2sM + 2s - 3\}$ as follows.

Two cases are considered here depending on n.

For j = 1,3,5,…s-1 if s is even and j = 1,3,5,…s if s is odd, define

$f(v_{i,j}) = 4i - 1 + 2n(j-1)$, $1 \leq i \leq n/2$ if n is even and $1 \leq i \leq (n-1)/2$ if n is odd

$f(v_{n-i,j}) = 4i + 1 + 2n(j-1)$, $0 \leq i \leq (n-2)/2$ if n is even and $0 \leq i \leq (n-1)/2$ if n is odd

For j = 2,4,6,…s if s is even and j = 2,4,6,…s-1 if s is odd, define

$f(v_{i,\,j}) = 2nj - 4i + 1$, $1 \leq i \leq n/2$ if n is even and $1 \leq i \leq (n-1)/2$ if n is odd

$f(v_{n-i,\,j}) = 2nj - 4i - 1$, $0 \leq i \leq (n-2)/2$ if n is even and $0 \leq i \leq (n-1)/2$ if n is odd

From the above it is seen that all the vertices labeled are distinct.

The edge set of G is given by $E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5$ where

$E_1 = \{v_{i,\,j}\, v_{i+1,\,j}$, $1 \leq i \leq (n-2)/2$ if n is even and $1 \leq i \leq (n-3)/2$ if n is odd and $1 \leq j \leq s\}$

$E_2 = \{v_{n-i,\,j}\, v_{n-(i+1),\,j}$, $0 \leq i \leq (n-4)/2$ if n is even and $0 \leq i \leq (n-3)/2$ if n is odd and $1 \leq j \leq s\}$

$E_3 = \{v_{i,\,j}\, v_{n-i,\,j}$, $1 \leq i \leq (n-2)/2$ if n is even and $1 \leq i \leq (n-1)/2$ if n is odd and $1 \leq j \leq s\}$

$E_4 = \{v_{0,\,j} v_{1,\,j} \& v_{n/2,\,j} v_{n/2+1,\,j}$ if n is even and $v_{0,\,j} v_{1,\,j}$ if n is odd and $1 \leq j \leq s\}$

$E_5 = \{v_{0,\,j} v_{0,\,j+1}$, $1 \leq j \leq s$-1$\}$.

The induced edge labeling of G is given as follows:

$f^*(v_{0,\,j}\, v_{0,\,j+1}) = 2nj$, $1 \leq j \leq s$-1

For j = 1,3,5,…s-1 if s is even and j = 1,3,5,…s if s is odd, define

$f^*(v_{i,\,j} v_{i+1,\,j}) = 4i + 1 + 2n(j-1)$, $1 \leq i \leq (n-2)/2$ if n is even and $1 \leq i \leq (n-3)/2$ if n is odd

$f^*(v_{n-i,\,j} v_{n-(i+1),\,j}) = 4i - 1 + 2n(j-1)$, $0 \leq i \leq (n-4)/2$if n is even and $0 \leq i \leq (n-3)/2$ if n is odd

$f^*(v_{i,\,j} v_{n-i,\,j}) = 4i + 2n(j-1)$, $1 \leq i \leq (n-2)/2$ if n is even and $1 \leq i \leq (n-1)/2$ if n is odd

$f^*(v_{0,\,j} v_{1,\,j}) = 2 + 2n(j-1)$

$f^*(v_{n/2,\,j} v_{n/2+1,\,j}) = 2nj - 2$ if n is even

For j = 2,4,6,…s if s is even and j = 2,4,6,…s-1 if s is odd, define

$f^*(v_{i,\,j} v_{i+1,\,j}) = 2nj - 4i - 1$, $1 \leq i \leq (n-2)/2$ if n is even and $1 \leq i \leq (n-3)/2$ if n is odd

$f^*(v_{n-i,\,j} v_{n-(i+1),\,j}) = 2nj - 4(i+1) + 1$, $0 \leq i \leq (n-4)/2$if n is even and $0 \leq i \leq (n-3)/2$ if n is odd

$f^*(v_{i,\,j} v_{n-i,\,j}) = 2nj - 4i$, $1 \leq i \leq (n-2)/2$ if n is even and $1 \leq i \leq (n-1)/2$ if n is odd

$f^*(v_{0,\,j} v_{1,\,j}) = 2nj - 2$

$f^*(v_{n/2,\,j} v_{n/2+1,\,j}) = 2n(j-1) + 2$ if n is even

From the above it is seen that the induced edge labels are in an increasing sequence for the successive copies and are distinct. Hence the graph is a vertex odd mean graph. An illustration is given in Fig.3a and Fig.3b

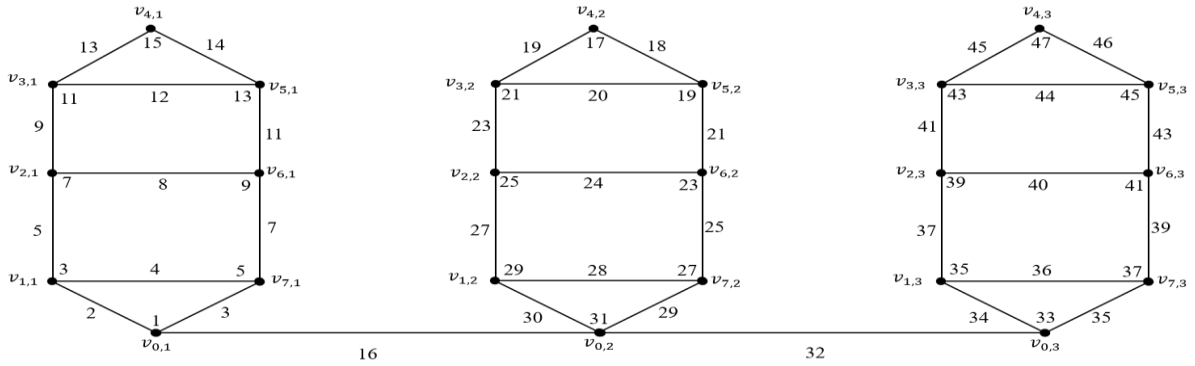


Fig 3a: Vertex odd mean labeling of path union of 3 copies of C_8 with parallel chords

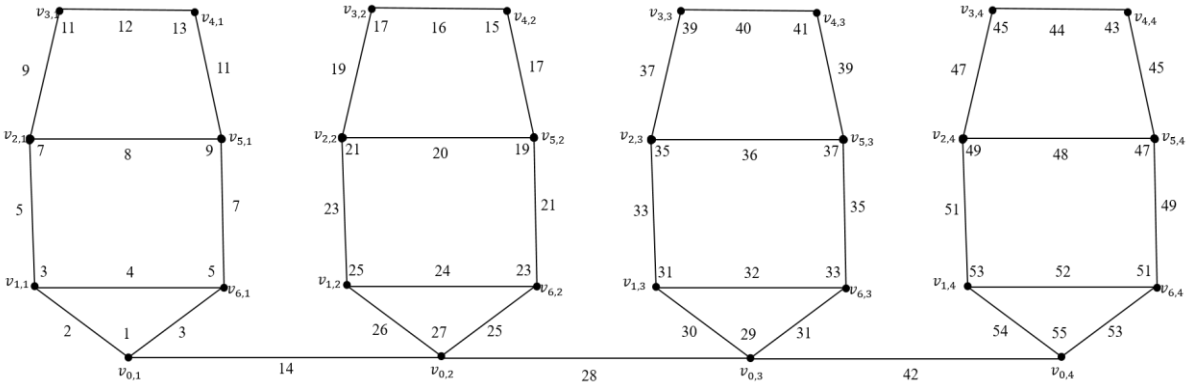


Fig 3b: Vertex odd mean labeling of path union of 4 copies of C_7 with parallel chords

Theorem 4: The Butterfly graph with parallel chords is a vertex odd mean graph

Proof: Consider two copies of even cycles C_n with parallel chords. Let $v_0, v_1, \ldots, v_{n-1}$ be the vertices of first copy and let $w_0, w_1, \ldots, w_{n-1}$be the vertices of second copy. Without loss of generality assume the common vertex as $v_{n/2}$ which is also w_0. Let the pendent edges be w_0d_i, $1 \le i \le g$. By definition 3, denote the resultant butterfly graph as B_n^gwhich has $2n + g - 1$ vertices and $2M + g$ edges where $M = (3n-2)/2$.

Define the vertex labeling $f : V(B_n^g) \rightarrow \{1,3,5\ldots. 4M + 2g - 1\}$as follows.

$f(v_0) = 1$

$f(v_i) = 4i - 1,\ 1 \le i \le n/2$

$f(v_{n-i}) = 4i + 1,\ 1 \le i \le n/2 - 1$

For the pendant vertices

$f(d_i) = 2n + 2i - 1,\ 1 \le i \le g$

$f(w_i) = 2n + 2g + 4i - 3,\ 1 \le i \le n/2$

$f(w_{n-i}) = 2n + 2g + 4i - 1,\ 1 \le i \le (n-2)/2$

From the above it is seen that all the vertices are labeled and are distinct.

The edge set is given by $E(B_n^g) = E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_1' \cup E_2' \cup E_3' \cup E_4' \cup E_5$ where

$E_1 = \{v_i\, v_{i+1},\ 1 \le i \le (n-2)/2\}$

$E_2 = \{v_i\, v_{i+1}, (n+2)/2 \le i \le n-1\}$

$E_3 = \{v_i\, v_{n-i},\ 1 \le i \le (n-2)/2\}$

$E_4 = \{v_0v_1, v_{\frac{n}{2}}v_{\frac{n+2}{2}}\}$

$E_1' = \{w_iw_{i+1},\ 1 \le i \le (n-2)/2\}$

$E_2' = \{w_{n-i}w_{n-(i+1)},\ 0 \le i \le (n-4)/2\}$

$E_3' = \{w_iw_{n-i},\ 1 \le i \le (n-2)/2\}$

$E_4' = \{w_0w_1, w_0w_{n-1}, w_{\frac{n}{2}}w_{\frac{n+2}{2}}\}$

$E_5 = \{d_1v_{n/2}, d_2\, v_{n/2}, \ldots, d_g\, v_{n/2}\}$.

Define the induced function $f^*: E(B_n^g) \rightarrow$ I (Set of positive integers) as follows.

$f^*(v_iv_{i+1}) = 4i + 1,\ 1 \le i \le (n-2)/2$

$f^*(v_iv_{i+1}) = 4n - 4i - 1,\ (n+2)/2 \le i \le n-1$

$f^*(v_iv_{n-i}) = 4i,\ 1 \le i \le (n-2)/2$

$f^*(v_0v_1) = 2$

$f^*(v_{n/2}v_{(n+2)/2}) = 2n - 2$
$f^*(w_iw_{i+1}) = 2n + 2g + 4i - 1,\ 1 \le i \le (n-2)/2$
$f^*(w_{n-i}w_{n-(i+1)}) = 2n + 2g + 4i + 1,\ 1 \le i \le (n-4)/2$
$f^*(w_iw_{n-i}) = 2n + 2g + 4i - 2,\ 1 \le i \le (n-2)/2$
$f^*(w_0 w_1) = 2n + g$
$f^*(w_0w_{n-1}) = 2n + g + 1$
$f^*(w_{\frac{n}{2}}w_{\frac{n+2}{2}}) = 5n - 4$
$f^*(v_{n/2}d_i) = 2n + i - 1,\ 1 \le i \le g$
From the above it is clear that all the edge labels are distinct. Hence the graph is a vertex odd mean graph. An illustration is given in Fig.4

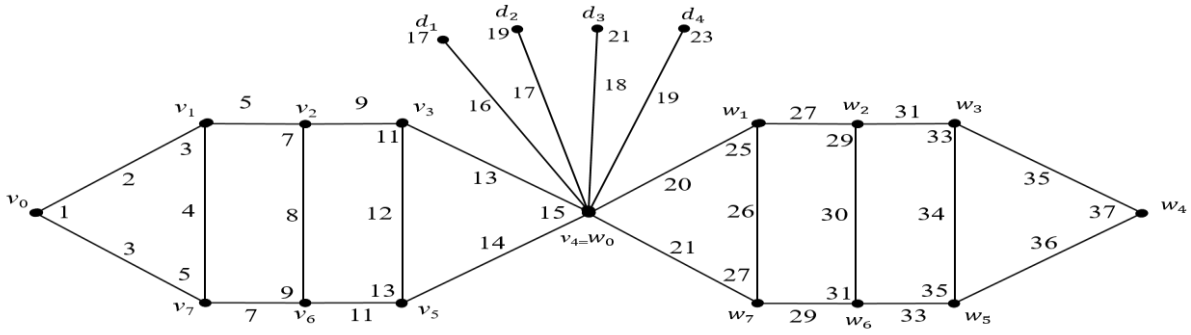


Fig.4: Vertex Odd Mean Labeling of B_8^4 with Parallel Chords

Theorem 5: Edge connected even cycles $C_n(n \ge 6)$ with parallel chords is a vertex odd mean graph.
Proof: Consider *s* copies of even cycle C_n ($n \ge 6$) with parallel chords. For j=1,2,3,……s let $v_{0,j}, v_{1,j}, \ldots v_{n-1,j}$ be the n vertices of jth copy of cycle C_n with parallel chords. Edge connected cycles with parallel chords is obtained by connecting a vertex of degree 2 in j^{th} copy of cycle $C_n$with a vertex of degree 2 in $(j+1)^{th}$ copy of cycle $C_n$by an edge for $j = 1, 2, ..., s - 1$ by definition 4. Without loss of generality, assume that $v_{n/2,\,j}$is connected to $v_{0,\,j+1}$ by an edge for $j = 1, 2, ..., s - 1$. Let the resultant graph be G. Now G has sn vertices and $sM + s$ -1edges where $M = (3n-2) / 2$. Define the vertex labeling $f : V(G) \to \{1, 3, 5, \ldots 2sM + 2s - 3\}$as follows
For $j = 1,2,3,\ldots s$ define
$f(v_{i,j}) = 4i - 1 + 2n(j-1),\ 1 \le i \le n/2$
$f(v_{n-i,j}) = 4i + 1 + 2n(j-1),\ 0 \le i \le (n-2)/2$
From the above it is seen that all the vertices labeled are distinct.
The edge set of G is given by $E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5$ where
$E_1 = \{v_{i,\,j}\, v_{i+1,\,j},\ 1 \le i \le (n-2)/2,\ 1 \le j \le s\}$
$E_2 = \{v_{n-i,\,j}\, v_{n-(i+1),\,j},\ 0 \le i \le (n-4)/2,\ 1 \le j \le s\}$
$E_3 = \{v_{i,\,j}\, v_{n-i,\,j},\ 1 \le i \le (n-2)/2,\ 1 \le j \le s\}$
$E_4 = \{v_{0,\,j}v_{1,\,j} \& v_{n/2,\,j}v_{n/2+1,\,j}\ 1 \le j \le s\}$
$E_5 = \{v_{n/2,\,j}v_{0,\,j+1},\ 1 \le j \le s-1\}$.
The induced edge labeling of G is given as follows:
$f^*(v_{i,\,j}v_{i+1,\,j}) = 4i + 1 + 2n(j-1),\ 1 \le i \le (n-2)/2,\ 1 \le j \le s$
$f^*(v_{n-i,\,j}v_{n-(i+1),\,j}) = 4i - 1 + 2n(j-1),\ 0 \le i \le (n-4)/2,\ 1 \le j \le s$
$f^*(v_{i,\,j}v_{n-i,\,j}) = 4i + 2n(j-1),\ 1 \le i \le (n-2)/2,\ 1 \le j \le s$
$f^*(v_{0,\,j}v_{1,j}) = 2 + 2n(j-1),\ 1 \le j \le s$
$f^*(v_{n/2,j}v_{n/2+1,\,j}) = 2nj - 2,\ 1 \le j \le s$
$f^*(v_{n/2,\,j}\, v_{0,\,j+1}) = 2nj,\ 1 \le j \le s-1$
From the above it is seen that the induced edge labels are in an increasing sequence for the successive copies and are distinct. Hence the graph is a vertex odd mean graph. An illustration is given in Fig. 5.

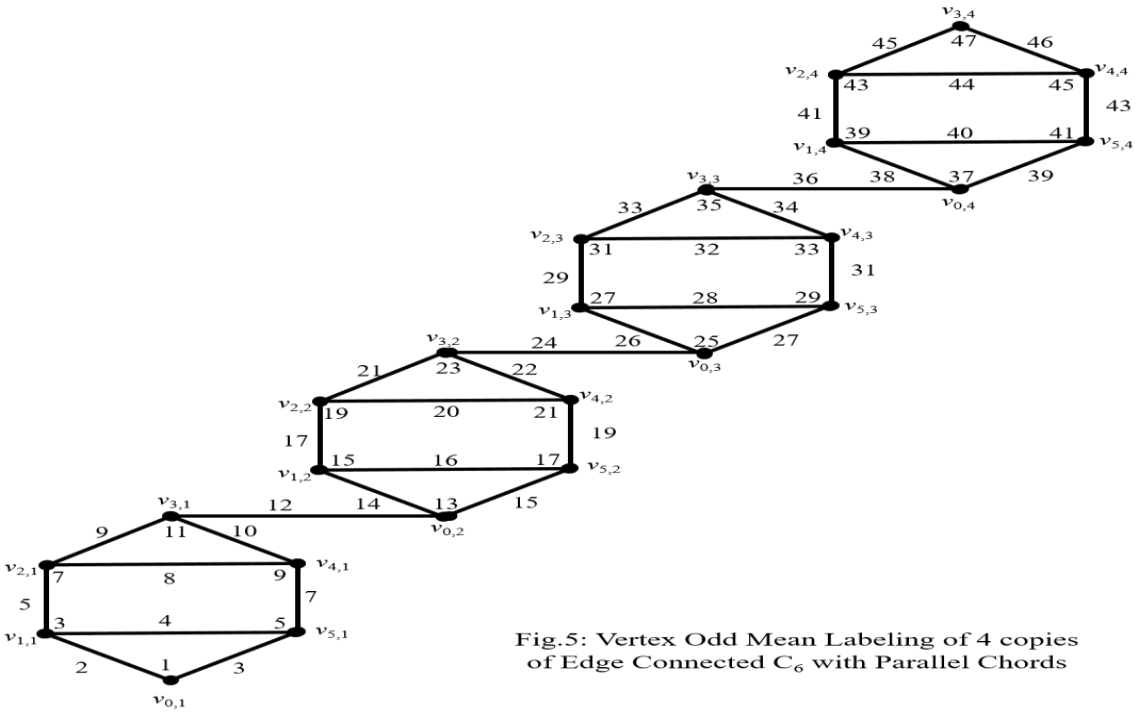


Fig.5: Vertex Odd Mean Labeling of 4 copies of Edge Connected C_6 with Parallel Chords

III. Conclusion

As cycle related graphs have been a major focus of attention, in this paper we have obtained new families of graphs that are vertex odd mean graphs using graph operations on cycles with parallel chords. Our future work is to extend this result on cycles with parallel P_3 chords and also to investigate yet another labeling on similar graphs.

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Adjacent Vertex Distinguishing Total Coloring Of Triangular Snake

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Abstract*: In this paper, we prove the existence of the adjacent vertex distinguishing total coloring (abbreviated as AVDTC) of the snake graphs like triangular snake, double triangular snake, alternate triangular snake and double alternate triangular snake. Also, we present an algorithm to obtain the AVDTC of these triangular graph family. The minimum number of colors needed for an AVDTC to the graph G is denoted by $\chi_{avt}(G)$.*

Keywords*: simple graph, chromatic number, triangular snake, double triangular snake, alternate triangular snake and double alternate triangular snake, adjacent vertex distinguishing total coloring.*

I. Introduction

All graphs in this paper are undirected and simple. A graph G consists of $V(G)$ a set of vertices and $E(G)$ a set of edges. The edge connecting two vertices is denoted by $uv \in E(G)$, for every vertex $u, v \in V(G)$. The degree of a vertex v denoted by $deg(v)$ of a graph is the number of edges incident to the vertex. The maximum degree of a graph G is denoted by $\Delta(G)$. For basic concepts of graph theory, we refer [1], [2], [3]. For graphs G_1 and G_2, we let $G_1 \cup G_2$ denotes their union, that is, $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$. Let $\lfloor n \rfloor$ stand for the largest integer less than or equal to n and $\lceil n \rceil$ stand for the smallest integer greater than or equal to n, for any real number n. The AV DTC conjecture, made by Zhang et al [9] says that every simple graph G has $\psi_{avt}(G) \leq \Delta(G) + 3$. It has been shown that, if two adjacent vertices of maximum degree of a simple graph G, then $\psi_{avt}(G) \geq \Delta(G) + 2$. Otherwise, if a simple graph G does not have two adjacent vertices of maximum degree, then $\psi_{avt}(G) = \Delta(G) + 1$. The tensor product of some special graph admits AVDTC in the literature [5], [6]. Also, quadrilateral snake admits AVDTC in the literature [7].

In this section, we write some basic definitions and results which are needed for next section.

Definition 1.1

The Triangular snake denoted by T_n is obtained from path P_n with the vertices $\{v_1, v_2, \cdots v_n\}$ by replacing each edge of the path P_n by the cycle C_3 with a new vertices $\{u_1, u_2, \cdots, u_{n-1}\}$.

$$V[T_n] = \left\{\left(\bigcup_{i=1}^{n} v_i\right) \cup \left(\bigcup_{i=1}^{n-1} u_i\right)\right\} \text{ and } E[T_n] = \left\{\bigcup_{i=1}^{n-1}(v_i v_{i+1}) \cup \bigcup_{i=1}^{n-1}(u_i v_i \cup u_i v_{i+1})\right\}$$

Here $deg(v_1) = 2$, $deg(v_n) = 2$, $deg(u_i) = 2$, $1 \leq i \leq n-1$ and $deg(v_i) = 4$, $2 \leq i \leq n-1$.

Definition 1.2 [4], [8] The Alternate Triangular snake denoted by $A(T_n)$ is obtained from P_n by replacing cycle C_3 in every alternate edges. Here, the triangle starts with either the vertex v_1 or with v_2 of the path P_n.

Definition 1.3 The Double Triangular snake is denoted by $D(T_n)$ which is obtained from the path P_n having the $\{v_1, v_2, \cdots v_n\}$ to new vertices u_i and u_i' by joining v_i and v_{i+1} for $1 \leq i \leq n-1$.

$V[D(T_n)] = \left\{\left(\bigcup_{i=1}^{n} v_i\right) \cup \left(\bigcup_{i=1}^{n-1}(u_i \cup u_i')\right)\right\}$; $E[D(T_n)] = \left\{\bigcup_{i=1}^{n-1}(v_i v_{i+1} \cup v_i u_i \cup v_i u_i' \cup u_i v_{i+1} \cup u_i' v_{i+1})\right\}$

Also, $deg(v_1) = 3,\ deg(v_i) = 6,\ for\ \ 2 \le i \le n-1,\ deg(u_i) = 2,\ deg(v_n) = 3,$

$deg(u'_i) = 2\ for\ \ 1 \le i \le n-1.$

Definition 1.4 The two alternate triangular snakes have a common path, which is denoted by the Double Alternate triangular snake $DA(T_n)$. This is obtained from a path $\{v_1, v_2, \cdots v_n\}$ by joining alternatively vi and vi+1 to two new vertices ui and u'_i.

Definition 1.5 A proper total k-coloring of G is a mapping $f : V(G) \cup E(G) \to \{1,2,\cdots,k\}$, $k \in Z^+$ which is an adjacent vertex distinguishing, if $C_f(u) \neq C_f(v)$ whenever $uv \in E(G)$, the color set of the vertex u (with respect to f), we denote $C_f(u)$ as $C(u)$.

$$C(u) = \{\{f(u)\} \cup \{f(uv) \mid uv \in E(G)\}\}$$
$$\overline{C}(u) = \{1,2,\cdots,k\} \setminus C(u)$$

II. AVDTC of T_n and AT_n

In this section, we present an algorithm for the AVDTC of Triangular snake (Tn) and Alternate Triangular snake A(Tn). Also, we discuss their color classes. Let the graph G having the vertices $\{v_1, v_2, \cdots v_n\}$. Define a function $f : V(G) \cup E(G) \to \{1,2,\cdots,k\}$.The vertices and edges are colored from the color set $\{1,2,\ldots k\}$ using the function f. Here $f(v_i)$ denotes the color of the vertex vi and f(vivj) denotes the color of the edge $v_i v_j$.

Algorithm 2.1 Procedure: Adjacent vertex distinguishing total coloring of Triangular snake T_n, for $n \ge 3$.

Input: $G(V(T_n), E(T_n))$

if n=3
for $i = 1,2,3$ do
$f(v_i) \leftarrow i$
end for

for $i = 1,2$ **do**
$f(v_i v_{i+1}) \leftarrow i+2;\ f(u_i) \leftarrow 4;\ f(v_i u_i) \leftarrow 5;\ f(v_{i+1} u_i) \leftarrow 1;$
$f(v_i) \leftarrow 2$
end for
else if $(n \ge 4)$
for $1 \le i \le n$ **do**
if $i \equiv 0\ (mod\ 2)$
$f(v_i) \leftarrow 2$
else
$f(v_i) \leftarrow 1$
end for
for $1 \le i \le n-1$ **do**
if $i \equiv 1\ (mod\ 2)$
$f(v_i v_{i+1}) \leftarrow 3;\ f(u_i) \leftarrow 3$
else

$$f(v_iv_{i+1}) \leftarrow 4;\ f(u_i) \leftarrow 4;\ \ f(u_iv_i) \leftarrow 5;\ f(u_iv_{i+1}) \leftarrow 6;$$

end for
end if
end procedure

Output: Adjacent vertex distinguishing total colored of T_n, for $n \geq 3$.

Theorem 2.1 The Triangular snake T_n admits $AVDTC$ and $\chi_{avt}(T_n) = \begin{cases} 5, n = 2,3 \\ 6, n \geq 4 \end{cases}$

Proof. From the definition (1.1), $|V(T_n)| = 2n-1$ and $|E(T_n)| = 3(n-1)$. We have colored the graph for $n \geq 4$,by using algorithm (2.1). Now the color classes for n = 3, we have

$C(v_1) = \{1,3,5\}, C(v_2) = \{1,2,3,4,5\}$ $\quad C(v_3) = \{1,3,4\}, C(u_i) = \{1,4,5\}$ $\quad for \quad 1 = 1,2$

$\therefore \quad \chi_{avt}(T_3) = 5.$

Now the color classes for $n \geq 4$, we have

$$C(v_1) = \{1,3,5\} and\ C(v_n) = \begin{cases} \{2,3,6\}, if\ n \equiv 0\ (mod\ 2) \\ \{1,4,6\}, if\ n \equiv 1\ (mod\ 2) \end{cases}$$

For $2 \leq i \leq n-1$, $\overline{C}(v_i) = \begin{cases} \{1\}, if\ i \equiv 0\ (mod\ 2) \\ \{2\}, if\ i \equiv 1\ (mod\ 2) \end{cases}$

For $2 \leq i \leq n-1$, $C(u_i) = \begin{cases} \{4,5,6\}, if\ i \equiv 0\ (mod\ 2) \\ \{3,5,6\}, if\ i \equiv 1\ (mod\ 2) \end{cases}$

Clearly, the color classes of any two adjacent vertices are not the same.

$$\therefore \qquad \chi_{avt}(T_n) = \begin{cases} 5, n = 2,3 \\ 6, n \geq 4 \end{cases}$$

Algorithm 2.2 Procedure: Adjacent vertex distinguishing total coloring of $A(T_n)$ for $n \geq 4$.

Input:

$$V[A(T_n)] \leftarrow \begin{cases} \left(\bigcup_{i=1}^{n} v_i\right) \cup \left(\bigcup_{i=1}^{\left\lfloor \frac{n}{2} \right\rfloor} u_i\right), & for\ n \equiv 1(mod\ 2) \\ \begin{cases} \left(\bigcup_{i=1}^{n} v_i\right) \cup \left(\bigcup_{i=1}^{\left(\frac{n}{2}\right)} u_i\right), & if\ deg(v_1) = deg(v_n) = 2 \\ \left(\bigcup_{i=1}^{n} v_i\right) \cup \left(\bigcup_{i=1}^{\left\lfloor \frac{n-1}{2} \right\rfloor} u_i\right), & if\ deg(v_1) = deg(v_n) = 1 \end{cases} & , \quad for\ n \equiv 0(mod\ 2) \end{cases}$$

if $n \equiv 1(\mathrm{mod}\, 2)$ **do**
for $i = 1\ to\ n$ do
if $i \equiv 0\ (mod\ 2)$
$f(v_i) \leftarrow 2$
else
$f(v_i) \leftarrow 1$
end for
for $i = 1\ to\ n-1$ **do**
if $i \equiv 1\ (mod\ 2)$
$f(v_iv_{i+1}) \leftarrow 3$
else

$f(v_i v_{i+1}) \leftarrow 4$

end for

for $i = 1 \, to \left\lfloor \frac{n}{2} \right\rfloor$ **do**

if $deg(v_n) = 1$

$f(u_i) \leftarrow 3; f(v_{2i-1}u_i) \leftarrow 2; f(v_{2i}u_i) \leftarrow 5$

else

$f(u_i) \leftarrow 4; f(v_{2i+1}u_i) \leftarrow 2; f(v_{2i}u_i) \leftarrow 5$

end for

else

for $i = 1 \, to \, n-1$ **do**

if $i \equiv 1 \, (mod \, 2)$

$f(v_i) \leftarrow 1; f(v_i v_{i+1}) \leftarrow 3$

else

$f(v_i) \leftarrow 2; (v_i v_{i+1}) \leftarrow 4; f(v_n) \leftarrow 4$

end for

if $deg(v_n) = \deg(v_1) = 2$

for $i = 1 \, to \, \frac{n}{2}$ **do**

$f(u_i) \leftarrow 3; \; f(v_{2i-1}u_i) \leftarrow 2; f(v_{2i}u_i) \leftarrow 5$

end for

else

for $i = 1 \, to \left\lfloor \frac{n-1}{2} \right\rfloor$ **do**

$f(u_i) \leftarrow 4; \; f(v_{2i+1}u_i) \leftarrow 2; f(v_{2i}u_i) \leftarrow 5$

end for

end procedure

Output: Adjacent vertex distinguishing total colored of $A(T_n)$, for $n \geq 4$.

Theorem 2.2 The Alternate Triangular snake $A(T_n)$ admits $AVDTC$ and

$$\chi_{avt}(A(T_n)) = \begin{cases} 4, n = 3 \\ 5, n \geq 4 \end{cases}$$

Proof. The vertex set of Alternate Triangular snake $A(T_n)$ is given in the algorithm (2.2). Here we have two cases for n is even or odd. In each case, the triangle starts with either the vertex of the path v_1 or v_2.

Case-1. $When \, n \equiv 1 \, (mod \, 2)$.

Suppose the triangle starts with v_1 and end with v_{n-1}, then $deg(v_n) = 1$ and if the triangle starts with v_2 and end with v_n, then $deg(v_1) = 1$. The edge set of $A(T_n)$ is given by

$$E(A(T_n)) = \begin{cases} \left(\bigcup_{i=1}^{n-1} v_i v_{i+1} \right) \cup \left(\bigcup_{i=1}^{\left\lfloor \frac{n}{2} \right\rfloor} ((v_{2i-1}u_i) \cup (v_{2i}u_i)) \right), \; if \; deg(v_n) = 1 \\ \left(\bigcup_{i=1}^{n-1} v_i v_{i+1} \right) \cup \left(\bigcup_{i=1}^{\left\lfloor \frac{n}{2} \right\rfloor} ((v_{2i+1}u_i) \cup (v_{2i}u_i)) \right), \; if \; deg(v_1) = 1 \end{cases}$$

$$|V(A(T_n))| = \frac{3n-1}{2} \; and \; |E(A(T_n))| = 2(n-1)$$

The color classes of $A(T_n)$ is given by

For $2 \le i \le n-1$, $\overline{C}(v_i) = \begin{cases} \{5\}, \; if \; i \equiv 1 \; (mod \; 2) \\ \{1\}, \; if \; i \equiv 0 \; (mod \; 2) \end{cases}$

If $deg(v_n) = 1$ then

$$C(v_1) = \{1,2,3\}, C(v_n) = \{1,4\}, \quad C(u_i) = \{2,3,5\} \; for \quad 1 = 1,2,\cdots \left\lfloor \frac{n}{2} \right\rfloor$$

If $deg(v_1) = 1$, then

$$C(v_1) = \{1,3\}, C(v_n) = \{1,2,4\}, \quad C(u_i) = \{2,4,5\} \; for \quad 1 = 1,2,\cdots \left\lfloor \frac{n}{2} \right\rfloor$$

Case-2. $When \; n \equiv 0 \; (mod \; 2)$.

Here, the triangle starts with either v_1 and ends with v_n, then $deg(v_1, v_n) = 2$. Suppose the triangle starts with v_2 and ends with v_n, then $deg(v_1, v_n) = 1$.

$$E(A(T_n)) = \begin{cases} \left(\bigcup_{i=1}^{n-1} v_i v_{i+1} \right) \cup \left(\bigcup_{i=1}^{\left(\frac{n}{2}\right)} ((v_{2i-1}u_i) \cup (v_{2i}u_i)) \right), \; if \; deg(v_1) = deg(v_n) = 2 \\ \left(\bigcup_{i=1}^{n-1} v_i v_{i+1} \right) \cup \left(\bigcup_{i=1}^{\lfloor \frac{n-1}{2} \rfloor} ((v_{2i+1}u_i) \cup (v_{2i}u_i)) \right), \; if \; deg(v_1) = deg(v_n) = 1 \end{cases}$$

If $deg(v_1) = deg(v_n) = 2$, then

$$|V(A(T_n))| = \frac{3n}{2} \; and \; |E(A(T_n))| = 2n-1$$

The color classes of $A(T_n)$ is given by

For $1,2,\cdots \frac{n}{2}$, $C(v_1) = \{1,2,3\}, C(v_n) = \{3,4,5\}, C(u_i) = \{2,3,5\}$

For $2 \le i \le n-1$, $\overline{C}(v_i) = \begin{cases} \{5\}, \; if \; i \equiv 1 \; (mod \; 2) \\ \{1\}, \; if \; i \equiv 0 \; (mod \; 2) \end{cases}$

If $deg(v_1) = deg(v_n) = 1$, then

$$|V(A(T_n))| = \frac{3n-2}{2} \; and \; |E(A(T_n))| = 2n-3$$

The color classes are for $i = 1,2,\cdots \left\lfloor \frac{n-1}{2} \right\rfloor$

$$C(v_1) = \{1,3\}, C(v_n) = \{3,4\}, \; C(u_i) = \{2,4,5\}$$

For $2 \le i \le n-1$, $\overline{C}(v_i) = \begin{cases} \{1\}, \; if \; i \equiv 0 \; (mod \; 2) \\ \{5\}, \; if \; i \equiv 1 \; (mod \; 2) \end{cases}$

Clearly, the color classes of any two adjacent vertices are different.

$$\chi_{avt}(A(T_n)) = \begin{cases} 4, n = 3 \\ 5, n \ge 4 \end{cases}$$

III. AVDTC of $D(T_n)$ and $DA(T_n)$

In this section, An algorithm has been presented to obtain the adjacent vertex distinguishing total chromatic number of Double Triangular snake $D(T_n)$ and Double Alternate Triangular snake $D(T_n)$ and also we discussed their color classes in detail.

Algorithm 3.1 Procedure: Adjacent vertex distinguishing total coloring of Double Triangular snake $D(T_n)$, for $n \geq 4$.

Input: $G(V(D(T_n)), E(D(T_n)))$

for $i = 1,2,\cdots,n$ **do**
  if $i \equiv 1(mod\ 2)$
    $f(v_i) \leftarrow 1$
  else
    $f(v_i) \leftarrow 2$
end for
for $1 \leq i \leq n-1$ **do**
  $f(u_i v_i) \leftarrow 5;\ f(u_i' u_i) \leftarrow 7;\ \ f(u_i u_{i+1}) \leftarrow 6;\ f(u'_i\, v_{i+1}) \leftarrow 8$
  if $i \equiv 1\,(mod\ 2)$
    $f(v_i v_{i+1}) \leftarrow 3;\ f(u_i) \leftarrow 3;\ f(u'_i) \leftarrow 3$
  else
    $f(v_i v_{i+1}) \leftarrow 4;\ f(u_i) \leftarrow 4;\ f(u'_i) \leftarrow 4$
end for
end procedure

Output: Adjacent vertex distinguishing total colored of $D(T_n)$, for $n \geq 4$.

Theorem 3.1 The Double triangular snake $D(T_n)$ admits $AVDTC$ and

$$\chi_{avt}(D(T_n)) = \begin{cases} 5, n = 2 \\ 7, n = 3 \\ 8, n \geq 4 \end{cases}$$

Proof. From the definition (1.3), it can be inferred that, the vertex and edge set of Double Triangular snake $D(T_n)$. We have $|V(D(T_n))| = 3n-2$ and $|E(D(T_n))| = 5(n-1)$. We have colored the graph using algorithm (3.1), for $n \geq 4$ and the color classes are given by

$$C(v_1) = \{1,3,5,7\}\ and\ C(v_n) = \begin{cases} \{2,3,6,8\}, if\ n \equiv 0\,(mod\ 2) \\ \{1,4,6,8\}, if\ n \equiv 1\,(mod\ 2) \end{cases}$$

For $2 \leq i \leq n-1$, $\quad \overline{C}(v_i) = \begin{cases} \{1\}, if\ i \equiv 0\,(mod\ 2) \\ \{2\}, if\ i \equiv 1\,(mod\ 2) \end{cases}$

For $1 \leq i \leq n-1$

$$C(u_i) = \begin{cases} \{3,5,6\}, if\ i \equiv 1\,(mod\ 2) \\ \{4,5,6\}, if\ i \equiv 0\,(mod\ 2) \end{cases} \qquad C(u'_i) = \begin{cases} \{3,7,8\}, if\ i \equiv 1\,(mod\ 2) \\ \{4,7,8\}, if\ i \equiv 0\,(mod\ 2) \end{cases}$$

Clearly, the color set of any two adjacent vertices are different.

$$\chi_{avt}(D(T_n)) = \begin{cases} 5, n = 2 \\ 7, n = 3 \\ 8, n \geq 4 \end{cases}$$

Algorithm 3.2 Procedure: Adjacent vertex distinguishing total coloring of $DA(T_n)$ for $n \geq 4$.

Input:

$$V[DA(T_n)] \leftarrow \begin{cases} \left(\bigcup_{i=1}^{n} v_i\right) \cup \left(\bigcup_{i=1}^{\lfloor \frac{n}{2} \rfloor} (u_i \cup u_i')\right), & \text{for } n \equiv 1 (mod\ 2) \\ \left(\bigcup_{i=1}^{n} v_i\right) \cup \left(\bigcup_{i=1}^{(\frac{n}{2})} (u_i \cup u_i')\right), \quad \text{if } deg(v_1) = deg(v_n) = 3 & \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad , & \text{for } n \equiv 0 (mod\ 2) \\ \left(\bigcup_{i=1}^{n} v_i\right) \cup \left(\bigcup_{i=1}^{\lfloor \frac{n-1}{2} \rfloor} (u_i \cup u_i')\right), \quad \text{if } deg(v_1) = deg(v_n) = 1 & \end{cases}$$

if $n \equiv 1 (mod\ 2)$

for $i = 1 to\ n$ **do**

if $i \equiv 1 (mod\ 2)$

$f(v_i) \leftarrow 1$

else

$f(v_i) \leftarrow 2$

end for

for $i = 1 to\ n-1$ **do**

if $i \equiv 1 (mod\ 2)$

$f(v_i v_{i+1}) \leftarrow 3$

else

$f(v_i v_{i+1}) \leftarrow 4$

end for

for $i = 1 to \left\lfloor \frac{n}{2} \right\rfloor$ **do**

if $deg(v_n) = 1$

$f(u_i) = f(u_i') \leftarrow 3;\ f(v_{2i-1}u_i) \leftarrow 2;\ f(v_{2i}u_i) = f(v_{2i-1}u'_i) \leftarrow 5;\ f(u_i'v_{2i}) \leftarrow 6$

else

$f(u_i) = f(u_i') \leftarrow 4;\ f(v_{2i+1}u_i) \leftarrow 2;\ f(v_{2i}u_i) = f(v_{2i+1}u'_i) \leftarrow 5;\ f(u_i'v_{2i}) \leftarrow 6$

end for

else

for $i = 1 to\ n-1$ do

if $i \equiv 1 (mod\ 2)$

$f(v_i) \leftarrow 1;\ f(v_i v_{i+1}) \leftarrow 3$

else

$f(v_i) \leftarrow 2;\ f(v_i v_{i+1}) \leftarrow 4;\ \ f(v_v) \leftarrow 4$

end for

if $deg(v_n) = \deg(v_1) = 3$

for $i = 1 to \frac{n}{2}$ **do**

$f(u_i) = f(u_i') \leftarrow 3;\ f(v_{2i-1}u_i) \leftarrow 2;\ f(v_{2i}u_i) = f(v_{2i-1}u'_i) \leftarrow 5;\ f(u_i'v_{2i}) \leftarrow 6$

end for

else

for $i = 1 to \left\lfloor \frac{n-1}{2} \right\rfloor$ **do**

$$f(u_i) = f(u_i') \leftarrow 4;\ f(v_{2i+1}u_i) \leftarrow 2;\ f(v_{2i}u_i) = f(v_{2i+1}u'_i) \leftarrow 5;\ f(u_i'v_{2i}) \leftarrow 6$$

end for

end procedure

Output: Adjacent vertex distinguishing total colored of $DA(T_n)$, for $n \geq 4$.

Theorem 3.2 The Double Alternate triangular snake $DA(T_n)$ admits *AVDTC* and $\chi_{avt}(DA(T_n)) = 6, \quad for\ n \geq 4.$

Proof. The vertex set of Double Alternate triangular snake is given in the algorithm (3.2). Here we discussed two cases that when n is even or odd. In each case, the double alternate triangular starts with either the vertex v_1 or with v_2.

Case-1. $When\ n \equiv 1\ (mod\ 2)$.

$$E[DA(T_n)] = \begin{cases} \left(\bigcup_{i=1}^{n-1} v_i v_{i+1}\right) \cup \left(\bigcup_{i=1}^{\lfloor \frac{n}{2} \rfloor} (v_{2i-1}u_i \cup v_{2i}u_i \cup u_i v_{2i} \cup v_{2i-1}u_i' \cup v_{2i}u_i)\right), & if\ deg(v_n) = 1 \\ \left(\bigcup_{i=1}^{n-1} v_i v_{i+1}\right) \cup \left(\bigcup_{i=1}^{\lfloor \frac{n}{2} \rfloor} (v_{2i+1}u_i \cup v_{2i}u_i \cup v_{2i+1}u'_i \cup v_{2i}u_i')\right), & if\ deg(v_1) = 1 \end{cases}$$

$$|V(DA(T_n))| = 2n-1 \ and\ |E(DA(T_n))| = 3(n-1)$$

Using algorithm (3.2), the color classes of $DA(T_n)$ for $n \geq 4$ is given by

For $2 \leq i \leq n-1$

$$\overline{C}(v_i) = \begin{cases} \{6\}, & if\ i \equiv 1\ (mod\ 2) \\ \{1\}, & if\ i \equiv 0\ (mod\ 2) \end{cases}$$

If $deg(v_n) = 1$ then

$$C(v_1) = \{1,2,3,5\}\ and\ C(v_n) = \{1,4\}$$

$$C(u_i) = \{2,3,5\}\ and\ C(u'_i) = \{3,5,6\} \quad for \quad i = 1,2,\cdots\left\lfloor \frac{n}{2} \right\rfloor$$

If $deg(v_1) = 1$, then

$$C(v_1) = \{1,3\}\ and\ C(v_n) = \{1,2,4,5\}$$

$$C(u_i) = \{2,4,5\}\ and\ C(u'_i) = \{4,5,6\} \quad for \quad i = 1,2,\cdots\left\lfloor \frac{n}{2} \right\rfloor$$

Case-2. $When\ n \equiv 0\ (mod\ 2)$.

$$E(DA(T_n)) = \begin{cases} \left(\bigcup_{i=1}^{n-1} v_i v_{i+1}\right) \cup \left(\bigcup_{i=1}^{(\frac{n}{2})} (v_{2i-1}u_i \cup v_{2i}u_i \cup v_{2i-1}u_i' \cup v_{2i}u'_i)\right), & if\ deg(v_1) = deg(v_n) = 3 \\ \left(\bigcup_{i=1}^{n-1} v_i v_{i+1}\right) \cup \left(\bigcup_{i=1}^{\lfloor \frac{n-1}{2} \rfloor} (v_{2i+1}u_i \cup v_{2i}u_i \cup v_{2i+1}u'_i \cup v_{2i}u_i')\right), & if\ deg(v_1) = deg(v_n) = 1 \end{cases}$$

If $deg(v_1) = deg(v_n) = 3$, then

$$|V(DA(T_n))| = 2n\ and\ |E(DA(T_n))| = 3n-1$$

The color classes of $DA(T_n)$ for $n \geq 4$ is given by

For $1 \le i \le \left(\frac{n}{2}\right)$, $C(u_i) = \{2,3,5\}, C(u_i') = \{3,5,6\}.$

For $2 \le i \le n-1$, $\overline{C}(v_i) = \begin{cases} \{6\}, & if\ i \equiv 1\ (mod\ 2) \\ \{1\}, & if\ i \equiv 0\ (mod\ 2) \end{cases}$

$$C(v_1) = \{1,2,3,5\}\ and C(v_n) = \{3,4,5,6\}$$

If $deg(v_1) = 1$ and $deg(v_n) = 1$, then

$$|V(DA(T_n))| = 2(n-1)\ and\ |E(DA(T_n))| = 3n-5$$

The color classes are $C(v_1) = \{1,3\}\ and C(v_n) = \{3,4\}$

for $i = 1,2,\cdots\left\lfloor \frac{n-1}{2} \right\rfloor$, $C(u_i) = \{2,4,5\}\ and \quad C(u_i') = \{4,5,6\}$

For $2 \le i \le n-1$, $\overline{C}(v_i) = \begin{cases} \{1\}, & if\ i \equiv 0\ (mod\ 2) \\ \{6\}, & if\ i \equiv 1\ (mod\ 2) \end{cases}$

Clearly, the color classes of any two adjacent vertices are different.

$$\therefore \chi_{avt}(DA(T_n)) = 6, \quad for\ n \ge 4.$$

Conclusion.

Thus, it has been found that the adjacent vertex distinguishing total chromatic number of Triangular snake family.

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On b-Coloring of Central Graph of Knödel Graph

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Abstract: *A b-coloring of G by k colors is a proper k-coloring of the vertices of G such that in each color class i there exists a vertex x_i having neighbors in all the other k-1 color classes. Such a vertex x_i is called a b-dominating vertex, and the set of vertices$\{x_1, x_2,\ldots,x_k\}$ is called a b-dominating system. The b-chromatic number of G, denoted by φ(G), is the maximum k for which G has a b-coloring by k colors. In this paper, we investigate the b-chromatic number of Central graph of Knödel graph which is denoted as φ[C($W_{\Delta,n}$)] respectively.*

Keywords: *Central graph, knödel graph, b-coloring*

I. Introduction

Graph theory is the theory of graphs dealing with nodes and connections or vertices and edges. This subject has experienced explosive growth, due in large measure to its role as an essential structure underpinning modern applied mathematics. Configurations of nodes and connections has great diversity of applications. They may represented physical networks, such as electrical circuits, roadways, or organic molecules. They are also used in representing less tangible interactions as might occur in ecosystem, sociological relationships, database, or in the flow of control in a computer program. It is a fastest growing field in Mathematics mainly because of its applications in distinct areas. There are plenty of works have been done in different topics, like decomposition, domination, factoring, orienting, coloring etc, in the past decades. A graph G = (V, E) is an ordered pair of two sets called V and E. Here elements of V are called vertices and elements of E are called edges. We consider the graphs here as undirected, finite and neither have multiple edges nor loops. The order of G is denoted by n and the size is denoted by m. All graph terminologies are referred from J.A. Bondy and U.S.R. Murty [1].

In Graph theory, Graph coloring is a well known area which is widely studied by many researchers. Here coloring means a vertex coloring of a graph. A k-coloring of a graph G = (V, E) is a mapping :
$V \rightarrow C$ where C is a set of k colors, thus k-coloring is an assignment of k colors to the vertices of G. Usually, the set C of colors is taken to be {1, 2, …, k}. A coloring C is proper if no two adjacent vertices are assigned the same color. Only loop less graphs admits proper coloring. The minimum k for which a graph G is k-colorable is called its chromatic number, and denoted χ (G). Ifχ(G) = k, the graph G is said to be k-chromatic.

The b-chromatic number of G, denoted by φ(G), is the maximum k for which G has a b-coloring by k colors. A b-coloring of G by k colors is a proper k-coloring of the vertices of G such that in each color class i there exists a vertex x_i having neighbors in all the other k-1 color classes. Such a vertex x_i is called a b-dominating vertex, and the set of vertices $\{x_1, x_2,\ldots,x_k\}$ is called a b-dominating system. The b-coloring was introduced by R.W.Irving and D.F.Manlove in [3]. They proved that determining φ(G) is NP-hard in general and polynomial for trees.

For a given graph G = (V, E), do an operation on G, by subdividing each edge exactly once and joining all the non-adjacent vertices of G. The graph obtained by this process is called Central graph[7] of G denote by C(G). The knödel graph[4] on $n \geq 2$ vertices (n is even) and of maximum degree $1 \leq \Delta \leq \lfloor \log_2 n \rfloor$is denotedby $W_{\Delta,n}$. The vertices of $W_{\Delta,n}$ are the pairs (i,j) with i = 1,2 and $0 \leq j \leq \frac{n}{2}$ -1. For every j, $0 \leq j \leq \frac{n}{2}$ -1, there is an edge between the vertex (1,j) and every vertex $(2,(j+2^k-1) \bmod \frac{n}{2})$, for k = 0, 1, …, Δ-1. Representing the

vertices (1,j) by $x_j$and (2,j) by y_j($0 \le j \le \frac{n}{2}$ -1). Knödel graph $W_{\Delta,n}$ ($1 \le \Delta \le \lfloor log_2 n \rfloor$) was introduced [5] as the topology underlying an optimal time algorithm for gossiping among n nodes.

However, it was formally defined in 2001[2]. Since then Knödel graphs have been widely studied as interconnection networks, mainly because of their good properties in terms of routing, broadcasting and gossiping. In this paper, we investigate the b-chromatic number of Central graph of Knödel graph which is denoted as $\varphi[C(W_{\Delta,n})]$ respectively.

II. b-Coloring of Central graph of Knödel graph

Theorem 2.1. Let n be an even positive integer atleast 4 and G be a knödel graph, then

$$\varphi[C(G)] = \begin{cases} n-1, & for\ n = 4 \\ n-2, & for\ 6 \le n \le 10 \\ n-3, & for\ 12 \le n \le 14 \\ n-4, & for\ n \ge 16 \end{cases}$$

Proof:

Consider the vertices of $W_{\Delta,n}$, (1,j) by x_i($0 \le i \le \frac{n}{2}$ -1)and (2,j) by y_j($0 \le j \le \frac{n}{2}$ -1).

Then in $W_{\Delta,n}$we have the vertex set

$$V(W_{\Delta,n}) = \bigcup_{i,j=0}^{\left(\frac{n}{2}\right)-1} \{x_i\} \cup \{y_j\}$$

and edge set

$$E(W_{\Delta,n}) = \bigcup_{i,j=0}^{\left(\frac{n}{2}\right)-1} \{e_{i,j} = x_i \ \ y_j : j = (i + 2^k - 1) mod \frac{n}{2}, 0 \le i \le \Delta - 1)\}, \{for\ k = 0,1, \dots, \Delta - 1\}.$$

Clearly $e_{ij} \ne e_{ji}, for\ i \ne j$.

From the definition of Central graph we subdivided the edge joining x_i($0 \le i \le \frac{n}{2}$ -1) and y_j($0 \le j \le \frac{n}{2}$ -1)

By the newly introduced vertex w_i($1 \le i \le$ n(Δ)) and then joining non adjacent vertices of $W_{\Delta,n}$.

Then the vertex set of Central graph of Knödel graph is,

$$V(C(W_{\Delta,n})) = \{ x_i: (0 \le i \le \tfrac{n}{2} -1)\} \cup \{ y_j(0 \le j \le \tfrac{n}{2} -1) \} \cup \{ w_i(1 \le i \le n(\Delta)) \}$$

Now we prove the results by the following four cases.

Case (i): For n = 4

Let us consider a set of color classes C_i($1 \le i \le$ n-1) and define a mapping $\Pi : V(C(W_{\Delta,n})) \to C_i$($1 \le i \le$ n-1). Assign the colors to $V(C(W_{\Delta,n}))$ as follows, For $0 \le i \le \frac{n}{2}$ -1, $\Pi(x_i) = c_{i+1}$.

For $0 \le j \le \frac{n}{2}$ -1, $\Pi(y_j) = c_{j+1}$. For $1 \le i \le$ n(Δ), $1 \le k \le$ n-1, $\Pi(w_i) = c_k$, $c_k \ne c(x_i)$ and $c(y_i)$.

Thus we have that $\varphi[C(W_{\Delta,n})] \ge$n-1. To prove the inequality, consider $V(C(W_{\Delta,n}))$. Here n vertices having degree n-1, and can maximize the colorsupto n colors. However the color class c_n miss the color class c_{n-1}, which cannot produce b-coloring. Therefore $\varphi[C(W_{\Delta,n})] \le$ n-1. Hence $\varphi[C(W_{\Delta,n})]$ =n-1.

Case (ii): For $6 \le n \le 10$

Define a mapping $\Pi : V(C(W_{\Delta,n})) \to C_i$($1 \le i \le$ n-2). Assign the colors to $V(C(W_{\Delta,n}))$ as follows, For $0 \le i \le \frac{n}{2}$ -1, $\Pi(x_i) = c_{i+1}$. For $2 \le j \le \frac{n}{2}$ -1, $\Pi(y_j) = c_{j+n/2-1}$. For $\Pi(y_0) = c_{n/2}$ and for $\Pi(y_1) = c_{n/2-1}$. For $1 \le i \le$ n(Δ), $1 \le k \le$ n-2, $\Pi(w_i) = c_k$. Thus we have that $\varphi[C(W_{\Delta,n})] \ge$n-2.

To prove the inequality, consider $V(C(W_{\Delta,n}))$. Here the vertex sets { x_i: ($0 \le i \le \frac{n}{2}$ -1)} and

{ y_j($0 \le j \le \frac{n}{2}$ -1) } separately forms two $\frac{n}{2}$ cliques. Then we can assign $\frac{n}{2}$ colors to the clique

{ x_i: ($0 \le i \le \frac{n}{2}$ -1)}, while assigning another $\frac{n}{2}$ colors to the clique { y_j($0 \le j \le \frac{n}{2}$ -1) }, the color classes

c_n and c_{n-1} miss the neighbors in the color classes c_3 and c_4, which contradicts the definition of b-coloring. Clearly to produce a b-coloring we can assign only n-2 colors to $V(C(W_{\Delta,n}))$.

Therefore $\varphi[C(W_{\Delta,n})] \leq n-2$. Hence $\varphi[C(W_{\Delta,n})] = n-2$.

Case (iii): For $12 \leq n \leq 14$

Let us define a mapping $\Pi : V(C(W_{\Delta,n})) \rightarrow C_i(1 \leq i \leq n-3)$ and assign the colors to $V(C(W_{\Delta,n}))$ as follows, For $0 \leq i \leq \frac{n}{2}-1$, $\Pi(x_i) = c_{i+1}$. For $0 \leq j \leq 2$, $\Pi(y_j) = c_{j+1}$ and for $3 \leq j \leq \frac{n}{2}-1$,

$\Pi(y_j) = c_{j+n/2-2}$. For $1 \leq i \leq n(\Delta)$, $1 \leq k \leq n-3$, $\Pi(w_i) = c_k$. Then $\varphi[C(W_{\Delta,n})] \geq n-3$. To prove the inequality, let us suppose that $\varphi[C(W_{\Delta,n})] > n-3$. i.e., $\varphi[C(W_{\Delta,n})] = n-2$. In $C(W_{\Delta,n})$ the vertex sets $\{ x_i: (0 \leq i \leq \frac{n}{2}-1)\}$ and $\{ y_j(0 \leq j \leq \frac{n}{2}-1) \}$ form an $\frac{n}{2}$ cliques separately. Then we can assign $\frac{n}{2}$ colors to the clique $\{ x_i: (0 \leq i \leq \frac{n}{2}-1)\}$ and remaining colors to the clique $\{ y_j(0 \leq j \leq \frac{n}{2}-1) \}$. If we try to maximize the colors as n-2, then the color class c_{n-2} does not have neighbors in the color classes $c_{n/2}$ and $c_{n/2-1}$, which does not satisfy the definition of b-coloring. Clearly$\varphi[C(W_{\Delta,n})] \leq n-3$.
Hence $\varphi[C(W_{\Delta,n})] = n-3$.

Case (iv): For $n \geq 16$

Let us define a mapping $\Pi : V(C(W_{\Delta,n})) \rightarrow C_i(1 \leq i \leq n-4)$. Assign the colors as follows, For $0 \leq i \leq \frac{n}{2}-1$, $\Pi(x_i) = c_{i+1}$. For $0 \leq j \leq 3$, $\Pi(y_j) = c_{j+1}$ and for $4 \leq j \leq \frac{n}{2}-1$, $\Pi(y_j) = c_{j+n/2-3}$.

For $\Pi(w_1) = c_{n-4}$. For $2 \leq i \leq n(\Delta)-1$, $1 \leq k \leq n-4$, $\Pi(w_i) = c_k$ and for $\Pi(w_{n(\Delta)}) = c_{n/2-1}$.
Thus $\varphi[C(W_{\Delta,n})] \geq n-4$. To prove the equality, suppose that $\varphi[C(W_{\Delta,n})] > n-4$. ie., $\varphi[C(W_{\Delta,n})] = n-3$. While assigning n-3 colors to $V(C(W_{\Delta,n}))$, the color class c_{n-3} does not have neighbors in the color classes c_{n-4} and c_{n-5}, which does not produce the b-coloring. Therefore $\varphi[C(W_{\Delta,n})] \leq n-4$.
Hence $\varphi[C(W_{\Delta,n})] = n-4$.

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Acyclic Coloring of Middle and Total graph of Extended Duplicate graph of Sunlet(S_n) Graph

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***Abstract**: In this paper we present acyclic coloring algorithms to color the vertices of middle graph and total graph of extended Duplicate graph of Sunlet(S_n). Also we obtain the chromatic number of the same.*

***Keywords**: Acyclic coloring, chromatic number, middle graph, total graph, extended duplicate graph, sunlet graph.*

I. Introduction

A proper coloring of a graph G is the coloring of the vertices of G such that no two neighbors in G are assigned the same color. Throughout this paper, by a graph we mean a finite, undirected, simple graph and the term coloring is used to denote vertex coloring of graphs.

A ***acyclic coloring*** of a graph G is the proper vertex coloring such that the subgraph induced by any two color classes does not contains a cycle. The notion of acyclic chromatic number was introduced by B.Grunbaum in 1973. The acyclic chromatic number of a graph G = G(V, E) is the minimum number of colors which are necessary to color G acyclically and is denoted by $a(G)$.

A n- ***sunlet graph*** S_n is the graph on 2n vertices obtained by appending n pendant edges to a cycle graph C_n

The concept of extended duplicate graph was introduced by Thirusangu et. al. A ***duplicate graph of G*** is DG = (V_1, E_1) where the vertex set $V_1 = V \cup V'$ and $V \cap V' = \phi$ and $f : V \to V'$ is bijective (for $v \in V$, we write $f(v) = v'$) and the edge set E_1 of DG is defined as follows. The edge uv is in E if and only if both uv′ and u′v are edges in E_1. The ***extended duplicate graph*** of DG, denoted by EDG, is defined as, adding an edge between any vertex from V to any other vertex in V', except the terminal vertices of V and V'. For convenience, we take $v_2 \in V$ and $v'_2 \in V$ and thus the edge $v_2v'_2$ is formed.

The ***middle graph of G***, denoted by M(G) was introduced by Michalak[2], in 1981 and is defined as follows. The vertex set of M(G) is $V(G) \cup E(G)$. Any two vertices x, y in M(G) are adjacent in M(G) if one of the following case holds.

(i) x and y are adjacent edges in G
(ii) x and y are incident in G.

The ***total graph*** (Michalak 1981, Harary 1969) of G, denoted by T(G), is defined as follows. The vertex set of T(G) is $V(G) \cup E(G)$. Any two vertices x, y in the vertex set of T(G) are adjacent in T(G) if one of the following cases holds.

(i) x, y are in V(G) and x is adjacent to y in G.
(ii) x, y are in E(G) and x, y are adjacent in G.
(iii) x is in V(G), y is in E(G) and x, y are incident in G.

The ***extended duplicate graph of S_n*** **is** denoted by ***EDG(S_n)***.

Acyclic Coloring of M[EDG(S_n)]

Coloring Algorithm 1 :

Input: M[EDG(S_n)], n > 3

$V \leftarrow \{v_1, v_2, \ldots, v_{2n}, v'_1, v'_2, \ldots, v'_{2n}, x_1, x_2, \ldots, x_{4n+1}\}$

{

for (k = 1 to 2n)

{

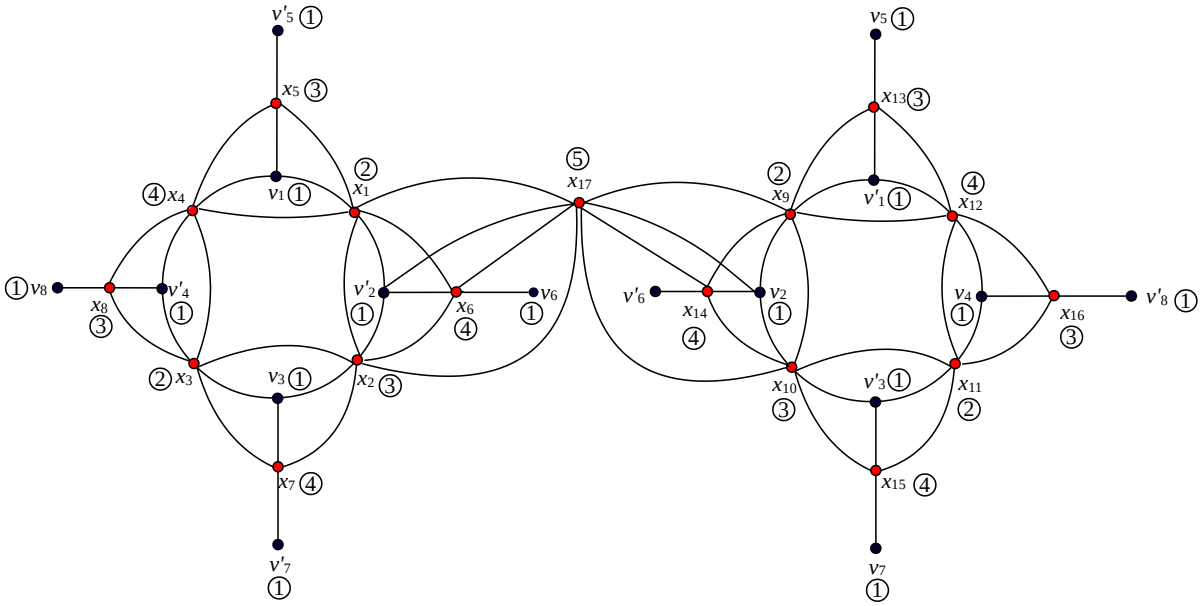
$v_k, v'_k \leftarrow 1;$

}
if $n \equiv 1 \pmod 2$
{
for (k =1 to n)
{
$x_{2k-1} \leftarrow 2;$
}
else
for (k =1 to $\left\lfloor \frac{n}{2} \right\rfloor$)
{
$x_{2k-1} \leftarrow 2;$
$x_{2n+2k-1} \leftarrow 2;$
}
}
if $n \equiv 1 \pmod 2$
{
for (k =1 to n-1)
{
$x_{2k} \leftarrow 3;$
}
$x_{2n+1}, x_{4n} \leftarrow 3;$
else
for (k =1 to $\left\lfloor \frac{n}{2} \right\rfloor$ -1)
{
$x_{2k} \leftarrow 3;$
$x_{2n+2k} \leftarrow 3;$
}
x_{n+1} , x_{2n} , x_{3n+1} , $x_{4n} \leftarrow 3;$
}
if $n \equiv 1 \pmod 2$
{
for (k =1 to 2(n-1))
{
$x_{2n} \leftarrow 4;$
$x_{2n+k+1} \leftarrow 4;$
}
else
for (k =1 to n-2)
{
$x_{n+k+1} \leftarrow 4;$
}
x_n , $x_{3n} \leftarrow 4;$
}
$x_{4n+1} \leftarrow 5;$
}
Output: vertex colored M[EDG(S_n)].

Theorem 1:
The acyclic chromatic number of middle graph of extended duplicate graph of sunlet graph S_n is given by $a(M[EDG(S_n)]) = 5$, $n > 3$.

Proof
Color the vertices of M[EDG(S_n)] as given in the algorithm 1.

The color class of 1 is $\{v_k, v'_k ; 1 \le k \le 2n\}$. The color class of 2 is $\{x_{2k-1} ; 1 \le k \le n\}$ when n is odd and $\{ x_{2k-1}, x_{2n+2k-1} ; 1 \le k \le \left\lfloor \frac{n}{2} \right\rfloor \}$ when n is even. The color class of 3 is $\{ x_{2n+1}, x_{4n}, x_{2k} ; \quad 1 \le k \le n-1\}$ when n is odd and $\{ x_{n+1}, x_{2n}, x_{3n+1}, x_{4n}, x_{2k}, x_{2n+2k}; 1 \le k \le \left\lfloor \frac{n}{2} \right\rfloor -1\}$ when n is even. The color class of 4 is $\{ x_{2n}, \ x_{2n+k+1} ; \quad 1 \le k \le 2(n-1)\}$ when n is odd and $\{ x_n, x_{3n}, x_{n+k+1}; 1 \le k \le n-2\}$ when n is even. The color class of 5 is x_{4n+1}



$a(M[EDG(S_4)]) = 5$

Case (i)
Consider the color classes of 1 and 2. The induced subgraph of the color classes of 1 and 2 is a collection of paths P_3 and isolated vertices, therefore, it is an acyclic graph.

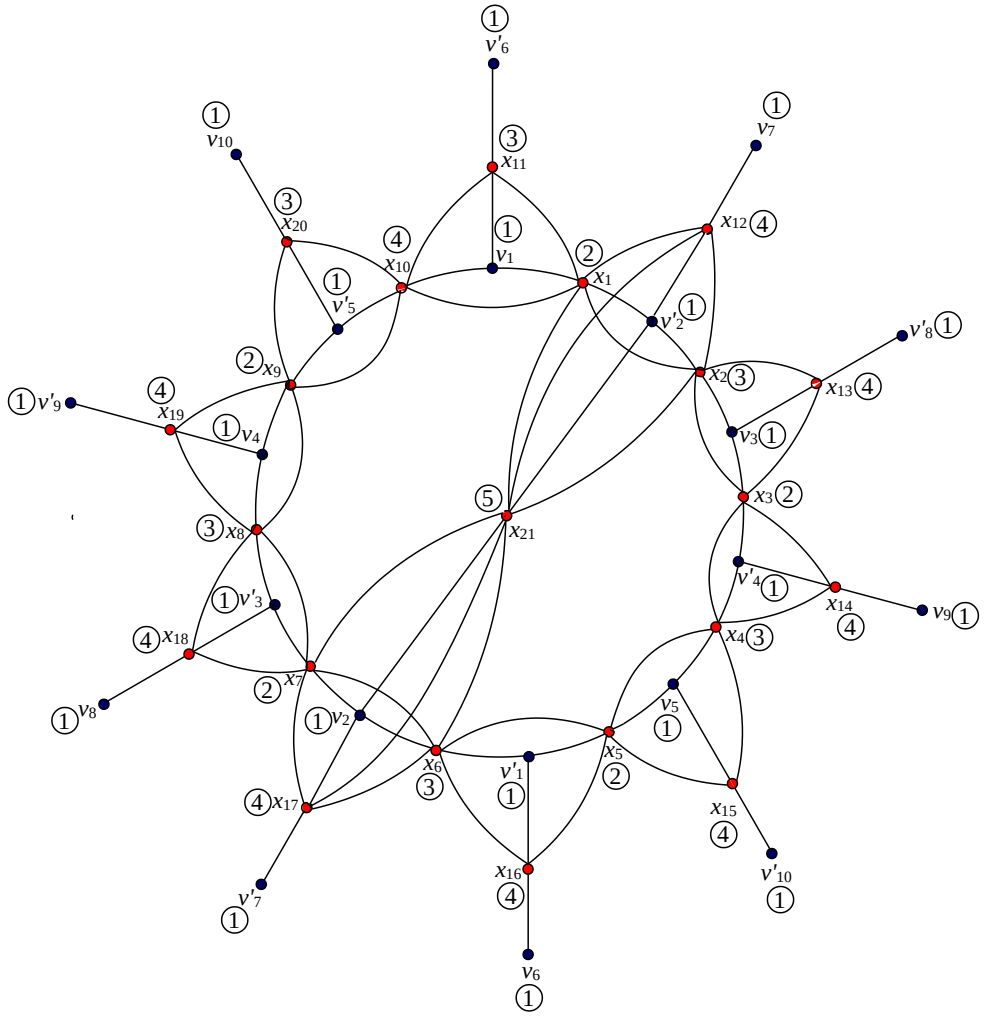
Case (ii)
Consider the color classes of 1 and 3. The induced subgraph of the color classes of 1 and 3 is a collection of paths P_3 and isolated vertices, therefore, it is an acyclic graph.

Case (iii)
Consider the color classes of 1 and 4. The induced subgraph of the color classes of 1 and 4 is a collection of paths P_3 and isolated vertices, therefore, it is an acyclic graph.

Case (iv)
Consider the color classes of 1 and 5. The induced subgraph of the color classes of 1 and 5 is a collection of path $v_2x_{4n+1}v'_2$ and isolated vertices, therefore, it is an acyclic graph.

Case (v)
Consider the color classes of 2 and 3. The induced subgraph of the color classes of 2 and 3 is a collection P_{2n+1} path when n is odd and P_{n+1} path when n is even therefore, it is an acyclic graph.

Case (vi)
Consider the color classes of 2 and 4. The induced subgraph of the color classes of 2 and 4 is a collection of paths P_3 and therefore, it is an acyclic graph.



a(M[EDG(S_5)]) = 5

Case (vii)
Consider the color classes of 2 and 5. The induced subgraph of the color classes of 2 and 5 is a collection of path $x_1x_{4n+1}x'_{n+2}$ when n is odd and $x_1x_{4n+1}x'_{2n+1}$ when n is even and therefore, it is an acyclic graph.

Case (viii)
Consider the color classes of 3 and 4. The induced subgraph of the color classes of 3 and 4 is a collection of paths P_3 and therefore, it is an acyclic graph.

Case (ix)
Consider the color classes of 3 and 5. The induced subgraph of the color classes of 3 and 5 is a collection of paths P_3 and isolated vertices, therefore, it is an acyclic graph.

Case (x)
Consider the color classes of 4 and 5. The induced subgraph of the color classes of 4 and 5 is a collection of paths P_3 and isolated vertices, therefore, it is an acyclic graph.

Thus, the induced subgraph of any two color classes is acyclic and therefore the coloring given in the algorithm 1, is an acyclic coloring. Hence ***a(M[EDG(S_n)]) = 5, n > 3.***

Acyclic Coloring of T[EDG(S_n)]

Coloring Algorithm 2 :

Input: $T[EDG(S_n)]$, $n \geq 3$

$V \leftarrow \{v_1, v_2, ..., v_{2n}, v'_1, v'_2, \ldots, v'_{2n}, x_1, x_2, ..., x_{4n+1}\}$

{

if $n \equiv 1 \pmod 2$

{

for (k =1 to $\left\lfloor \frac{n+2}{2} \right\rfloor$)

{

$v_{2k-1}, x_{2n+2k} \leftarrow 1;$

}

for (k =1 to $\left\lfloor \frac{n}{2} \right\rfloor$)

{

$x_{3n+2k} \leftarrow 1;$

}

for (k =1 to $\left\lfloor \frac{n}{2} \right\rfloor - 1$)

{

$v'_{2k+1}, x_{4n} \leftarrow 1;$

}

}

else

for (k =1 to $\left\lfloor \frac{n}{2} \right\rfloor$)

{

$v_{2k-1}, x_{n+2k} \leftarrow 1;$

}

for (k =1 to $\left\lfloor \frac{n-1}{2} \right\rfloor$)

{

$v_{2k}, x_{3n-1}, x_{3n+2k-1} \leftarrow 1;$

}

$v_{2n-1}, v'_{2n} \leftarrow 1;$

}

if $n \equiv 1 \pmod 2$

{

for (k =1 to $\left\lfloor \frac{n}{2} \right\rfloor$)

{

$v'_{2k}, v'_1, x_{2n+2k+1} \leftarrow 2;$

}

for (k =1 to $\left\lfloor \frac{n}{2} \right\rfloor - 1$)

{

$v_{2k+2}, x_{2n}, x_{n+2} \leftarrow 2;$

}

for (k =1 to $\left\lfloor \frac{n}{2} \right\rfloor - 2$)

{

$x_{3n+2k+3} \leftarrow 2;$

}

$v'_{n+1}, v'_{n+2}, v_{n+3}, v_{2n} \leftarrow 2;$

else

for (k =1 to $\left\lfloor \frac{n}{2} \right\rfloor - 1$)

{

$v'_{2k}, x_{n-1}, x_{n+2k-1} \leftarrow 2;$

}

for (k =1 to $\left\lfloor \frac{n}{2} \right\rfloor$)

{

$v'_{2k-1}, x_{4n-2k+2} \leftarrow 2;$

}

$v'_{2n-1}, v_{2n} \leftarrow 2;$

}

if $n \equiv 1 \pmod 2$

{

for (k =1 to $\left\lfloor \frac{n}{2} \right\rfloor + 1$)

{

$v_2, x_{2k-1} \leftarrow 3;$

}

for (k =1 to $\left\lfloor \frac{n}{2} \right\rfloor - 1$)

{

$x_{n+2k+1}, v'_{n+2k+2}, v'_n \leftarrow 3;$

}

for (k =1 to $\left\lfloor \frac{n}{2} \right\rfloor - 2$)

{

$v_{n+1}, v_{n+2k+3} \leftarrow 3;$

}

for (k =1 to $\left\lfloor \frac{n}{2} \right\rfloor$)

{

$v_{n+2k}, v'_{n+2k+1} \leftarrow 3;$

}

else

for (k =1 to $\left\lfloor \frac{n}{2} \right\rfloor - 1$)

{

$x_{2k-1}, x_{2n+2k-1}, v_{n+2k-1}, v'_{n+2k-1}, v_{n+2k}, v'_{n+2k} \leftarrow 3;$

}

$v_n, v'_n, x_{2n-1}, x_{4n-1} \leftarrow 3;$

}

if $n \equiv 1 \pmod 2$

{

for (k =1 to $\left\lfloor \frac{n+1}{2} \right\rfloor$)

{

$x_{2k} \leftarrow 4;$

}

for (k =1 to $\left\lfloor \frac{n}{2} \right\rfloor - 1$)

{

x_{n+2k+2} ← 4;

}

x_{2n+1} x_{3n+3} ← 4;

}

else

for (k =1 to $\left\lfloor \frac{n}{2} \right\rfloor$)

{

x_{2k}, x_{2n+2k} ← 4;

}

x_{4n+1} ← 5;

}

Output: vertex colored T[EDG(S_n)].

Theorem 2:

The acyclic chromatic number of total graph of extended duplicate graph of sunlet graph S_n is given by a(T[EDG(S_n)]) = 5, n > 3.

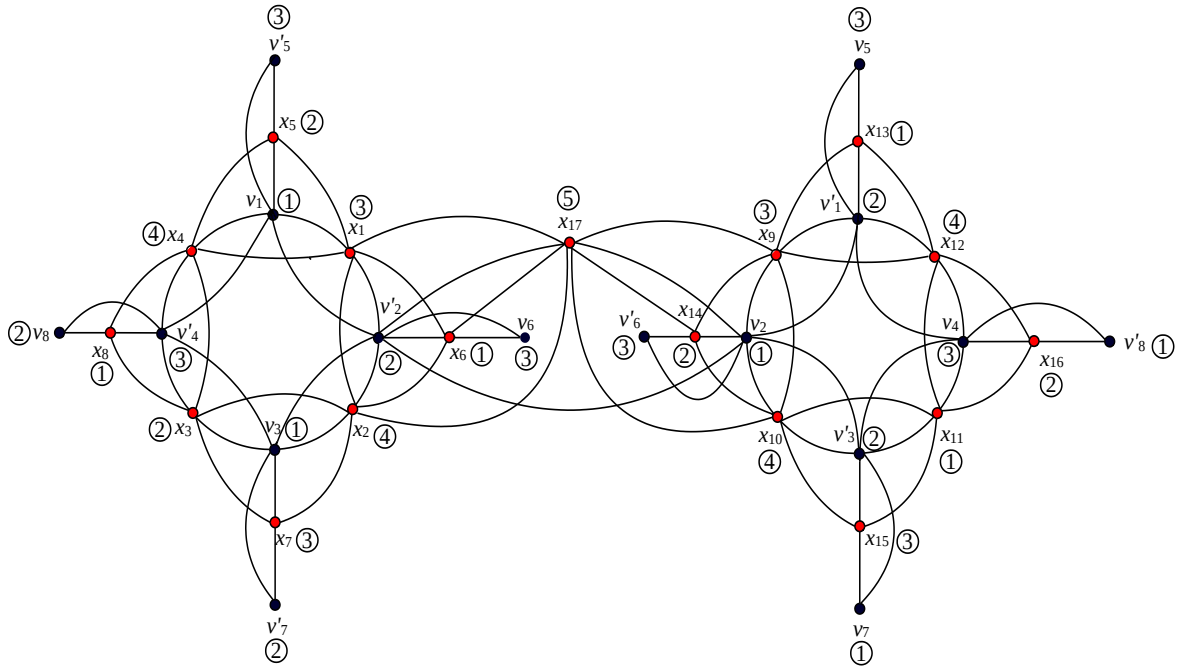
Proof

Color the vertices of M[EDG(S_n)] as given in the algorithm 2.

The color class of 1 is { v_{2k-1}, x_{2n+2k} ; $1 \le k \le \left\lfloor \frac{n+2}{2} \right\rfloor$}, {$x_{3n+2k}$; $1 \le k \le \left\lfloor \frac{n}{2} \right\rfloor$} and { v'_{2k-1}, x_{4n} ; $1 \le k \le \left\lfloor \frac{n}{2} \right\rfloor - 1$ } when n is odd and {v_{2k-1}, x_{n+2k}; $1 \le k \le \left\lfloor \frac{n}{2} \right\rfloor$}, {$v_{2k}$, x_{3n-1}, $x_{3n+2k-1}$; $1 \le k \le \left\lfloor \frac{n-1}{2} \right\rfloor$}, and {$v_{2n-1}$, v'_{2n}} when n is even. The color class of 2 is {v'_{2k}, v'_1, $x_{2n+2k+1}$; $1 \le k \le \left\lfloor \frac{n}{2} \right\rfloor$}, {$v_{2k+2}$, x_{2n}, x_{n+2} ; $1 \le k \le \left\lfloor \frac{n}{2} \right\rfloor - 1$ }, { $x_{3n+2k+3}$; $1 \le k \le \left\lfloor \frac{n}{2} \right\rfloor - 2$ } and {v'_{n+1}, v'_{n+2}, v_{n+3}, v_{2n}} when n is odd and {v'_{2k}, x_{n-1}, x_{n+2k-1} ; $1 \le k \le \left\lfloor \frac{n}{2} \right\rfloor - 1$ }, { v'_{2k-1}, $x_{4n-2k+2}$; $1 \le k \le \left\lfloor \frac{n}{2} \right\rfloor$} and {$v'_{2n-1}$, v_{2n}} when n is even. The color class of 3 is { v_2, x_{2k-1} ; $1 \le k \le \left\lfloor \frac{n}{2} \right\rfloor + 1$ },{ x_{n+2k+1}, v'_{n+2k+2}, v'_n ; $1 \le k \le \left\lfloor \frac{n}{2} \right\rfloor - 1$ }, { x_{n+1}, x_{2n}, x_{3n+1}, x_{4n}, x_{2k}, x_{2n+2k}; $1 \le k \le \left\lfloor \frac{n}{2} \right\rfloor$-1},{ v_{n+1}, v_{n+2k+3} ; $1 \le k \le \left\lfloor \frac{n}{2} \right\rfloor - 2$ } and {v_{n+2k}, v'_{n+2k+1} ; $1 \le k \le \left\lfloor \frac{n}{2} \right\rfloor$} when n is odd and {$x_{2k-1}$, $x_{2n+2k-1}$,v_{n+2k-1}, v'_{n+2k-1},v_{n+2k},v'_{n+2k}; $1 \le k \le \left\lfloor \frac{n}{2} \right\rfloor$-1} and {$v_n$, v'_n, x_{2n-1}, x_{4n-1}} when n is even. The color class of 4 is {x_{2k} ; $1 \le k \le \left\lfloor \frac{n+1}{2} \right\rfloor$)}, {x_{n+2k+2} ; $1 \le k \le \left\lfloor \frac{n}{2} \right\rfloor$-1} and { x_{2n+1}, x_{3n+3}} when n is odd and {x_{2k}, x_{2n+2k} ; $1 \le k \le \left\lfloor \frac{n}{2} \right\rfloor$} when n is even. The color class of 5 is x_{4n+1}.

Case (i)

Consider the color classes of 1 and 2. The induced subgraph of the color classes 1 and 2 is the collection of paths and isolated vertices when n is odd and the collection of stars K_1 when n is even, therefore, it is an acyclic graph.



a(T[EDG(S_4)]) = 5

Case (ii)
Consider the color classes of 1 and 3. The induced subgraph of the color classes of 1 and 3 is the collections of path P_4 when n is odd and the collection of stars K_1 when n is even, therefore, it is an acyclic graph.

Case (iii)
Consider the color classes of 1 and 4. The induced subgraph of the color classes of 1 and 4 is the collection of paths P_3 and P_2 when n is odd and a collection of paths P_3 and isolated vertices when n is even, therefore, it is an acyclic graph.

Case (iv)
Consider the color classes of 1 and 5. The induced subgraph of the color classes of 1 and 5 is a collection of path P_3 and isolated vertices, therefore, it is an acyclic graph.

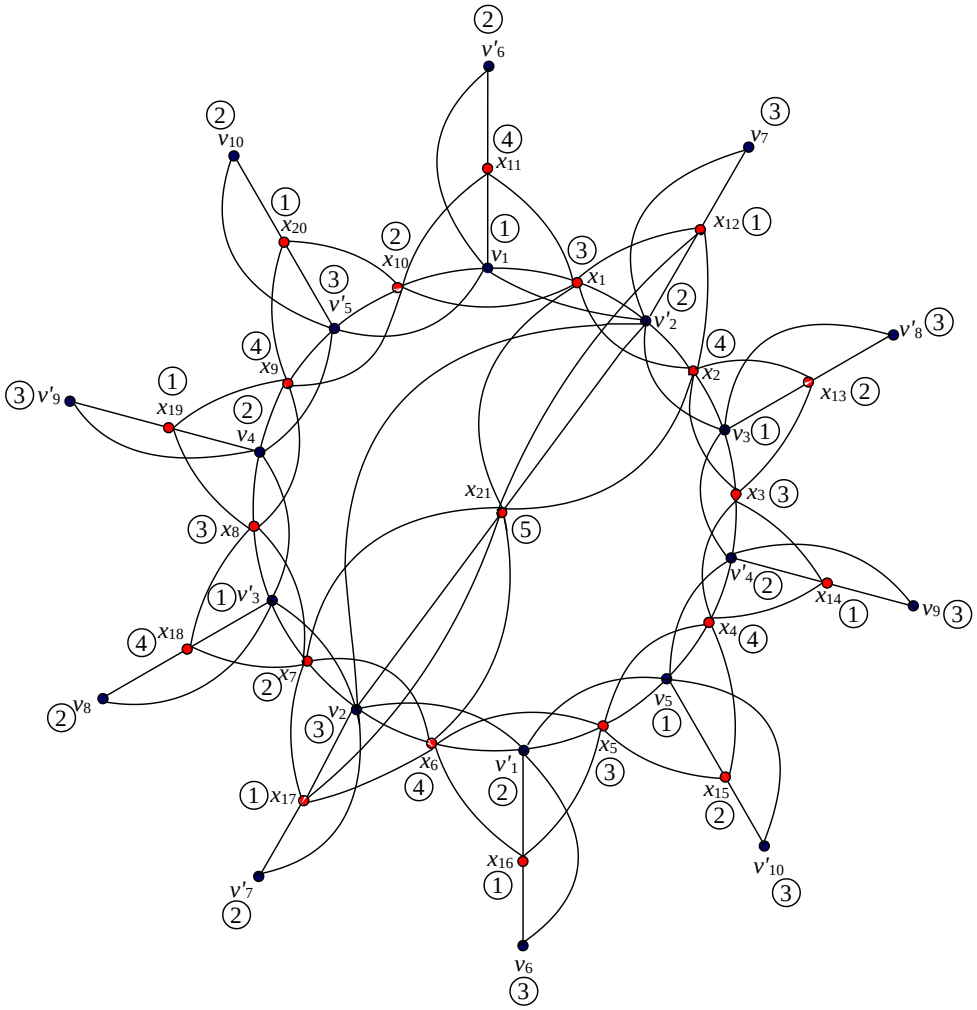
Case (v)
Consider the color classes of 2 and 3. The induced subgraph of the color classes of 2 and 3 is a collection of paths and therefore, it is an acyclic graph.

Case (vi)
Consider the color classes of 2 and 4. The induced subgraph of the color classes of 2 and 4 is a collection of paths P_3 and therefore, it is an acyclic graph.

Case (vii)
Consider the color classes of 2 and 5. The induced subgraph of the color classes of 2 and 5 is a collection of path P_3 and isolated vertices, therefore, it is an acyclic graph.

Case (viii)
Consider the color classes of 3 and 4. The induced subgraph of the color classes of 3 and 4 is a collection of paths and isolated vertices, therefore, it is an acyclic graph.

Case (ix)
Consider the color classes of 3 and 5. The induced subgraph of the color classes of 3 and 5 is a collection of path P_3 and isolated vertices, therefore, it is an acyclic graph.



a(T[EDG(S_5)]) = 5

Case (x)

Consider the color classes of 4 and 5. The induced subgraph of the color classes of 4 and 5 is a collection of path P_3 and isolated vertices, therefore, it is an acyclic graph.

Thus, the induced subgraph of any two color classes is acyclic and therefore the coloring given in the algorithm 2 is an acyclic coloring. Hence ***a(T[EDG(S_n)]) = 5, n > 3.***

II. Conclusion

In this paper, we obtained the acyclic chromatic number of middle graph and total graph of Extended Duplicate graph of Sunlet graph S_n.

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MINIMUM PENDANT DOMINATING ENERGY OF A GRAPH

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Abstract: Let G be any graph. A dominating set S in G is called a pendant dominating set if $< S >$ contains at least one pendant vertex. The least cardinality of a pendant dominating set in G is called the pendant domination number of G, denoted by $\gamma_{pe}(G)$. In this paper, we define and study the minimum pendant dominating energy$E_{pe}(G)$. The exact value of $E_{pe}(G)$ is calculated for some standard graphs. Further, the bounds are established for $E_{pe}(G)$.

Keywords: Dominating set, Pendant Dominating set, Minimum Pendant Dominating set.

I. Introduction

Let $G = (V, E)$ be a graph with n vertices and m edges. The degree of v_i writtenby $d(v_i)$ is the number of edges incident with v_i. The maximum vertex of degree is denoted by$\Delta(G)$. The adjacency matrix $A_{pe}(G)$ of G is defined by its entries as $a_{ij} = 1$ if $v_i v_j \in E(G)$ *or* $v_i \in D$ *if* $(i = j)$ where D is a pendant dominating set of G and 0 otherwise. The eigen values of graph G are the eigenvalues of its adjacency matrix $A_{pe}(G)$, denoted by $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. A graph G is said to be singular if at least one of its eigenvalues is equal to zero. For singular graphs, evidently, $detA = 0$. A graph is nonsingular if all its eigenvalues are different from zero. A graph G is said to be k-regular if every vertex in G has degree k.

The energy of a graph G is defined as $E(G) = \sum_{i=1}^{n} | \lambda_i |$. This concept was introduced by I. Gutman in 1978 [4]. Initially, the graph energy concept did not attract any noteworthy attention of mathematicians, but later they did realize its value and worldwide mathematical research of graph energy started. Nowadays, in connection with graph energy, energy like quantities was considered also for other matrices. In this paper, we are defining a matrix, called the minimum pendant dominating matrix denoted by $A_{pe}(G)$ and we study its eigenvalues and the energy. Further, we study the mathematical aspects of the minimum pendant dominating energy of a graph. It may be possible that the minimum pendant dominating energy which we are considering in this paper have applications in other areas of science such as chemistry and so on. The graphs we are considering are assumed to be finite, simple, undirected having no isolated vertices, and of order at least two.

Definitions

Definition 1.1. A subset S of $V(G)$ is a dominating set of G if each vertex $u \in V - S$ is adjacent to a vertex in S. The least cardinality of a dominating set is called the domination number of G and is usually denoted by $\gamma(G)$.

Definition 1.2. A dominating set S is called a pendant dominating set if $< S >$ contains at least one pendant vertex. The least cardinality of a pendant dominating set in G is called the pendant domination number of G, denoted by$\gamma_{pe}(G)$.

For more details on the terms used in this paper refer [8].

We recall the following results required for our study:

Lemma 1.1. [5] Let G be a nonempty graph with maximum vertex degrees Δ. Then $\lambda_1 \geq \sqrt{\Delta}$.

Lemma 1.2. [5] If G is a connected graph with n vertices, then $\lambda_1 \geq 2\cos\left(\frac{\pi}{n+1}\right)$.

II. The Minimum Pendant Dominating Energy of a Graph

In this section, we define the minimum pendant dominating energy of a graph and study some basic properties of the minimum pendant dominating energy. Let D be a minimum pendant dominating set of a graph$G$. The minimum pendant dominating matrix of G is an n-square matrix defined by $A_{pe}(G) = (\mathrm{a_{ij}})$, where

$$\mathrm{a_{ij}} = \begin{cases} 1, & \text{if } \mathrm{v_i v_j} \in \mathrm{E}, \\ 1, & \text{if } \mathrm{i = j \text{ and } v_i} \in \mathrm{D}, \\ 0, & \text{otherwise.} \end{cases}$$

The characteristic polynomial of $A_{pe}(G)$ is denoted by $f_n\left(G, \lambda\right)$ and is defined by $f_n\left(G, \lambda\right) = \det(\lambda I - A_{pe}(G))$. The minimum pendant dominating eigenvalues of the graph G are the eigenvalues of the matrix $A_{pe}(G)$. We note that these eigenvalues are real numbers since $A_{pe}(G)$ is real and symmetric. So, we can label them in the non-increasing order $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. The minimum pendant dominating energy of G is defined to be the sum of the absolute values of the eigenvalues of $A_{pe}(G)$. In symbols, we write

$$E_{pe}(G) = \sum_{i=1}^{n} | \lambda_i |$$

Observation 2.1. For any Graph G $E_{pe}(G) \leq E(G)$.

Example 2.1: Consider a graph as shown in Figure 1. The possible minimum pendant dominating sets are:
$(i) D_1 = \{v_1, v_2, v_4\}$ (ii) $D_2 = \{v_2, v_3, v_5\}$ (iii) $D_3 = \{v_2, v_3, v_6\}$ $(iv) D_4 = \{v_2, v_3, v_4\}$
(v) $D_5 = \{v_1, v_4, v_3\}$ (vi) $D_6 = \{v_5, v_6, v_2\}$

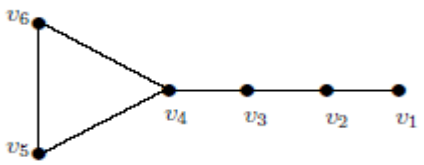


Figure 1

$$A_{pe,D_1}(G) = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

So, $f_n\left(G, \lambda\right) = \lambda^6 - 3\lambda^5 - 3\lambda^4 + 9\lambda^3 + 5\lambda^2 - 4\lambda - 1 = 0.$
The minimum pendant dominating energy, $E_{pe}(G) \approx 7.9681$
Now, suppose if we choose another pendant dominating set, $D_2 = \{v_2, v_3, v_5\}$. Then

$$A_{pe,D_2}(G) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

Clearly, $f_n\left(G, \lambda\right) = \lambda^6 - 3\lambda^5 - 3\lambda^4 + 9\lambda^3 + 5\lambda^2 - 4\lambda - 2 = 0.$
The minimum pendant dominating energy, $E_{pe}(G) \approx 8.1301$
Therefore, it is clear from the above example that the minimum pendant dominating energy of a graph G depends on the minimum pendant dominating set.

Theorem 2.1. Let G be a graph with vertex set, edge set E and a minimum pendant dominating set D. Let $f_n\left(G, \lambda\right) = a_o\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_n$ be the characteristic polynomial of G. Then,

- $a_0 = 1$
- $a_1 = -|D|$
- $a_3 = \binom{D}{2} - |E|$

Theorem 2.2. Let G be a graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$, edge set E and $D = \{u_1, u_2, \ldots, u_m\}$ be a minimum pendant dominating set. If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigen values of a minimum pendant dominating matrix $A_{pe}(G)$ then,

1. $\sum_{i=1}^{n} \lambda_i = \gamma_{pe}(G).$

2. $\sum_{i=1}^{n} \lambda^2{}_i = 2m + \gamma_{pe}(G).$

Theorem 2.3. If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $A_{pe}(G)$. Then $\sum_{i=1}^{n} \lambda^2{}_i = 2m + \gamma_{pe}(G)$.

III. Upper and Lower Bounds

Theorem 3.1. Let G be a nonempty and nonsingular graph with n vertices, m edges. Then
$E_{pe}(G) \geq \sqrt{\Delta} + (n-1) + \ln|detA| - \ln\sqrt{\Delta}.$
Proof. Since G is non-singular all its eigenvalue are different from zero. Consider a function $f_1(x) = x - 1 - \ln x$, for $x > 0$. It is elementary to prove that $f_1(x)$ is an increasing function for $x \geq 1$ and decreasing for $0 < x \leq 1$. Consequently, $f_1(x) \geq f_1(0) = 0$, implying that $x \geq 1 + \ln x$ for $x > 0$. Using this result, we get

$$E_{pe}(G) = \lambda_1 + \sum_{i=2}^{n} |\lambda_i| \quad (3.1)$$
$$\geq \lambda_1 + n - 1 + \sum_{i=2}^{n} \ln|\lambda_i| \quad (3.2)$$
$$= \lambda_1 + n - 1 + \ln\prod_{i=2}^{n} \lambda_i \quad (3.3)$$
$$= \lambda_1 + n - 1 + \ln|detA| - \ln\lambda_i \quad (3.4)$$

Now, by Lemma 1.1, we must have $\lambda_1 \geq \sqrt{\Delta}$. Further, for $1 \leq x \leq n$, the function $g_1(x) = x + n - 1 + \ln|detA| - \ln x$, is monotonically increasing. Therefore, we may conclude that $g_1(x) \geq g_1\left(\sqrt{\Delta}\right) = \sqrt{\Delta} + (n-1) + \ln|detA| - \ln\sqrt{\Delta}$, for $x \geq \Delta$. Combining the above result with (3.4), we get the required result.

Theorem 3.2. Let G be a nonempty nonsingular and connected graph with n vertices, m edges. Then

$$E_{pe}(G) \geq 2cos\left(\frac{\pi}{n+1}\right) + (n-1) + \ln|detA| - \ln\left(2cos\left(\frac{\pi}{n+1}\right)\right)$$

Proof. Since G is non-singular, all its eigenvalues are different from zero. Consider a function $g_2(x) = x - 1 - \ln x$, for $x > 0$. It is elementary to prove that $g_2(x)$ is an increasing function for $x \geq 1$ and decreasing for $0 < x \leq 1$. Consequently, $g_2(x) \geq g_2(0) = 0$, implying that $x \geq 1 + \ln x$ for $x > 0$. Using this result, we get

$$E_{pe}(G) = \lambda_1 + \sum_{i=2}^{n} |\lambda_i| \quad (3.5)$$
$$\geq \lambda_1 + n - 1 + \sum_{i=2}^{n} \ln|\lambda_i| \quad (3.6)$$
$$= \lambda_1 + n - 1 + \ln\prod_{i=2}^{n} \lambda_i \quad (3.7)$$
$$= \lambda_1 + n - 1 + \ln|detA| - \ln\lambda_1 \quad (3.8)$$

Now by using Lemma 1.2. $\lambda_1 \geq 2cos\left(\frac{\pi}{n+1}\right)$. Further, for $1 \leq x \leq n$, the function $g_2(x) = x + n - 1 + \ln | detA | - \ln x$, is monotonically increasing. Therefore, we may conclude that $g_2(x) \geq g_2\left(2cos\left(\frac{\pi}{n+1}\right)\right) = 2cos\left(\frac{\pi}{n+1}\right) + (n-1) + \ln | detA | - \ln\left(2cos\left(\frac{\pi}{n+1}\right)\right)$ for $x \geq 2cos\left(\frac{\pi}{n+1}\right)$. Combining the above result with (3.8), we get the required result.

Theorem 3.3. If G is k −regular graph with n vertices and m edges and let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ denotes the eigenvalues of G. Then $E_{pe}(G) \leq \lambda_1 + \sqrt{(n-1)(nk + \gamma_{pe} - (| \lambda_1 |)^2)}$.

Proof. Let G be a $k-$regular graph with n vertices and m edges. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the eigenvalues of G. Then taking $a_i = 1, b_i = | \lambda_i |$ in Cauchy's-Schwarz inequality, we get

$$\left(\sum_{i=2}^{n} | \lambda_i |\right)^2 \leq (n-1)\left(\sum_{i=2}^{n} \lambda^2{}_i\right) \quad (3.9)$$

and $$E_{pe}(G) - | \lambda_1 | = \sum_{i=2}^{n} | \lambda_i | \quad (3.10)$$

Also $$2m + \gamma_{pe} = \sum_{i=1}^{n} | \lambda_i |^2 = | \lambda_1 |^2 + \sum_{i=2}^{n} | \lambda_i |^2 \quad (3.11)$$

Hence

$$2m + \gamma_{pe} - | \lambda_1 |^2 = \left(\sum_{i=2}^{n} | \lambda_i |^2\right)$$

Now, from (3.9), (3.10) and (3.11)

$$\left(E_{pe}(G) - | \lambda_1 |\right)^2 \leq (n-1)(2m + \gamma_{pe} - | \lambda_1 |^2)$$

Since $2m = nk$, we obtain that $E_{pe}(G) \leq \left|\lambda_1\right| + \sqrt{(n-1)(nk + \gamma_{pe} - (| \lambda_1 |)^2)}$.

Theorem 3.4. Let G be a connected nonsingular graph of order n, with m edges. Then
$E_{pe}(G) \leq 2m\left(1 - \frac{2m}{n^2} + \frac{1}{n}\right) - \ln\left(\frac{n|detA|}{2m}\right)$

Proof. Since G is nonsingular, we have $\lambda_i > 0, i = 1,2,3, \ldots, n$. Thus

$$| detA | = \prod_{i=1}^{n} | \lambda_i | > 0 \quad (3.12)$$

If G has no isolated vertices, then

$$2m = \sum_{i=1}^{n} d_i \geq n \quad i.e., \frac{2m}{n} \geq 1 \quad (3.13)$$

Consider the function $g_3(x) = x^2 - x - \ln x \; x > 0$. Thus, $g_3(x)$ is an increasing function on $x \geq 1$ and decreasing function on $0 < x \leq 1$. Thus, $g_3(x) \geq g_3(1) = 0$, it implies $x \leq x^2 - \ln x$ for $x > 0$. Equality holiding if and only $x = 1$.

$$E_{pe}(G) = \lambda_1 + \sum_{i=2}^{n} | \lambda_i |$$

$$\leq \lambda_1 + \sum_{i=2}^{n}\left(\lambda^2{}_i - \ln \lambda_i\right)$$

$$= \lambda_1 + 2m - \lambda^2{}_1 - \ln \prod_{i=2}^{n} \ln \lambda_1$$

$$= 2m + \lambda_1 - \lambda^2{}_1 + \ln | detA | + \ln \lambda_1 \quad (3.14)$$

From Lemma 1.1, we know that $\lambda_1 \geq \sqrt{\Delta} = \frac{2m}{n}$. Further for $0 < x \leq 1$, the function $g(x) = 2m + x + x^2 + \ln | detA | + \ln x$ is monotonically increasing function and a decreasing on $x \geq 1$, and since $x \geq \frac{2m}{n} \geq 1$,
We have

$$g(x) \leq g\left(\frac{2m}{n}\right) = 2m + \left(\frac{2m}{n}\right) - \left(\frac{2m}{n}\right)^2 + \ln | detA | + \ln\left(\frac{2m}{n}\right)$$

Combining this with 3.14, we get the required result.

Theorem 3.5. Let G be any graph. Then $E_{pe}(G) \leq \sqrt{2mn + n\gamma_{pe}}$.

Proof. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the eigenvalues of $A_{pe}(G)$ arranged in the non-increasing order. By Cauchy-Schwarz inequality, we have

$$\left(\sum_{i=1}^{n} a_1 b_i\right)^2 \leq \left(\sum_{i=1}^{n} {a_i}^2\right)\left(\sum_{i=1}^{n} {b_i}^2\right)$$

If we take $a_i = 1, b_i = \mid \lambda_i \mid$ then

$$\left(\sum_{i=1}^{n} \mid \lambda_i \mid\right)^2 \leq \left(\sum_{i=1}^{n} 1\right)\left(\sum_{i=1}^{n} {\lambda_i}^2\right)$$

Therefore by Theorem2.2. We obtain,

$$\left(E_{pe}(G)\right)^2 \leq n\left(\sum_{i=1}^{n} {\lambda_i}^2\right) = n(2m + \gamma_{pe})$$

$i.e., E_{pe}(G) \leq \sqrt{2mn + n\gamma_{pe}}$.

Theorem 3.6. Let G be any graph having n vertices and m edges. Then

$$E_{pe}(G) \geq \sqrt{2m + \gamma_{pe} + n(n-1)P^{\frac{2}{n}}}$$

Where $P = detA_{pe}(G) > 0$.

Proof. By definition we have

$$\left(E_{pe}(G)\right)^2 = \left(\sum_{i=1}^{n} \mid \lambda_i \mid\right)\left(\sum_{j=1}^{m} \mid \lambda_j \mid\right)$$

$$= \sum_{i=1}^{n} \mid \lambda_i \mid^2 + \sum_{i \neq j} \mid \lambda_i \mid \mid \lambda_j \mid$$

Now, by using arithmetic and geometric mean inequality, we get

$$\frac{1}{n(n-1)} \sum_{i \neq j} \mid \lambda_i \mid \mid \lambda_j \mid \geq \left[\prod_{i \neq j} \mid \lambda_i \mid \mid \lambda_j \mid\right]^{\frac{1}{n(n-1)}}$$

$$E_{pe}(G) \geq \sum_{i=1}^{n} \mid \lambda_i \mid^2 + n(n-1)\left(\prod_{i \neq j} \mid \lambda_i \mid \mid \lambda_j \mid\right)^{\frac{1}{n(n-1)}}$$

$$\geq \sum_{i=1}^{n} \mid \lambda_i \mid^2 + n(n-1)\left[\prod_{i=1} \mid \lambda_i \mid^{2(n-1)}\right]^{\frac{1}{n(n-1)}}$$

$$= \sum_{i=1}^{n} \mid \lambda_i \mid^2 + n(n-1)\left|\prod_{i=1}^{n} \lambda_i\right|^{\frac{2}{n}}$$

$$= \left(2m + \gamma_{pe}(G)\right) + n(n-1)P^{\frac{2}{n}}, \text{ by Theorem2.3.}$$

Therefore

$$E_{pe}(G) \geq \sqrt{2m + \gamma_{pe} + n(n-1)P^{\frac{2}{n}}}$$

IV. The Minimum Pendant Dominating Energy of Some Standard Graphs

Theorem 4.1. For $n \geq 3$, the minimum pendant dominating energy of a complete graph K_n is $(n-3) + \sqrt{n^2-2n+9}$.

Proof. Let K_n be the complete graph with the vertex set $V = \{v_1, v_2, \dots, v_n\}$. Clearly, $D = \{v_1, v_2\}$ is a minimum pendant dominating set of K_n. Then characteristic polynomial is

$$A_{pe}\left(K_n, \lambda\right) = \begin{pmatrix} \lambda-1 & 1 & 1 & . & . & . & 1 & 1 \\ 1 & \lambda-1 & 1 & . & . & . & 1 & 1 \\ 1 & 1 & \lambda & . & . & . & 1 & 1 \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ 1 & 1 & 1 & . & . & . & \lambda & 1 \\ 1 & 1 & 1 & . & . & . & 1 & \lambda \end{pmatrix}$$

Characteristic equation is. $\lambda(\lambda+1)^{n-3}(\lambda^2-(n-1)\lambda-2)=0$

The minimum Pendant dominating eigen values are;

$\lambda = 0, \lambda = -1$ [(n-3) times], $\lambda = \dfrac{(n-1) \pm \sqrt{n^2-2n+9}}{2}$ [one time each]

Minimum pendant dominating energy is

$$E_{pe}(K_n) = 0 + |-1|(n-3) + \left|\frac{(n-1)+\sqrt{(n^2-2n+9)}}{2}\right| + \left|\frac{(n-1)-\sqrt{(n^2-2n+9)}}{2}\right|$$

$$E_{pe}(K_n) = (n-3) + \sqrt{n^2-2n+9}.$$

Theorem 4.2. For $n \geq 2$, the minimum pendant dominating energy of a star $K_{1,n-1}$ is at most $2 + 2\sqrt{(n-2)}$. Equality holds if and only if $n = 2$.

Proof. Let $K_{1,n-1}$ be a star with the vertex set $V = \{v_1, v_2, \dots, v_n\}$ having the vertex v_n at the center. The minimum pendant dominating set is $D = \{v_1, v_n\}$. Then characteristic polynomial is

$$A_{pe}\left(K_{1,n-1}, \lambda\right) = \begin{pmatrix} \lambda-1 & 1 & 1 & . & . & . & 1 & 1 \\ 1 & \lambda & 0 & . & . & . & 0 & 0 \\ 1 & 0 & \lambda & . & . & . & 0 & 0 \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ 1 & 0 & 0 & . & . & . & \lambda & 0 \\ 1 & 0 & 0 & . & . & . & 0 & \lambda-1 \end{pmatrix}$$

$$f_n\left(G, \lambda\right) = \lambda^n - 2\lambda^{n-1} - (n-2)\lambda^{n-2} + \lambda^{n-3}$$

$$= \lambda^{n-3}\left(\lambda^3 - 2\lambda^2 - (n-2)\lambda + (n-2)\right)$$

$$= \lambda^{n-3}\left[\lambda^3 - 2\lambda^2 - (n-2)\lambda + 2(n-2) - (n-2)\right]$$

$$\leq \lambda^{n-3}\left[\lambda^3 - 2\lambda^2 - (n-2)\lambda + 2(n-2)\right]$$

$$= \lambda^{n-3}\left[(\lambda-2)\left(\lambda^2-(n-2)\right)\right]$$

The minimum pendant dominating eigen values are:

$\lambda = 0$ [(n-3) times], $\lambda = 2$ [1 time] $\lambda = \pm\sqrt{(n-2)}$ [one time each]. Minimum pendant dominating energy is

$E_{pe}(K_{1,n-1}) = 0 + |2|(1) + \left|\sqrt{(n-2)}\right| + \left|-\sqrt{(n-2)}\right|$

$E_{pe}(K_{1,n-1}) = 2\left(1+\sqrt{(n-2)}\right)$.

Therefore $E_{pe}(K_{1,n-1}) \leq 2\left(1+\sqrt{(n-2)}\right)$.

Theorem 4.3. For$n > 2$, the minimum pendant dominating energy of a complete bipartite graph $K_{n,n}$ is $(n+1)+\sqrt{n^2+2n-3}$.

Proof. Let $K_{n,n}$ be a complete bipartite graph and let $V(K_{m,n}) = \{u_i, v_j\ 1 \leq i \leq m, 1 \leq j \leq n\}$. Then the minimum pendant dominating set of $K_{m,n}$ is $D = \{u_1, v_1\}$. Hence the characteristic polynomial

$$A_{pe}\left(K_{n,n}, \lambda\right) = \begin{pmatrix} \lambda-1 & 0 & 0 & \cdots & 1 & 1 \\ 0 & \lambda & 0 & \cdots & 1 & 1 \\ 1 & 0 & \lambda & \cdots & 1 & 1 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 1 & 1 & 1 & \cdots & \lambda & 0 \\ 1 & 1 & 1 & \cdots & 0 & \lambda-1 \end{pmatrix}$$

$$f_n\left(K_{n,n}, \lambda\right) = \lambda^n\left[\left(\lambda^{n-2}-(n+1)\lambda^{n-3}+(n-1)\lambda^{n-4}\right)\left(\lambda^{n-2}+(n-1)\lambda^{n-3}-(n-1)\lambda^{n-4}\right)\right]$$

$$= \lambda^{2n-4}\left[\left(\lambda^2-(n+1)\lambda+(n-1)\right)\left(\lambda^2+(n-1)\lambda-(n-1)\right)\right].$$

The minimum pendant dominating eigen values are:

$\lambda = 0$ [(2n-4) times], $\lambda = \dfrac{(n-1)\pm\sqrt{\left(n^2-2n+5\right)}}{2}$ [one time each], $\lambda = \dfrac{-(n-1)\pm\sqrt{\left(n^2+2n-3\right)}}{2}$ [one time each]

$$E_{pe}(K_{n,n}) = 0 + \left|\frac{(n+1)+\sqrt{(n^2-2n+5)}}{2}\right| + \left|\frac{(n+1)-\sqrt{(n^2-2n+5)}}{2}\right| + \left|\frac{-(n-1)+\sqrt{(n^2+2n-3)}}{2}\right| - \left|\frac{-(n-1)-\sqrt{(n^2+2n-3)}}{2}\right|$$

Therefore

$E_{pe}(K_{n,n}) = (n+1)+\sqrt{n^2+2n-3}$.

Theorem 4.4. For $n > 3$, the minimum pendant dominating energy of a cocktail party graph $K_{n\times 2}$ is

$$2(n-3)+\sqrt{5}+\sqrt{4n^2-4n+13} \leq E_{pe}(K_{n\times 2}) \leq (2n-5)+\sqrt{5}+2\sqrt{n^2+1}$$

Proof. Let $K_{n\times 2}$ be the cocktail party graph with vertex set $V = \{u_i, v_i\ 1 \leq i \leq n\}$ and the minimum pendant dominating set of $K_{n\times 2}$ is $D = \{u_1, u_2\}$. Then the characteristic polynomial is

$$A_{pe}\left(K_{n\times 2},\lambda\right)=\begin{pmatrix}\lambda-1 & 1 & 1 & . & . & . & 1 & 0\\ 1 & \lambda-1 & 1 & . & . & . & 0 & 1\\ 1 & 1 & \lambda & . & . & . & 1 & 1\\ . & . & . & . & . & . & . & .\\ . & . & . & . & . & . & . & .\\ . & . & . & . & . & . & . & .\\ 1 & 0 & 1 & . & . & . & \lambda & 1\\ 0 & 1 & 1 & . & . & . & 0 & \lambda\end{pmatrix}$$

$$f_n\left(K_{n\times 2},\lambda\right)=\lambda^{n-2}(\lambda+2)^{n-3}\left(\lambda^2+\lambda-1\right)\left(\lambda^3-(2n-3)\lambda^2-(2n+1)\lambda+(2n-2)\right)$$

$$\leq \lambda^{n-2}(\lambda+2)^{n-3}\left(\lambda^2+\lambda-1\right)\left(\lambda^3-(2n-3)\lambda^2-(2n+1)\lambda+(2n-2)+1\right)$$

$$=\lambda^{n-2}(\lambda+2)^{n-3}\left(\lambda^2+\lambda-1\right)(\lambda-1)\left(\lambda^2-(2n-4)\lambda-(4n-3)\right)$$

The minimum pendant dominating eigen values are:

$\lambda=0$ [(n-2) times], $\lambda=-2$ [(n-3) times], $\lambda=\frac{-1\pm\sqrt{5}}{2}$ [one time each], $\lambda=1$ [one time], $\lambda=(n-2)\pm\sqrt{n^2+1}$ [one time each].

$$E_{pe}(K_{n\times 2})=0+|-2|(n-3)+\left|\frac{-1+\sqrt{5}}{2}\right|+\left|\frac{-1-\sqrt{5}}{2}\right|+|1|+\left|(n-2)+\sqrt{n^2+1}\right|+\left|(n-2)-\sqrt{n^2+1}\right|.$$

Therefore

$$E_{pe}(K_{n\times 2})\leq(2n-5+\sqrt{5}+2\sqrt{n^2+1}) \qquad (4.1)$$

On the other hand

$$f_n\left(K_{n\times 2},\lambda\right)=\lambda^{n-2}(\lambda+2)^{n-3}\left(\lambda^2+\lambda-1\right)\left(\lambda^3-(2n-3)\lambda^2-(2n+1)\lambda+(2n-2)\right)$$

$$\geq \lambda^{n-2}(\lambda+2)^{n-3}\left(\lambda^2+\lambda-1\right)\left(\lambda^3-(2n-3)\lambda^2-(2n+1)\lambda\right)$$

$$=\lambda^{n-1}(\lambda+2)^{n-3}\left(\lambda^2+\lambda-1\right)\left(\lambda^2-(2n-3)\lambda-(2n+1)\right).$$

The minimum pendant dominating eigen values are:

$\lambda=0$ [(n-2) times], $\lambda=-2$ [(n-3) times], $\lambda=\frac{-1\pm\sqrt{5}}{2}$ [one time each], $\lambda=1$ [one time], $\lambda=\frac{(2n-3)\pm\sqrt{4n^2-4n+13}}{2}$ [one time each].

$$E_{pe}(K_{n\times 2})=0+|-2|(n-3)+\left|\frac{-1+\sqrt{5}}{2}\right|+\left|\frac{-1-\sqrt{5}}{2}\right|+\left|\frac{(2n-3)+\sqrt{4n^2-4n+13}}{2}\right|+\left|\frac{(2n-3)-\sqrt{4n^2-4n+13}}{2}\right|.$$

Therefore

$$E_{pe}(K_{n\times 2})\geq 2(n-3)+\sqrt{5}+\sqrt{4n^2-4n+13} \qquad (4.2)$$

Combining the equations 4.1 and 4.2 we obtain

$$2(n-3)+\sqrt{5}+\sqrt{4n^2-4n+13}\leq E_{pe}(K_{n\times 2})\leq(2n-5)+\sqrt{5}+2\sqrt{n^2+1}$$

Theorem 4.5. Let G be a graph with a minimum pendant dominating set D. If the minimum pendant dominating energy $E_{pe}(G)$ of G is a rational number, then $E_{pe}(G) \equiv |D| \ (mod\ 2)$

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Fuzzification on Large Graph Coloring Using α-cuts

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Abstract*: In this paper we use the definition of fuzzy set, fuzzy graph, chromatic number of fuzzy graph, fuzzy k-vertex coloring of fuzzy graph to reduce the number of colors in coloring the graph through α-cuts. In this paper we use α-cuts for fuzzification of large graphs.*

Keywords*: Fuzzy set, fuzzy graphs, chromatic number of fuzzy graph, α-cuts*.

I. Introduction

Graph theory is proved to be tremendously useful in modeling the essential features of systems with finite components. Graphical models are used to represent telephone network, communication problems, traffic networks etc. Many problems of practical interest that can be modeled as graph theoretic problems may be uncertain. To deal with this uncertainty the concept of fuzzy theory was applied to graph theory.

The notion of fuzzy sets was introduced by L.A. Zadeh in 1965. It involves the concept of a membership function defined on a universal set. The value of the membership function lies in [0,1] [5]. The concept of the fuzzy graph was introduced by A. Rosenfeld in 1975 [3]. We now that a fuzzy graph is symmetric binary fuzzy relation on a fuzzy subset. The fuzzy graph was first defined by Kaufmann in 1973 based on Zadeh's fuzzy relations [6]. The applications of fuzzy relations are widespread and important concept in the field of clustering analysis neural networks, computer networks, Pattern recognition, Decision making and expert systems.

Fuzzy graph coloring is one of the most important problems of fuzzy graph theory , the concept of chromatic number of fuzzy graph was introduced by Munoz [2]. The authors considered fuzzy graphs with crisp vertex set, i.e., fuzzy graphs for which $\sigma(u)=1 \forall\, u \in V$ and edges with membership degree in [0, 1]. [5]; it is mainly studied in combinatorial optimization like traffic light control, exam scheduling, register allocation, storage problem [1] etc.

In this paper we explain the storage problem of chemicals using fuzzy coloring through α-cuts[4]. Since in crisp graph we cannot coloring the large graph for storing chemicals in different compartments with minimum colors. So we fuzzify the large graph with fuzzy edges to allocating the chemicals in different ways with minimum number of compartments.

II. Preliminaries

Definition 2.1:

A fuzzy set A defined on a non empty set X is the family $A=\{(x,\mu_A(x)/x\in X\}$ where $\mu_A : X \to I$ the membership function is. In classical fuzzy set I is usually defined as the interval [0, 1] such that $\mu_A(x)=\begin{cases}0; x\notin A\\ 1; x\in A.\end{cases}$

Definition: 2.2

A fuzzy graph is an ordered triple $G:(V,\sigma,\mu)$ where V is a set of vertices $\{u_1,u_2,...,u_n\}$. σ is a fuzzy subset of V i.e., $\sigma : V \to [0,1]$ and is denoted by $\sigma=\{(u_1,\sigma(u_1)),(u_2,\sigma(u_2)),...,(u_n,\sigma(u_n))\}$ and μ is a fuzzy relation on σ , i.e., $\mu(u,v)\le\sigma(u)\wedge\sigma(v)\,\forall\, u,v\in V$. We consider fuzzy graph G with no loops and assume that V is finite and nonempty.

Definition: 2.3

If $G=(V,\sigma,\mu)$ is such a fuzzy graph where $V=\{1,2,3,...,n\} and\ \mu$ is a fuzzy number on the set of all subsets of $V\times V$. Assume $I=A\cup\{0\} where\ A=\{\alpha_1<\alpha_2<,...,<\alpha_k\}$ is the fundamental set (level set) of G. For each $\alpha\in I, G_\alpha$ denote the crisp graph $G_\alpha=(V,E_\alpha) where\ E_\alpha=\{ij/1\le i<j\le n, \mu(i,j)\ge\alpha\}$ and $\chi_\alpha=\chi(G_\alpha)$ denote the chromatic number of crisp graph G_α. By this definition the chromatic number of fuzzy graph G is the fuzzy number $\chi(G)=\{(i,V(i))/i\in X\} where V(i)=\max\{\alpha\in I/i\in A_\alpha\}$ and $A_\alpha=\{1,2,3,...,\chi_\alpha\}$.

Definition: 2.4

A fuzzy set A defined on X can be characterized from its family of α-cuts

$A_\alpha=\{x\in X/\mu_A(x)\ge\alpha\}\ \alpha\epsilon I$.

α-cut set of fuzzy set is crisp set. In this paper α-cut depends on edge membership value.

Definition: 2.5

Let G be a fuzzy graph. The edge connectivity of G denoted by $\lambda(G)$ is defined to be the minimum weight of cut-sets of G.

Definition: 2.6

Let $G:(\sigma,\mu)$ be a fuzzy graph. The strong degree of a vertex $v\in\sigma^*$ is defined as the sum of membership values of all strong edges incident at v. It is denoted by $D_s(v)$. Also if $n_s(v)$ denote the set of all strong neighbours of v, then $D_s(v)=\sum_{u\in n_s\frown v}\mu(u,v)$.

III. A Storage Problem

A company manufactures n chemicals $C_1, C_2, \dots C_n$. Certain pairs of these chemicals are incompatible and would cause explosions if brought into contact with each other. As a precautionary measure the company wishes to partition its warehouse into compartments, and store incompatible chemicals in different compartments. What is the least number of compartments into which the warehouse should be partitioned?

We obtain a fuzzy graph G on the vertex set $\{v_1, v_2, \dots, v_n\}$ by joining two vertices v_i and v_j if and only if the chemicals C_i and C_j are incompatible, and take the edges as fuzzy edges for the number of compartments to accommodate the chemicals which are incompatible. It is easy to see that the least number of compartments into which the warehouse should be partitioned is equal to the chromatic number of a fuzzy graph χ_FG.

The solution of many problems of practical interest involves finding the chromatic number of a fuzzy graph G. Since the chromatic number of a fuzzy graph is the least number of independent sets into which its vertex set can be partitioned, we start by describing a method for listing all the independent sets in a fuzzy graph through α-cuts. Because every independent set is a subset of a maximal independent set, it sufficient to determine all the maximal independent sets. In fact, our procedure first determines fuzzy α-cuts of maximal independent sets, i.e., minimal coverings.

Observe that a fuzzy subset S of V is a minimal covering of fuzzy graph G is and only if, for each vertex v belongs to S (but not both). This provides us with a procedure for finding minimal coverings.

In crisp graph the seven chemicals are colored by three colors which means they categorized they chemicals as {b, d, f}, {a, e, g} and {c}. Here they assign the chemicals in three compartments as in the name of coloring. The chemical {c} assigned in Blue compartment which carries highly incompatible, chemicals {a, e, g} are assigned in Red compartment which carries medium incompatible and chemicals {b, d, f} are assigned Green compartment which carries low incompatible. Now we assign the edge value as fuzzy number in between of two chemicals. That fuzzy numbers are shown than the compatible chemicals are brought together. We classify the chemicals as highly incompatible, medium incompatible, low medium incompatible, low incompatible and very low incompatible. Here we have chemicals named as {a, b, c, d, e, f, g}. The seven chemicals are categorized by the above five classification.

Now we have to minimize the colors by using α-cuts in fuzzy edges. The fuzzy edges are avoiding the explosions of chemicals and also minimize the number of compartments. The membership values of the fuzzy edges are defined and also fuzzy edge are the distance between the chemicals.

Classification of chemicals	Distance between the Incompatible chemicals
Highly incompatible	0.6
Medium incompatible	0.5
Low Medium incompatible	0.4
Low incompatible	0.3
Very low incompatible	0.2

Consider the fuzzy graph $G=(v,\mu)$ where $v=\{a,b,c,d,e,f,g\}$ and the μ is defined as

μ(a, d) = 0.2 = μ(c, e);
μ(c, d) = μ(d, g) = μ(d, e) = μ(b, g) = 0.3;
μ(c, f) = μ(e, f) = 0.4;
μ(a, b) = μ(e, f) = 0.5;
μ(b, c) = μ(b, e) = 0.6.

In the following diagrams we explain the α- cuts, how to reduce the number of colors in the fuzzified large graph G.

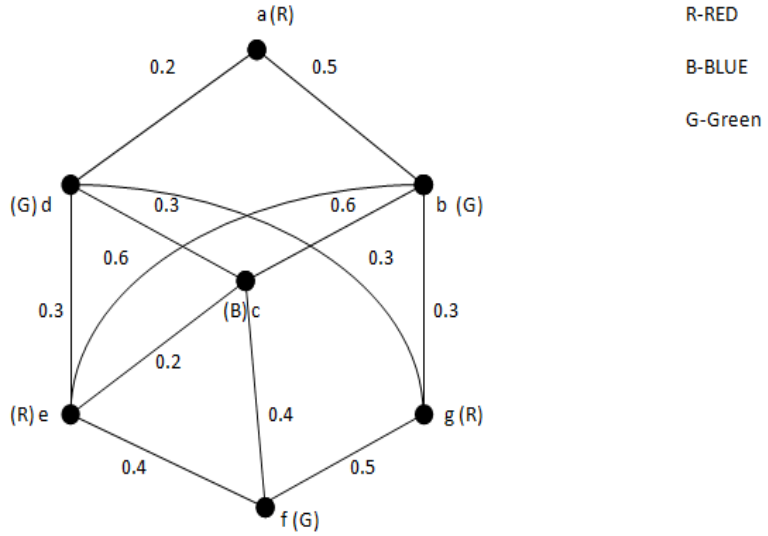


Figure 1 Fuzzified large graph G.

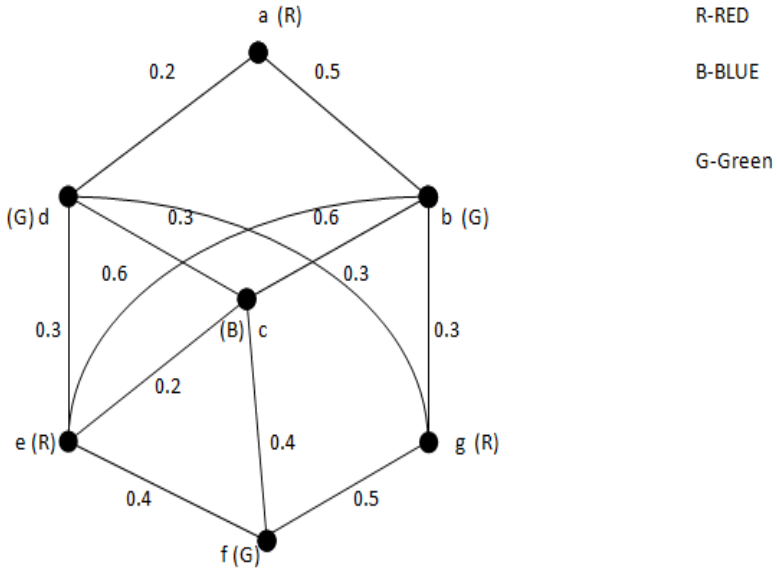


Figure 2 If α=0.2, then $\chi_{F_\alpha}(G)$ = 3.

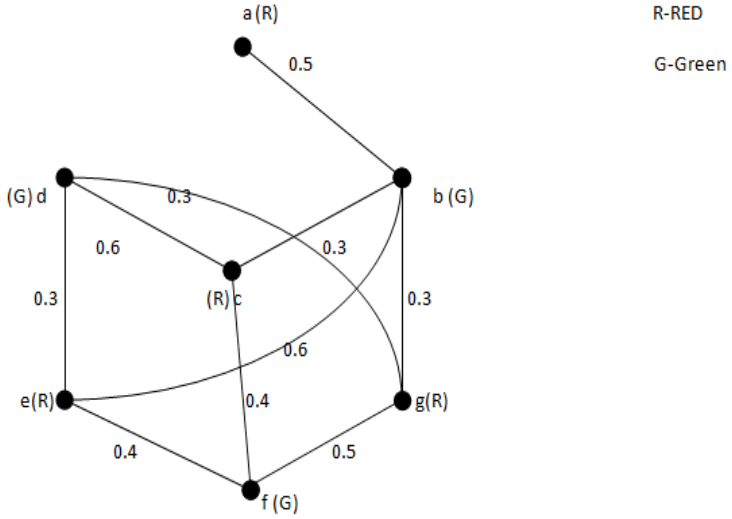


Figure 3 If α=0.3, then $\chi_{F_\alpha}(G) = 2$.

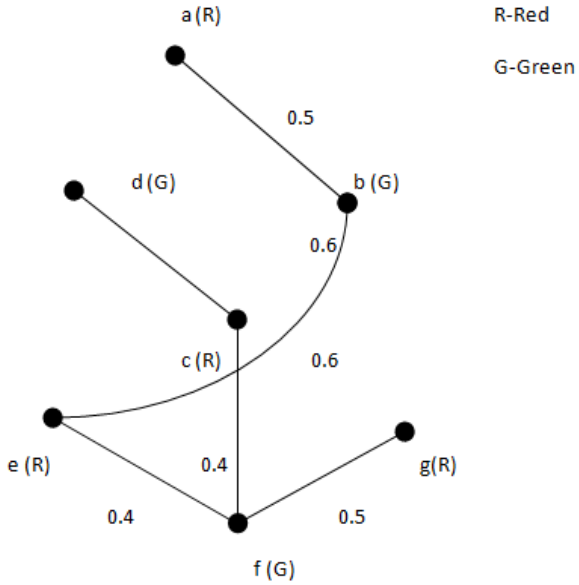


Figure 4 If α=0.4, then $\chi_{F_\alpha}(G) = 2$

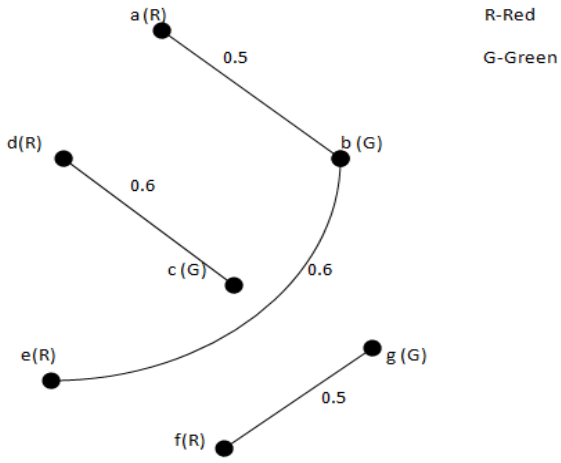


Figure 5 If α=0.5, then $\chi_{F_\alpha}(G) = 2$

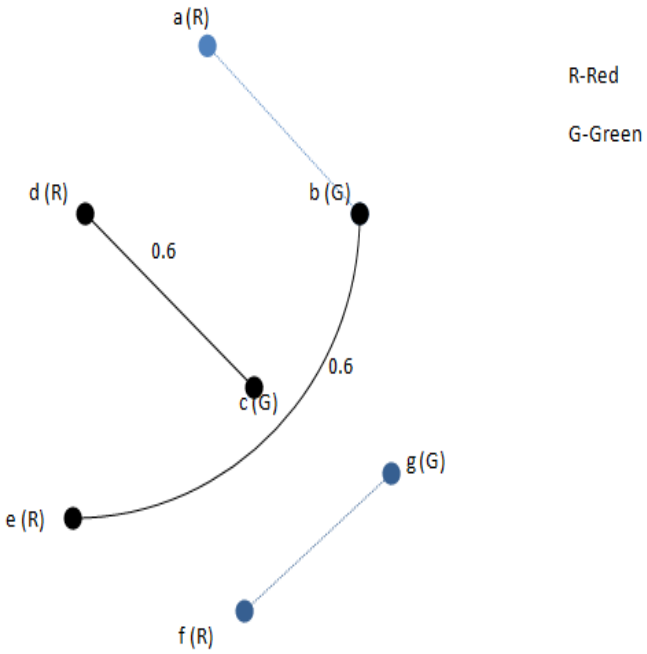


Figure 6 If α=0.6, then $\chi_{F_\alpha}(G) = 2$

The chromatic number of Fuzzified large graph G is χ_{F_α}(G) = {(1, 1) (2, 0.3, 0.4, 0.5 and 0.6) (3, 0.2)}.

IV. Conclusion

In this paper we determine the problem of chemical storage in the warehouse to optimize the utility of space with incompatible chemicals in a safe manner by using the concept of coloring in fuzzified large graphs by chromatic number through α-cuts. We obtain the minimum number of fuzzy k-vertex coloring through α-cuts $\chi_{F_\alpha}(G)$ for fuzzified large graph G. So the minimum number of coloring is 2.

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Scheduling a Round-Robin Tournament Using Independent Edge Domination

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Abstract*: In this paper work, we wish to Schedule Round-Robin Tournament with the help of graph using independent edge domination. We construct even number and odd number of team scheduling using independent edge domination.*

Keywords: *Domination; edge domination; independent edge domination; Schedule; Round-Robin*

I. Introduction.

Domination in graphs is the most extremely interesting area for many researchers in graph theory all over the world. The reason for this fascination is not very difficult to figure out. The concept of domination can be applied to form various Mathematical models for practical problems. Several areas like communication network, decision making process, administration, social net work theory etc. use domination theory

Sports tournaments draw attention of millions of people around the world. Lot of money invests by the broadcaster and organizer in sports events. So they are main economic activities around the world. In tournament, the schedule plays the important roll. On one hand, there are multiple decision makers, the broadcasters, the team, the organizer and the government for their difficulty of scheduling. Sports tournament have gained considerable amount of attention in recent years among the operation research constantly.

A round-robin is a sports competition in which each player or team plays against every other player or team. In this paper work we wish to Schedule Round-Robin Tournament with the help of graph using independent edge domination. We construct even number and odd number of team scheduling using independent edge domination.

1.1 Definition

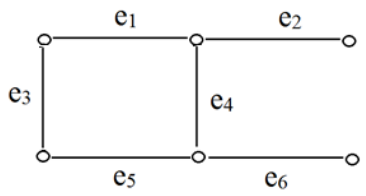
A set D of vertices in a graph G = (V, E) is called a dominating set of G if every vertex in V – D is adjacent to some vertex in D. In other words, A Set D of vertices of G is a dominating set if every vertex of G is dominated by at least one vertex of D. The cardinality of minimum dominating set is called domination number it is denoted by γ (G).

1.2 Definition

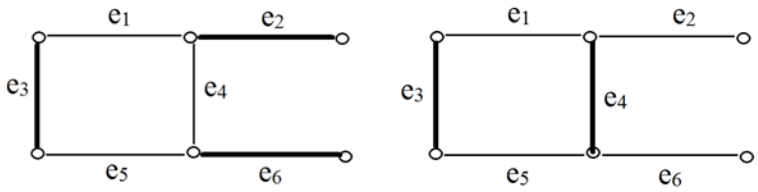
A set F of edges in a graph G = (V,E) is called an edge dominating set of G if every edge in E – F is adjacent to at least one edge in F and is an independent edge dominating set if edges are independent.

A edge dominating set is called minimal edge dominating set if no proper subset of S is a edge dominating set. A minimal edge dominating set of minimum cardinality is minimum edge dominating set.

***1.3 Example*:** Let e_1, e_2, e_3, e_4, e_5, e_6 are edges in the graph



Minimal edge dominating sets : $\{e_1,e_4\},\{e_1,e_5\},\{e_1,e_6\},\{e_2,e_5\}\{e_3,e_4\},\{e_2, e_3, e_6\}$
Minimum edge dominating sets : $\{e_1,e_4\},\{e_1,e_5\},\{e_1,e_6\},\{e_2,e_5\}\{e_3,e_4\}$
Independent edge dominating sets : $\{e_1,e_5\},\{e_1,e_6\},\{e_2,e_5\}\{e_3,e_4\},\{e_2, e_3, e_6\}$



This two graph are example of independent edge dominating set $\{e_2, e_3, e_6\}$ and $\{e_3,e_4\}$.

1.4 Difficult in scheduling round robin tournaments

Suppose we have four teams named 1,2,3,4 and we want to schedule round robin tournament. All we need to do is to choose an opponent for team 1 on the day 1 and another opponent for team 1 on the day 2. All other games are determined by these choices.

If we choose the game 1-2 for day One, 1-3 for day Two and 1-4 for day Three. Then, the remaining teams must meet on those days such as Team 3 & 4 on day One, Team 2 & 4 on day Two and Team 2 & 3 for day Three.

	Day One	**Day Two**	**Day Three**
Game 1	1-2	1-3	1-4
Game 2	3-4	2-4	2-3

Table 1: four team tournament schedule

Now let us try to schedule for six teams. If we choose the game 1-2 for day One, 1-3 for day Two, 1-4 for day Three, 1-5 for day Four and 1-6 for day Five. Then with some effort, using the method of trial and error and an exhaustive search, we would find a schedule.

	Day One	**Day Two**	**Day Three**	**Day Four**	**Day Five**
Game 1	1-2	1-3	1-4	1-5	1-6
Game 2	3-4	2-5	2-6	2-4	2-3
Game 3	5-6	4-6	3-5	3-6	4-5

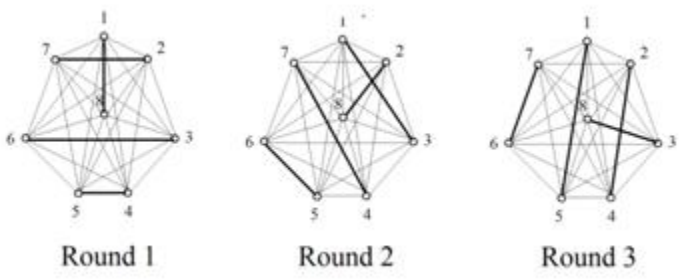
Table 2: Six team tournament schedule

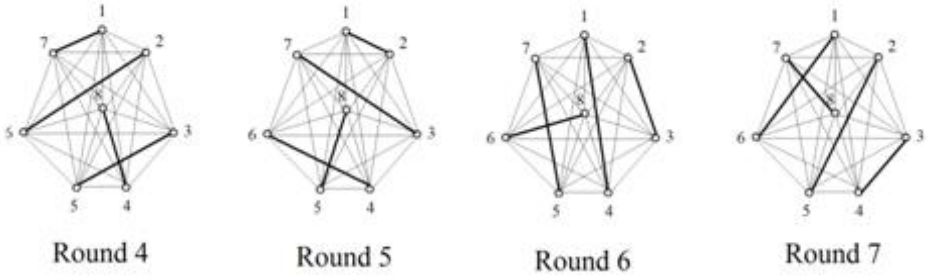
However, scheduling for eight teams using an exhaustive search could take many hours. It is obvious that we need a better method than trial and error. To find one, we use graph theory. A graph consists of a set of points called vertices and a set of lines called edges. Each edge joins two vertices. A graph in which every vertex is joined by an edge to every other vertex is called a complete graph. A complete graph can be viewed as a model of a round robin tournament in a natural way. Each vertex represents one team, and an edge joining two vertices, i and j, represents a game between the corresponding teams i - j.

II. Scheduling a Round-Robin Tournament

2.1 Even number of teams scheduling using independent edge domination

The round robin tournament can be schedule using independent edge dominating method. First we construct even team scheduling using independent edge dominating method. Example: 8 teams. Construct a complete graph k_8. And find the independent edge dominating sets. There are seven various independent edge dominating sets be results. Each sets be consider as rounds as shown below.





All rounds are the independent edge dominating sets. From this we can schedule round robin tournament.

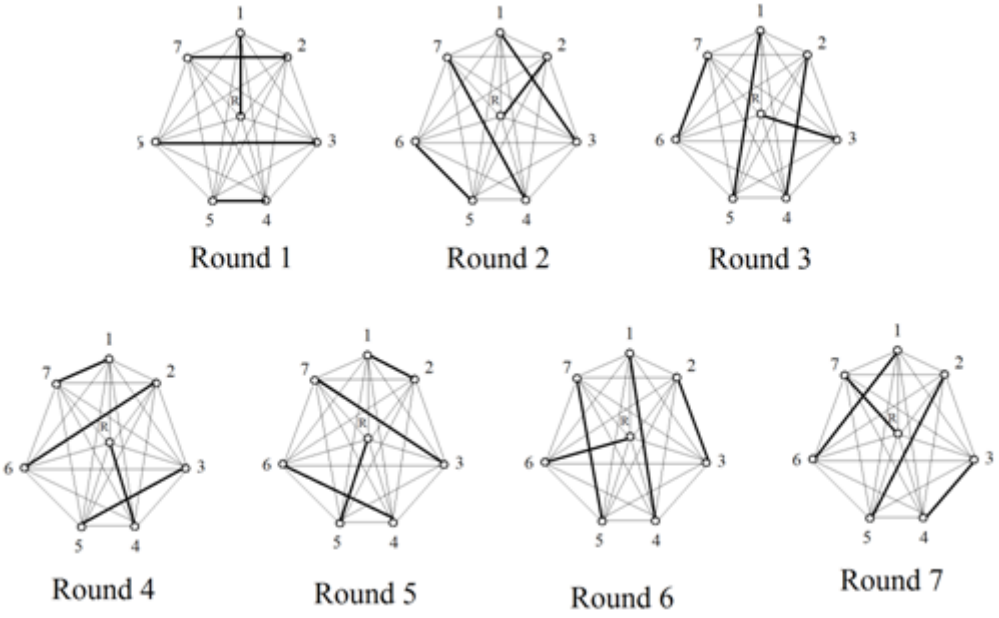
	Round 1	Round 2	Round 3	Round 4	Round 5	Round 6	Round 7
Game 1	1-8	2-8	3-8	4-8	5-8	6-8	7-8
Game 2	2-7	3-1	4-2	5-3	6-4	7-5	1-6
Game 3	3-6	4-7	5-1	6-4	7-3	1-4	2-5
Game 4	4-5	5-6	6-7	7-1	1-2	2-3	3-4

Table 3: eight team tournament schedule

In this schedule each team plays against every other team. This is the round robin schedule for 8 teams.

2.2 Odd number of teams scheduling using independent edge domination

Now, odd team scheduling using independent edge dominating set. Example: 7 teams 1,2,3,4,5,6,7, R are the vertices where R means rest position. Construct a complete graph k_8. And find the independent edge dominating sets. There are seven various independent edge dominating sets be results. Each sets be consider as rounds as shown below.



All rounds are the independent edge dominating sets. From this we can schedule round robin tournament

	Round 1	Round 2	Round 3	Round 4	Round 5	Round 6	Round 7
Rest	1	2	3	4	5	6	7
Game 1	2-7	3-1	4-2	5-3	6-4	7-5	1-6
Game 2	3-6	4-7	5-1	6-4	7-3	1-4	2-5
Game 3	4-5	5-6	6-7	7-1	1-2	2-3	3-4

Table 4: seven team tournament schedule

In this schedule each team plays against every other team. This is round robin schedule for 7 teams.

III. Conclusion

Schedule Round-Robin Tournament with the help of graph using independent edge domination is easy way than current round robin tournament method.

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ON r- DYNAMIC COLORING OF SPLITTING GRAPHS

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Abstract: : An r-dynamic coloring of a graph G is a proper coloring c of the vertices such that$|c(N(v))| \leq min\{r, d(v)\}$, for each v in $V(G)$. The r-dynamic chromatic number of a graph G is the minimum k such that G has an r-dynamic coloring with k colors. In this paper, we obtain the r-dynamic chromatic number for splitting graph of paths, cycles, complete and complete bipartite graphs.

Keywords: Cycle, Complete graph, Complete bipartite graphs, Path, Splitting graph and $\boldsymbol{r}$-dynamic coloring.

I. Introduction

Graph theory as a subject recognized as one of the fastest growing fields of research with diversified applications Graphs are useful to describe any complex situation to simple one. There are many interesting areas of research within graph theory. The coloring problem of graphs is one of the classical research areas in graph theory. Coloring of graphs are very extended areas of research. A coloring of a graph can be described by a function that maps elements of a graph (vertices or edges or both) into some set of numbers (possibly N, Z or even R) usually called colors such that some property is satisfied. The r-dynamic chromatic number was first introduced by Montgomery [4]. An r -dynamic coloring of a graph G is a proper coloring and it maps c from $V(G)$ to the set of colors such that (i) if $uv \in E(G)$, then $c(u) \neq c(v)$ and (ii) for each vertex $v \in V(G)$, $|c(N(v))| \geq min\{r, d(v)\}$, where $N(v)$ denotes the set of vertices adjacent to v, $d(v)$ its degree and r is a positive integer. The r -dynamic chromatic number of a graph G, written as $\chi_r(G)$, is the minimum k such that G has an r -dynamic proper k-coloring. In this paper we consider only the graphs which are simple, finite, loop less and connected. For all terms and definition which are not specifically defined in this paper, we refer to [3]. The r -dynamic chromatic number has been studied by many authors, for instance in [1], [2], [3], [5], [6], [7], [8], [9], [10], [11], [12].

II. Preliminaries

In this paper, we consider the r -dynamic coloring of some splitting graphs. For a given graph G the splitting graph [13] $S(G)$ of graph G is obtained by adding a new vertex v' corresponding to each vertex v of G such that $N(v) = N(v')$, where $N(v)$ is the neighborhood of vertex v.

III. r-Dynamic Coloring of Splitting graphs of Some Graphs

3.1. Theorem

Let P_n be a path on $n \geq 3$ vertices, the r-dynamic chromatic number of $S(P_n)$ is

$$\chi_r(S(P_n)) = \begin{cases} 2, r = 1, \\ 4, r = 2, 3, \\ 5, r = \Delta. \end{cases}$$

Proof

The graph $S(P_n)$ is connected graph with vertex set $V(S(P_n)) = \{v_i, v'_i : 1 \leq i \leq n\}$ and edge set $E(S(P_n)) = \{v_i v_{i+1}, v'_i v_{i+1}, v_i v'_{i+1} : 1 \leq i \leq n-1\}$.

The order of graph $S(P_n)$ is $|V(S(P_n))| = 2n$ and the size of graph $S(P_n)$ is $|E(S(P_n))| = 3(n-1)$, thus $\Delta(S(P_n)) = 4$.

Case 1: $r = 1$

Consider the color function $C_1 : V(S(P_n)) \to \{1,2, \dots \dots k\}$ defined as follows.

$C_1(v_1, v_2 \dots \dots v_n) = C_1(v'_1, v'_2 \dots . . v'_n) = (1,2,1,2 \dots \dots .)$

It is clear that C_1 is a r- dynamic coloring and hence $\chi_r(S(P_n)) = 2$, for $r = 1$.

Case 2: $r = 2,3$

Consider the color function $C_2 : V(S(P_n)) \to \{1,2, \dots \dots k\}$ defined as follows.

$C_2(v_1, v_2 \dots \dots v_n) = (1,2,3,1,2,3 \dots \dots .)$

For $1 \le i \le n$, $C_2(v'_i) = 4$

It is clear that C_2 is a r- dynamic coloring and hence $\chi_r(S(P_n)) = 4$, for $r = 2,3$.

Case 3: $r = \Delta$

Consider the color function $C_3 : V(S(P_n)) \to \{1,2, \dots \dots k\}$ defined as follows.

$C_3(v_1, v_2 \dots \dots v_n) = (1,2,3,1,2,3 \dots \dots .)$

$C_3(v'_1, v'_2 \dots . . v'_n) = (4,4,5,5,4,4,5,5 \dots \dots .)$

It is clear that C_3 is a r- dynamic coloring and hence $\chi_r(S(P_n)) = 5$, for $r = \Delta$.

3.2. Theorem

Let C_n be a cycle graph on $n \ge 3$ vertices, the r-dynamic chromatic number of $S(C_n)$ is

$$\chi_r(S(C_n)) = \begin{cases} 2, r = 1, n \text{ is even}, \\ 3, r = 1, n \text{ is odd}, \\ 3, r = 2, n \equiv 0 \ (mod\ 3), \\ 5, r = 2, n = 5, \\ 4, r = 2, otherwise, \\ 4, r = 3, \ n \equiv 0 \ (mod\ 3), \\ 5, r = 3, otherwise, \\ 8, r = \Delta, \ n = 5, \\ 6, \ r = \Delta, \ otherwise. \end{cases}$$

Proof

The graph $S(C_n)$ is connected graph with vertex set $V(S(C_n)) = \{v_i, v'_i : 1 \le i \le n\}$ and edge set $E(S(C_n)) = \{v_i v_{i+1}, v_n v_1 : 1 \le i \le n-1\} \cup \{v'_i v_{i+1}, v_i v'_{i+1}, v'_n v_1, v_n v'_1 : 1 \le i \le n-1\}$.

The order of graph $S(C_n)$ is $|V(S(C_n))| = 2n$ and the size of graph $S(C_n)$ is $|E(S(C_n))| = 3n$, thus $\Delta(S(C_n)) = 4$.

Case 1: $r = 1$, n is even

Consider the color function $C_1 : V(S(C_n)) \to \{1,2, \dots \dots k\}$ defined as follows.

$C_1(v_1, v_2 \dots \dots v_n) = C_1(v'_1, v'_2 \dots . . v'_n) = (1,2,1,2 \dots \dots .)$

It is clear that C_1 is a r- dynamic coloring and hence $\chi_r(S(C_n)) = 2$, for $r = 1$, n is even.

Case 2: $r = 1$, n is odd

Consider the color function $C_2 : V(S(C_n)) \to \{1,2, \dots \dots k\}$ defined as follows.

$C_2(v_1, v_2 \dots \dots v_n) = C_2(v'_1, v'_2 \dots . . v'_n) = (1,2,1,2 \dots \dots .1,2,3)$

It is clear that C_2 is a r- dynamic coloring and hence $\chi_r(S(C_n)) = 3$, for $r = 1$, n is odd.

Case 3: $r = 2, n \equiv 0 \ (mod\ 3)$

Consider the color function $C_3 : V(S(C_n)) \to \{1,2, \dots \dots k\}$ defined as follows.

$C_3(v_1, v_2 \dots \dots v_n) = C_3(v'_1, v'_2 \dots . . v'_n) = (1,2,3,1,2,3 \dots \dots .)$

It is clear that C_3 is a r- dynamic coloring and hence $\chi_r(S(C_n)) = 3$, for $r = 2, n \equiv 0 \ (mod\ 3)$.

Case 4: $r = 2, n = 5$

Consider the color function $C_4 : V(S(C_n)) \to \{1,2, \dots \dots k\}$ defined as follows.

$C_4(v_1, v_2 \dots \dots v_n) = C_4(v'_1, v'_2 \dots . . v'_n) = (1,2,3,4,5)$

It is clear that C_4 is a r- dynamic coloring and hence $\chi_r(S(C_n)) = 5$, for $r = 2, n = 5$.

Case 5: $r = 2, otherwise$

Consider the color function $C_5 : V(S(C_n)) \to \{1,2, \dots \dots k\}$ defined as follows.

If $n \equiv 1 \ (mod\ 3)$

$C_5(v_1, v_2 \dots \dots v_n) = C_5(v'_1, v'_2 \dots . . v'_n) = (1,2,3,1,2,3 \dots \dots .1,2,3,4)$

If $n \equiv 2 \ (mod\ 3)$

$C_5(v_1, v_2 \dots \dots v_n) = C_5(v'_1, v'_2 \dots . . v'_n) = (1,2,3,4,1,2,3,4,1,2,3 \dots \dots .1,2,3)$

It is clear that C_5 is a r- dynamic coloring and hence $\chi_r(S(C_n)) = 4$, for $r = 2, n \equiv 1,2 \ (mod\ 3)$.

Case 6: $r = 3, n \equiv 0 \ (mod\ 3)$

Consider the color function $C_6 : V(S(C_n)) \rightarrow \{1,2, \ldots \ldots k\}$ defined as follows.
$C_6(v_1, v_2 \ldots \ldots v_n) = (1,2,3,1,2,3 \ldots \ldots .)$
For $1 \leq i \leq n$, $C_6(v'_i) = 4$
It is clear that C_6 is a r- dynamic coloring and hence $\chi_r(S(C_n)) = 4$, for $r = 3$, $n \equiv 0 \ (mod\ 3)$.
Case 7: $r = 3$, $otherwise$
Consider the color function $C_7 : V(S(C_n)) \rightarrow \{1,2, \ldots \ldots k\}$ defined as follows.
If $n = 5$
$C_7(v_1, v_2 \ldots \ldots v_n) = (1,2,3,4,5)$
$C_7(v'_1, v'_2 \ldots \ldots v'_n) = (3,4,5,1,2)$
If $n \equiv 1, 2\ (mod\ 3)$
$C_7(v_1, v_2 \ldots \ldots v_n) = \begin{cases} (1,2,3,4,1,2,3 \ldots \ldots .1,2,3) & , n \equiv 1\ (mod\ 3) \\ (1,2,3,4,1,2,3,4,1,2,3 \ldots \ldots .1,2,3) & , n \equiv 2\ (mod\ 3) \end{cases}$
For $1 \leq i \leq n$, $C_7(v'_i) = 5$
It is clear that C_7 is a r- dynamic coloring and hence $\chi_r(S(C_n)) = 5$, for $r = 3$, $n = 5$ and $n \equiv 1, 2\ (mod\ 3)$.
Case 8: $r = \Delta$, $n = 5$
Consider the color function $C_8 : V(S(C_n)) \rightarrow \{1,2, \ldots \ldots k\}$ defined as follows.
$C_8(v_1, v_2 \ldots \ldots v_n) = (1,2,3,4,5)$
$C_8(v'_1, v'_2 \ldots \ldots v'_n) = (6,6,7,7,8)$
It is clear that C_8 is a r- dynamic coloring and hence $\chi_r(S(C_n)) = 8$, for $r = \Delta$, $n = 5$.
Case 9: $r = \Delta$, $otherwise$
Consider the color function $C_9 : V(S(C_n)) \rightarrow \{1,2, \ldots \ldots k\}$ defined as follows.
If $n \equiv 0\ (mod\ 4)$
$C_9(v_1, v_2 \ldots \ldots v_n) = (1,2,3,4,1,2,3,4 \ldots \ldots .)$
$C_9(v'_1, v'_2 \ldots \ldots v'_n) = (5,5,6,6,5,5,6,6 \ldots \ldots \ldots)$
If $n \equiv 0\ (mod\ 3)$
$C_9(v_1, v_2 \ldots \ldots v_n) = (1,2,3,1,2,3 \ldots \ldots . .)$
$C_9(v'_1, v'_2 \ldots \ldots v'_n) = (4,5,6,4,5,6 \ldots \ldots \ldots .)$
If $n \equiv 1, 2\ (mod\ 3)$
$C_9(v_1, v_2 \ldots \ldots v_n) = \begin{cases} (1,2,3,4,1,2,3 \ldots \ldots .1,2,3) & , n \equiv 1\ (mod\ 3) \\ (1,2,3,4,1,2,3,4,1,2,3 \ldots \ldots .1,2,3) & , n \equiv 2\ (mod\ 3) \end{cases}$
$C_9(v'_1) = C_9(v'_n) = 4$ and $C_9(v'_2, \ldots \ldots \ldots v'_{n-1}) = (5,5,6,6,5,5,6,6 \ldots \ldots .)$
It is clear that C_9 is a r- dynamic coloring and hence $\chi_r(S(C_n)) = 6$, for $r = \Delta$, $otherwise$.

3.3. Theorem
Let K_n be a complete graph on $n \geq 3$ vertices, the r-dynamic chromatic number of $S(K_n)$ is

$$\chi_r(S(K_n)) = \begin{cases} n\ ,\ 1 \leq r \leq n-1\ , \\ r+1\ ,\ n \leq r \leq n + \frac{n-2}{2}\ , n\ is\ even\ , \\ r+1\ ,\ n \leq r \leq n + \frac{n-3}{2}\ , n\ is\ odd\ , \\ r+2\ ,\ n + \frac{n-2}{2} + 1 \leq r \leq \Delta - 2\ , n\ is\ even\ , \\ r+2\ ,\ n + \frac{n-3}{2} + 1 \leq r \leq \Delta - 2\ , n\ is\ odd\ , \\ 2n-1\ ,\ r = \Delta - 1\ , \\ 2n\ ,\ r = \Delta\ . \end{cases}$$

Proof
The graph $S(K_n)$ is connected graph with vertex set $V(S(K_n)) = \{v_i, v'_i : 1 \leq i \leq n\}$ and edge set $E(S(K_n)) = \{v_i v_j\ (i < j)\ , v'_i v_j\ (i \neq j) : 1 \leq i \leq n\ , 1 \leq j \leq n\}$.
The order of graph $S(K_n)$ is $|V(S(K_n))| = 2n$ and the size of graph $S(K_n)$ is $|E(S(K_n))| = nC_2 + n^2 - n$, thus $\Delta(S(K_n)) = 2(n-1)$.
Case 1: $1 \leq r \leq n-1$
Consider the color function $C_1 : V(S(K_n)) \rightarrow \{1,2, \ldots \ldots k\}$ defined as follows.
$C_1(v_1, v_2 \ldots \ldots v_n) = C_1(v'_1, v'_2 \ldots \ldots v'_n) = (1,2 \ldots \ldots \ldots n)$
It is clear that C_1 is a r- dynamic coloring and hence $\chi_r(S(K_n)) = n$, for $1 \leq r \leq n-1$.
Case 2: $n \leq r \leq n + \frac{n-2}{2}$, $n\ is\ even$
Consider the color function $C_2 : V(S(K_n)) \rightarrow \{1,2, \ldots \ldots k\}$ defined as follows.
$C_2(v_1, v_2 \ldots \ldots v_n) = (1,2 \ldots \ldots \ldots n)$
$C_2(v'_1, v'_2 \ldots \ldots v'_n) = (n+1, \ldots \ldots r+1, n+1 \ldots \ldots r+1, \ldots \ldots)$

It is clear that C_2 is a r- dynamic coloring and hence $\chi_r(S(K_n)) = r+1$, for $n \le r \le n + \frac{n-2}{2}$, $n\ is\ even$.

Case 3: $n \le r \le n + \frac{n-3}{2}$, $n\ is\ odd$

Consider the color function $C_3 : V(S(K_n)) \to \{1,2, \dots \dots k\}$ defined as follows.

$C_3(v_1, v_2 \dots \dots v_n) = (1,2 \dots \dots \dots n)$

$C_3(v'_1, v'_2 \dots \dots v'_n) = (n+1, \dots \dots r+1, n+1 \dots \dots r+1, \dots \dots)$

It is clear that C_3 is a r- dynamic coloring and hence $\chi_r(S(K_n)) = r+1$, for $n \le r \le n + \frac{n-3}{2}$, $n\ is\ odd$.

Case 4: $n + \frac{n-2}{2} + 1 \le r \le \Delta - 2$, $n\ is\ even$

Consider the color function $C_4 : V(S(K_n)) \to \{1,2, \dots \dots k\}$ defined as follows.

$C_4(v_1, v_2 \dots \dots v_n) = (1,2 \dots \dots \dots n)$

$C_4(v'_1, v'_2 \dots \dots v'_n) = (n+1, \dots \dots r+1, r+2, n+1, n+2 \dots \dots \dots)$

It is clear that C_4 is a r- dynamic coloring and hence $\chi_r(S(K_n)) = r+2$, for $n + \frac{n-2}{2} + 1 \le r \le \Delta - 2$, $n\ is\ even$.

Case 5: $n + \frac{n-3}{2} + 1 \le r \le \Delta - 2$, $n\ is\ odd$

Consider the color function $C_5 : V(S(K_n)) \to \{1,2, \dots \dots k\}$ defined as follows.

$C_5(v_1, v_2 \dots \dots v_n) = (1,2 \dots \dots \dots n)$

$C_5(v'_1, v'_2 \dots \dots v'_n) = (n+1, \dots \dots r+1, r+2, n+1, n+2 \dots \dots \dots)$

It is clear that C_5 is a r- dynamic coloring and hence $\chi_r(S(K_n)) = r+2$, for $n + \frac{n-3}{2} + 1 \le r \le \Delta - 2$, $n\ is\ odd$.

Case 6: $r = \Delta - 1$

Consider the color function $C_6 : V(S(K_n)) \to \{1,2, \dots \dots k\}$ defined as follows.

$C_6(v_1, v_2 \dots \dots v_n) = (1,2 \dots \dots \dots n)$

$C_6(v'_1, v'_2 \dots \dots v'_n) = (n+1, n+1, n+2 \dots \dots 2n-1)$

It is clear that C_6 is a r- dynamic coloring and hence $\chi_r(S(K_n)) = 2n-1$, for $r = \Delta - 1$.

Case 7: $r = \Delta$

Consider the color function $C_7 : V(S(K_n)) \to \{1,2, \dots \dots k\}$ defined as follows.

$C_7(v_1, v_2 \dots \dots v_n) = (1,2 \dots \dots \dots n)$

$C_7(v'_1, v'_2 \dots \dots v'_n) = (n+1, n+2, \dots \dots \dots 2n)$

It is clear that C_7 is a r- dynamic coloring and hence $\chi_r(S(K_n)) = 2n$, for $r = \Delta$.

3.4. Theorem

Let $K_{m,n}$ be a Complete Bipartite Graph on mn vertices, for $m \ge 3$and $m \le n$, the r-dynamic chromatic number of $S(K_{m,n})$ is

$$\chi_r\left(S(K_{m,n})\right) = \begin{cases} 2\ , r = 1\ , \\ 2r\ , 2 \le r \le 2m\ , if\ 2m \le n\ , \\ 2m + r\ , 2m + 1 \le r \le n\ , if\ 2m \le n\ , \\ 2m + n,\ n + 1 \le r \le m + n\ , if\ 2m \le n\ , \\ m + r\ , m + n + 1 \le r \le \Delta\ , if\ 2m \le n\ , \\ 2r\ , 2 \le r \le n\ , if\ 2m > n\ , \\ n + r\ , n + 1 \le r \le 2m\ , if\ 2m > n\ , \\ 2m + n\ ,\ 2m + 1 \le r \le m + n\ , if\ 2m > n\ , \\ m + r\ , m + n + 1 \le r \le \Delta\ , if\ 2m > n\ . \end{cases}$$

Proof

The graph $S(K_{m,n})$ is connected graph with vertex set $V\left(S(K_{m,n})\right) = \left\{x_i, y_j, x'_i, y'_j : 1 \le i \le m, 1 \le j \le n\right\}$ and edge set $E\left(S(K_{m,n})\right) = \left\{x_i y_j, x'_i y_j, x_i y'_j, : 1 \le i \le m\ , 1 \le j \le n\right\}$.

The order of graph $S(K_{m,n})$ is $\left|V\left(S(K_{m,n})\right)\right| = 2(m+n)$ and the size of graph $S(K_{m,n})$ is $\left|E\left(S(K_{m,n})\right)\right| = 3mn$, thus $\Delta\left(S(K_{m,n})\right) = 2n$.

Case 1: $r = 1$

Consider the color function $C_1 : V(S(K_{m,n})) \to \{1,2, \dots \dots k\}$ defined as follows.

$C_1(x_1, x_2 \dots \dots x_m) = C_1(x'_1, x'_2 \dots \dots x'_m) = (1,1, \dots \dots)$

$C_1(y_1, y_2 \dots \dots y_n) = C_1(y'_1, y'_2 \dots \dots y'_n) = (2,2, , \dots \dots)$

It is clear that C_1 is a r- dynamic coloring and hence $\chi_r\left(S(K_{m,n})\right) = 2$, for $r = 1$.

Case 2: $2 \leq r \leq 2m$, if $2m \leq n$,
Consider the color function $C_2 : V(S(K_{m,n})) \rightarrow \{1,2, \ldots \ldots k\}$ defined as follows.
Subcase (i) : $r = 2$
$C_2(y'_1, y'_2, \ldots . y'_n) = (1,1, \ldots \ldots)$
$C_2(x_1, x_2 \ldots \ldots . x_m) = C_2(x'_1, x'_2, \ldots . . x'_m) = (2,3,2,3 \ldots \ldots)$
$C_2(y_1, y_2 \ldots \ldots . y_n) = (1,4,1,4 \ldots \ldots)$
Subcase (ii) : : $3 \leq r \leq m$
$C_2(x_1, x_2 \ldots \ldots . x_m) = C_2(x'_1, x'_2, \ldots . . x'_m) = (1,2,3 \ldots \ldots r, 1,1 \ldots . . .)$
$C_2(y_1, y_2 \ldots \ldots . y_n) = C_2(y'_1, y'_2, \ldots . . y'_n) = (r+1 \ldots \ldots 2r, r+1, r+1 \ldots \ldots)$
Subcase (iii) : : $m + 1 \leq r \leq 2m$
$C_2(x_1, x_2 \ldots \ldots . x_m) = (1,2,3 \ldots \ldots \ldots m)$
$C_2(y_1, y_2 \ldots \ldots . y_n) = C_2(y'_1, y'_2, \ldots . . y'_n) = (m+1, m+2, \ldots \ldots m+r, m+1, m+1 \ldots \ldots)$
$C_2(x'_1, x'_2, \ldots . . x'_m) = (m+r+1, m+r+2 \ldots \ldots 2r, m+r+1, m+r+1 \ldots \ldots . .)$
It is clear that C_2 is a r- dynamic coloring and hence $\chi_r\left(S(K_{m,n})\right) = 2r$, for $2 \leq r \leq 2m$, if $2m \leq n$.

Case 3: $2m + 1 \leq r \leq n$, if $2m \leq n$,
Consider the color function $C_3 : V(S(K_{m,n})) \rightarrow \{1,2, \ldots \ldots k\}$ defined as follows.
$C_3(x_1, x_2 \ldots \ldots . x_m) = (1,2,3 \ldots \ldots \ldots m)$
$C_3(y_1, y_2 \ldots \ldots . y_n) = C_3(y'_1, y'_2, \ldots . . y'_n) = (m+1, m+2, \ldots \ldots m+r, m+1, m+1 \ldots \ldots)$
$C_3(x'_1, x'_2, \ldots . . x'_m) = (m+r+1, m+r+2 \ldots \ldots 2m+r)$
It is clear that C_3 is a r-dynamic coloring.
Hence $\chi_r\left(S(K_{m,n})\right) = 2m + r$, for $2m + 1 \leq r \leq n$, if $2m \leq n$.

Case 4: $n + 1 \leq r \leq m + n$, if $2m \leq n$,
Consider the color function $C_4 : V(S(K_{m,n})) \rightarrow \{1,2, \ldots \ldots k\}$ defined as follows.
$C_4(x_1, x_2 \ldots \ldots . x_m) = (1,2,3 \ldots \ldots \ldots m)$
$C_4(y_1, y_2 \ldots \ldots . y_n) = (m+1, m+2, \ldots \ldots m+n)$
$C_4(x'_1, x'_2, \ldots . . x'_m) = (m+n+1, m+n+2 \ldots \ldots 2m+n)$
$C_4(y'_1, y'_2, \ldots . . y'_n) = (m+n+1, m+n+2, \ldots \ldots m+r, m+n+1, m+n+1 \ldots \ldots)$
It is clear that C_4 is a r- dynamic coloring.
Hence $\chi_r\left(S(K_{m,n})\right) = 2m + n$, for $n + 1 \leq r \leq m + n$, if $2m \leq n$.

Case 5: $m + n + 1 \leq r \leq \Delta$, if $2m \leq n$,
Consider the color function $C_5 : V(S(K_{m,n})) \rightarrow \{1,2, \ldots \ldots k\}$ defined as follows.
$C_5(x_1, x_2 \ldots \ldots . x_m) = (1,2,3 \ldots \ldots \ldots m)$
$C_5(y_1, y_2 \ldots \ldots . y_n) = (m+1, m+2, \ldots \ldots m+n)$
$C_5(x'_1, x'_2, \ldots . . x'_m) = (m+n+1, m+n+2 \ldots \ldots 2m+n)$
$C_5(y'_1, y'_2, \ldots . . y'_n) = (m+n+1, m+n+2, \ldots \ldots m+r, m+n+1, m+n+1 \ldots \ldots)$
It is clear that C_5 is a r- dynamic coloring.
Hence $\chi_r\left(S(K_{m,n})\right) = m + r$, for $m + n + 1 \leq r \leq \Delta$, if $2m \leq n$.

Case 6: $2 \leq r \leq n$, if $2m > n$,
Consider the color function $C_6 : V(S(K_{m,n})) \rightarrow \{1,2, \ldots \ldots k\}$ defined as follows.
Subcase (i) : $r = 2$
$C_6(y'_1, y'_2, \ldots . y'_n) = (1,1, \ldots \ldots)$
$C_6(x_1, x_2 \ldots \ldots . x_m) = C_2(x'_1, x'_2, \ldots . . x'_m) = (2,3,2,3 \ldots \ldots)$
$C_6(y_1, y_2 \ldots \ldots . y_n) = (1,4,1,4 \ldots \ldots)$
Subcase (ii) : $3 \leq r \leq m$
$C_6(x_1, x_2 \ldots \ldots . x_m) = C_6(x'_1, x'_2, \ldots . . x'_m) = (1,2,3 \ldots \ldots r, 1,1 \ldots . . .)$
$C_6(y_1, y_2 \ldots \ldots . y_n) = C_6(y'_1, y'_2, \ldots . . y'_n) = (r+1 \ldots \ldots 2r, r+1, r+1 \ldots \ldots)$
Subcase (iii) : $m + 1 \leq r \leq n$
$C_6(x_1, x_2 \ldots \ldots . x_m) = (1,2,3 \ldots \ldots \ldots m)$
$C_6(y_1, y_2 \ldots \ldots . y_n) = C_6(y'_1, y'_2, \ldots . . y'_n) = (m+1, m+2, \ldots \ldots m+r, m+1, m+1 \ldots \ldots)$
$C_6(x'_1, x'_2, \ldots . . x'_m) = (m+r+1, m+r+2 \ldots \ldots 2r, m+r+1, m+r+1 \ldots \ldots . .)$
It is clear that C_6 is a r- dynamic coloring and hence $\chi_r\left(S(K_{m,n})\right) = 2r$, for : $2 \leq r \leq n$, if $2m > n$.

Case 7: $n + 1 \leq r \leq 2m$, if $2m > n$,
Consider the color function $C_7 : V(S(K_{m,n})) \rightarrow \{1,2, \ldots \ldots k\}$ defined as follows.
$C_7(x_1, x_2 \ldots \ldots . x_m) = (1,2,3 \ldots \ldots \ldots m)$

$C_7(y_1, y_2 \dots \dots . y_n) = (m+1, m+2, \dots \dots m+n)$
$C_7(x'_1, x'_2, \dots . . x'_m) = (m+n+1, m+n+2 \dots \dots n+r, m+n+1, \dots \dots \dots)$
$C_7(y'_1, y'_2, \dots . . y'_n) = (m+n+1, m+n+2, \dots \dots m+r, m+n+1, m+n+1 \dots \dots)$
It is clear that C_7 is a r- dynamic coloring and hence $\chi_r\left(S(K_{m,n})\right) = n+r$, for $n+1 \leq r \leq 2m$, if $2m > n$.

Case 8: $2m+1 \leq r \leq m+n$, if $2m > n$,
Consider the color function $C_8 : V(S(K_{m,n})) \to \{1,2, \dots \dots k\}$ defined as follows.
$C_8(x_1, x_2 \dots \dots . x_m) = (1,2,3 \dots \dots \dots m)$
$C_8(y_1, y_2 \dots \dots . y_n) = (m+1, m+2, \dots \dots m+n)$
$C_8(x'_1, x'_2, \dots . . x'_m) = (m+n+1, m+n+2 \dots \dots 2m+n)$
$C_8(y'_1, y'_2, \dots . . y'_n) = (m+n+1, m+n+2, \dots \dots m+r, m+n+1, m+n+1 \dots \dots)$
It is clear that C_8 is a r- dynamic coloring and hence $\chi_r\left(S(K_{m,n})\right) = 2m+n$, for $2m+1 \leq r \leq m+n$, if $2m > n$.

Case 9: $m+n+1 \leq r \leq \Delta$, if $2m > n$,
Consider the color function $C_9 : V(S(K_{m,n})) \to \{1,2, \dots \dots k\}$ defined as follows.
$C_9(x_1, x_2 \dots \dots . x_m) = (1,2,3 \dots \dots \dots m)$
$C_9(y_1, y_2 \dots \dots . y_n) = (m+1, m+2, \dots \dots m+n)$
$C_9(x'_1, x'_2, \dots . . x'_m) = (m+n+1, m+n+2 \dots \dots 2m+n)$
$C_9(y'_1, y'_2, \dots . . y'_n) = (m+n+1, m+n+2, \dots \dots m+r, m+n+1, m+n+1 \dots \dots)$
It is clear that C_9 is a r- dynamic coloring.
Hence $\chi_r\left(S(K_{m,n})\right) = m+r$, for $m+n+1 \leq r \leq \Delta$, if $2m > n$.

IV. Conclusion

In this paper, we discussed the r -dynamic coloring of some splitting graphs. We obtain the $\boldsymbol{r}$-dynamic chromatic number for splitting graph of paths, cycles, complete and complete bipartite graphs.

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Fuzzy G*amma* Preopen and Fuzzy G*amma* Preclosed sets in Fuzzy Bitopological Spaces

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Abstract: *In this paper, we introduce the concept of fuzzy gamma preopen (respectively fuzzy gamma preclosed) set in fuzzy bitopological spaces, which is weaker than the concept of fuzzy strongly semiopen (respectively fuzzy strongly semiclosed) set and fuzzy preopen (respectively fuzzy preclosed) set. We further investigate that the concept of fuzzy gamma preopen (respectively fuzzy gamma preclosed) sets are independent of the concept of fuzzy semiopen (respectively fuzzy semiclosed) sets, fuzzy gamma open (respectively fuzzy gamma closed) sets, fuzzy gamma semiopen (respectively fuzzy gamma semiclosed) sets and fuzzy semi preopen (respectively fuzzy semi preclosed) sets in fuzzy bitopological spaces. We study their characteristics and properties along with examples and discuss the relationship between these concepts and relevant concepts. We introduce the notion of fuzzy gamma pre neighbourhood and fuzzy gamma pre q neighbourhood. We also define fuzzy gamma preinterior and fuzzy gamma preclosure and investigate their fundamental properties.*

Keywords: *Fuzzy bitopological spaces; (δ_i, δ_j) F-γ-open; (δ_i, δ_j) F-γ-preopen; (δ_i, δ_j) F-γ-preclosed; (δ_i, δ_j) F-γ-pre neighbourhood; (δ_i, δ_j) F-γ-pre q neighbourhood; (δ_i, δ_j) F-γ-preinterior; (δ_i, δ_j) F-γ-preclosure.*

I. Introduction

The concept of preopen set was introduced by A.S. Mashour et al. [8] and various topological properties have been discussed. The generalization of preopen sets in topological spaces was discussed by H. Maki et al. [7]. The notion of preopenness was extended to bitopological spaces by S. Sampath Kumar [15] and was called ij preopen set. Fuzzy preopen sets were defined and their properties in fuzzy topological spaces were investigated by M. K. Singal and Niti Prakash [18]. Fuzzy γ-open sets in fuzzy topological spaces were introduced by I.M. Hanafy [4] and their properties were discussed.

The notion of fuzzy bitopological spaces (Fbts, in short) was introduced by A. Kandil and M.E. El-Shafee [5] in 1989, which is an extension of fuzzy topological spaces and generalization of bitopological spaces. In fuzzy bitopological spaces, the concept of fuzzy-γ-open sets and fuzzy-γ-continuity were introduced by F.S. Mahmoud, M.A. Fath Alla and M.M. Khalaf [10] and related properties were studied.

In this paper, we extend the notion of preopen sets to Fuzzy γ-open sets to obtain (δ_i, δ_j) Fuzzy γ-preopen sets to examine their behaviour in fuzzy bitopological spaces. This assists to study the nature of (δ_i, δ_j) Fuzzy γ-preopen sets in fuzzy bitopological spaces and their applications and to develop Fuzzy γ-preopen pairwise continuous functions.

In section 3, we establish their related characteristics along with examples. In section 4, we define (δ_i, δ_j) Fuzzy-γ-preinterior and (δ_i, δ_j) Fuzzy- γ-preclosure along with their properties. Throughout this paper (X, δ_i, δ_j) (or simply X), denote fuzzy bitopological spaces (Fbts, in short). For a fuzzy set A in a fuzzy bitopological space X, δ_i-cl(A) and δ_i-int(A) denote the closure and interior with respect to the topology δ_i respectively.

II. Preliminaries

Definition 2.1. Let X be a nonempty set and I= [0, 1]. A fuzzy set (briefly F-set) A in X is a mapping from X to I. A fuzzy set A of X is contained in a fuzzy set B of X denoted by $A \leq B$ if and only if $A(x) \leq B(x)$ for each $x \in X$. A fuzzy point [19] with singleton support $x \in X$ and the value $\alpha \in [0, 1]$ is denoted by x_α. The complement A' of a fuzzy set X is 1−A defined by (1−A) (x)= 1−A(x) for each $x \in X$. A fuzzy point $x_\beta \in A$ if and only if $\beta \leq A(x)$.

Definition 2.2. [14] A fuzzy set A is the union of all fuzzy points which belong to A. A fuzzy point x_β is said to be quasicoincident with the fuzzy set A denoted by $x_\beta q A$ if and only if $\beta + A(x) > 1$.

A fuzzy set A is said to be quasicoincident [14] with B denoted by AqB if and only if there exists $x \in X$ such that $A(x) + B(x) > 1$. $A \leq B$ if and only if $\text{'I}(AqB')$.

Definition 2.3. Let λ be a fuzzy set of a fuzzy topological space (X, δ). Then λ is called
(a) a F semiopen (briefly FSO) set of X if $\lambda \leq$ cl (int (λ)) [1];
(b) a F preopen (briefly FPO) set of X if $\lambda \leq$ int (cl (λ)) [9];
(c) a F strongly semiopen (briefly FSSO) set of X if $\lambda \leq$ int (cl (int (λ))) [2];
(d) a F semi-preopen (briefly FSPO) set of X if $\lambda \leq$ cl (int (cl (λ))) [12];
The set of all F-so (resp. F-sc), F-po (resp. F-pc), F-sso (resp. F-ssc), F-spo (resp. F-spc) of a fuzzy topological space will be denoted by FSO(X) (resp. FSC(X)), FPO(X) (resp. FPC (X)), F-SSO(X) (resp. F-SSC(X)), F-SPO(X) (resp. F-SPC(X)).

Definition 2.4 [4]. Let (X, δ) be a fuzzy topological space. Then v is called a F - γ open (F - γ closed) set of X if $v \leq$ int (cl (v)) $\vee$ cl (int (v)) ($v \geq$ cl (int (v)) $\wedge$ int (cl (v))).
The family of all F-γ open (respectively F-γ closed) sets of X is denoted by F-γ O(X) (resp. F-γ C(X)).

Lemma 2.5 [1]. For a family $\{\lambda_\alpha\}$of fuzzy sets of a Fts X, $\vee$ cl(λ_α) $\leq$ cl($\vee\lambda_\alpha$) and $\vee$int(λ_α) $\leq$ int($\vee\lambda_\alpha$)

Lemma 2.6 [1]. For a fuzzy set λ of a F-ts X, (i) (int (λ))′ = cl (λ') and (ii) (cl(λ))′ = int(λ')

Lemma 2.7 [1]. For a fuzzy set λ of a F-ts X, (a) 1–int λ=cl (1–λ) and (b) 1–cl λ=int (1–λ).

Definition 2.8 [5]. A set X on which are defined two (arbitrary) F-topologies δ_1 and δ_2 is called a F-bitopological space (briefly F-bts) and denoted by (X, δ_1,δ_2). As to the notions, we shall write δ_i-int(λ) and δ_i-cl(λ) to mean respectively the interior and closure of a F-set λ with respect to the F-topology δ_i in F-bts (X, δ_i, δ_j), with δ_i-F-o set and δ_i-F-c set, we mean respectively δ_i-F-open and δ_i-F-closed set. The indices i and j take values {1,2} throughout this paper and i ≠ j, i=j gives the known results in F-ts.

Definition 2.9 [17]. Let λ be a fuzzy set of a F-bts (X, δ_i, δ_j). Then λ is called
(a) a (δ_i, δ_j) F semiopen (briefly (δ_i, δ_j) F-so) set of X if $\lambda \leq \delta_j$-cl (δ_i-int (λ));
(b) a (δ_i, δ_j) F semiclosed (briefly (δ_i, δ_j) F-sc) set of X if $\lambda \geq \delta_j$-int (δ_i-cl (λ));
The set of all (δ_i, δ_j) F-so, (resp. (δ_i, δ_j) F-sc) sets of a F-bts X will be denoted by (δ_i, δ_j) FSO(X), (resp. (δ_i, δ_j) FSC(X)).

Definition 2.10 [16]. Let λ be a fuzzy set of a F-bts (X, δ_i, δ_j).Then λ is called
(a) a (δ_i, δ_j) F strongly semiopen (briefly (δ_i, δ_j) F-sso) set of X if $\lambda \leq \delta_i$-int (δ_j-cl (δ_i-int (λ)));
(b) a (δ_i, δ_j)F strongly semiclosed (briefly(δ_i, δ_j) F-ssc) set of X if $\lambda \geq \delta_i$-cl (δ_j-int ($\delta_i$cl(λ)));
(c) a (δ_i, δ_j) F preopen (briefly (δ_i, δ_j) F-po) set of X if $\lambda \leq \delta_i$-int (δ_j-cl (λ));
(d) a (δ_i, δ_j) F preclosed (briefly (δ_i, δ_j) F-pc) set of X if $\lambda \geq \delta_i$-cl (δ_j-int (λ));
The set of all (δ_i, δ_j) F-sso, (δ_i, δ_j) F-ssc, (δ_i, δ_j) F-po, (δ_i, δ_j) F-pc sets of a F-bts X will be denoted by (δ_i, δ_j) FSSO(X), (δ_i, δ_j) FSSC(X), (δ_i, δ_j) FPO(X) and (δ_i, δ_j) FPC(X) respectively.

Definition 2.11 [13]. Let λ be a fuzzy set of a F-bts (X, δ_i, δ_j). Then λ is called
(a) a (δ_i, δ_j) F semi-preopen (briefly (δ_i, δ_j)F-spo) set of X if $\lambda \leq \delta_j$-cl (δ_i-int (δ_j-cl (λ)));
(b) a (δ_i, δ_j) F semi-preclosed (briefly (δ_i, δ_j)F-spc) set of X if $\lambda \geq \delta_j$-int (δ_i-cl (δ_j-int (λ)));
The set of all (δ_i, δ_j) F-spo, (respectively (δ_i, δ_j) F-spc) sets of a F-bts X will be denoted by (δ_i, δ_j) FSPO(X), (respectively (δ_i, δ_j) FSPC(X)).

Definition 2.12 [10]. Let λ be a fuzzy set of a F-bts (X, δ_i, δ_j).Then λ is called a (δ_i, δ_j) F γ open (respectively (δ_i, δ_j) F γ closed), briefly (δ_i, δ_j) F-γo (respectively (δ_i, δ_j) F-γ c) if $\lambda \leq \delta_i$-int (δ_j- cl (λ)) $\vee$ δ_j-cl (δ_i-int (λ)), respectively $\lambda \geq \delta_i$-cl (δ_j-int (λ)) $\wedge$ δ_j-int (δ_i-cl (λ)).
The family of all (δ_i, δ_j) F-γ o (respectively (δ_i, δ_j) F-γ c) sets of X is denoted by (δ_i, δ_j) F-γ O(X) and (respectively (δ_i, δ_j) F-γ C(X)).

Remark 2.13 [10] (i) The union of (δ_i, δ_j) F-γo sets is a (δ_i, δ_j) F-γo set.
(ii) The intersection of (δ_i, δ_j) F-γc sets is a (δ_i, δ_j) F-γc set.

Definition 2.14 [10]. Let λ be a fuzzy set of a F-bts (X, δ_i, δ_j). Then the (δ_i, δ_j) γ-closure ((δ_i, δ_j) γ-cl for short) and (δ_i, δ_j) γ-interior ((δ_i, δ_j) γ-int for short) of λ are defined as
(δ_i, δ_j) γ-cl(λ) = $\wedge$ {v : v is (δ_i, δ_j) F-γ closed and $\lambda \leq v$}
(δ_i, δ_j) γ-int(λ) = $\vee${v : v is (δ_i, δ_j) F-γ open and $v \leq \lambda$}

Remark 2.15 (i) (δ_i, δ_j) γ-cl (λ) is the intersection of all (δ_i, δ_j) F-γ c sets of X containing λ.
(ii) (δ_i, δ_j) γ-int (λ) is the union of all (δ_i, δ_j) F-γo sets of X contained in λ.
(iii) δ_i-γ cl (λ) is the intersection of all (δ_i, δ_j) F-γc sets of X containing λ with respect to δ_i.
(iv) δ_i-γ int (λ) is the union of all (δ_i, δ_j) F-γo sets of X contained in λ with respect to the δ_i.

Definition 2.16 [11]. Let A be a fuzzy set of a F-bts (X, δ_i, δ_j). Then A is called a
(i) (δ_i, δ_j) Fuzzy-γ-semiopen (briefly (δ_i, δ_j) F-γ-so) set if A $\leq \delta_j$-cl (δ_i -γ int(A))
(ii) (δ_i, δ_j) Fuzzy-γ-semiclosed (briefly (δ_i, δ_j) F-γ-sc) set if A $\geq \delta_j$- int (δ_i -γ cl(A))
The family of all (δ_i, δ_j) Fγ so (respectively (δ_i, δ_j) F-γ sc)sets of X is denoted by (δ_i, δ_j) F-γ SO(X) and respectively (δ_i, δ_j) F-γ SC(X)).

III. (δ_i, δ_j) F-γ Preopen and (δ_i, δ_j) F-γ Preclosed set

Definition 3.1. Let A be a fuzzy set of a fuzzy bitopological space (X, δ_i, δ_j). Then A is called a
(i) (δ_i, δ_j) F-γ preopen (briefly (δ_i, δ_j) F-γ po) set if A $\leq$ δ_i int (δ_j γ cl(A))
(ii) (δ_i, δ_j) F-γ preclosed (briefly (δ_i, δ_j) F-γ pc set if A $\geq$ δ_i cl (δ_j γ int(A))
The family of all (δ_i, δ_j) F-γ po (respectively (δ_i, δ_j) F-γ pc) sets of X is denoted by (δ_i, δ_j) F-γ PO(X) and (respectively (δ_i, δ_j) F-γ PC(X)).

Example 3.2. Let (X, δ_1, δ_2) be a F-bts with X = {a, b}, $\delta_1 = \{\tilde{0}, \tilde{1}, A\}$, $\delta_2 = \{\tilde{0}, \tilde{1}, B\}$ and fuzzy sets A = $\{a_{0.3}, b_{0.6}\}$, B = $\{a_{0.2}, b_{0.4}\}$. Here (δ_1, δ_2) F-γ o sets = $\{\tilde{0}, \tilde{1}, A, B\}$. The sets A and B (resp. A' and B') are (δ_1, δ_2) F-γ-po (resp. (δ_1, δ_2) F-γ-pc).

Theorem 3.3. Let (X,δ_i,δ_j) be a F-bts and let A be a fuzzy subset of X. Then A is (δ_i,δ_j) F-γ-preclosed if and only if A′ is (δ_i, δ_j) F-γ-preopen.

Proof: Follows from Definition 3.1 and Lemma 2.4 [1].

Remark 3.4. The concepts of (δ_i, δ_j) F-γ-preopen (resp. (δ_i, δ_j) F - γ - preclosed) and (δ_j, δ_i)F-γ-preopen (resp. (δ_j, δ_i) F-γ-preclosed) set are independent. The following example illustrates this.

Example 3.5. Let (X, δ_1, δ_2) be a F-bts with X = {a, b}, $\delta_1 = \{\tilde{0}, \tilde{1}, A\}$ and $\delta_2 = \{\tilde{0}, \tilde{1}, B\}$ where fuzzy sets A = $\{a_{0.2}, b_{0.5}\}$ and B = $\{a_{0.5}, b_{0.4}\}$. Here A (resp. A') is (δ_1, δ_2)F-γ-po (resp. (δ_1, δ_2) F-γ-pc). But A (resp. A') is not (δ_2, δ_1) F-γ-po ((δ_2, δ_1) F-γ-pc). Also, B (resp. B') is not (δ_1, δ_2)F-γ-po (resp. (δ_1, δ_2) F-γ-pc) but (δ_2, δ_1) F-γ-po ((resp. (δ_2, δ_1) F-γ-pc))

Theorem 3.6. A subset B of X is (δ_i,δ_j)Fγpo if and only if there exists a δ_iFo set H in X such that B$\leq$H$\leq\delta_j\gamma$cl(B).

Proof: Let B be a (δ_i, δ_j)Fγ po. By Definition 3.1, B $\leq$ $\delta_i$int($\delta_j\gamma$cl(B)). Let H=δ_i int($\delta_j\gamma$cl(B)). Then H is δ_iFo and δ_i int($\delta_j\gamma$cl(B)) $\leq$ $\delta_j\gamma$cl(B) and so, B $\leq$ δ_i int(δ_j γcl(B)) $\leq$ $\delta_j\gamma$cl(B). Thus B $\leq$ H $\leq$ $\delta_j\gamma$cl(B). Conversely, suppose there exists a δ_iFo set H in X such that B $\leq$ H $\leq$ $\delta_j\gamma$cl (B). Define H= δ_i int($\delta_j\gamma$cl (B)). Then B $\leq$ δ_i int($\delta_j\gamma$cl(B)).

Theorem 3.7. A subset B of a F-bts X is (δ_i, δ_j) F-γ pc if and only if there exists a δ_i F c set F in X such that δ_j γ int (B) $\leq$ F $\leq$ B.

Proof: Let B be a (δ_i, δ_j) F-γ pc. Then B $\geq$ δ_i cl(δ_j γint(B)). Let F = δ_i cl (δ_j γint(B)). Then F is δ_i F c and δ_j γint(B) $\leq$ δ_i cl(δ_j γint(B)) $\leq$ B.That is δ_j γint(B) $\leq$ F $\leq$ B. Suppose these exists a δ_i F c set F in X such that δ_j γint(B) $\leq$ F$\leq$ B. Define F = δ_i cl (δ_j γint (B)). Then B is (δ_i, δ_j) Fγpc.

Remark 3.8. The above theorems also hold when the set B (resp. B') is (δ_i, δ_j) F-γ po (resp. (δ_i, δ_j) F-γ-pc) but not (δ_i, δ_j) F-γ-o (resp. (δ_i, δ_j) F-γ-c).

Example 3.9. Let (X, δ_1, δ_2) be a F-bts with X = {a, b}, δ_1 = $\{\tilde{0}, \tilde{1}, A, H\}$, δ_2 = $\{\tilde{0}, \tilde{1}, B\}$ and A, H, B are fuzzy sets defined in X as, A = $\{a_{0.3}, b_{0.6}\}$, H = $\{a_{0.4}, b_{0.8}\}$, B = $\{a_{0.4}, b_{0.5}\}$. (δ_1, δ_2) F-γ-o sets = $\{\tilde{0}, \tilde{1}, A, H\}$. B (resp. B') is not ($\delta_1$, δ_2) F-γ-o (resp. (δ_1, δ_2) F-γ-c) but B (resp. B') is (δ_1, δ_2) F-γ-po (resp. (δ_1, δ_2) F-γ-pc). Then there exists a δ_i F-o (resp. δ_1 F-c) set H (resp. F) in X such that B $\leq$ H $\leq$ $\delta_j\gamma$cl(B) (resp. δ_j γint(B′) $\leq$ F$\leq$ B′)

Theorem 3.10. Let (X, δ_i, δ_j) be a F-bts. Then a fuzzy subset A of X is (δ_i, δ_j) F-γ po if and only if there exists a δ_j F - γ c set U such that A $\leq$ δ_i int (U) $\leq$ U.

Proof: Let A be (δ_i, δ_j)F-γ po. Then A $\leq$ δ_i int(δ_j γcl(A)). Let U = δ_j γcl(A) then U is δ_j F-γc and A $\leq$ δ_i int(U) $\leq$ U. Suppose these exists a δ_j F-γ c set U such that A $\leq$ δ_i int(U) $\leq$ U. Define U = $\delta_j\gamma$cl(A), then A $\leq$ δ_i int(δ_j γcl(A)). Thus, A is (δ_i, δ_j) F-γ po.

Theorem 3.11. Let (X, δ_i, δ_j) is a F-bts. Then a fuzzy subset A of X is (δ_i, δ_j) F-γ pc if and only if there exists a δ_j F - γ o set U such that U $\leq$ δ_i cl (U) $\leq$ A.

Proof: Follows from Definition 3.1

Proposition 3.12. The union of two (δ_i, δ_j) F-γ po sets is a (δ_i, δ_j) F-γ po set in a F-bts (X, δ_i, δ_j).

Remark 3.13. Intersection of two (δ_i, δ_j) F-γ po sets is not a (δ_i, δ_j) F-γ po set and the intersection of a (δ_i, δ_j) F-γ po set and a δ_iFo set is not necessarily (δ_i, δ_j) Fγpo and the intersection of a (δ_i, δ_j)Fγpo and a (δ_i, δ_j) Fsso is not necessarily (δ_i, δ_j) Fγpo which is shown in the following example.

Example 3.14. Let (X, δ_1, δ_2) be a F-bts with X = {a, b, c}, $\delta_1 = \{\tilde{0}, \tilde{1}, A\}$ and $\delta_2 = \{\tilde{0}, \tilde{1}, B\}$ where A = $\{a_{0.5}, b_{0.6}, c_{0.3}\}$ and B = $\{a_{0.7}, b_{0.2}, c_{0.4}\}$. ($\delta_1$, δ_2) Fγpo sets = $\{\tilde{0}, \tilde{1}, A, B\}$. Let C = $\{a_{0.3}, b_{0.9}, c_{0.7}\}$ and C is (δ_1, δ_2) Fγ po. Now A $\wedge$ C = $\{a_{0.3}, b_{0.6}, c_{0.3}\}$ and A$\wedge$C $\not\leq$ 0, so A $\wedge$ C is not (δ_1, δ_2) Fγ po. Thus, intersection of two (δ_i, δ_j) Fγpo sets need not be (δ_i, δ_j) F-γpo.

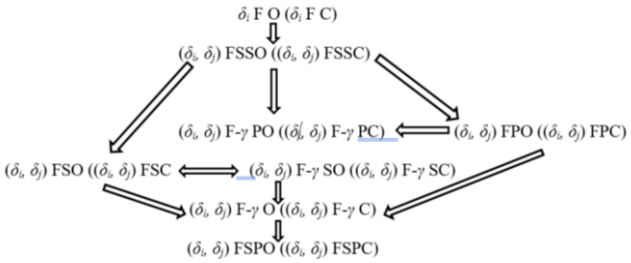
The intersection of a (δ_i, δ_j) F-γ po set and a δ_i Fo set is not necessarily (δ_i, δ_j) F-γ po. Here A is δ_i Fo and C is (δ_1, δ_2) F-γpo. But A $\wedge$ C is not (δ_1, δ_2) F-γpo.

The intersection of a (δ_i, δ_j) F-γ po and a (δ_i, δ_j) Fsso is not necessarily (δ_i, δ_j) F-γ po. C is (δ_1, δ_2) F-γpo and A is (δ_1, δ_2) Fsso but A $\wedge$ C is not (δ_1, δ_2) F-γpo.

Theorem 3.15. An arbitrary union of (δ_i, δ_j) F-γ po sets is a (δ_i, δ_j) F-γ po set in a F-bts (X, δ_i, δ_j).

Proof: Let $\{A_\alpha\}_{\alpha\in A}$ be a collection of (δ_i, δ_j) F-γ po sets in a F-bts (X, δ_i, δ_j). Then for each $\alpha\in$A, A_α is (δ_i, δ_j)F-γ po. That is $\vee_{\alpha\in\Delta} A_\alpha \leq \vee_{\alpha\in\Delta}$ δ_i int (δ_j γcl (A)) $\leq$ δ_i int ($\vee_{\alpha\in\Delta}$ δ_j γcl (A)) = δ_i int (δ_j γcl ($\vee_{\alpha\in\Delta} A_\alpha$)). Thus, $\{\vee_{\alpha\in\Delta} A_\alpha\}$ is (δ_i, δ_j) F-γ po

Proposition 3.16. The intersection of two (δ_i, δ_j) Fγ pc sets is a (δ_i, δ_j) Fγ pc set in a F-bts (X, δ_i, δ_j).
Remarks 3.17. The union of two (δ_i, δ_j)Fγpc sets need not be (δ_i, δ_j)Fγpc which is illustrated below.
Example 3.18. Let (X, δ_1, δ_2) be a F-bts with X = {a, b ,c}, $\delta_1 = \{\tilde{0}, \tilde{1}, A\}$ and $\delta_2 = \{\tilde{0}, \tilde{1}, B\}$ where A and B are fuzzy sets defined in X as $A = \{a_{0.5}, b_{0.8}, c_{0.4}\}$ and $B = \{a_{0.8}, b_{0.1}, c_{0.6}\}$. ($\delta_1$, δ_2) Fγpc sets= $\{\tilde{0}, \tilde{1}, A', B'\}$. Let $C = \{a_{0.2}, b_{0.7}, c_{0.8}\}$ then $C' = \{a_{0.8}, b_{0.3}, c_{0.2}\}$ and $A' \vee C' = \{a_{0.8}, b_{0.3}, c_{0.6}\}$. Here $A' \vee C'$ is not (δ_1,δ_2) Fγpc. Thus, union of two (δ_i, δ_j) Fγpc sets need not be (δ_i, δ_j) Fγpc.
Theorem 3.19. An arbitrary intersection of (δ_i,δ_j)Fγ pc sets is a (δ_i,δ_j)Fγ pc set in a F-bts (X,δ_i,δ_j).
Proof: Let $\{A_\alpha\}_{\alpha\in\Delta}$ be a collection of (δ_i, δ_j) F-γ-pc sets in (X, δ_i, δ_j). Then for each $\alpha\in\Delta$, A_α is (δ_i, δ_j) F-γ-pc. That is, for each $\alpha \in \Delta$, A_α' is (δ_i, δ_j) F-γ-po which implies $\vee_{\alpha\in\Delta}$ A_α' is (δ_i, δ_j) F-γ-po. By Theorem 3.3, ($\vee_{\alpha\in\Delta}$ A_α')' is (δ_i, δ_j) F-γ-pc. $\wedge_{\alpha\in\Delta}(A_\alpha')'$ is (δ_i, δ_j) F-γ-pc. Thus, $\wedge_{\alpha\in\Delta} A_\alpha$ is (δ_i, δ_j) F-γ-pc
Theorem 3.20. In a F-bts (X, δ_i, δ_j) every δ_i F open is (δ_i, δ_j) F-γ po and every δ_i Fγo is (δ_i, δ_j) F-γ po.
Proof: Follows from Definition 3.1 and δ_i int (A) = δ_i γ int (A) [11]
Theorem 3.21. Let (X, δ_i, δ_j) be a F-bts, every (δ_i, δ_j) Fsso set is (δ_i, δ_j) F-γ po.
Proof: By Definition 2.10 [16] and from [11].
Theorem 3.22. Let (X, δ_i, δ_j) be a F-bts every (δ_i, δ_j) Fpo set is (δ_i, δ_j) F-γ po
Proof: Let A be (δ_i, δ_j)Fpo. Then A $\leq$ δ_i int(δ_j cl(A). That is A $\leq$ δ_i int ($\delta_j\gamma$cl (A). Thus, A is (δ_i, δ_j) F-γ po.
Remark 3.23. The converse of the above theorems need not be true, which is illustrated by example. This example also shows that (δ_i,δ_j)Fγ po (resp. (δ_i,δ_j)Fγ pc) need not be (δ_i,δ_j)Fspo (resp. (δ_i,δ_j)Fspc).
Example 3.24. Let (X, δ_1, δ_2) be a F-bts where X = {a, b, c}and $\delta_1 = \{\tilde{0}, \tilde{1}, A\}$, $\delta_2 = \{\tilde{0}, \tilde{1}, B\}$ be the topologies on X. Let A, B be fuzzy sets defined in X as $A = \{a_{0.8}, b_{0.4}, c_{0.6}\}$ and $B = \{a_{0.3}, b_{0.2}, c_{0.4}\}$. Here A (resp. A') is ($\delta_1$, δ_2)F-γ o (resp. (δ_1, δ_2)F-γ c) and (δ_1, δ_2)F po (resp. (δ_1, δ_2)F pc) whereas B (resp. B') is not (δ_1, δ_2)F-γ o (resp. (δ_1, δ_2)F-γ c) and not (δ_1, δ_2)F po (resp. (δ_1, δ_2)F pc). Thus, B (resp. B') is (δ_1, δ_2)F-γpo (resp. (δ_1, δ_2)F-γ pc) but not δ_1Fo (resp. δ_1Fc) and not δ_1Fγo (resp. δ_1Fγc). Also, B (resp. B') is (δ_1, δ_2)F-γpo (resp. (δ_1, δ_2)F-γ pc) but not (δ_1, δ_2)F po (resp. (δ_1, δ_2)F pc) as well as not (δ_1, δ_2)F-γ o (resp. (δ_1, δ_2)F-γ c). Further B (resp. B') is (δ_1, δ_2)F-γ po (resp. (δ_1, δ_2)F-γ pc) but not (δ_1, δ_2)Fspo (resp. (δ_1, δ_2)Fspc) as well as not (δ_1, δ_2)Fsso (resp. (δ_1, δ_2) Fssc)
Remark 3.25. The following example shows that (δ_i, δ_j) F-γo (resp. (δ_i, δ_j) F-γc) need not be (δ_i, δ_j) F-γ po (resp. (δ_i, δ_j) F-γ pc) and (δ_i, δ_j) Fspo (resp. (δ_i, δ_j) Fspc) need not be (δ_i, δ_j) F-γ po (resp. (δ_i, δ_j) F-γ pc).
Example 3.26. Let (X, δ_1, δ_2) be a F-bts where X ={a, b, c}, $\delta_1 = \{\tilde{0}, \tilde{1}, A\}$ and $\delta_2 = \{\tilde{0}, \tilde{1}, B\}$ with fuzzy sets $A = \{a_{0.4}, b_{0.2}, c_{0.3}\}$ and $B = \{a_{0.5}, b_{0.2}, c_{0.4}\}$. Here ($\delta_1$, δ_2) Fγo sets = $\{\tilde{0}, \tilde{1}, A, B\}$. B (resp. B') is ($\delta_1$, δ_2) Fγo (resp. (δ_1, δ_2) F-γ c) but not (δ_1, δ_2) Fγpo (resp. (δ_1, δ_2) F-γ pc). Also, B (resp. B') is (δ_1, δ_2) Fspo (resp. (δ_1, δ_2) F-γ sc) but not (δ_1, δ_2) Fγpo (resp. (δ_1, δ_2) F-γ pc).
Remark 3.27. A (δ_i, δ_j) Fγpo set need not be (δ_i, δ_j) Fγso and vice versa which is illustrated below.
Example 3.28. Let (X, δ_1, δ_2) be a F-bts where X = {a, b, c}, $\delta_1 = \{\tilde{0}, \tilde{1}, A\}$ and $\delta_2 = \{\tilde{0}, \tilde{1}, B\}$ with fuzzy sets $A = \{a_{0.4}, b_{0.2}, c_{0.1}\}$ and $B = \{a_{0.6}, b_{0.4}, c_{0.3}\}$.Here B (resp. B') is ($\delta_1$,$\delta_2$)F$\gamma$ po (resp. (δ_1,δ_2)Fγ pc) but not (δ_1,δ_2) Fγso (resp. (δ_1, δ_2)Fγ sc). Let $C = \{a_{0.4}, b_{0.5}, c_{0.3}\}$. C (resp. C') is ($\delta_1$, δ_2)Fγso (resp. (δ_1, δ_2)Fγ sc) but not (δ_1, δ_2) Fγpo (resp. (δ_1, δ_2) Fγ pc)



Remark 3.29. Thus, we have arrived that (δ_i, δ_j) F-γ po sets are weaker than the concepts of (δ_i, δ_j) Fsso and (δ_i,δ_j) Fpo set. We have the above diagram.
Proposition 3.30. Each (δ_i, δ_j) F-γ-o set which is δ_i Fo is (δ_i, δ_j) F-γ-po.
Proposition 3.31. If A is (δ_i, δ_j) F-γo and A is not (δ_i, δ_j) Fso then A is (δ_i, δ_j) F-γ-po
Proposition 3.32. If A is (δ_i, δ_j) F-γo and if δ_j cl (δ_i int (A)) = 0 then A is (δ_i, δ_j) F-γ-po
Proposition 3.33. Each (δ_i, δ_j) F-γ-po which is δ_i Fo is (δ_i, δ_j) Fsso
Proposition 3.34. Each (δ_i, δ_j) Fspo which is δ_i Fo is (δ_i, δ_j) F-γ-po
Theorem 3.35. Let (X, δ_i, δ_j) be a F-bts and A be a fuzzy set of X. Then A is (δ_i, δ_j) Fγpo if and only if each fuzzy point $x_\beta \in A$ there exists a (δ_i, δ_j) Fγ po set H such that $x_\beta \in A \leq H$.
Proof: Let A be (δ_i, δ_j) Fγ po. Assume $x_\beta \in A$. By the Theorem 3.6, there exists a δ_i Fo set H such that A $\leq$ H. Using the Theorem 3.20, H is (δ_i, δ_j) Fγpo. Thus $x_\beta \in A \leq H$. Conversely suppose for every fuzzy point $x_\beta \in A$, there exists a (δ_i, δ_j) Fγpo set H_β such that $x_\beta \in A \leq H_\beta$. Then $\{H_\beta\}$ is a collection of (δ_i, δ_j) Fγpo sets such that,

for every $x_\beta \in A$, $x_\beta \in A \leq H_{\beta i}$, $\beta_i \in \Delta$ and further $\bigcup_{\beta i \in \Delta} H_{\beta i} = A$. Now, $H_{\beta i}$ is (δ_i, δ_j) Fγ po which implies that $\bigcup_{\beta i \in \Delta} H_{\beta i} = A$ is (δ_i, δ_j) Fγpo.

IV. (δ_i, δ_j) Fuzzy-γ-preinterior and (δ_i, δ_j) Fuzzy-γ-preclosure

Definition 4.1. Let A be a fuzzy set of a F-bts (X, δ_i, δ_j). Then the (δ_i, δ_j) γ-preclosure ($(\delta_i, \delta_j)\gamma$-pcl for short) and (δ_i, δ_j) γ-preinterior ($(\delta_i, \delta_j)\gamma$-pint for short) of A are defined as

(δ_i, δ_j) γ - p cl (A) = $\wedge$ { B : B is (δ_i, δ_j) F-γ-p closed and $A \leq B$}

(δ_i, δ_j) γ - p int (A) = $\vee$ { B : B is (δ_i, δ_j) F-γ-p open and $B \leq A$}

Proposition 4.2. Let A be a fuzzy set of a F-bts (X, δ_i, δ_j). Then

(i) (δ_i, δ_j) γ-pcl (A′) = $((\delta_i, \delta_j)$ γ-pint (A))′

(ii) (δ_i, δ_j) γ-pint (A′) = $((\delta_i, \delta_j)$ γ-pcl (A))′

***Proof*:** Follows from Definition 4.1

Definition 4.3. Let (X, δ_i, δ_j) be a fuzzy bitopological space and x_β is a fuzzy point of X. A fuzzy set A of X is called

(a) (δ_i, δ_j) F-γ pre neighbourhood (briefly (δ_i, δ_j) F - γ - pre nbhd) of x_β if there exists a (δ_i, δ_j) F-γ-po set O such that $x_\beta \in O \leq A$

(b) (δ_i, δ_j) F-γ pre q neighbourhood (briefly (δ_i, δ_j) F-γ-pre q nbhd)of x_β if there exists a (δ_i, δ_j) F-γ-po set O such that x_β q $O \leq A$

A. Properties of (δ_i, δ_j) F-γ-Preinterior and (δ_i, δ_j) F-γ-Preclosure Operators

Theorem 4.4. Let (X, δ_i, δ_j) be a F-bts.Then for any fuzzy sets A and B of X,

(i) (δ_i, δ_j) γ-p int($\tilde{0}$) = $\tilde{0}$ and (δ_i, δ_j) γ-p int($\tilde{1}$) = $\tilde{1}$

(ii) δ_i int(A) $\leq$ (δ_i, δ_j) γ-p int(A) $\leq$ A

(iii) A is (δ_i, δ_j) F-γ po if and only if A = (δ_i, δ_j) γ-p int(A)

(iv) (δ_i, δ_j) γ-p int(A) is (δ_i, δ_j) F-γ-po set and (δ_i, δ_j) γ- p int ((δ_i, δ_j) γ-p int (A)) = (δ_i, δ_j) γ-p int (A)

(v) If $A \leq B$, then (δ_i, δ_j) γ-p int (A) $\leq$ (δ_i, δ_j) γ-p int (B)

***Proof*:** (i), (ii) and (iii) Follows from Definition 4.1

(iv) From Definition 4.1 (δ_i, δ_j) γ-pint (A) is (δ_i, δ_j)F-γ-po. From (iii) other result holds.

(v) Let $A \leq B$. From (ii), (δ_i, δ_j) γ-p int (A) $\leq A \leq B$. By (iv), (δ_i, δ_j) γ-p int (A) $\leq$ (δ_i, δ_j) γ-p int (B).

Proposition 4.5. Let (X, δ_i, δ_j) be a F-bts and A and B be any two fuzzy sets of X. Then

(i) (δ_i, δ_j) γ-p int (A $\wedge$ B) = (δ_i, δ_j) γ-p int (A) $\wedge$ (δ_i, δ_j) γ-pint (B)

(ii) (δ_i, δ_j) γ-p int (A $\vee$ B) $\geq$ (δ_i, δ_j) γ-p int (A) $\vee$ (δ_i, δ_j) γ-pint (B)

***Proof*:** (i) By Theorem 4.4 , (δ_i, δ_j) γ-pint (A $\wedge$ B) $\leq$ (δ_i, δ_j) γ-pint (A), (δ_i, δ_j) γ-pint (A $\wedge$ B) $\leq$ (δ_i, δ_j) γ pint (B). Then, (δ_i, δ_j) γ-pint(A $\wedge$ B) $\leq$ (δ_i, δ_j) γ-pint(A) $\wedge$ (δ_i, δ_j) γ-pint(B). Let C $\in$ [(δ_i, δ_j) γ-pint (A) $\wedge$ (δ_i, δ_j) γ-pint (B)]. Then C is a (δ_i, δ_j) F-γ po set and C $\leq$ A $\wedge$ B. Then C $\leq$ (δ_i, δ_j) γ-pint (A $\wedge$ B). Thus, [(δ_i, δ_j) γ-pint(A) $\wedge$ (δ_i, δ_j) γ-pint (B)] $\leq$ (δ_i, δ_j) γ -pint (A $\wedge$ B). Hence (δ_i, δ_j) γ-pint (A $\wedge$ B) = (δ_i, δ_j) γ-pint (A) $\wedge$ (δ_i, δ_j) γ-pint (B).

(ii) By Theorem 4.4, (δ_i, δ_j) γ-pint (A$\vee$ B) $\geq$ (δ_i, δ_j) γ-pint (A) $\vee$ (δ_i, δ_j) γ-pint (B).

Remark 4.6. Equality need not hold in Proposition 4.5, which is shown by the example below.

Example 4.7. Let (Z, δ_1, δ_2) be a F-bts with Z = {a, b, c}, δ_1 ={$\tilde{0}$, $\tilde{1}$, A}, δ_2 ={$\tilde{0}$, $\tilde{1}$, B} and fuzzy sets A={$a_{0.5}$, $b_{0.6}$, $c_{0.7}$}, B ={$a_{0.5}$, $b_{0.3}$, $c_{0.2}$}, C ={$a_{0.3}$, $b_{0.7}$, $c_{0.8}$}, D ={$a_{0.5}$, $b_{0.5}$, $c_{0.7}$}, C$\vee$D = {$a_{0.5}$,$b_{0.7}$,$c_{0.8}$}=B'. Then (δ_1, δ_2) F-γ-po sets = {$\tilde{0}$, $\tilde{1}$, A, B} and (δ_1, δ_2) γ-pint(C $\vee$ D) = A, (δ_1, δ_2) γ-pint(C) = B, (δ_1, δ_2) γ-pint(D) = B. Thus, (δ_1, δ_2) γ-p int (A) $\vee$ (δ_1, δ_2) γ-pint (B) = B which implies (δ_1, δ_2) γ-pint (C $\vee$ D) $\geq$ (δ_1, δ_2) γ-pint (C) $\vee$ (δ_1, δ_2) γ-pint (D).

Theorem 4.8. Let (X, δ_i, δ_j) be a F-bts. Then for fuzzy sets A and B of X, the following holds:

(i) (δ_i, δ_j) γ-p cl($\tilde{0}$) = $\tilde{0}$ and (δ_i, δ_j) γ-p cl($\tilde{1}$) = $\tilde{1}$

(ii) $A \leq (\delta_i, \delta_j)$ γ-p cl(A) $\leq$ δ_i cl(A)

(iii) A is (δ_i, δ_j) F-γ pc if and only if A = (δ_i, δ_j) γ-p cl(A)

(iv) (δ_i, δ_j) γ-p cl(A) is (δ_i, δ_j) F-γ-pc set and (δ_i, δ_j) γ-p cl((δ_i, δ_j) γ-p cl(A)) = (δ_i, δ_j) γ-p cl(A)

(v) If $A \leq B$, then (δ_i, δ_j) γ-p cl(A) $\leq$ (δ_i, δ_j) γ-p cl(B)

***Proof*:** Follows from Definition 4.1

Proposition 4.9. Let (X, δ_i, δ_j) be a F-bts and A and B be any two fuzzy sets of X. Then

(i) (δ_i, δ_j) γ-p cl(A $\vee$ B) = (δ_i, δ_j) γ-p cl(A) $\vee$ (δ_i, δ_j) γ-p cl(B)

(ii) (δ_i, δ_j) γ-p cl(A $\wedge$ B) $\leq$ (δ_i, δ_j) γ-p cl(A) $\wedge$ (δ_i, δ_j) γ-p cl(B)

***Proof*:** (i) Consider (δ_i, δ_j) γ-pcl(A$\vee$B) = [(δ_i, δ_j) γ-pint(A'$\wedge$ B')]' = [(δ_i, δ_j) γ-pint(A') $\wedge$ (δ_i, δ_j) γ-pint(B')]'
= [(δ_i, δ_j) γ-p int(A')]' $\vee$ [(δ_i, δ_j) γ-p int(B')]' = (δ_i, δ_j) γ-p cl(A) $\vee$ (δ_i, δ_j) γ-p cl(B).

(ii) Consider (δ_i, δ_j) γ - p cl(A $\wedge$ B) = [(δ_i, δ_j) γ- p int(A $\wedge$ B)']' = [(δ_i, δ_j) γ- p int(A' $\vee$ B') $\geq$ [(δ_i, δ_j) γ- p int(A') $\vee$ (δ_i, δ_j) γ-p int (B')] $\leq$ [(δ_i, δ_j) γ-p int(A')]' $\wedge$ [(δ_i, δ_j) γ-p int(B')]' = [(δ_i, δ_j) γ-p cl(A)']' $\wedge$ [(δ_i, δ_j) γ-p cl(B)']'
= (δ_i, δ_j) γ-p cl(A) $\wedge$ (δ_i, δ_j) γ-p cl(B).

Remark 4.10. Equality need not hold in Proposition 4.9 which is illustrated as follows.

Example 4.11. Let (Z, δ_1, δ_2) be a F-bts with Z ={a, b, c}, δ_1 ={$\tilde{0}$, $\tilde{1}$, A}, δ_2 = {$\tilde{0}$, $\tilde{1}$, B} with fuzzy sets A={$a_{0.6}$, $b_{0.5}$, $c_{0.6}$}, B ={$a_{0.3}$, $b_{0.5}$, $c_{0.2}$}, C ={$a_{0.4}$, $b_{0.3}$, $c_{0.7}$}, D ={$a_{0.6}$, $b_{0.4}$, $c_{0.2}$} and C $\wedge$ D ={$a_{0.4}$, $b_{0.3}$, $c_{0.2}$}. Then (δ_1, δ_2) F-γ-po sets = {$\tilde{0}$,$\tilde{1}$, A, B, D, C $\wedge$ D}. Then (δ_1, δ_2) γ-p cl (C $\wedge$ D) = A′, (δ_1, δ_2) γ-p cl(C) = B′ $\wedge$ D′ and (δ_1,δ_2) γ-pcl(D) = D′.

Then $(\delta_1,\delta_2)\gamma$-p cl(C) ∧ $(\delta_1, \delta_2)\gamma$-p cl(D) = { $a_{0.4}$, $b_{0.5}$, $c_{0.8}$} and A′≤ { $a_{0.4}$, $b_{0.5}$, $c_{0.8}$}.Thus, (δ_i, δ_j) γ-p cl(C ∧ D) ≤ (δ_i, δ_j) γ-p cl(C) ∧ (δ_i, δ_j) γ-p cl(D).

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ALWAYS αgs-R_o AND αgs-R_1 SPACES

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***Abstract**: In this paper, we introduce two new classes of topological spaces called always αgs- R_o and αgs - R_1 spaces in terms of the concept of αgs-open sets and investigate some of their fundamental properties.*

***Keywords**: Topological spaces, semi open-sets, semi-closed sets, αgs -closed sets, αgs-open sets, αgs-kernel and αgs-closure of a set.*

I. Introduction

In 1963 and 1970 N. Levine [4,5] had introduced the concepts of semi-open sets and g-closed sets, respectively .Since then these notions have been studied extensively in the recent years by many topologists. In 1965 O. Njastad [8] introduced α-open sets in a topological space and studied some of its properties. In 2003 M. Caldas et al. [3] have studied the low separation axioms utilizing α-closures, α-open and α-closed sets. In 2004 Rajamani et al. [9] have introduced the concepts of αgs-open and αgs-closed sets and studied their basic properties. In 2005 G.B. Navalagi et al.[6] has studied the αgs-neighbouhoods . Recently in the year 2008 G.B. Navalagi et al. [7] has studied the concepts of αgs-T_0 , αgs-T_1 , and αgs-T_2 via αgs-open sets.

In this paper we offer some new separation axioms via αgs-open sets and αgs-closure [10] operators, called always αgs-R_o and αgs-R_1 spaces. We also characterize their fundamental properties.

II. Preliminaries

Through out this paper **(**X**,**τ) and **(**Y**,**σ **)** (or simply X and Y) denote the topological spaces on which no separation axioms are assumed unless explicitly stated. If A is any subset of space X, then Cl (A), Int (A), denote the closure of A and the interior of A in X respectively.

The following definitions are useful in the sequel

DEFINITION 2.1: A subset A of space X is called

(i) a semi-open set [4] if A ⊆ Cl (Int (A))

(ii) a semi-closed set [1] if Int (Cl (A)) ⊆ A

(iii) α-open set [8] if A ⊆ Int(Cl(Int(A)))

DEFINITION 2.2: A subset A of a space X is called

(i) a generalized-closed(briefly, g-closed) set [5] if Cl (A) ⊆ U whenever A ⊆ U and U is open in X .

(ii) an α-genaralized semi closed (briefly, αgs-closed) set [9] if αCl (A) ⊆ U whenever A ⊆ U and U is semi – open in X

(iii) The αgs-closure [10] of a subset A of X is the intersection of all αgs-closed sets containing A and it is denoted by αgsCl (A)..

The complement of αgs-closed is called αgs- open. The family of all αgs-open sets is denoted by αGSO (X,τ) and the family of all αgs-closed sets is denoted by αGSC (X,τ).

DEFINITION 2.3: Let X be a topological space and A ⊂ X. Then A is called semi-neighborhood [2] of a point x in X, if there exists semi- open set U in X such that x ∈U ⊂ A.

(i) A subset A of a topological space (X,τ) is called a αgs- neighborhood [6] (in short αgs-nbd.) of a point x of X if there exists αgs-open set U in X containing x such that $A \subseteq U$.

DEFINITION 2.4 [7]**:** A space X is said to be
(i) αgs-T_o (briefly, αgs-T_o) if for each pair of distinct points x, y of X, there exists a αgs-open set containing one but not other.
(ii) αgs-T_1 (briefly , αgs- T_1) if for each pair of distinct points x ,y of X, there exists a pair of αgs-open sets , one containing x but not y and the other containing y but not x.
(iii) αgs-T_2 (briefly, αgs- T_2) if for each pair of distinct points x ,y of X, there exists a disjoint pair of αgs-open sets ,one containing x and the other the other containing y .

LEMMA 2.5 [10]: Let (X,τ) be a topological space and A, B subsets of X. Then the following hold:
(i) $x \in Cl(A)$ if and only if $A \cap V \neq \phi$ for every $V \in \alpha GSO(X, \tau)$, $x \in V$.
(ii) A is αgs-closed in (X,τ) if and only if A = αgsCl (A).
(iii) αgsCl (A) $\subset$ αgsCl (B) if $A \subset B$.
(iv) αgsCl (αgsCl (A)) = αgsCl(A).

III. Always αgs-R_o space

In this section we define the notion of α - generalized semi- kernel of $A \subset X$ (briefly, αgs-ker (A)) in order to establish some more results on always αgs-R_o spaces.

DEFINITION 3.1: Let X be a topological space and $A \subset X$. Then the α-generalized semi kernel of A denoted by αgs-ker (A), is defined to be the set αgs-ker (A) = $\cap \{G \in \alpha GSO (X, \tau) / A \subset G \}$.

LEMMA 3.2: Let X be a topological space and A be a subset of X. Then αgs-ker (A) = $\{x \in X / \alpha gsCl (\{x\}) \cap A \neq \varnothing \}$.

PROOF: Let $x \in$ αgs-ker (A) and suppose αgsCl ($\{x\}$) $\cap$ A = ϕ .Hence $x \notin X \setminus \alpha gsCl (\{x\})$ which is a αgs-open set containing A. This is absurd, since $x \in$ αgs-ker (A).Consequently, αgsCl ($\{x\}$) $\cap$ A $\neq \varnothing$. Next, let αgsCl ($\{x\}$) $\cap$ A $\neq \varnothing$ and suppose that $x \notin$ αgs-ker (A). Then there exists a αgs-open set U containing A and $x \notin U$. Let $y \in$ αgsCl ($\{x\}$) $\cap$ A. Hence, U is a αgs-neighbourhood of y where $x \notin U$. But this is a contradiction. Therefore $x \in$ αgs-ker (A) and the claim.

LEMMA 3.3: Let X be a topological space and $x \in X$. Then $y \in$ αgs-ker ($\{x\}$) if and only if $x \in$ αgsCl ($\{y\}$).

PROOF: Suppose that $y \notin$ αgs-ker ($\{x\}$). Then there exists a αgs-open set V containing x such that $y \notin V$. Therefore we have $x \notin$ αgsCl ($\{y\}$).

The proof of the converse can be done similarly.

Now, we define the following;

DEFINITION 3.4: A topological space X is said to be always αgs-R_o space if αgsCl ($\{x\}$) $\subseteq$ U whenever U is αgs-open and $x \in U$.

THEOREM 3.5: For any topological space X the following properties are equivalent:
(i) X is always αgs-R_o space;
(ii) For any $F \in \alpha GS (X ,\tau)$ $x \notin F \Rightarrow F \subset U$ and $x \notin U$ for some $U \in \alpha GSO (X,\tau)$;
(iii) For any $F \in \alpha GSC (X ,\tau)$ $x \notin F \Rightarrow F \cap$ αgsCl ($\{x\}$) $= \varnothing$;
(iv) For any distinct points x and y either αgsCl ($\{x\}$) = αgsCl ($\{y\}$) or αgsCl ($\{x\}$) $\cap$ αgsCl ($\{y\}$) = $\varnothing$.

PROOF: (i) → (ii); Suppose $F \in \alpha GSC (X,\tau)$ and $x \notin F$. Then by (i) αgsCl ($\{x\}$) $\subset X \setminus F$. Set U = X \ αgsCl ({x} then $U \in \alpha GSO (X, \tau)$, $F \subset U$ and $x \notin U$.
(ii) → (iii); Let $F \in \alpha GSC (X, \tau)$, $x \notin F$. Therefore, there exists $U \in \alpha GSO (X,\tau)$ such that $F \subset U$ and $x \notin U$. Since $U \in \alpha GSO (X, \tau)$, U $\cap$ αgsCl ($\{x\}$) = $\varnothing$ and F $\cap$ αgs Cl ($\{x\}$) = $\varnothing$.

(iii) → (iv); Suppose that αgsCl ({x}) ≠ αgsCl ({y}) for distinct points x , y ∈ X. There exists z ∈ αgsCl ({x}) such that z ∉ αgsCl ({y}). One can also assume that z ∈ αgsCl ({y}) such that z ∉ αgsCl ({x}). There exists V ∈ αGSO (X,τ) such that y ∉ V and z ∈ V. Hence x ∈V. Therefore we obtain x ∉ αgsCl ({y}). By (iii) we obtain αgsCl ({x}) ∩ αgsCl ({y}) = ∅ .The proof of otherwise is similar.
(iv) → (i); Let V ∈ αGSO(X, τ) and x ∈ V. For each y ∉ V, x ≠ y and x ∉ αgsCl ({y}) This shows that αgsCl ({x}) ≠ αgsCl ({y}). By (iv) αgsCl ({x}) ∩ αgsCl ({y}) = ∅ for each y ∈ X \ V. Hence αgsCl ({x}) ∩ (∪{αgsCl ({y}) / y ∈X \ V) = ∅ . On the Other hand, since V ∈ αGSO (X,τ) and y ∈ X \ V. We have αgsCl ({y} ⊂ X \ V). Therefore X \ V = ∪ { αgsCl ({y}) / y ∈ X \ V}. Therefore we obtain (X \ V) ∩ αgsCl ({x}) = ∅ and αgsCl ({x}) ⊂ V. Hence (X,τ) is always αgs-R_o space**.**

THEOREM 3.6: A topological space is always sg-R_o space if and only for any x and y, αgsCl ({x}) ≠ αgsCl ({y}) implies αgsCl ({x}) ∩ αgsCl ({y}) = ∅

PROOF: It follows from Theorem 3.5.

LEMMA 3.7: The following statements are equivalent for any points x and y in a topological space X:
(i) αgs-ker ({x}) ≠ αgs-ker ({y})
(ii) αgsCl ({x}) ≠ αgsCl({y}).

PROOF: (i) → (ii). Suppose that αgs-ker ({x}) ≠ αgs-ker ({y}), then there exists a point z in X such that z ∈ αgs-ker ({x}) and z ∉ αgs-ker ({y}). From z ∈ αgs-ker ({x}) it follows that {x} ∩ αgsCl ({z}) ≠ ∅ which implies x ∈ αgsCl ({z}). By z ∉αgs-ker ({y}), we have {y} ∩ αgsCl ({z}) = ∅. Since x ∈ αgsCl ({z}), αgsCl ({x}) ⊂ αgsCl ({z}) and {y} ∩ αgsCl ({x}) = ∅. Therefore it follows that αgsCl ({x}) ≠ αgsCl ({y}). Now αgs-ker ({x}) ≠ αgs-ker ({y}) implies that αgsCl ({x}) ≠ αgsCl ({y}).
(ii) → (i). Suppose that αgsCl ({x}) ≠ αgsCl ({y}). It follows that there exists a point z in X such that z ∈ αgsCl ({x}) and z ∉ αgsCl ({y}). Then there exists a αgs -open set
containing z and therefore x but not y, namely, y ∉ αgs-ker ({x}). Hence αgs-ker ({x}) ≠ αgs-ker ({y}).

THEOREM 3.8: A topological space X is always αgs-R_o space if and only if for any x, y ∈ X, αgs-ker ({x}) ≠ αgs-ker ({y}) implies αgs-ker ({x}) ∩ αgs-ker ({y}) = ∅.

PROOF: Suppose X is always αgs-R_o space. Thus by Lemma 3.7 for any points x , y ∈ X if αgs-ker ({x}) ≠ αgs-ker ({y}) then αgsCl ({x}) ≠ αgsCl ({y}). Now we prove
that αgs-ker ({x}) ∩ αgs-ker ({y}) = ∅ . Assume that z ∈ αgs-ker ({x}) ∩ αgs-ker({y}). By z ∈ αgs-ker ({x}), it follows that x ∈ αgsCl ({z}). Since x ∈ αgsCl ({x}),
by Theorem 3.6 αgsCl ({x}) = αgsCl ({z}). Similarly, we have αgsCl ({y}) = αgsCl ({z}) = αgsCl ({x}). But this is a contradiction. Thus we have αgs-ker ({x}) ∩ αgs-ker
({y}) = ∅.
Conversely, let (X, τ) be a topological space such that for any points x and y in X, αgsCl ({x}) ≠ αgsCl ({y}) , then by Lemma 3.2 , αgs-ker ({x}) ≠ αgs-ker ({y}). Hence by hypothesis αgs ker ({x}) ∩ αgs-ker ({y}) = ∅ which implies αgsCl ({x}) ∩ αgsCl ({y}) = ∅. Because z ∈ αgsCl ({x}) implies that x ∈ αgs-ker ({z}) and therefore αgs-ker ({x}) ∩ αgs-ker ({y}) ≠ ∅. By hypothesis, we therefore have αgs- ker ({x}) = αgs-ker ({z}) = αgs-ker ({y}). This is contradiction. Therefore by Theorem 3.6 (X,τ) is always αgs -Ro space.

THEOREM 3.9: For a topological space X , the following properties are equivalent:
(i) X is a always αgs-R_o space;
(ii) For any A ≠ ∅ and G ∈ αGSO(X,τ) such that A ∩ G ≠ ∅, there exists F ∈ αGSC (X ,τ) such that A ∩ F ≠ ∅ and F⊂ G;
(iii) Any G ∈ αGSO (X,τ), G = ∪{F ∈ αGSC(X,τ) / F ⊂ G};
(iv) Any F ∈ αGSC **(X,**τ), F = ∩ {G ∈ αGSO(X,τ) / F⊂ G};
(v) For any x ∈ X, αgsCl ({x}) ⊂ αgs-ker ({x}).

PROOF: (i) → (ii). Let A be a non-empty set of X and G ∈ αGSO (X,τ) such that A ∩ G ≠ ∅. There exist x ∈ A ∩ G. Since x ∈ G ∈ αGSO (X,τ), αgsCl ({x}) ⊂ G. Set

F = αgsCl ({x}), then F ∈ αGSC (X, τ) , F ⊂ G and A ∩ F ≠∅

(ii) → (iii). Let G ∈ αGSO(X, τ). Then G ⊃ ∪ {F ∈ αGSC(X, τ) / F ⊂ G}.Let x be any point of G then there exists F ∈ αGSC (X,τ) such that x ∈ F and F⊂ G. Therefore we have x ∈ F ⊂ ∪ {F ∈ αGSC (X,τ) / F ⊂ G} and hence G = ∪{F ∈ αGSC (X,τ) / F⊂ G}.

(iii) → (iv). Obvious.

(iv) → (v). Let x be any point of X and y ∉ αgs-ker ({x}).Then there exists V∈ αGSO (X,τ) such that x ∈ V and y ∉ V. Hence αgsCl ({y}) ∩ V = ϕ . By (iv) (∩ {G ∈ αGSO (X ,τ) / αgsCl ({y}) ⊂ G}) ∩ V = ∅ and there exists G ∈ αGSO (X,τ) such that x ∉ G and αgsCl ({y}) ⊂ G .Therefore αgsCl ({x}) ∩ G = ∅ and y ∉ αgsCl ({y}). Consequently, we obtain αgsCl ({x}) ⊂ αgs-ker ({x}).

(v) → (i). Let G ∈ αGSO(X, τ) and x ∈ G. Suppose y ∈ αgs-ker ({x}), then x ∈ αgsCl ({y}) and y ∈ G .This implies αgsCl ({x}) ⊂ αgs-ker ({x}) .This show the (X, τ) is a always αgs-R_o space.

COROLLARY 3.10: For a topological space X , the following properties are equivalent:

(i) X is always αgs-R_o space;

(ii) αgsCl ({x}) = αgs-ker ({x}) for all x ∈ X.

PROOF: (i) → (ii). Suppose (X, τ) is a always αgs-R_o space. By the Theorem 3.9 αgsCl ({x}) ⊂ αgs-ker ({x}) for each x ∈ X Let y ∈ αgs-ker ({x}) then x ∈ αgsCl ({y}) and by Theorem 3.6 αgsCl ({x}) = αgsCl ({y}).Therefore y ∈ αgsCl ({x}) and hence αgs-ker ({x}) ⊂ αgsCl ({x}). This shows that αgsCl ({x}) = αgs-ker ({x})

(ii) → (i).This is obvious by Theorem 3.7.

THEOREM 3.11: For a topological space X, the following properties are equivalent:

(i) X is always αgs-R_o space.

(ii) x ∈ αgsCl ({x}) if and only if y ∈ αgsCl ({x}) for any points x and y in X.

PROOF: (i) →(ii). Assume that X is always αgs-R_o space. Let x ∈ αgsCl ({y}) and U be any αgs-open set such that y ∈ U. Now by hypothesis x ∈ U. Therefore, every αgs -open set which contain y contains x. Hence y ∈ αgsCl ({x})

(ii) → (i). Let U be a αgs-open set and x ∈ U if y ∉ U then x ∉ αgsCl({y}) and hence y ∉ αgsCl({x}).This implies that αgsCl ({x}) ⊂ U Hence (X, τ) is a always αgs-R_o space

THEOREM 3.12: For a topological pace X, the following properties are equivalent:

(i) X is always αgs-R_o space;

(ii) If F is a αgs-closed, then F = αgs-ker (F);

(iii) If F is a αgs-closed and x ∈ F, then αgs-ker ({x}) ⊂ F;

(iv) If x ∈ X, then αgs-ker ({x}) ⊂ αgsCl ({x}).

PROOF: (i) → (ii). Let F be αgs-closed and x ∉ F .Thus X - F is a αgs-open set containing x. Since (X,τ) is a always αgs-R_o space, αgsCl ({x}) ⊂ X – F. Thus αgsCl ({x}) ∩ F = ∅ and by the Lemma 3.1 x ∉ αgs-ker(F).Therefore αgs-ker (F) = F

(ii) → (iii). It is obvious that A ⊂ B implies αgs-ker (A) ⊂ αgs-ker (B).Therefore it follows from (ii) that αgs-ker ({x}) ⊂ αgs-ker ({F}) = F

(iii) → (iv). Since x ∈ αgsCl ({x}) and αgsCl ({x}) is αgs-closed by (iii) αgs-ker ({x}) ⊂ αgs-ker (F) = F.

(iv) → (i). We show the implication by the Theorem 3.11. Let x ∈ αgsCl ({y}). Then by the Lemma 3.2 y ∈ αgs-ker ({x}). Since x ∈ αgsCl({x}) and αgsCl({x}) is αgs-closed , by (iv) we obtain y ∈ αgs-ker({x}) ⊂ αgsCl({x}) .Therefore x ∈ αgsCl({y}) implies y ∈ αgsCl({x}) the converse is obvious and (X, τ) is always αgs-R_o space.

IV. αsg-R_1 space

DEFINITION 4.1: A topological space X is said to be αgs-R_1 space if for x ,y in X with αgsCl({x}) ≠ αgsCl ({y}), there exists disjoint sg-open sets U and V such that αgs Cl ({x}) is a subset of U and αgsCl ({y}) is a subset of V.

PROPOSITION 4.2: If X is αgs-R_1, then X always αgs-R_o space.

PROOF: Let U be a αgs-open and $x \in U$. If $y \notin U$, then since $x \notin \alpha gsCl(\{y\})$, $\alpha gsCl(\{x\}) \neq \alpha gsCl(\{y\})$. Hence, there exists a αgs-open V such that $\alpha gsCl(\{y\}) \subset V$ and $x \notin V$, which implies $y \notin \alpha gsCl(\{x\})$. Thus $\alpha gsCl(\{y\}) \subset U$. Therefore (X,τ) is always αgs-R_o space.

THEOREM 4.3: For a topological space X, the following statements are equivalent:
(i) X is αgs-T_2;
(ii**)** X is αgs-R_1.

PROOF: (i) → (ii). Since X is αgs-T_2 and αgs-T_1 and hence αgs-T_o. If $x, y \in X$ such that $\alpha gsCl(\{x\}) \neq \alpha gsCl(\{y\})$, then $x \neq y$. Then there exists disjoint αgs-open sets U and V such that $x \in U$ and $y \in V$. Thus $\alpha gsCl(\{x\}) = \{x\} \subset U$ and $\alpha gsCl(\{y\}) = \{y\} \subset V$. Hence X is αgs-$R_1$.
(ii) → (i). Suppose that (X,τ) is αgs-R_1. By Theorem 4.1 X is always αgs-R_o space. Therefore it is αgs-T_1 by Theorem 3.12. Let $x, y \in X$ such that $x \neq y$. Since $\alpha gsCl(\{x\}) = \{x\} \neq \{y\} = \alpha gsCl(\{y\})$, there exists disjoint αgs-open sets U and V such that $x \in U$ and $y \in V$. It follows that X is αgs-T_2.

THEOREM 4.4: If a topological space (X, τ) is αgs-R_1 then if $x, y \in X$ such that $\alpha gsCl(\{x\}) \neq \alpha gsCl(\{y\})$, there exists αgs-closed sets F_1 and F_2 such that $x \in F_1$, $y \notin F_1$, $y \in F_2$, $x \notin F_2$ and $X = F_1 \cup F_2$

PROOF: Let $x, y \in X$ such that $\alpha gsCl(\{x\}) \neq \alpha gsCl(\{y\})$. By Theorem 4.2, X is αgs -T_2 and hence $x \neq y$. Therefore, there exists disjoint αgs-open sets U_1 and U_2 such that $x \in U_1$ and $y \in U_2$. Then $F_1 = X - U_2$ and $F_2 = X - U_1$ are αgs-closed sets such that $x \in F_1$, $y \notin F_1$, $y \in F_2$, $x \notin F_2$ and $X = F_1 \cup F_2$.

THEOREM 4.5: A topological space is always αgs-R_1 space if and only for $x, y \in X$, $\alpha gs\text{-}ker(\{x\}) \neq \alpha gs\text{-}ker(\{y\})$, there exists disjoint αgs-open sets U and V such that $\alpha gsCl(\{x\}) \subset U$ and $\alpha gsCl(\{y\}) \subset V$.

PROOF: It follows from Lemma 3.7.

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Magnified Translation of L-Fuzzy Bi-Ideal of a Ring

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Abstract: *The aim of this paper is to introduce the concept of magnified translation of L-fuzzy bi-ideal in a ring R.*

Keywords: *Fuzzy ideal, Fuzzy bi-ideal, L-fuzzy bi-ideal, fuzzy magnified translation, Fuzzy multiplication.*

I. INTRODUCTION

The theory of fuzzy set plays an important role in modern mathematics. The idea of L-fuzzy set was introduced by J.A Goguen [1] as a generalization of zadeh's [6] fuzzy sets. Lajos and szasz [3] initiated the idea of bi-ideals in a ring. The concept of fuzzy translation and fuzzy magnified translation in fuzzy group theory was introduced by Kandasamy [2] and Majumder and Sardar [4] respectively. Also, Sanjib kumar Datta [5] introduced the notion of fuzzy magnified translation in ring. In this paper we introduce the concept of fuzzy magnified translation of L-fuzzy bi-ideals in rings and established some of its properties in detail.

II. PRELIMINARIES

2.1 Definition:

Let R be a ring .and μ be a fuzzy subset of R. μ is called a fuzzy ideal of R if

(i) $\mu(x-y) \geq min\{\mu(x), \mu(y)\}$

(ii) $\mu(xy) \geq max\{\mu(x), \mu(y)\}$

2.2 Definition:

A non empty fuzzy subset μ of a ring R is called an fuzzy bi-ideal of R if

(i) $\mu(x-y) \geq min\{\mu(x), \mu(y)\}$

(ii) $\mu(xy) \geq min\{\mu(x), \mu(y)\}$

(iii) $\mu(xyz) \geq min\{\mu(x), \mu(z)\} \quad \forall x, y, z \in R.$

2.3 Definition:

Let μ be a fuzzy subset of a ring R. Im μ is defined as,

Im μ= $\{t \in [0,1] / \mu(x) = t \; for \; some \; x \in R\}$

2.4 Definition:

A fuzzy subset μ of ring R is called L-fuzzy left (right) ideal of R, if for every $x, y \in R$

(i) μ is a L-fuzzy subgroup of (R,+), ie, $\mu(x-y) \geq \mu(x) \wedge \mu(y)$

(ii) $\mu(xy) \geq \mu(y) \quad \mu(xy) \geq \mu(x)$

If μ is both an L-fuzzy left ideal and L-fuzzy right ideal of R, then it is L-fuzzy ideal of R.

2.5 Definition:

Let R be a ring and L be a lattice. A fuzzy set μ of R is said to be L-fuzzy bi-ideal of R if

(i) $\mu(x-y) \geq \mu(x) \wedge \mu(y)$

(ii) $\mu(xy) \geq \mu(x) \wedge \mu(y)$

(iii) $\mu(xyz) \geq \mu(x) \wedge \mu(z) \qquad \forall x, y, z \in R$

Example:

Consider the fuzzy set μ of R by

$\mu_A(x) = \begin{Bmatrix} 0.9 \; if \; x \; is \; rational \\ 0.5 \; if \; x \; is \; irrational \end{Bmatrix}$

Then μ is an L-fuzzy bi-ideal of R.

III. FUZZY MAGNIFIED TRANSLATION OF L-FUZZY BI-IDEAL

3.1 Definition:

Let μ be a non empty fuzzy subset of a ring R (ie, $\mu(x) \neq 0 for \; some \; x \in R$)and also

let $\alpha \in [0,1 - sup\{\mu(x): x \in R\}], \beta \in [0,1]$.Then the fuzzy magnified translation $\mu^C_{\beta\alpha}(x)$ of μ in R is defined as, $\mu^C_{\beta\alpha}(x) = \beta.\mu(x) + \alpha \; for \; all \; x \in R$.It is also a fuzzy subset of R.

In particular, if $\beta = 1$, then μ^T_α is called the fuzzy translation of μ.

Ie, $\mu^T_\alpha(x) = \mu(x) + \alpha \; for \; all \; x \in R$

Also when $\alpha = 0$ then μ^M_β is called the fuzzy multiplication of μ.ie, $\mu^M_\beta(x) = \beta\mu(x) for \; all \; x \in R$.

3.2 Theorem:

Let μ be an L-fuzzy bi-ideal of a ring R.Then the fuzzy magnified translation $\mu^C_{\beta\alpha}(x)$ is also an L-fuzzy bi-ideal of R.

Proof:

Let μ be a L-fuzzy bi-ideal of a ring R. Now for all $x, y, z \in R$,

$$\mu^C_{\beta\alpha}(x - y) = \beta.\mu(x - y) + \alpha$$
$$\geq \beta.\{\mu(x) \wedge \mu(y)\} + \alpha$$
$$= (\beta\mu(x) + \alpha) \wedge (\beta\mu(y) + \alpha)$$
$$\mu^C_{\beta\alpha}(x - y) \geq \mu^C_{\beta\alpha}(x) \wedge \mu^C_{\beta\alpha}(y)$$

Again,

$$\mu^C_{\beta\alpha}(xy) = \beta.\mu(xy) + \alpha$$
$$\geq \beta.\{\mu(x) \wedge \mu(y)\} + \alpha$$
$$= (\beta\mu(x) + \alpha) \wedge (\beta\mu(y) + \alpha)$$
$$\mu^C_{\beta\alpha}(xy) \geq \mu^C_{\beta\alpha}(x) \wedge \mu^C_{\beta\alpha}(y)$$

Also,

$$\mu^C_{\beta\alpha}(xyz) = \beta.\mu(xyz) + \alpha$$
$$\geq \beta.\{\mu(x) \wedge \mu(z)\} + \alpha$$
$$= (\beta\mu(x) + \alpha) \wedge (\beta\mu(z) + \alpha)$$
$$\mu^C_{\beta\alpha}(xyz) \geq \mu^C_{\beta\alpha}(x) \wedge \mu^C_{\beta\alpha}(z)$$

Thus, $\mu^C_{\beta\alpha}$ is an L-fuzzy bi-ideal of R.

3.3 Corollary:

If μ be an L-fuzzy bi-ideal of a ring R then the fuzzy translation μ^T_α of μ is also an L-fuzzy bi-ideal of R.

Proof:

Taking $\beta = 1$ the above corollary follows from Theorem 3.1.

3.4 Corollary:

Let μ be an L-fuzzy bi-ideal of a ring R. Then the fuzzy multiplication μ^M_β of μ is also an L-fuzzy bi-ideal of R.

Proof:

Taking $\alpha = 0$, corollary 2 follows from theorem 3.1.

3.5 Theorem:

If μ be an L-fuzzy left (right, two sided) ideal of a ring R, then the fuzzy magnified translation $\mu^C_{\beta\alpha}$ of μ is an L-fuzzy bi-ideal of R.

Proof:

Let μ be a L-fuzzy left ideal of a ring R.

Then for all $x, y \in R$

$$\mu(x - y) \geq \mu(x) \wedge \mu(y)$$
$$\mu(xy) \geq \mu(x) \wedge \mu(y)$$

Also let $x, y, z \in R$

Then,

$$\mu(xyz) \geq \mu((xy)z) \geq \mu(z) \geq \mu(x) \wedge \mu(z)$$

Thus μ will be an L-fuzzy bi-ideal of R.

Now for all $x, y \in R$

$$\mu^C_{\beta\alpha}(x - y) = \beta.\mu(x - y) + \alpha$$
$$\geq \beta.\{\mu(x) \wedge \mu(y)\} + \alpha$$
$$= (\beta\mu(x) + \alpha) \wedge (\beta\mu(y) + \alpha)$$
$$\mu^C_{\beta\alpha}(x - y) \geq \mu^C_{\beta\alpha}(x) \wedge \mu^C_{\beta\alpha}(y)$$

Again,

$$\mu^C_{\beta\alpha}(xy) = \beta.\mu(xy) + \alpha$$
$$\geq \beta.\{\mu(x) \wedge \mu(y)\} + \alpha$$

$$= (\beta\mu(x) + \alpha) \wedge (\beta\mu(y) + \alpha)$$

$$\mu^C_{\beta\alpha}(xy) \geq \mu^C_{\beta\alpha}(x) \wedge \mu^C_{\beta\alpha}(y)$$

Also let $x, y, z \in R$

Now

$$\mu^C_{\beta\alpha}(xyz) = \beta.\mu(xyz) + \alpha$$
$$\geq \beta\mu(z) + \alpha$$
$$= \mu^C_{\beta\alpha}(z) \geq \mu^C_{\beta\alpha}(x) \wedge \mu^C_{\beta\alpha}(z)$$

So $\mu^C_{\beta\alpha}$ is an L-fuzzy bi-ideal of R.

3.6 Corollary:

Let μ be an L-fuzzy left ideal of a ring R. Then the fuzzy translation μ^T_α of μ is a fuzzy bi-ideal of R.

Proof:

Taking $\beta = 1$,corollary 3.6 follows from theorem 3.5.

3.7 Corollary:

If μ be an L-fuzzy left ideal of a ring R then the fuzzy multiplication μ^M_β of μ is an L-fuzzy bi-ideal of R.

Proof:

Taking $\alpha = 0$, corollary 3.7 follows from theorem 3.5.

3.8 Theorem:

The fuzzy magnified translation of the intersection of an arbitrary collection of L-fuzzy bi ideal of a ring R is an L-fuzzy bi-ideal of R if it is non empty.

Proof:

Let $\mu_i (i \in \mathrm{I})$ be an arbitrary collection of L-fuzzy bi-ideals of R.

$$Let\ \mu = \bigcap_{i\in \mathrm{I}} \mu_i\ \ be\ \ non\ empty.\ Let\ x, y, z \in R.\ \text{Then}$$

$$\mu(x-y) = \left(\bigcap_{i\in \mathrm{I}} \mu_i\right)(x-y) = \inf_{i\in I}\{\mu_i(x-y)\} \geq \inf_{i\in I}\{\mu_i(x) \wedge \mu_i(y)\}$$
$$= [\inf_{i\in I}\{\mu_i(x)\} \wedge \inf_{i\in I}\{\mu_i(y)\}]$$
$$= \left(\bigcap_{i\in \mathrm{I}} \mu_i\right)(x) \wedge \left(\bigcap_{i\in \mathrm{I}} \mu_i\right)(y)$$

And

$$\mu(xy) = \left(\bigcap_{i\in \mathrm{I}} \mu_i\right)(xy) = \inf_{i\in I}\{\mu_i(xy)\} \geq \inf_{i\in I}\{\mu_i(x) \wedge \mu_i(y)\}$$
$$= [\inf_{i\in I}\{\mu_i(x)\} \wedge \inf_{i\in I}\{\mu_i(y)\}]$$
$$= \left(\bigcap_{i\in \mathrm{I}} \mu_i\right)(x) \wedge \left(\bigcap_{i\in \mathrm{I}} \mu_i\right)(y)$$

Again for all $x, y, z \in R$,

$$\mu(xyz) = \left(\bigcap_{i\in \mathrm{I}} \mu_i\right)(xyz) = \inf_{i\in I}\{\mu_i(xyz)\} \geq \inf_{i\in I}\{\mu_i(x) \wedge \mu_i(z)\}$$
$$= [\inf_{i\in I}\{\mu_i(x)\} \wedge \inf_{i\in I}\{\mu_i(z)\}]$$
$$= \left(\bigcap_{i\in \mathrm{I}} \mu_i\right)(x) \wedge \left(\bigcap_{i\in \mathrm{I}} \mu_i\right)(z)$$

Hence arbitrary collection of L-fuzzy bi-ideal of R is an L-fuzzy bi-ideal of R.

3.9 Corollary:

The fuzzy translation of the intersection of an arbitrary collection of L-fuzzy bi-ideals of a ring R is an L-fuzzy bi-ideal of R if it is not empty.

3.10 Corollary:

The fuzzy multiplication of the intersection of an arbitrary collection of L-fuzzy bi-ideals of a ring R is an L-fuzzy bi-ideal of R if it is not empty.

3.11 Theorem:

Let R and R' be any two rings. Then the homomorphic image of an L-fuzzy bi-ideals of magnified translation of an L-fuzzy bi-ideals of R is an L-fuzzy bi-ideals of R'.

Proof:

Let R and R'be any two rings and $f: R \to R'$ be a ring homomorphism.

Therefore $f(x+y) = f(x) + f(y)$ and

$f(xy) = f(x)f(y)\ for\ all\ x\ and\ y\ \in R$

Let $\mu^C_{\beta\alpha}(x)$ is a fuzzy magnified translation of L-fuzzy bi-ideal of R.

We show that $\mu^C_{\beta\alpha}(x)$ is an L-fuzzy bi-ideal of R'

Now, for $f(x)$ and $f(y)$ in R', we have

$$\begin{aligned}\mu^C_{\beta\alpha}[f(x)-f(y)] &= \mu^C_{\beta\alpha}[f(x-y)]\\ &= \mu^C_{\beta\alpha}(x-y)\\ &= \beta.\mu(x-y)+\alpha\\ &\geq \beta.\{\mu(x)\wedge\mu(y)\}+\alpha\\ &= (\beta\mu(x)+\alpha)\wedge(\beta\mu(y)+\alpha)\\ &= \mu^C_{\beta\alpha}(f(x))\wedge\mu^C_{\beta\alpha}(f(y))\end{aligned}$$

Thus, $\mu^C_{\beta\alpha}[f(x)-f(y)] \geq \mu^C_{\beta\alpha}(f(x))\wedge\mu^C_{\beta\alpha}(f(y))$

$$\begin{aligned}\mu^C_{\beta\alpha}[f(x)f(y)] &= \mu^C_{\beta\alpha}[f(xy)]\\ &= \mu^C_{\beta\alpha}(xy)\\ &= \beta.\mu(xy)+\alpha\\ &\geq \beta.\{\mu(x)\wedge\mu(y)\}+\alpha\\ &= (\beta\mu(x)+\alpha)\wedge(\beta\mu(y)+\alpha)\\ &= \mu^C_{\beta\alpha}(f(x))\wedge\mu^C_{\beta\alpha}(f(y))\end{aligned}$$

Thus, $\mu^C_{\beta\alpha}[f(x)f(y)] \geq \mu^C_{\beta\alpha}(f(x))\wedge\mu^C_{\beta\alpha}(f(y))$

$$\begin{aligned}\mu^C_{\beta\alpha}[f(x)f(y)f(z)] &= \mu^C_{\beta\alpha}[f(xyz)]\\ &= \mu^C_{\beta\alpha}(xyz)\\ &= \beta.\mu(xyz)+\alpha\\ &\geq \beta.\{\mu(x)\wedge\mu(z)\}+\alpha\\ &= (\beta\mu(x)+\alpha)\wedge(\beta\mu(z)+\alpha)\\ &= \mu^C_{\beta\alpha}(f(x))\wedge\mu^C_{\beta\alpha}(f(z))\end{aligned}$$

Thus, $\mu^C_{\beta\alpha}[f(x)f(y)f(z)] \geq \mu^C_{\beta\alpha}\big(f(x)\big)\wedge\mu^C_{\beta\alpha}\big(f(z)\big)$

Hence the homomorphic image of an L-fuzzy bi-ideals of magnified translation of an L-fuzzy bi-ideal of R is an L-fuzzy bi-ideals of R'.

IV. CONCLUSION

In this article authors have been discussed fuzzy magnified translation of L-fuzzy bi-ideal in ring. This concept can further be generalized to intuitionistic fuzzy set for new results in our future work.

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WEAKER FORM OF REGULAR CONTINUOUS FUNCTION

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Abstract*: In this paper we introduce a new class of continuousfunction called regular b – continuous functions (briefly rb – continuous function). rb-functions are weaker than the regular continuous function and stronger than the b-continuous function and generalized continuous functions. Also we study some characterization and basic properties of regular b – continuous functions.*

Keywords*: Continuous function, b– Continuous function, r– continuous function, rb – continuous function, rbcl(A), rbint(A)*

I. Introduction

Nagaveni and Narmadha [14] introduced the concepts of regular b- closed sets in a topological space and investigated some of their properties. Ekici [8] introduced and studied the concept of b-continuous functions in topological spaces. The regular continuous function was first introduced by Arya and Gupta [2]. Later Palaniappan and Rao[16]studied the concept of regular generalized continuous function.. The purpose of this paper is to introduce a new class of function called regular b – continuous functions (briefly rb – continuous function). These functions are weaker than the regular continuous function and stronger than the b-continuous function and also generalized continuous functions.

In this paper we introduce new class of functions which are weaker form of regular continuous function. We observe that these functions are stronger than the b-continuous function and generalized continuous function.

Also we study some characterization of these new class of continuous functions. Throughout this paper, the space (X,τ) and (Y,σ) (or simply XY) always means a topological space on which no separation axioms are assumed unless explicitly stated. For a subset A of X, the closure of A and interior of A are denoted by cl(A) and int(A) respectively

II. Preliminaries

First we recall the following definitions which are used in our study.

Definition 2.1:A subset A of a space (X, τ) is called

(1) a regular open set[12] if int(cl(A)) = A and a regular closed set if cl(int(A)) = A.

(2) a b-open[7] or sp-open if $A \subseteq cl(int(A)) \cup Int(cl(A))$

(3) asemi-open set[9] if $A \subseteq cl(int(A))$ and a semi-closed set if $int(cl(A)) \subseteq A$.

(4) apreopen set[10] if $A \subseteq int(cl(A))$ and a preclosedset if $cl(int(A)) \subseteq A$.

(5) an α-open set [1]if $A \subseteq int(cl(int(A)))$ and a α-closed set if $cl(int(cl(A))) \subseteq A$.

(6) asemi-preopen set (=β-open)[7] if $A \subseteq cl(int(cl(A)))$ and a semi-preclosedset

(=β-closed [7]) if $int(cl(int(A))) \subseteq A$.

Definition 2.2: A subset A of a topological space (X,τ) is regular b-closed ((briefly rb – closed) if rcl A $\subseteq$Uwhenever $A \subseteq U$ and U is b-open in X.

Definition 2.3:A map $f : X \to Y$ is called

1) Continuous if $f^{-1}(V)$ is a closed set in (X,τ) for every closed set V in (Y,σ).
2) b-continuous if $f^{-1}(V)$ is a b-closed set in (X,τ) for every closed set V in (Y,σ).
3) semi-continuous if $f^{-1}(V)$ is a semi-closed set in (X,τ) for every closed set V in (Y,σ).
4) δ -continuous if $f^{-1}(V)$ is a δ -closed set of (X,τ) for every closed set V of (Y,σ).
5) α -continuous if $f^{-1}(V)$ is a α -closed set in (X,τ) for every closed set V in (Y,σ).
6) g-continuous if $f^{-1}(V)$ is a g-closed set in (X,τ) for every closed set V in (Y,σ).

III. REGULAR b-CONTINUOUS FUNCTIONS IN TOPOLOGICAL SPACES

In this section we introduce a new class of function called regular b – continuous functions in topological space and study characterization and basic properties of regular b – continuous functions. Also, we study another types of generalized regular continuous function and we study the relation among them.

Definition 3.1: A function $f : (X, \tau) \rightarrow (Y, \sigma)$. from a topological space X into a topological space Y is called regular b – continuous functions (briefly rb-continuous) if $f^{-1}(V)$is a rb-closed set in (X,τ) for every closed set V in (Y,σ).

Theorem 3.2: A function $f : (X, \tau) \rightarrow (Y, \sigma)$. from a topological space X into a topological space Y is regular b – continuous functions (briefly rb-continuous) if and only if $f^{-1}(V)$ is a rb-open set in (X,τ) for every open set V in (Y,σ).

Proof:

It is obvious.

Example 3.3:

Let X = Y = {a, b, c}with $\tau = \{X, \emptyset, \{a\}, \{b, c\}\}, \sigma = \{X, \emptyset, \{a\}\}$. Let $f : X \rightarrow Y$be the identity map.it is clear that f is continuous and rb-continuous. Since $f^{-1}(\emptyset) = \emptyset, f^{-1}(X) = X, f^{-1}(\{a\}) = \{a\}$ are closed and rb-closed in(X,τ).

Theorem 3.4: A function $f : (X, \tau) \rightarrow (Y, \sigma)$. from a topological space X into a topological space Y is regular b – continuous, then $f(rbcl(A) \subseteq cl(f(A))$for every subset A of X.

Proof:

Since $f(A) \subseteq cl(f(A)) \Rightarrow A \subseteq f^{-1}\left(cl(f(A))\right)$. Since $cl(f(A))$ is closed set in Y and f is rb-continuous, then $f^{-1}\left(cl(f(A))\right)$ is a rb-closed set in X containing A.

Hence $rbcl(A) \subseteq f^{-1}\left(cl(f(A))\right)$.Therefore$f(rbcl(A) \subseteq cl(f(A))$.

Theorem 3.5:

A function $f : (X, \tau) \rightarrow (Y, \sigma)$ from a topological space X into a topological space Y. Then the following conditions are equivalent.

(i)each point x in X and each open set V in Y with $f(x) \epsilon V, there\ is\ a$ rb-open set U in X such that $x \in U$ and $f(U) \subseteq V$.

(ii)for each subset A of X $f(rbcl(A) \subseteq cl(f(A))$

(iii) for subset B of Y, $rbcl(f^{-1}(B)) \subseteq f^{-1}(cl(B))$.

Proof:

(i)⇒(ii)

Suppose that (i) holds and let $y \epsilon f(rbcl(A))$ and V be any open neighborhood of Y. Since $y \in f(rbcl(A) \Rightarrow$ there exist $x \in rbcl(A))$. Such that $f(x) = y$. Since $f(x) \in V$, then by (i) there exist a rb-open set U in X. Such that $x \in U$and $f(U) \subseteq V$. Since $x \in rbcl(A))$. Then $U \cap A \neq \emptyset$ and hence $f(a) \cap V \neq \emptyset$.

Therefore we have $y = f(x) \in cl(f(A))$. Hence $f(rbcl(A)) \subseteq cl(f(A))$.

(ii)⇒(i)

If (ii) holds and let $x \in X$and V any open set in Y containing f(x).

Let $A = f^{-1}(V^c) \Rightarrow x$ does not belongs to A. Since $f\big(rbcl(A)\big) \subseteq cl\big(f(a)\big) \subseteq V^c \Rightarrow rbcl(A) \subseteq f^{-1}(V^c) = A$. Since x does not belongs to A $\Rightarrow$ x does not belongs to $rbcl(A)$ then there exist a rb-open set U containing x such that $U \cap A \neq \emptyset$. And hence $f(U) \subseteq f(A^c) \subseteq V$

(ii)⇒(iii)

Suppose that (ii)holds and let B be an subset of Y replacing A by $f^{-1}(B)$ we get from (ii) $f\big(rbcl(f^{-1}(B))\big) \subseteq cl\left(f\big(f^{-1}(B)\big)\right) \subseteq cl(B)$. Hence $rbcl(f^{-1}(B)) \subseteq f^{-1}(cl(B))$.

(iii) ⇒ (ii)

Suppose that (iii) holds , let $B = f(A)$ where A is a subset of X. then we get from (iii) $rbcl(A) \subseteq rbcl\big(f^{-1}(f(A))\big) \subseteq f^{-1}(cl(A))$.

Therefore$f\big(rbcl(A)\big) \subseteq cl(f(A))$.

Theorem 3.6:

Every rb-continuous function is b-continuous function but not conversely.

Proof:

Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a rb-continuous function.let V be any closed set in Y. f is rb-continuous function $f^{-1}(V)$ is rb-closed in (X, τ). Then by [14] rb-closed set is b-closed. Hence $f^{-1}(V)$ is β-closed in (X, τ). Therefore f is b- continuous.

Converse need not be true as seen from the following example

Example 3.7:

Let $X = Y = \{a, b, c\}, \tau = \big\{X, \emptyset, \{a\}, \{a, b\}\big\}, \sigma = \{X, \emptyset, \{b, c\}\}$.Let$f : (X, \tau) \rightarrow (Y, \sigma)$ be a function defined by :$f(a) = c, f(b) = b$ and $f(c) = a$ it is clear that f is b-continuous function but not rb- continuous function. Since $\{b\}$ is closed in (Y, σ),$f^{-1}(\{b\}) = \{b\}$ is rb-closed in (X, τ).

Theorem 3.8:

Every rb-continuous function is g-continuous function but not conversely.

Proof:

Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a rb-continuous function.let V be any closed set in Y. f is rb-continuous function $f^{-1}(V)$ is rb-closed in(X, τ). Then by [14] rb-closed set is g-closed. Hence $f^{-1}(V)$ is g-closed in (X, τ). Therefore f is g- continuous.

Converse need not be true as seen from the following example.

Example 3.9:

Let$X = Y = \{a, b, c\}, \tau = \big\{X, \emptyset, \{a\}, \{a, b\}, \{a, c\}\big\}, \sigma = \{X, \emptyset, \{a, c\}\}$. Let$f : (X, \tau) \rightarrow (Y, \sigma)$ be a function defined by :$f(a) = a, f(b) = b$ and $f(c) = c$ it is clear that f is g-continuous function but not rb-continuous function. Since $\{b\}$ is closed in (Y, σ),$f^{-1}(\{b\}) = \{b\}$ is rb-closed in (X, τ).

Theorem 3.10:

Every rcontinuous function is rb-continuous function.

Proof:

Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a r-continuous function. Let V be any closed set in Y. f is rb-continuous function $f^{-1}(V)$ is r-closed in (X, τ). Then by [14] r-closed set is rb-closed. Hence $f^{-1}(V)$ is rb-closed in (X, τ). Therefore f is rb- continuous.

Remark 3.11:

Converse of the above theorem need not be true as seen from the following example.

Example 3.12:

Let $X = Y = \{a, b, c\}, \tau = \{X, \emptyset, \{a\}, \{a, b\}, \{a, c\}\}, \sigma = \{X, \emptyset, \{a\}\}$. Let$f : (X, \tau) \rightarrow (Y, \sigma)$ be a function defined by: $f(a) = a, f(b) = b$ and $f(c) = c$ it is clear that f is r-continuous function but not rb- continuous function. Since $\{b, c\}$ is closed in (Y, σ),$f^{-1}(\{b, c\}) = \{b, c\}$ is rb-closed in (X, τ).

Theorem 3.13:

Every rb-continuous function is continuous function but not conversely.

Proof:

Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a rb-continuous function.let V be any closed set in Y. f is rb-continuous function $f^{-1}(V)$ is rb-closed in (X, τ). Then by [14] rb-closed set is closed. Hence $f^{-1}(V)$ is closed in (X, τ). Therefore f is continuous.

Converse need not be true as seen from the following example

Example 3.14:

Let $X = Y = \{a, b, c\}, \tau = \{X, \emptyset, \{a\}, \{a, b\}\}, \sigma = \{X, \emptyset, \{a, b\}\}$.Let$f : (X, \tau) \to (Y, \sigma)$ be a function defined by identity mapping it is clear that f is continuous function but not rb- continuous function. Since $\{c\}$ is closed in (Y, σ),$f^{-1}(\{c\}) = \{c\}$ is not rb-closed in (X, τ).

Theorem 3.15:

Every rb-continuous function is αcontinuous function.

Proof:

Let $f : (X, \tau) \to (Y, \sigma)$ be a rb-continuous function.let V be any closed set in Y. f is rb-continuous function $f^{-1}(V)$ is rb-closed in (X, τ). Then by [14] rb-closed set is α-closed. Hence $f^{-1}(V)$ is α-closed in (X, τ). Therefore f is α- continuous.

Remark 3.16:

Converse of the above theorem need not be true as seen from the following example.

Example 3.17:

Let $X = Y = \{a, b, c\}$, $\tau = \{X, \emptyset, \{a\}, \}, \sigma = \{X, \emptyset, \{a, c\}\}$.Let$f : (X, \tau) \to (Y, \sigma)$ be a function defined by $f(a) = a, f(b) = b$ and $f(c) = c$ it is clear that f is α-continuous function but not rb- continuous function. Since $\{b\}$ is closed in(Y, σ),$f^{-1}(\{b\}) = \{b\}$is rb-closed in(X, τ).

Theorem 3.18:

Every rb-continuous function is β-continuous function.

Proof:

Let $f : (X, \tau) \to (Y, \sigma)$ be a rb-continuous function.let V be any closed set in Y. f is rb-continuous function $f^{-1}(V)$ is rb-closed in (X, τ). Then by [14] rb-closed set is β-closed. Hence $f^{-1}(V)$ is β-closed in (X, τ). Therefore f is β- continuous.

Remark 3.19:

Converse of the above theorem need not be true as seen from the following example.

Example 3.20:

Let $X = Y = \{a, b, c\}, \tau = \{X, \emptyset, \{b\}\}, \sigma = \{X, \emptyset, \{c\}\}$.Let$f : (X, \tau) \to (Y, \sigma)$ be a function defined by :$f(a) = b, f(b) = a$ and $f(c) = c$ it is clear that f isβ-continuous function but not rb- continuous function. Since $\{b\}$ is closed in (Y, σ),$f^{-1}(\{b\}) = \{a\}$ is rb-closed in (X, τ).

Remark 3.21:

The above results can be represented in the following figure:

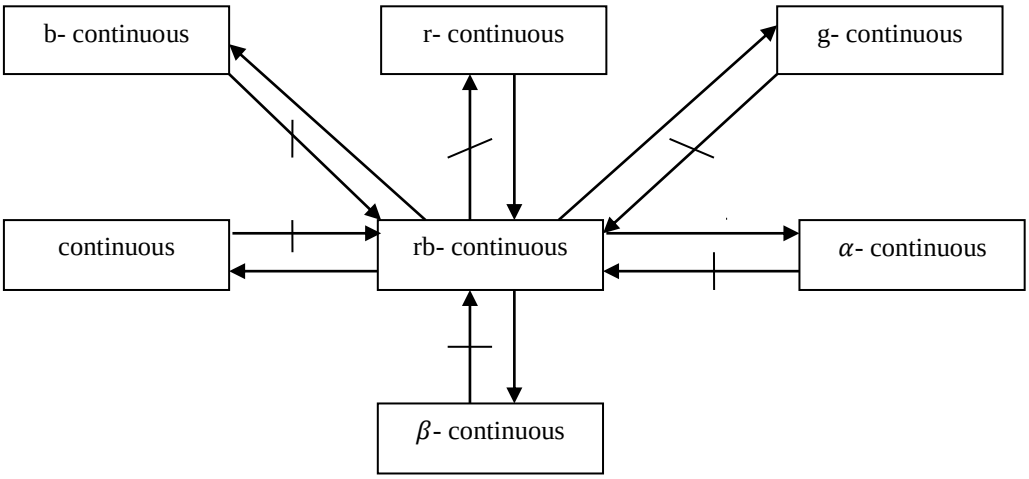


Figure 3.1 Implied relationship of rb-Continuous function

Where $A \to B$ represent A implies B

$A \nrightarrow B$ Represent A does not implies$B$

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Neutrosophic $b^*g\alpha$ - Interior and Neutrosophic $b^*g\alpha$- Closure

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Abstract: *In this paper, we have introduced Neutrosophic $b^*g\alpha$-Interior point and Neutrosophic $b^*g\alpha$ neighbourhood point in Neutrosophic topological spaces. By using this concept, we have discussed some properties based on Neutrosophic $b^*g\alpha$ -Interior and Neutrosophic $b^*g\alpha$ -Closure in Neutrosophic topological spaces and studied some of their properties*.

Keywords: *Neutrosophic $b^*g\alpha$-closed sets; Neutrosophic $b^*g\alpha$-Interior; Neutrosophic $b^*g\alpha$-Interior point; Neutrosophic $b^*g\alpha$-Closure; Neutrosophic $b^*g\alpha$ neighbourhood.*

I. Introduction

In 1965 Zadeh[1] introduced the fuzzy set values, it is based on only a membership values. In addition to the fuzzy set values non-membership values are commenly named as an intuitionistic fuzzy set and this was introduced by Atanassov[3]. Generalization of intuitionistic fuzzy set was named as a neutrosophic set, in addition of membership values and nonmembership values there will be an indeterminacy values are takes place in the neutrosophic set. All the membership, non-membership and indeterminacy values will lie only in [0, 1]. Smarandache[11] introduced the neutrosophic set, neutrosophic logic, neutrosophic probability and neutrosophic statistics in 1998. By using the concept of neutrosophic set Salama and Alblowi[6] was introduced neutrsophic topological spaces in the year 2012. The Author has also introduced neutrosophic closed set and neutrosophic continuous functions in 2014.

Neutrosophic α-closed sets in neutrosophic topological space was introduced by Arokiarani et al[7]. In 2017 Generalized neutrosophic closed sets was introduced by Dhavaseelan and SaeidJafari[8]. In 2012, Kalamani et al.[10] introduced generalized alpha closed sets in intuitionistic fuzzy topological spaces. In 2018, Vigneshwaran and Saranya[2] was introduced $b^*g\alpha$ -closed sets in topological spaces. The Author's[9] also introduced neutrosophic $b^*g\alpha$ -closed sets, neutrosophic $b^*g\alpha$ -interior; neutrosophic $b^*g\alpha$ -closure in neutrosophic topological spaces.

In this paper we have discussed some of the properties based on neutrosophic $b^*g\alpha$ -interior; neutrosophic $b^*g\alpha$-closure and $b^*g\alpha$ neighbourhood in neutrosophic topological spaces. We also give some statements based on neutrosophic $b^*g\alpha$-interior; neutrosophic $b^*g\alpha$-closure and $b^*g\alpha$ neighbourhood.

II. Preliminaries

In this section, we recollect some of the basic definitions which was already defined by various authors.

Definition: 2.1[6]

Let X be a non empty fixed set. A neutrosophic set E is an object having the form

$$E = \{< x, mv(E(x)), iv(E(x)), nmv(E(x)) > \text{for all } x \in X\},$$

where $\mathrm{mv}(\mathrm{E(x)})$ represents the degree of membership, $\mathrm{iv}(\mathrm{E(x)})$ represents the degree of indeterminacy and $\mathrm{nmv}(\mathrm{E(x)})$ represents the degree of non-membership functions of each element $x \in X$ to the set E.

Remark: 2.2[6]

A neutrosophic set $E = \{< x, mv(E(x)), iv(E(x)), nmv(E(x)) > \text{for all } x \in X\}$ can be identified to an ordered triple $< \mathrm{mv}(\mathrm{E(x)}), \mathrm{iv}(\mathrm{E(x)}), \mathrm{nmv}(\mathrm{E(x)}) >$ in $]^{-}0,1[^{+}$ on$X$.

Definition: 2.3 [6]

Let E and F be two neutrosophic sets of the form,
$E = \{< x, mv(E(x)), iv(E(x)), nmv(E(x)) > \text{for all } x \in X\}$and
$F = \{< x, mv(\mathrm{F(x)}), \mathrm{iv}(\mathrm{F(x)}), \mathrm{nmv}(\mathrm{F(x)}) > \text{for all } x \in X\}$.
Then,
i) $E \subseteq F$if and only if $mv(E(x)) \leq mv(F(x))$, $iv(E(x)) \leq iv(F(x))$and
$nmv(E(x)) \leq nmv(F(x))$for all $x \in X$,
ii) $E = F$ if and only if $E \subseteq F$ and $F \subseteq E$,
iii) $A^c = \{< x, nmv(E(x)), 1 - iv(E(x)), mv(E(x)) > \text{for all } x \in X\}$,
iv) $E \cup F = \{x, max[mv(E(x)), mv(F(x))],\ min[iv(E(x)), iv(F(x))],$
$min[nmv(E(x)), nmv(F(x))]$for all $x \in X\}$,
v) $E \cap F = \{x, min[mv(E(x)), mv(F(x))], max[iv(E(x)), iv(F(x))],$
$max[nmv(E(x)), nmv(F(x))]$for all $x \in X\}$.

Definition: 2.4 [6]

A neutrosophic topology on a non empty set X is a family τ of neutrosophic subsets in X satisfying the following axioms:

i) $0_N, 1_N \in \tau$,
ii) $G_1 \cap G_2 \in \tau$for any $G_1, G_2 \in \tau$,
iii) $\cup G_i \in \tau$for all $\{G_i : i \in J\} \subseteq \tau$.

Then the pair (X, τ) or simply X is called a neutrosophic topological space.

Definition: 2.5[7]

A neutrosophic set E in a neutrosophic topological space (X, τ) is called
1) aneutrosophic semi-closed ($Ns\text{-}closed$. .In short) set if $Nint(Ncl(E)) \subseteq E$
2) aneutrosophic pre-closed ($Np\text{-}closed$. In short) set if $Ncl(Nint(E)) \subseteq E$
3) aneutrosophic regular closed($Nr\text{-}closed$. In short) set if $E = Ncl(Nint(A))$
4) aneutrosophicα-closed($N\alpha\text{-}closed$. In short) set if $Ncl(Nint(Ncl(E))) \subseteq E$
5) aneutrosophic semi-pre($N\beta\text{-}closed$. In short) closed set if $Nint\left(Ncl(Nint(E))\right) \subseteq E$.

Then the complements of neutrosophic semi-closed, neutrosophic pre-closed, neutrosophic regular closed, neutrosophicα-closed and neutrosophic semi-pre closed sets are neutrosophic semi-open, neutrosophic pre-open, neutrosophic regular open, neutrosophic α-open and neutrosophic semi-pre open sets of neutrosophic topological space (X, τ).

Definition: 2.6[9]

Let E be a subset of a neutrosophic topological space (X, τ). Then E is called a neutrosophic b-closed ($N_b\text{-}closed$. In short) set if

$$[NCl(NInt(E))] \cup [NInt(NCl(E))] \subseteq E.$$

Definition: 2.7[9]

Let E be a subset of a neutrosophic topological space (X, τ). Then E is called a neutrosophic $g\alpha$-closed ($N_{g\alpha}\text{-}closed$. In short) set if $N\alpha cl(E) \subseteq V$ whenever $E \subseteq V$ and V is neutrosophic α-open in (X, τ).

Definition: 2.8[9]

Let E be a subset of a neutrosophic topological space (X, τ). Then E is called a neutrosophic ${}^{*}g\alpha$-closed ($N_{{}^{*}g\alpha}$-$closed$. In short) set if $Ncl(E) \subseteq V$ whenever $E \subseteq V$ and V is neutrosophic $g\alpha$-open in (X, τ).

Definition: 2.9[9]

Let E be a subset of a neutrosophic topological space (X, τ). Then E is called a neutrosophic$b^{*}g\alpha$-closed ($N_{b^{*}g\alpha}$-$closed$. In short) set if $Nbcl(E) \subseteq V$ whenever $E \subseteq V$ and V is neutrosophic ${}^{*}g\alpha$-open in (X, τ).

Definition: 2.10[9]

Let E be a subset of a neutrosophic topological space (X, τ). Then,

$N_{b^{*}g\alpha}$-$int(E)$ = $\cup \{F: F\ is\ neutrosophic\ b^{*}g\alpha$-$open\ set\ and\ F \subset E\}$.

Definition: 2.11[9]

Let E be a subset of a neutrosophic topological space (X, τ). Then,

$N_{b^{*}g\alpha}$-$cl(E)$ = $\cap \{F: F\ is\ neutrosophic\ b^{*}g\alpha$-closed $set\ and\ E \subset F\}$.

III. Neutrosophic $b^{*}g\alpha$- Interior andNeutrosophic $b^{*}g\alpha$- Closure

Definition: 3.1

Let (X, τ) be a neutrosophic topological space and let $x \in X$. A subset E of X is said to be a neutrosophic $b^{*}g\alpha$ neighbourhood of x if there exists aneutrosophic$b^{*}g\alpha$-open set F such that $x \in F \subset E$.

Definition: 3.2

Let Ebe a subset of a neutrosophic topological space (X, τ). A point $x \in E$ is said to be a neutrosophic $b^{*}g\alpha$-interior point of E if$E$ is a neutrosophic$b^{*}g\alpha$ neighbourhood of x.

Remark: 3.3

Let Ebe a subset of a neutrosophic topological space (X, τ). Then the set of all neutrosophic $b^{*}g\alpha$-interior points of E is called the neutrosophic $b^{*}g\alpha$-interiorof E.

Theorem: 3.4

Let E and F be any two subsets of a neutrosophic topological space (X, τ). Then,

i) $N_{b^{*}g\alpha}$-$int(X) = X$ and $N_{b^{*}g\alpha}$-$int(\varphi) = \varphi$
ii) If F is any $N_{b^{*}g\alpha}$-open set contained in E, then $F \subset N_{b^{*}g\alpha}$-$int(E)$
iii) If $E \subset F$, then $N_{b^{*}g\alpha}$-$int(E) \subset N_{b^{*}g\alpha}$-$int(F)$
iv) $N_{b^{*}g\alpha}$-$int(N_{b^{*}g\alpha}$-$int(E)) = N_{b^{*}g\alpha}$-$int(E)$.

Proof:

i) Since X and φ are $N_{b^{*}g\alpha}$ -open sets, then by Definition 2.8 $N_{b^{*}g\alpha}$ - $int(X)$ = $\cup \{H: H\ is\ neutrosophic\ b^{*}g\alpha$-$open\ set\ and\ H \subset X\} = X \cup \{E$ is a$N_{b^{*}g\alpha}$-open sets} = X. That is, $N_{b^{*}g\alpha}$-$int(X)$ = X. Since φ is the only $neutrosophic\ b^{*}g\alpha$-$open\ set$ contained in φ, which implies $N_{b^{*}g\alpha}$-$int(\varphi) = \varphi$.

ii) Let F be any neutrosophic $b^{*}g\alpha$ -open set, such that $F \subset E$. Let $x \in F$. Since F is a neutrosophic$b^{*}g\alpha$-open set contained in E. $x$is a neutrosophic$b^{*}g\alpha$-interior point of E. That is $x \in N_{b^{*}g\alpha}$-$int(E)$. Hence F$\in N_{b^{*}g\alpha}$-$int(E)$.

iii). Let E and F be any two subsets of (X,τ) such that $E \subset F$. Let $x \in N_{b^*g\alpha}\text{-}int(E)$. Then x is a neutrosophic$b^*g\alpha$-interior point of E and so E is a neutrosophic$b^*g\alpha$-neighbourhood of x. Since $F \supset E$, F is also a neutrosophic$b^*g\alpha$-nbhd of x. This implies that $x \in Nb^*g\alpha\text{-}int(F)$. Thus we have shown that $x \in Nb^*g\alpha\text{-}int(E) \Rightarrow x \in Nb^*g\alpha\text{-}int(F)$. Hence $Nb^*g\alpha\text{-}int(E) \subset Nb^*g\alpha\text{-}int(F)$.

iv) Let E be any subset of (X,τ). By definition of $Nb^*g\alpha$-interior, $Nb^*g\alpha\text{-}int(E)$ is $Nb^*g\alpha$- open and hence $N_{b^*g\alpha}\text{-}int(N_{b^*g\alpha}\text{-}int(E)) = N_{b^*g\alpha}\text{-}int(E)$.

Theorem: 3.5

If a subset E of a topological space (X,τ) isneutrosophic$b^*g\alpha$-open, then $Nb^*g\alpha\text{-}int(E) = E$.

Proof.

Let Ebe a neutrosophic$b^*g\alpha$-open subset of (X,τ). We know that $Nb^*g\alpha\text{-}int(E) \subset E$. Also, $A$is neutrosophic$b^*g\alpha$-open set contained in E. Then by Theorem: 3.4 (ii), we have $E \subset Nb^*g\alpha\text{-}int(E)$. Hence $Nb^*g\alpha\text{-}int(E) = E$.

Theorem: 3.6

Let E and F be any two subsets of a neutrosophic topological space (X,τ). Then,

i) $N_{b^*g\alpha}\text{-}cl(X) = X$ and $N_{b^*g\alpha}$-cl$(\varphi) = \varphi$
ii) If F is any $N_{b^*g\alpha}$-closed set containing E, then $N_{b^*g\alpha}\text{-}cl(E) \subset F$
iii) If $E \subset F$, then $N_{b^*g\alpha}\text{-}cl(E) \subset N_{b^*g\alpha}$-cl$(F)$
iv) $N_{b^*g\alpha}\text{-}cl(N_{b^*g\alpha}$-cl$(E)) = N_{b^*g\alpha}$-cl$(E)$.

Proof:

i) By the definition of $N_{b^*g\alpha}$-closure, x is the only $N_{b^*g\alpha}$-closed set containing X. Therefore $N_{b^*g\alpha}\text{-}cl(X)$= Intersection of all the $N_{b^*g\alpha}$-closed sets containing $X = \cap\{X\}=X$. That is $N_{b^*g\alpha}\text{-}cl(X)= X$. By the definition of $N_{b^*g\alpha}$-closure, $N_{b^*g\alpha}\text{-}cl(\varphi)$ = Intersection of all the $N_{b^*g\alpha}$-closed sets containing $\varphi = \varphi \cap \{N_{b^*g\alpha}$-closed sets containing$\varphi\} = \varphi$. That is $N_{b^*g\alpha}\text{-}cl(\varphi)= \varphi$.

ii) Let F be any neutrosophic$b^*g\alpha$-closed set containing E. Since $N_{b^*g\alpha}\text{-}cl(E)$ is the intersection of all $N_{b^*g\alpha}$-closed sets containing E, $N_{b^*g\alpha}cl(E)$ is contained in every $N_{b^*g\alpha}$-closed set containing E. Hence in particular $N_{b^*g\alpha}\text{-}cl(E) \subset F$.

iii). Let E and F be any two subsets of (X,τ).Since $E \subset F$. By the definition, $N_{b^*g\alpha}\text{-}cl(F) = \cap\{H: F \subset H \in N_{b^*g\alpha}\text{-}C(X)\}$. If $F \subset H \in N_{b^*g\alpha}\text{-}C(X)$, then $N_{b^*g\alpha}\text{-}cl(F) \subset H$.Since $E \subset F$, $E \subset F \subset H \in N_{b^*g\alpha}\text{-}C(X)$, we have $N_{b^*g\alpha}\text{-}cl(E) \subset H$. Therefore $N_{b^*g\alpha}\text{-}cl(E) \subset \cap\{H: F \subset H \in N_{b^*g\alpha}\text{-}C(X)\} = N_{b^*g\alpha}\text{-}cl(F)$. Hence $N_{b^*g\alpha}\text{-}cl(E) \subset N_{b^*g\alpha}$-cl$(F)$.

iv). Let F be a $N_{b^*g\alpha}$-closed set containing E. Then by the definition, $N_{b^*g\alpha}\text{-}cl(E) \subset F$. Since F is $N_{b^*g\alpha}$-closed set and contains $N_{b^*g\alpha}\text{-}cl(E)$ which contained in every $N_{b^*g\alpha}$-closed set containing E. It follows that $N_{b^*g\alpha}\text{-}cl(N_{b^*g\alpha}$-cl$(E)) \subset N_{b^*g\alpha}$-cl$(E)$.. Therefore $N_{b^*g\alpha}\text{-}cl(N_{b^*g\alpha}$-cl$(E)) = N_{b^*g\alpha}$-cl$(E)$.

Theorem: 3.7

If a subset E of a neutrosophic topological space (X,τ) isneutrosophic$b^*g\alpha$-closed, then it is a $N_{b^*g\alpha}\text{-}cl(E)$.

Proof.

Let Ebe neutrosophic$b*g\alpha$-closed subset of (X,τ). We know that $E \subset Nb^{*}g\alpha\text{-}cl(E)$. Also, $E \subset E$ and $E$is neutrosophic$b^{*}g\alpha$-closed. Then by Theorem: 3.6 (ii), we have $Nb^{*}g\alpha\text{-}cl(E) \subset E$. Hence $Nb^{*}g\alpha\text{-}cl(E) = E$.

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On Supra N* Topological Space

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***Abstract**: The aim of this paper is to introduce the concept of supra N* topological spaces and discuss about supra separation axioms in supra N* topological spaces.*

***Keywords**: supra topological space, supra N* topological space, supra $N\tau^*$ open set , supra $N\tau^*$-closure , supra $N\tau^*$-T_0space, supra $N\tau^*$-T_1space and supra $N\tau^*$-T_2space.*

I. Introduction

The concept of N topological space was introduced by M. Lellis Thivagar, V. Ramesh and M. ArockiaDasan[5]. A non-empty set X equipped with N-arbitrary topologies $\tau_1, \tau_2, \dots, \tau_N$ is called N topological spaces. The supra topological spaces have been introduced by A.S. Mashhour[7] in 1983. In topological space the arbitrary union condition is enough to have a supra topological space. Here every topological space is a supra topological space but the converse is not always true. In this paper we introduce and study the concept of supra N*topological spaces and investigate some new separation axioms called supra $N\tau^*$-T_0space, supra $N\tau^*$-T_1space and supra $N\tau^*$-T_2space. Also we study some of their basic properties in supra N*topological spaces.

II. Preliminaries

Definition 2.1: (X,τ) is said to be a Supra topological space if it satisfies the following conditions:

a) $X, \emptyset \in \tau$
b) The union of any number of sets in τ belongs to τ.

Definition 2.2: Each element $A \in \tau$ is called a Supra open set in (X, τ) and its complement is called a Supra closed set in (X, τ).

Definition 2.3: If (X, τ) is a supra topological space, $A \subseteq X$, $A \neq \emptyset$, τ_A is the class of all intersection of A with each element in τ, then (A, τ_A) is called a Supra topological subspace of (X, τ).

Definition 2.4: The Supra closure of the set A is denoted by Supra cl(A) and is defined as Supra-$cl(A) = \cap \{ B : B \text{ is Supra closed and } A \subseteq B\}$.

Definition 2.5: The Supra interior of the set A is denoted by Supra int(A) and is defined as Supra-$\text{int}(A) = \cup \{B: B \text{ is Supra closed and } B \subseteq A\}$.

Definition 2.6: A Supra open function means the image of any supra open set in (X,τ) is a supra open set in (X^*,τ^*) where these are two supra topological spaces.

III. Supra N* topological spaces

Definition 3.1: If $\tau_1^*, \tau_2^*, \dots, \tau_N^*$ or $N\tau^*$ are N-arbitrary Supra N* topologies on a non-empty set X, then $(X, N\tau^*)$ is said to be a Supra N* topological space.

Definition 3.2: Each element of $N\tau^*$ is called a Supra $N\tau^*$ open set in $(X, N\tau^*)$ and its complement is called a Supra $N\tau^*$ closed set in $(X, N\tau^*)$.

Definition 3.3: If (X, Nτ*) is a supra N* topological space, Y ⊆ X, Y ≠ ∅, then (Y,Nτ*#) is called a Supra N* topological subspace of (X, Nτ*) if Nτ*# = {U ∩ Y: U is a Supra Nτ* open in X}

Definition 3.4: The Supra Nτ*-closure of the set A is denoted by Supra Nτ*-cl(A) and is defined as Nτ*-$cl(A) = \cap$ { B : B is a Supra Nτ* − closed and A ⊆ B}.

Definition 3.5: The Supra Nτ*-interior of the set A is denoted by Supra Nτ*-int(A) and is defined as Supra Nτ*-int(A) = ∪ {B: B is a Supra Nτ*-closed and B ⊆ A}.

Definition 3.6: Let (X, Nτ*) and (Y, Nτ*) be N* topological spaces. A function f: X→Y is a S-N* Continuous function if the inverse image of each Nτ*-open set in Y is Supra Nτ*-open in X.

Definition 3.7: Let $(X, N\tau^*)$ and $(Y, N\tau^*)$ be N* topological spaces and Nτ** be associated supra N* topologies with Nτ*. Let f: X→Y be a function. Then f is S*-N*continuous if the inverse image of each supra Nτ*-open set is Supra Nτ*-open.

From the above definition, one can draw the following diagram,

N*-CONTINUITY⇒S-N* CONTINUITY⇒S*-N* CONTINUITY.

IV. Separation Axioms in Supra N* topological space

4.1. Supra $N\tau^*$-T_0 space:

Definition 4.1.1: A supra N* topological space $(X, N\tau^*)$ is said to be supra Nτ*-T_0 space if and only if for each pair of distinct points x,y∈ X, there exist a Nτ*-open set U such that either x∈U and y∉U or x∉U and y∈U.

Example 4.1.2: Let $X = \{a, b, c\}$. For N* =2, $\tau_1^* = \{X, \emptyset, \{a\}, \{a, c\}\}, \tau_2^* = \{X, \emptyset, \{b, c\}\}$. Supra-2τ* $O(X) = \{X, \emptyset, \{a\}, \{c\}, \{b, c\}, \{a, c\}\}$. Let $a,b \in X$. Then there exist a supra-2τ* open set $U = \{a, c\}$ such that $a \in U$ and b∉U. Let b,c∈ X. Then there exist a supra-2τ* open set $U = \{a, c\}$ such that $c \in U$ and $b \notin U$. Let $a, c \in X$. Then there exist a supra-2τ* open set $U = \{b, c\}$ such that $c \in U$ and $a \notin U$. For N* =3, $\tau_1^* = \{X, \emptyset, \{a, c\}\}, \tau_2^* = \{X, \emptyset\}, \tau_3^* = \{X, \emptyset, \{b, c\}\}$.Supra-3τ* $O(X) = \{X, \emptyset, \{c\}, \{b, c\}, \{a, c\}\}$. Let $a,b \in X$. Then there exist a supra-3τ* open set $V = \{a, c\}$ such that $a \in V$ and b∉V. Let b,c∈ X. Then there exist a supra-3τ* open set $V = \{a, c\}$ such that $c \in V$ and $b \notin V$. Let $a, c \in X$. Then there exist a supra-3τ* open set $V = \{b, c\}$ such that $c \in V$ and $a \notin V$. For N* = 4, $\tau_1^* = \{X, \emptyset, \{a\}\}, \tau_2^* = \{X, \emptyset, \{a\}, \{a, b\}\}, \tau_3^* = \{X, \emptyset, \{a, b\}\}, \tau_3^*$ = Indiscrete topology. Supra-4τ* $O(X) = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$. Let $a,b \in X$. Then there exist a supra-4τ* open set $W = \{a\}$ such that $a \in W$ and b∉W. Let b,c∈ X. Then there exist a supra-4τ* open set $W = \{a, b\}$ such that $b \in W$ and $c \notin W$. Let $a, c \in X$. Then there exist a supra-4τ* open set $W = \{a, b\}$ such that $a \in W$ and $c \notin W$. And so on. Therefore (X,Nτ*) is a Supra Nτ*-T_0 space.

Theorem 4.1.3: If (X,Nτ*) is a supra Nτ*-T_0space and (Y,Nτ*#) is a supra N* topological subspace of (X,Nτ*) then (Y,Nτ*#) is also a supra Nτ*-T_0space.

Proof: Suppose $x, y \in Y$, $x \neq y$, since Y⊆X, then $x, y \in X$. Since (X,Nτ*) is a supra Nτ*-T_0space, there exist a supra Nτ* open set U⊆ X such that $x \in U$ and $y \notin U$. Then $U \cap Y$ is supra Nτ*# open set in Y such that $x \in U \cap Y, y \notin U$. Hence (Y,Nτ*#) is also a supra Nτ*-T_0space.

Theorem 4.1.4: If $(X, N\tau^*), (X^\#, N\tau^\#)$ are two supra N* topological spaces, $(X, N\tau^*)$ is a supra $N\tau^*$-T_0 space and f is a supra open function and bijective then $(X^\#, N\tau^\#)$ is a supra $N\tau^*$-T_0 space.

Proof: Suppose that $(X, N\tau^*)$ is a supra $N\tau^*$-T_0space. *Let* $x^\#, y^\# \in X^\#$, $x^\# \neq y^\#$, since f is a bijective function then there exist $x, y \in X$ such that $x^\# = f(x)$, $y^\# = f(y)$ and $x \neq y$. Since $(X, N\tau^*)$ is a supra $N\tau^*$-T_0space, then there exists $U \subseteq X$ is a supra $N\tau^*$-open set such that $x \in U$ and $y \notin U$. We obtain that $f(U) \subseteq X^\#$ is a supra $N\tau^*$-open set in $X^\#$ because f is a supra open function. Also $f(x) \in f(U)$, $f(y) \notin f(U)$. Hence $(X^\#, N\tau^\#)$ is a supra $N\tau^*$-T_0 space.

Theorem 4.1.5: If $\{x\}$ is a supra $N\tau^*$-open set for some $x \in X$, then $x \notin$ supra $N\tau^*$-$cl(\{y\})$, for all $y \neq x$.

Proof: Let $\{x\}$ be a supra $N\tau^*$open set for some $x \in X$, then $X - \{x\}$ is a supra $N\tau^*$-closed set and $x \notin X - \{x\}$. If $x \in$ supra $N\tau^*$-$cl(\{y\})$, for some $y \neq x$, then y, x both are in all the supra $N\tau^*$-closed sets containing y. Therefore $x \in X - \{x\}$ which is a contradiction. Hence $x \notin$ supra $N\tau^*$-$cl(\{y\})$, for all $y \neq x$.

Theorem 4.1.6: In any supra N^* topological space X, any distinct points have distinct supra $N\tau^*$closures.
Proof: Let $x, y \in X$ with $x \neq y$ and let $A = \{x\}^c$ and hence supra $N\tau^*$-$cl(A) = A$ *or* X. Now if $N\tau^*$-$cl(A) = A$, then A is supra $N\tau^*$-closed so $X - A = \{x\}$ is supra $N\tau^*$-open and not containing y. So by the above theorem $x \notin$ supra $N\tau^*$-$cl(\{y\})$ and $y \in$ supra $N\tau^*$-$cl(\{y\})$ which implies that supra $N\tau^*$-$cl(\{y\})$ and supra $N\tau^*$-$cl(\{x\})$ are distinct. If $N\tau^*$-$cl(A) = X$, then A is supra $N\tau^*$-open, hence $\{x\}$ is supra $N\tau^*$-closed, that is supra $N\tau^*$-$cl(\{x\}) = \{x\}$ which is not equal to supra $N\tau^*$-$cl(\{y\})$.

Theorem 4.1.7: In any supra N^* topological space X, if distinct points have supra $N\tau^*$ T_0-closures, then X is supra $N\tau^*$- T_0space.
Proof: Let $x, y \in X$ with $x \neq y$, with supra $N\tau^*$-$cl(\{y\}) \neq$ supra $N\tau^*$-$cl(\{x\})$, hence there exists $v \in X$ such that $v \in$ supra $N\tau^*$-$cl(\{x\})$, but $v \notin$ supra $N\tau^*$-$cl(\{y\})$ or $v \in$ supra $N\tau^*$-$cl(\{y\})$,but $v \notin$ supra $N\tau^*$-$cl(\{x\})$. Now let us assume without loss of generality $v \in$ supra $N\tau^*$-$cl(\{x\})$. Then $v \in$ supra $N\tau^*$-$cl(\{y\})$. If $x \in$ supra $N\tau^*$-$cl(\{y\})$, then supra $N\tau^*$-$cl(\{x\})$ is contained in supra $N\tau^*$-$cl(\{y\})$ and hence $v \notin$ supra $N\tau^*$-$cl(\{y\})$ which is a contradiction, that is $x \notin$ supra $N\tau^*$-$cl(\{y\})$. Hence $x \in$ supra $N\tau^*$-$cl(\{y\}^c)$ and therefore X is supra $N\tau^*$-T_0space.

Remark 4.1.8: Every $N\tau^*$ T_0-space is a supra $N\tau^*T_0$-space, but the converse is not true.

Example 4.1.9: From Example 4.1.2, we can see that $(X, N\tau^*)$ is a supra $N\tau^*$-T_0 space but not a $N\tau^*$-T_0 space.

4.2. Supra $N\tau^*$-T_1space:

Definition 4.2.1: A supra N^* topological space X is said to be supra $N\tau^*$-T_1 space if and only if for a given pair of distinct points x and y of X, then there exists two supra $N\tau^*$-open sets U, V such that $x \in U, y \notin U$ and $y \in V, x \notin V$.

Example 4.2.2: Let $X = \{a, b, c\}$. For $N^* = 3$, $\tau_1^* = \{X, \emptyset, \{a\}\}$, $\tau_2^* = \{X, \emptyset, \{a, b\}, \{b, c\}\}$, $\tau_3^* = \{X, \emptyset, \{a, c\}\}$. Then supra-3$\tau^*$ O(X)= $\{X, \emptyset, \{a\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}\}$. Let $a, b \in X$. Then there exist a supra 3τ^*-open set $U = \{a, c\}$ and $V = \{b, c\}$ such that $a \in U, b \notin U$ and $b \in V, a \notin V$. Let b,c $\in X$. Then there exist a supra 3τ^*-open set $U = \{a, b\}$ and $V = \{a, c\}$ such that $b \in U, c \notin U$ and $c \in V, b \notin V$. Let $a, c \in X$. Then there exist a supra 3τ^*-open set $U = \{a, b\}$ and $V = \{b, c\}$ such that $a \in U, c \notin U$ and $c \in V, a \notin V$. Therefore $(X, 3\tau^*)$ is a supra 3 τ^*T_1 -space. For $N^* = 4$, $\tau_1^* = \{X, \emptyset, \{a\}, \{a, c\}\}$, $\tau_2^* = \{X, \emptyset, \{a\}, \{a, b\}\}$, $\tau_3^* = \{X, \emptyset, \{a, b\}, \{b, c\}\}$, $\tau_4^* =$ Indiscrete Topology. Then supra-4τ^* O(X)= $\{X, \emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}\}$. Let $a, b \in X$. Then there exist a supra 4τ^*-open set $U = \{b, c\}$ and $V = \{a, c\}$ such that $b \in U, a \notin U$ and $a \in V, b \notin V$. Let b,c $\in X$. Then there exist a supra 4 τ^*-open set $U = \{a, b\}$ and $V = \{a, c\}$ such that $b \in U, c \notin U$ and $c \in V, b \notin V$. Let $a, c \in X$. Then there exist a supra 4τ^*-open set $U = \{a, b\}$ and $V = \{b, c\}$ such that $a \in U, c \notin U$ and $c \in V$, $a \notin V$. Therefore $(X,4\tau^*)$ is a supra 4τ^*T_1-space and so on.

Theorem 4.2.3: If $(X, N\tau^*)$ is a supra $N\tau^*$-T_1space and $(Y, N\tau^{*\#})$ is a supra N* topological subspace of (X,Nτ*) then $(Y, N\tau^{*\#})$ is also a supra Nτ*- T_1space.
Proof: Suppose $x, y \in Y$, $x \neq y$,since Y$\subseteq$X, then $x, y \in X$. Since (X,Nτ*) is a supra Nτ*-T_1space, there exists two supra Nτ*-open sets U and V such that $x \in U, y \notin U$ and $y \in V$, $x \notin V$. Then $U \cap Y, V \cap Y$ are supra N$\tau^{*\#}$ open set in Y respectively such that $x \in U \cap Y, y \notin U \cap Y$ and $y \in V \cap Y, x \notin V \cap Y$. Hence (Y,N$\tau^{*\#}$) is also a supra N$\tau$*-$T_1$space.

Theorem 4.2.4: If $(X, N\tau^*), (X^\#, N\tau^\#)$ are two supra N* topological spaces, $(X, N\tau^*)$ is a supra $N\tau^*$-T_1space and f is a supra open function and bijective then $(X^\#, N\tau^\#)$ is a supra $N\tau^*$-T_1space.
Proof: Suppose that $(X, N\tau^*)$ is a supra $N\tau^*$- T_1space. *Let* $x^\#, y^\# \in X^\#$, $x^\# \neq y^\#$, since f is a bijective function then there exist $x, y \in X$ such that $f(x) = x^\#$, $f(y) = y^\#$and $x \neq y$. Since $(X, N\tau^*)$ is a supra $N\tau^*$-T_1space, then there exists two supra Nτ*- open sets U and V such that $x \in U$, $y \notin U$ and $y \in V$, $x \notin V$. Clearly we obtain that $f(U) \subseteq X^\#$ and $f(V) \subseteq X^\#$, since f is a supra open function, $f(U)$ and $f(V)$ is a supra $N\tau^\#$-open set in $X^\#$. Also $f(x) \in U, f(y) \notin U$ and $f(y) \in V, f(x) \notin V$. Hence $(X^\#, N\tau^\#)$ is a supra $N\tau^*$-T_1space.

Remark 4.2.5: Every supra $N\tau^*$-T_1space is a supra $N\tau^*$-T_0space, but the converse is not true as shown in the following example.

Example 4.2.6: Let $X = \{a, b, c\}$. For $N^* = 3$, $\tau_1^* = \{X, \emptyset, \{a, c\}\}$, $\tau_2^* = \{X, \emptyset\}$, $\tau_3^* = \{X, \emptyset, \{b, c\}\}$. Then supra -3$\tau^*$O(X) = {X, $\emptyset$,{c},{b,c},{a,c}}. Let $a, b \in X$. Then there exists a supra-3τ^*open set $V = \{a, c\}$ such that

$a \in V, b \notin V$. Let $b, c \in X$. Then there exists a supra -3 τ^* open set $V = \{a, c\}$ such that $b \notin V, c \in V$. Let $a, c \in X$. Then there exists a supra -3τ^*open set $V = \{b, c\}$ such that $a \notin V, c \in V$. Hence $(X, 3\tau^*)$ is a supra $N\tau^*$-T_0 space but $(X, 3\tau^*)$ is not a supra $3\tau^*$-T_1 space. For $N^* = 4$, $\tau_1^* = \{X, \emptyset, \{a\}\}$, $\tau_2^* = \{X, \emptyset, \{a\}, \{a, b\}\}$, $\tau_3^* = \{X, \emptyset, \{a, b\}\}$ and τ_4^* = Indiscrete Topology . Then supra-4 τ^*O(X) = {X,∅,{a},{b},{a,b}}. Let $a, b \in X$. Then there exists a supra-4 τ^* open set $W = \{a\}$ such that $a \in W, b \notin W$. Let $b, c \in X$. Then there exists a supra-4τ^*open set $W = \{a, b\}$ such that $b \in W, c \notin W$. Let $a, c \in X$. Then there exists a supra-4τ^* open set $W = \{a\}$ such that $a \in W, c \notin W$. Hence $(X, 4\tau^*)$ is a supra $4\tau^*$-T_0space but $(X, 4\tau^*)$ is not a supra $4\tau^*$-T_1space and so on. Therefore $(X, N\tau^*)$ is a supra $N\tau^*$-T_0space but need not be a supra $N\tau^*$-T_1space.

4.3. Supra $N\tau^*$-T_2space:

Definition 4.3.1: A supra N* topological space X is said to be supra $N\tau^*$-T_2space (or supra $N\tau^*$ Hausdorff space) if for each pair of distinct points $x, y \in X, x \neq y$, there are two supra $N\tau^*$-open sets U and V such that $x \in U$ and $y \in V, U \cap V = \emptyset$.

Example 4.3.2:
Let $X = \{a, b, c\}$. For $N^* = 3$, Let $\tau_1^* = \{X, \emptyset, \{b\}, \{a, c\}\}$, $\tau_2^* = \{X, \emptyset, \{a\}\}$, $\tau_3^* = \{X, \emptyset, \{b, c\}\}$. Then supra-$3\tau^*O(X) = \{X, \emptyset, \{a\}, \{b\}, \{c\}, \{a, c\}, \{b, c\}\}$. Let $a, b \in X$. Then there exists a supra $3\tau^*$-open set $U = \{a\}$ and $V = \{b, c\}$ such that $a \in U$ and $b \in V, U \cap V = \emptyset$. Let $b, c \in X$. Then there exists a supra $3\tau^*$-open set $U = \{b\}$ and $V = \{c\}$ such that $b \in U$ and $c \in V, U \cap V = \emptyset$. Let $a, c \in X$. Then there exists a supra $3\tau^*$-open set $U = \{a\}$ and $V = \{b, c\}$ such that $a \in U$ and $c \in V, U \cap V = \emptyset$. Therefore $(X, 3\tau^*)$ is a supra $3\tau^*$-T_2space. For $N^* = 4$, Let $\tau_1^* = \{X, \emptyset, \{a\}, \{b, c\}\}$, $\tau_2^* = \{X, \emptyset, \{b\}\}$, $\tau_3^* = \{X, \emptyset, \{c\}, \{a, c\}\}$ *and* τ_4^* = Indiscrete Topology . Then supra-$4\tau^*O(X) = \{X, \emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}\}$. Let $a, b \in X$. Then there exists a supra $4\tau^*$-open set $U = \{a\}$ and $V = \{b, c\}$ such that $a \in U$ and $b \in V, U \cap V = \emptyset$. Let $b, c \in X$. Then there exists a supra $4\tau^*$-open set $U = \{a, b\}$ and $V = \{c\}$ such that $b \in U$ and $c \in V, U \cap V = \emptyset$. Let $a, c \in X$. Then there exists a supra $4\tau^*$-open set $U = \{a, b\}$ and $V = \{c\}$ such that $a \in U$ and $c \in V, U \cap V = \emptyset$. Therefore $(X, 4\tau^*)$ is a supra $4\tau^*$-T_2 space and so on.

Theorem 4.3.3: If $(X, N\tau^*)$ is a supra $N\tau^*$-T_2 space and $(Y, N\tau^{*\#})$ is a supra N* topological subspace of (X,Nτ*), then $(Y, N\tau^{*\#})$ is also a supra Nτ*-T_2 space.
Proof: Suppose $x, y \in Y$, $x \neq y$, since Y⊆X, then $x, y \in X$. Since (X,Nτ*) is a supra Nτ* T_2-space, there exists two supra Nτ*- open sets U and V such that $x \in U$, $y \in V$, $U \cap Y = \emptyset$.Then $U \cap Y$, $V \cap Y$ are supraNτ*$^{\#}$ open set in Y respectively such that $x \in U \cap Y, y \notin U \cap Y$ and $y \in V \cap Y, x \notin V \cap Y$, $U \cap V = (U \cap Y) \cap (V \cap Y) = (U \cap Y) \cap Y = \emptyset \cap Y = \emptyset$. Hence (Y,Nτ*$^{\#}$) is also a supra Nτ*-$T_2$ space.

Theorem 4.3.4: If $(X, N\tau^*), (X^{\#}, N\tau^{\#})$ are two supra N* topological spaces, $(X, N\tau^*)$ is a supra $N\tau^*$-T_2 space and f is a supra open function and bijective then $(X^{\#}, N\tau^{\#})$ is a supra $N\tau^*$-T_2 space.
Proof: Suppose that $(X, N\tau^*)$ is a supra $N\tau^*$-T_2space. *Let* $x^{\#}, y^{\#} \in X^{\#}$, $x^{\#} \neq y^{\#}$, since f is a bijective function then there exist $x, y \in X$ such that $f(x) = x^{\#}$, $f(y) = y^{\#}$and $x \neq y$. Since $(X, N\tau^*)$is a supra $N\tau^*$-T_2space, then there exists two supra Nτ*- open sets U and V such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$. Clearly we obtain that $f(U) \subseteq X^{\#}$ and $f(V) \subseteq X^{\#}$, since f is a supra open function, $f(U)$ and $f(V)$ are supra $N\tau^{\#}$-open sets in $X^{\#}$. Also $f(x) \in f(U)$ and $f(y) \in f(V), f(U) \cap f(Y) = \emptyset$. Hence $(X^{\#}, N\tau^{\#})$ is a supra $N\tau^*$-T_2space.

Remark 4.3.5: Every supra $N\tau^*$-T_2 space is a supra $N\tau^*$-T_1 space, but the converse is not true.

Example 4.3.6: From Example 4.2.2, we get $(X, N\tau^*)$ is a supra $N\tau^*$-T_1 space but not a supra $N\tau^*$-T_2 space.

Result 4.3.7:
From all the above remarks we draw the following diagram:

$$\begin{array}{ccccc} N\tau^*\text{-}T_2 space & \Rightarrow & N\tau^*\text{-}T_1 space & \Rightarrow & N\tau^*\text{-}T_0 space \\ \Downarrow & & \Downarrow & & \Downarrow \\ S\,N\tau^*\text{-}T_2 space & \Rightarrow & S\,N\tau^*\text{-}T_1 space & \Rightarrow & S\,N\tau^*\text{-}T_0 space \end{array}$$

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On intuitionistic supra generalized α - homeomorphism

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Abstract: *The aim of this paper is to define intuitionistic supra gα - closed maps, intuitionistic supra gα - open maps, intuitionistic supra gα - homeomorphism and investigate their relationship with other function.*

Keywords*: intuitionistic supra gα - closed maps, intuitionistic supra gα - open maps, intuitionistic supra gα- homeomorphism.*

1. Introduction

In 1970, Levine[4] introduced the concept of generalized closed sets in topological spaces. Njaˆstad.O [7] and Maki.H et al [5] introduced α - closed sets and gα - closed sets in topological spaces. The concept of intuitionistic set and intuitionistic topological space was introduced by Coker [2][3]. Supra topology was introduced by Mashhour.A.S et al [6]. The concept of intuitionistic supra closed maps, intuitionistic supra open maps and intuitionistic supra homeomorphism was introduced by L.Vidyarani and M.Vigneshwaran [8][9].

II. Preliminaries

Definition 2.1. **[10]** An intuitionistic supra topology on a non-empty set X is a family μ of IS's in X satisfying the following axioms:

(1) $\underset{\sim}{X}, \underset{\sim}{\phi} \in \mu$.

(2) $\cup A_i \in \mu$ for any arbitrary family $\{A_i : i \in J\} \subseteq \mu$.

The pair (X, μ) is called Intuitionistic supra topological space (ISTS in short) and IS in μ is known as an intuitionistic supra open set (ISOS in short) in X, the complement of ISOS is called intuitionistic supra closed set (ISCS in short).

Definition 2.2. [10] Let (X, μ) be an ISTS and A = <X, A_1, A_2> be an IS in X, then the intuitionistic supra closure and intuitionistic supra interior of A are defined by:

$Icl^{\mu}(A) = \cap\{K : K \text{ is an ISCS in X and } A \subseteq K\}$ and $Iint^{\mu}(A) = \cup\{K : K \text{ is an ISOS in X and } A \supseteq K\}$.

Definition 2.3. Let (X, μ) be an intuitionistic supra topological space. An intuitionistic set A is called

(1) Intuitionistic supra α - closed set [10] (ISαCS in short) if $Icl^{\mu}(Iint^{\mu}(Icl^{\mu}(A))) \subseteq A$.

(2) Intuitionistic supra regular - closed set [10] (ISRCS in short) if $A = Icl^{\mu}(Iint^{\mu}(A))$.

(3) Intuitionistic supra g - closed set [1] (ISgCS in short) if $Icl^{\mu}(A) \subseteq U$, whenever $A \subseteq U$, U is intuitionistic supra open in (X, μ) .

(4) Intuitionistic supra gα - closed set [1] (ISgαCS in short) if $I\alpha cl^{\mu}(A) \subseteq U$, whenever $A \subseteq U$, U is intuitionistic supra α - open in (X, μ) .

The compliment of intuitionistic supra α - closed set is intuitionistic supra α - open set(ISαOS in short).
The compliment of intuitionistic supra regular - closed set is intuitionistic supra regular - open set(ISROS in short).
The compliment of intuitionistic supra g - closed set is intuitionistic supra g - open set(ISgOS in short).
The compliment of intuitionistic supra gα - closed set is intuitionistic supra gα - open set(ISgαOS in short).

Definition 2.4. [10] Let (X, μ) and (Y, σ) be two intuitionistic supra topological space. A function f:(X, τ) → (Y, σ) is called

(1) intuitionistic supra regular closed maps if image of each ISCS in X is ISCS in Y.
(2) intuitionistic supra α - closed maps if image of each ISCS in X is ISαCS in Y.
(3) intuitionistic supra regular closed maps if image of each ISOS in X is ISOS in Y.
(4) intuitionistic supra α - closed maps if image of each ISOS in X is ISαOS in Y.

Definition 2.5. [11] Let (X, μ) and (Y, σ) be two intuitionistic supra topological space. A bijective mapping f:(X, μ) → (Y, σ) is called an intuitionistic supra homeomorphism if f and f^{-1} are intuitionistic supra continuous mapping.

III. Intuitionistic supra generalized α - closed maps

Definition 3.1. Let (X, μ) and (Y, σ) be two intuitionistic supra topological space. A function f:(X, μ) → (Y, σ) is called intuitionistic supra regular closed maps if image of each ISCS in X is ISRCS in Y.

Definition 3.2. Let (X, μ) and (Y, σ) be two intuitionistic supra topological space. A function f:(X, μ) → (Y, σ) is called intuitionistic supra generalized closed maps if image of each ISCS in X is ISgCS in Y.

Definition 3.3. Let (X, μ) and (Y, σ) be two intuitionistic supra topological space. A function f:(X, μ) → (Y, σ) is called intuitionistic supra generalized closed maps if image of each ISCS in X is ISgαCS in Y.

Theorem 3.4. Every intuitionistic supra closed map is intuitionistic supra generalized α - closed map.
Proof: Let f:(X, μ) → (Y, σ) be an intuitionistic supra closed map. Let V be ISCS in X. Since f is intuitionistic supra closed map, then f(V) is ISCS in Y. We know that every ISCS is ISgαCS. Implies f(V) is ISgαCS in Y. Hence f is intuitionistic supra generalized α - closed map.
The converse of the above theorem need not be true as seen from the following example.

Example 3.5. Let X={a, b, c}, μ = {$\underset{\sim}{X}$, $\underset{\sim}{\phi}$,A_1, A_2, A_3}, where A_1=<X, {a}, {b}>, A_2=<X, {a, b}, ϕ>, and A_3 = <X, {a, b}, {c}>. σ = {$\underset{\sim}{Y}$, $\underset{\sim}{\phi}$, B_1, B_2, B_3}, where B_1 = <Y, {a}, {b}>, B_2 = <Y, {c}, {a}>, and B_3 = <Y, {a, c}, ϕ >. Let f : (X, μ) →(Y, σ) be a function defined by f(a)=a, f(b)=b and f(c)=c. Here V = <X, ϕ,{a, b}> is ISCS in X and f(v) = <Y, ϕ, {a, b}> is ISgαCS but not ISCS in Y. Hence f is not intuitionistic supra closed map.

Theorem 3.6. Every intuitionistic supra α - closed map is intuitionistic supra generalized α - closed map.
Proof: Let f:(X, μ) → (Y, σ) be an intuitionistic supra α - closed map. Let V be ISCS in X. Since f is intuitionistic supra closed map, then f(V) is ISαCS in Y. We know that every ISαCS is ISgαCS. Implies f(V) is ISgαCS in Y. Hence f is intuitionistic supra generalized α - closed map.
The converse of the above theorem need not be true as seen from the following

Example 3.7. Let X={a, b, c}, μ = {$\underset{\sim}{X}$, $\underset{\sim}{\phi}$,A_1, A_2, A_3}, where A_1 = <X, {a}, {b}>, A_2 = <X, {a, b}, □>, and A_3 = <X, {a, b}, {c}>. σ = {$\underset{\sim}{Y}$, $\underset{\sim}{\phi}$, B_1, B_2, B_3}, where B_1 = <Y, {a}, {b}>, B_2 = <Y, {c}, {a}>, and B_3 = <Y, {a, c}, ϕ >. Let f : (X, μ) →(Y, σ) be a function defined by f(a)=a, f(b)=b and f(c)=c. Here V = <X, {a, b}, {c}> is ISCS in X and f(v) = <Y, {a, b}, {c}> is ISgαCS but not ISαCS in Y. Hence f is not intuitionistic supra α - closed map.

Example 3.8. Every intuitionistic supra regular closed map is intuitionistic supra generalized α - closed map.
Proof: Let f:(X, μ) → (Y, σ) be an intuitionistic supra regular closed map. Let V be ISCS in X. Since f is intuitionistic supra closed map, then f(V) is ISRCS in Y. We know that every ISRCS is ISgαCS. Implies f(V) is ISgαCS in Y. Hence f is intuitionistic supra generalized α - closed map.
The converse of the above theorem need not be true as seen from the following example.

Example 3.9. Let X={a, b, c}, μ = {$\underset{\sim}{X}$, $\underset{\sim}{\phi}$,A_1, A_2, A_3}, where A_1 = <X, {a}, {b}>, A_2 = <X, {a, b}, □>, and A_3 = <X, {a, b}, {c}>. σ = {$\underset{\sim}{Y}$, $\underset{\sim}{\phi}$, B_1, B_2, B_3}, where B_1 = <Y, {a}, {b}>, B_2 = <Y, {c}, {a}>, and B_3 = <Y, {a, c}, ϕ >. Let f : (X, μ) →(Y, σ) be a function defined by f(a) =a, f(b)=b and f(c)=c. Here V = <X, ϕ, {a,b}> is ISCS in X and f(v) = <Y, ϕ , {a, b}> is ISgαCS but not ISRCS in Y. Hence f is not intuitionistic supra regular closed map.

Theorem 3.10. Every intuitionistic supra g - closed map is intuitionistic supra generalized α - closed map.
Proof: Let f:(X, μ) → (Y, σ) be an intuitionistic supra g - closed map. Let V be ISCS in X. Since f is intuitionistic supra closed map, then f(V) is ISgCS in Y. We know that every ISgCS is ISgαCS. Implies f(V) is ISgαCS in Y. Hence f is intuitionistic supra generalized α - closed map.
The converse of the above theorem need not be true as seen from the following example.

Example 3.11. Let X={a, b, c}, $\mu = \{\underset{\sim}{X}, \underset{\sim}{\phi}, A_1, A_2, A_3\}$, where A_1 = <X, {a}, {b}>, A_2 = <X, {a, b}, ϕ >, and A_3 = <X, {a, b}, {c}>. $\sigma = \{\underset{\sim}{Y}, \underset{\sim}{\phi}, B_1, B_2, B_3\}$, where B_1 = <Y, {a}, {b}>, B_2 = <Y, {c}, {a}>, and B_3 = <Y, {a, c}, ϕ>. Let f : (X, μ) →(Y, σ) be a function defined by f(a)=a, f(b)= b and f(c)=c. Here V = <X, ϕ, {a,b}> is ISCS in X and f(v) = <Y, ϕ, {a, b}> is ISgαCS but not ISgCS in Y. Hence f is not intuitionistic supra g - closed map.

IV. Intuitionistic supra generalized α - open maps

Definition 4.1. Let (X, μ) and (Y, σ) be two intuitionistic supra topological space. A function f:(X, μ) → (Y, σ) is called intuitionistic supra regular open map if image of each ISOS in X is ISROS in Y.

Definition 4.2. Let (X, μ) and (Y, σ) be two intuitionistic supra topological space. A function f:(X, μ) → (Y, σ) is called intuitionistic supra generalized open map if image of each ISOS in X is ISgOS in Y.

Definition 4.3. Let (X, μ) and (Y, σ) be two intuitionistic supra topological space. A function f:(X, μ) → (Y, σ) is called intuitionistic supra generalized α - open map if image of each ISOS in X is ISgαOS in Y.

Theorem 4.4. Every intuitionistic supra open map is intuitionistic supra generalized α - open map.

Proof: Let f:(X, μ) → (Y, σ) be an intuitionistic supra closed map. Let V be ISOS in X. Since f is intuitionistic supra closed map, then f(V) is ISOS in Y. We know that every ISOS is ISgαOS. Implies f(V) is ISgαCS in Y. Hence f is intuitionistic supra generalized α - open map.

The converse of the above theorem need not be true as seen from the following example.

Example 4.5. Let X={a, b, c}, $\mu = \{\underset{\sim}{X}, \underset{\sim}{\phi}, A_1, A_2, A_3\}$, where A_1 = <X, {a}, {b}>, A_2 = <X, {a, b}, ϕ >, and A_3 = <X, {a, b}, {c}>. $\sigma = \{\underset{\sim}{Y}, \underset{\sim}{\phi}, B_1, B_2, B_3\}$, where B_1 = <Y, {a}, {b}>, B_2 = <Y, {c}, {a}>, and B_3 = <Y, {a,c}, ϕ >. Let f : (X, μ) →(Y, σ) be a function defined by f(a) = a, f(b) = b and f(c) = c. Here V = <X, ϕ, {a,b}> is ISCS in X and f(v) = <Y, {a,b}, ϕ > is ISgαOS but not ISOS in Y. Hence f is not intuitionistic supra open map.

Theorem 4.6. Every intuitionistic supra α - open map is intuitionistic supra generalized α - open map.

Proof: Let f:(X, μ) → (Y, σ) be an intuitionistic supra α - closed map. Let V be ISOS in X. Since f is intuitionistic supra closed map, then f(V) is ISαOS in Y. We know that every ISαOS is ISgαOS. Implies f(V) is ISgαCS in Y. Hence f is intuitionistic supra generalized α - open map.

The converse of the above theorem need not be true as seen from the following example.

Example 4.7. Let X={a, b, c}, $\mu = \{\underset{\sim}{X}, \underset{\sim}{\phi}, A_1, A_2, A_3\}$, where A_1 = <X, {a}, {b}>, A_2 = <X, {a, b}, ϕ >, and A_3 = <X, {a, b}, {c}>. $\sigma = \{\underset{\sim}{Y}, \underset{\sim}{\phi}, B_1, B_2, B_3\}$, where B_1 = <Y, {a}, {b}>, B_2 = <Y, {c}, {a}>, and B_3 = <Y, {a,c}, ϕ >. Let f : (X, μ) →(Y, σ) be a function defined by f(a) = a, f(b) = b and f(c) = c. Here V = <X, {c}, {a, b}> is ISOS in X and f(v) = <Y, {c}, {a, b}> is ISgαOS but not ISαOS in Y. Hence f is not intuitionistic supra α - open map.

Theorem 4.8. Every intuitionistic supra regular open map is intuitionistic supra generalized α - open map.

Proof: Let f:(X, μ) → (Y, σ) be an intuitionistic supra regular open map. Let V be ISOS in X. Since f is intuitionistic supra open map, then f(V) is ISROS in Y. We know that every ISROS is ISgαOS. Implise f(V) is ISgαOS in Y. Hence f is intuitionistic supra generalized α - open map.

The converse of the above theorem need not be true as seen from the following example.

Example 4.9. Let X={a, b, c}, $\mu = \{\underset{\sim}{X}, \underset{\sim}{\phi}, A_1, A_2, A_3\}$, where A_1 = <X, {a}, {b}>, A_2 = <X, {a, b}, ϕ >, and A_3 = <X, {a, b}, {c}>. $\sigma = \{\underset{\sim}{Y}, \underset{\sim}{\phi}, B_1, B_2, B_3\}$, where B_1 = <Y, {a}, {b}>, B_2 = <Y, {c}, {a}>, and B_3 = <Y, {a,c}, ϕ >. Let f : (X, μ) →(Y, σ) be a function defined by f(a)=a, f(b)=b and f(c)=c. Here V = <X, {a, b}, ϕ > is ISCS in X and f(v) = <Y, {a, b}, ϕ > is ISgαOS but not ISROS in Y. Hence f is not intuitionistic supra regular open map.

Theorem 4.10. Every intuitionistic supra g - open map is intuitionistic supra generalized α - open map.

Proof: Let f:(X, μ) → (Y, σ) be an intuitionistic supra g - open map. Let V be ISOS in X. Since f is intuitionistic supra open map, then f(V) is ISgOS in Y. We know that every ISgOS is ISgαOS.Implise f(V) is ISgαOS in Y. Hence f is intuitionistic supra generalized α - open map.

The converse of the above theorem need not be true as seen from the following example.

Example 4.11. Let X={a, b, c}, $\mu = \{\underset{\sim}{X}, \underset{\sim}{\phi}, A_1, A_2, A_3\}$, where A_1 = <X, {a}, {b}>, A_2 = <X, {a, b}, ϕ >, and A_3 = <X, {a, b}, {c}>. $\sigma = \{\underset{\sim}{Y}, \underset{\sim}{\phi}, B_1, B_2, B_3\}$, where B_1 = <Y, {a}, {b}>, B_2 = <Y, {c}, {a}>, and B_3 = <Y, {a,c}, ϕ >. Let f : (X, μ) →(Y, σ) be a function defined by f(a)=a, f(b)=b and f(c)=c. Here V = <X, {a, b}, ϕ > is ISCS in X and f(v) = <Y, {a, b}, ϕ > is ISgαOS but not ISOS in Y. Hence f is not intuitionistic supra g - open map.

V. Intuitionistic supra generalized α - homeomorphism

Definition 5.1. Let (X, μ) and (Y, σ) be two intuitionistic supra topological space. A bijective mapping f:(X, μ) → (Y, σ) is called an intuitionistic supra generalized α - homeomorphism if f and f^{-1} are intuitionistic supra generalized α - continuous mapping.

Theorem 5.2. Every intuitionistic supra homeomorphism is an intuitionistic supra generalized α - homeomorphism.

Proof: Let f:(X, μ) → (Y, σ) be an intuitionistic supra homeomorphism. Then f and f^{-1} are intuitionistic supra continuous mapping. Then f and f^{-1} are intuitionistic supra generalized α - continuous mapping. Hence the mapping f is an intuitionistic supra generalized α - homeomorphism.

The converse of the above theorem need not be true as seen from the following example.

Example 5.3. Let X={a, b, c}, μ = {$\underset{\sim}{X}$, $\underset{\sim}{\phi}$,A_1, A_2, A_3}, where A_1 = <X, {a}, {b}>, A_2 =<X, {a, b}, ϕ >, and A_3= <X, {a, b}, {c}>. σ = {$\underset{\sim}{Y}$, $\underset{\sim}{\phi}$,B_1, B_2, B_3}, where B_1 =<Y, {a}, {b}>, B_2 = <Y, {c}, {a}>, and B_3= <Y, {a,c}, ϕ >. Let f : (X, μ) →(Y, σ) be a function defined by f(a)= a, f(b)= b and f(c)=c. Here V = <Y, ϕ, {a, c}> is ISCS in Y and f^{-1}(V) = <X, ϕ, {a, c}> is ISgαCS but not ISCS in X. Hence f is not intuitionistic supra homeomorphism.

Theorem 5.4. Let f:(X, μ) → (Y, σ) a bijective mapping. If f is an intuitionistic supra generalized α - continuous mapping. Following statements are equivalent

(1) f is an intuitionistic supra generalized α - closed map.
(2) f is an intuitionistic supra generalized α - open map.
(3) f is an intuitionistic supra generalized α - homeomorphism.

Proof: (1) ⇒ (2)

Let f:(X, μ) → (Y, σ) a bijective mapping and intuitionistic supra generalized α- closed map. Let V be ISOS in X. Then f(V^C) is ISCS in X. Since f is an intuitionistic supra generalized α - closed map f(V^C) is an ISgαCS in Y. Then f(V) is ISgαOS in Y. Hence f is an intuitionistic supra generalized α - open map.

(2) ⇒ (3)

Let f:(X, μ) → (Y, σ) be a intuitionistic supra generalized α - open map. Then f^{-1} is intuitionistic supra generalized α - continuous mapping. Hence f and f^{-1} are intuitionistic supra generalized α - continuous mapping. Hence f is an intu- itionistic supra generalized α - homeomorphism.

(3) ⇒ (1)

Let f:(X, μ) → (Y, σ) be a intuitionistic supra generalized α - homeomorphism. Then f and f^{-1} are intuitionistic supra generalized α - continuous mapping. Hence f is an intuitionistic supra generalized α - closed map.

VI. Conclusion

In this paper we have defined intuitionistic supra gα - closed maps and intuitionistic supra gα - open maps. Also we investigated intuitionistic supra gα – homeomorphism in intuitionistic supra topological spaces.

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ON ψ*gα-CLOSED SETS IN TOPOLOGICAL SPACES

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Abstract: In this paper we introduce a new class of sets namely ψ*gα-closed sets in topological spaces and derive the properties of ψ*gα-closed sets. Also we find the relationship between ψ*gα-closed sets and other existing sets, Moreover we introduce ψ*gα-continuous function and ψ*gα-irresolute function.

Keywords: ψ*gα-closed sets, ψ*gα-continuity, ψ*gα-irresolute.

I. Introduction

The study of generalized closed sets in a topological space was initiated by Levine. N [6]. In 1937 Stone. M [13] introduced regular closed sets. Bhattacharya. P and Lahiri B.K [4], Arya S.P and Nour. T [3], Maki. H et al [8, 9] introduced semi generalized-closed sets, generalized semi closed sets, α-generalized closed sets and generalized α-closed sets respectively. Veera Kumar M.K.R.S [14, 15] introduced g*-closed sets and ψ-closed sets. Vigneshwaran. M, et al [16, 17] introduced *gα-closed sets and **gα closed sets in topological spaces.
In this paper we introduce a new class of sets namely ψ*gα-closed sets in topological spaces and derive the properties of ψ*gα-closed sets. Also we find the relationship between ψ*gα-closed sets and other existing sets, we introduce ψ*gα-continuous function and ψ*gα-irresolute function.

II. Preliminaries

Throughout this paper (X,τ), (Y,σ) and (Z,η), represent topological space on no sepration axioms are assumed unless otherwise to be mentioned. For a subsets A of (X,τ), cl(A) and int(A) denote the closure and the interior of A in (X,τ) respectively.
Let us recall the following definitions, which are useful in the sequel.

Definition 2.1: A subset A of a topological space (X,τ) is called
(1) a pre-open set [10] if A ⊆ int(cl(A)) and a pre-closed set if cl(int(A)) ⊆ A.
(2) a semi-open set [7] if A ⊆ cl(int(A)) and a semi-closed set if int(cl(A)) ⊆ A.
(3) an α-open set [12] if A ⊆ int(cl(int(A))) and an α-closed set if cl(int(cl(A))) ⊆ A.
(4) a semi pre-open set [1] (= β-open) if A ⊆ cl(int(cl(A))) and a semi pre-closed set (=β-closed) int(cl(int(A))) ⊆ A.
(5) a regular-open set [13] if A = int(cl(A)) and a regular-closed set if A = cl(int(A)).

Definition 2.2: A subset A of a topological space (X,τ) is called
(1) a generalized closed set (briefly g-closed) [6] if cl(A) ⊆ U whenever A ⊆ U and U is open in (X,τ).
(2) a semi-generalized closed set (briefly sg-closed) [4] if scl(A) ⊆ U whenever A ⊆ U and U is semi-open in (X,τ).
(3) a generalized semi-closed set (briefly gs-closed) [3] if scl(A) ⊆ U whenever A ⊆ U and U is open in (X,τ).
(4) a generalized α-closed set (briefly gα-closed) [9] if αcl(A) ⊆ U whenever A ⊆ U and U is α-open in (X,τ).
(5) an α-generalized closed set (briefly αg-closed) [8] if αcl(A) ⊆ U whenever A ⊆ U and U is α-open in (X,τ).
(6) a generalized semi pre-closed set (briefly gsp-closed) [5] if spcl(A) ⊆ U whenever A ⊆ U and U is open in (X,τ).
(7) a g*-closed [14] if cl(A) ⊆ U whenever A ⊆ U and U is g-open in (X,τ).
(8) a *gα-closed [16] set if cl(A) ⊆ U whenever A ⊆ U and U is gα-open in (X,τ).
(9) a **gα-closed [17] set if cl(A) ⊆ U whenever A ⊆ U and U is *gα-open in (X,τ)
(10)a ψ-closed set [15] if scl(A) ⊆ U whenever A ⊆ U and U is sg-open in (X,τ).

Definition 2.3: A function f:(X,τ)→(Y,σ) is said to be
(1) semi-continuous [7] if f^{-1}(V) is semi-closed in (X,τ) for every closed set V of (Y,σ).

(2) α-continuous [11] if f^{-1}(V) is α-closed in (X,τ) for every closed set V of (Y,σ).
(3) gsp-continuous [5] if f^{-1}(V) is gsp-closed in (X,τ) for every closed set V of (Y,σ).
(4) g*-continuous [14] if f^{-1}(V) is g*-closed in (X,τ) for every closed set V of (Y,σ).
(5) *gα-continuous [16] if f^{-1}(V) is *gα-closed in (X,τ) for every closed set V of (Y,σ).
(6) gp-irresolute [2] if f^{-1}(V) is gp-closed in (X,τ) for every gp-closed set V of (Y,σ).
(7) sp-irresolute [5] if f^{-1}(V) is gsp-closed in (X,τ) for every gsp-closed set V of (Y,σ).

III. BASIC PROPERTIES OF ψ*gα-CLOSED SETS

We introduce the following definition**.**

Definition 3.1: A subset A of a topological space (X,τ) is said to be ψ*gα-closed if ψcl(A) ⊆ U whenever A ⊆ U and U is *gα-open in (X,τ) and is denoted by ψ*GαC(X,τ).

Theorem 3.2: Every closed set is a ψ*gα-closed set.

Proof: Let A ⊆ U and U is *gα-open set in X. Since A is closed, cl(A) = A.Then cl(A) = A ⊆ U implies cl(A) ⊆ U.We know that ψcl(A) ⊆ cl(A) ⊆ U. Therefore ψcl(A) ⊆ U. Hence A is ψ*gα- closed set.

The converse of the above theorem need not be true. It can be seen from the following example.

Example 3.3: Let X={a,b,c} with τ = {X,ϕ,{b}}; τ^c = {X,ϕ,{a,c}}, ψ*gα-closed sets of (X,τ) = {X,ϕ,{a},{c},{a,b},{b,c},{a,c}}.

Here {b,c} is a ψ*gα-closed set but not a closed set of (X,τ).

Theorem 3.4: Every regular-closed set is a ψ*gα-closed set.

Proof: Let A ⊆ U and U is *gα-open set in X. Since A is regular-closed, rcl(A) = A. Then rcl(A) = A ⊆ U. We know that ψcl(A) ⊆ rcl(A) ⊆ U implies ψcl(A) ⊆ U. Hence A is ψ*gα-closed set.

The converse of the above theorem need not be true. It can be seen from the following example.

Example 3.5: Let X = {a,b,c} with τ = {X,ϕ,{a,b}}; τ^c = {X,ϕ,{c}}, ψ*gα-closed sets of (X,τ) = {X,ϕ,{c},{b,c},{a,c}}, regular-closed sets of (X,τ) = {X,ϕ}.

Here {a,c} is a ψ*gα-closed set but not a regular-closed set of (X,τ).

Theorem 3.6: Every α-closed set is ψ*gα-closed set.

Proof: Let A ⊆ U and U is *gα-open set in X. Since A is α-closed, αcl(A) = A. We know that ψcl(A) ⊆ αcl(A) ⊆ U. Therefore ψcl(A) ⊆ U. Hence A is ψ*gα-closed set.

The converse of the above theorem need not be true. It can be seen from the following example.

Example 3.7: Let X = {a,b,c} with τ = {X,ϕ,{a,b}}; τ^c = {X,ϕ,{c}}, α-closed sets of (X,τ) = {X,ϕ,{c}}, ψ*gα-closed sets of (X,τ) = {X,ϕ,{c},{b,c},{a,c}}.

Here {b,c} is a ψ*gα-closed set but not a α-closed set of (X,τ).

Theorem 3.8: Every semi-closed set is ψ*gα-closed set.

Proof: LetA ⊆ U and U is *gα-open set in X. Since A is semi-closed, scl(A) = A. We know that ψcl(A) ⊆ scl(A) ⊆ U. Therefore ψcl(A) ⊆ U. Hence A is ψ*gα-closed set.

The converse of the above theorem need not be true. It can be seen from the following example.

Example 3.9: Let X = {a,b,c} with τ = {X,ϕ,{a},{a,b}}; τ^c = {X,ϕ,{a,c},{c}}, semi-closed sets of (X,τ) = {X,ϕ,{b},{c},{b,c}}, ψ*gα-closed sets of (X,τ) = {X,ϕ,{b},{c},{b,c},{a,c}}.

Here {a,c} is a ψ*gα-closed set set but not a semi-closed set of (X,τ).

Theorem 3.10: Every *gα-closed set is a ψ*gα-closed set.

Proof: Let A ⊆ U and U is *gα-open set in X. Since every *gα-open set is gα-open then U is gα- open set. Since A is *gα-closed, cl(A) ⊆ U. We know that ψcl(A) ⊆ cl(A) ⊆ U. Therefore ψcl(A) ⊆ U. Hence A is ψ*gα-closed set.

The converse of the above theorem need not be true. It can be seen from the following example.

Example 3.11: Let X = {a,b,c} with τ = {X,ϕ,{a},{a,b}}; τ^c = {X,ϕ,{b,c},{c}}, *gα-closed sets of (X,τ) = {X,ϕ,{c},{b,c}}, ψ*gα-closed sets of (X,τ) = {X,ϕ,{b},{c},{b,c},{a,c}}.

Here {a,c} is a ψ*gα-closed set but not a *gα-closed set of (X,τ).

Theorem 3.12: Every **gα-closed set is a ψ*gα-closed set.

Proof: Let A ⊆ U and U is *gα-open set in X. Since A is *gα-closed, cl(A) ⊆ U. We know that ψcl(A) ⊆ cl(A) ⊆ U. Therefore ψcl(A) ⊆ U. Hence A is ψ*gα-closed set.

The converse of the above theorem need not be true. It can be seen from the following example.

Example 3.13: Let X = {a,b,c} with τ = {X,ϕ,{b},{c},{b,c}}; τ^c = {X,ϕ,{a,c},{a,b} {a}}, **gα-closed sets of (X,τ) = {X, ϕ,{a},{a,b},{a,c}}, ψ*gα-closed sets of (X,τ) = {X,ϕ,{a},{b},{c},{a,b},{a,c}}.

Here {c} is a ψ*gα-closed set but not a **gα-closed set of (X,τ).

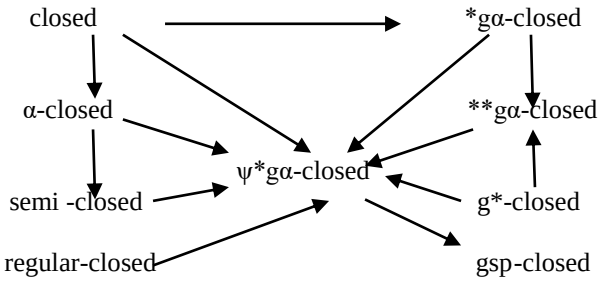
Theorem 3.14: Every g*-closed set is ψ*gα-closed set.

Proof: Let A ⊆ U and U is *gα-open set in X. Since every *gα-open set is g-open then U is g-open set. Since A is g*-closed, cl(A) ⊆ U. We know that ψcl(A) ⊆ cl(A) ⊆ U, implies ψcl(A)⊆ U. Hence A is ψ*gα-closed set.

The converse of the above theorem need not be true. It can be seen from the following example.

Example 3.15: Let X = {a,b,c} with τ = {X,ϕ,{b}}; τ^c = {X,ϕ,{a,c}}, g*-closed sets of (X,τ) = {X,ϕ,{a,c}}, ψ*gα-closed sets of (X,τ) = {X,ϕ,{a},{c},{a,b},{b,c},{a,c}}.
Here {b,c} is a ψ*gα -closed set set but not a g*-closed set of (X,τ).
Theorem 3.16: Every ψ*gα-closed set is gsp-closed set.
Proof: Let A $\subseteq$ U and U is open set in X. Since every open set is *gα-open then U is *gα-open. Since A is ψ*gα-closed, ψcl(A) $\subseteq$ U.We know that spcl(A) $\subseteq$ ψcl(A) $\subseteq$ U. Therefore spcl(A) $\subseteq$ U. Hence A is gsp-closed set.
The converse of the above theorem need not be true. It can be seen from the following example.
Example 3.17: Let X = {a,b,c} with τ = {X,ϕ,{a,b}}; τ^c = {X,ϕ,{c}}, gsp-closed sets of (X,τ) = {X,ϕ,{a},{b},{c},{b,c},{a,c}}, ψ*gα-closed sets of (X,τ) = {X,ϕ,{c},{b,c},{a,c}}.
Here {a} is a gsp-closed set set but not a ψ*gα-closed set of (X,τ).
Remark 3.18: The union (intersection) of two ψ*gα-closed sets need not be ψ*gα-closed sets.
Example 3.19: Let X = {a,b,c} with τ = {X, ϕ,{a},{c},{a,c}}; τ^c ={X, ϕ,{b,c},{a,b},{b}}, ψ*gα-closed sets of (X,τ) = {X, ϕ,{a},{b},{c},{a,b},{b,c}}.
Here {a} and {c} are ψ*gα-closed sets but {a}$\cup${c} = {a,c} is not a ψ*gα-closed set.
Example 3.20: Let X ={a,b,c} with τ = {X,ϕ,{b}}; τ^c = {X,ϕ,{a,c}}, ψ*gα-closed sets of (X,τ) = {X,ϕ,{a},{c},{a,b},{b,c},{a,c}}.
Here {a,b} and {b,c} are ψ*gα-closed sets but {a,b}$\cap${b,c} = {b} is not a ψ*gα closed set.
Theorem 3.21: Let A be a ψ*gα-closed set of (X,τ) then ψcl(A)-A does not contain any non-empty *gα-closed set.
Proof: Let F be a *gα-closed set contained in (X,τ), ψcl(A)-A , ψcl(A)$\subseteq$ X-F, since X-F is *gα-open set with A $\subseteq$ X-F and A is a ψ*gα-closed set then F $\subseteq$ (X-ψcl(A)) $\cap$ (ψcl(A)-A) $\subseteq$ (X-ψcl(A)) $\cap$ ψcl(A) = ϕ, F = ϕ.
Theorem 3.22: Let A$\subseteq$ B $\subseteq$ ψcl(A) and A is a ψ*gα-closed set of (X,τ) then B is also a ψ*gα-closed set of (X,τ).
Proof: Let A be a ψ*gα-closed set of X such that A $\subseteq$ B $\subseteq$ ψcl(A) and U be a *gα-open set of X such that B $\subseteq$ U. Then A $\subseteq$ U. But A is ψ*gα-closed, ψcl(A) $\subseteq$ U. Hence ψcl(B) $\subseteq$ ψcl(ψcl(A)) = ψcl(A) $\subseteq$ U implies ψcl(B) $\subseteq$ U. Hence B is ψ*gα-closed set in X.

Remark 3.23: The following theorem shows that the relationship between ψ*gα-closed sets and some other sets in theorem 3.2, 3.4, 3.6, 3.8, 3.10, 3.12, 3.14, 3.16.



IV. ψ*gα-CONTINUOUS AND IRRESOLUTE FUNCTIONS

We introduce the following definition
Definition 4.1: A function f: (X,τ)$\rightarrow$(Y,σ) is called ψ*gα-continuous if f^{-1}(V) is ψ*gα-closed in (X,τ) for every closed set V of (Y,σ).
Theorem 4.2: Every continuous map is ψ*gα-continuous.
Proof: Let V be a closed set of (Y,σ). Since f is continuous f^{-1}(V) is closed in (X,τ). But every closed set is ψ*gα-closed set. Hence f^{-1}(V) is ψ*gα-closed in (X,τ). Thus f is ψ*gα-continuous.

The converse of the above theorem need not be true. It can be seen from the following example.
Example 4.3: Let X = {a,b,c} = Y, τ = {X,ϕ,{a}}, σ = {Y,ϕ,{b,c}}.
Define f: (X,τ) $\rightarrow$ (Y, σ) by f(a) = c, f(b) = b, f(c) = a, ψ*gα-closed sets of (X,τ) = {X,ϕ,{b},{c},{a,b},{b,c},{a,c}}.
Here f^{-1}({a}) = {c} is not a closed set in (X,τ). Therefore f is not continuous. However f is ψ*gα-continuous.
Theorem 4.4: Every semi-continuous map is ψ*gα-continuous.
Proof: Let V be a closed set of (Y,σ). Since f is semi-continuous f^{-1}(V) is semi-closed in (X,τ). But every semi-closed set is ψ*gα-closed set. Hence f^{-1}(V) is ψ*gα-closed in (X,τ). Thus f is ψ*gα-continuous.
The converse of the above theorem need not be true. It can be seen from the following example.

Example 4.5: Let X = {a,b,c} = Y, τ ={X,ϕ{a},{b,c}}, σ = {Y,ϕ,{a}}.
Define f: (X,τ) → (Y, σ) by f(a) = b, f(b) = c, f(c) = a, ψ*gα-closed sets of (X,τ) = {X,ϕ,{a},{b},{c},{a,b},{b,c},{a,c}}, Semi-closed sets of (X,τ) = {X,ϕ,{a},{b,c}}
Here f^{-1}({b,c}) = {a,b} is not semi-closed set in (X,τ). Therefore f is not semi-continuous. However f is ψ*gα-continuous.
Theorem 4.6: Every α-continuous map is ψ*gα-continuous.
Proof: Let V be a closed set of (Y,σ). Since f is α-continuous f^{-1}(V) is α-closed in (X,τ). But every α-closed set is ψ*gα-closed set. Hence f^{-1}(V) is ψ*gα-closed in (X,τ). Thus f is ψ*gα-continuous.
The converse of the above theorem need not be true. It can be seen from the following example.
Example 4.7: Let X = {a,b,c} = Y, τ = {X,ϕ,{c},{a,c}}, σ = {Y,ϕ,{b}}.
Define f: (X,τ)→(Y,σ) by f(a) = b, f(b) = a, f(c) = c, ψ*gα-closed sets of (X,τ) = {X,ϕ,{a},{b}, {a,b},{b,c}}, α-closed sets of (X,τ) = {X,ϕ,{a},{b},{a,b}}.
Here f^{-1}({a,c}) = {b,c} is not α-closed set in (X,τ). Therefore f is not α-continuous. However f is ψ*gα-continuous
Theorem 4.8: Every g*-continuous map isψ*gα-continuous.
Proof: Let V be a closed set of (Y,σ). Since f is g*-continuous f^{-1}(V) is g*-closed in (X,τ). But every g*-closed set is ψ*gα-closed set. Hence f^{-1}(V) is ψ*gα-closed in (X,τ). Thus f is ψ*gα-continuous.
The converse of the above theorem need not be true. It can be seen from the following example.
Example 4.9: Let X = {a,b,c} = Y, τ = {X,ϕ,{a},{a,c}}, σ = {Y,ϕ,{a,b}}.
Define f: (X,τ)→(Y,σ) by f(a) = a, f(b) = b, f(c) = c, ψ*gα-closed sets of (X,τ) = {X,ϕ,{b},{c}, {a,b},{b,c}}, g*-closed sets of (X,τ) = {X,ϕ,{b},{a,b},{b,c}}
Here f^{-1}({c}) = {c} is not g*-closed set in (X, τ). Therefore f is not g*-continuous. However f is ψ*gα-continuous.
Theorem 4.10: Every ψ*gα-continuous map is gsp-continuous.
Proof: Let V be a closed set of (Y,σ). Since f is ψ*gα-continuous, f^{-1}(V) is ψ*gα-closed set in (X,τ). But every ψ*gα-closed set is gsp-closed set. Hence f^{-1}(V) is gsp-closed set in (X,τ). Thus f is gsp-continuous.
The converse of the above theorem need not be true. It can be seen from the following example.
Example 4.11: Let X = {a,b,c} = Y, τ = {X,ϕ,{a,c}}, σ = {Y,ϕ,{a}}
Define f: (X,τ)→(Y,σ) by f(a) = b, f(b) = c, f(c) = a, ψ*gα-closed sets of (X,τ) = {X,ϕ,{b},{b,c},{a,b}}, gsp-closed sets of (X,τ) = {X,ϕ,{a},{b},{c},{b,c},{a,b}}
Here f^{-1}({a}) = {c} is not a ψ*gα-closed set in (X,τ). Therefore f is not ψ*gα-continuous. However f is gsp-continuous.
Theorem 4.12: Every *gα-continuous map is ψ*gα-continuous.
Proof: Let V be a closed set of (Y,σ). Since f is *gα-continuous, f^{-1}(V) is *gα-closed in (X,τ). But every *gα-closed set is ψ*gα-closed set. Hence f^{-1}(V) is ψ*gα-closed in (X,τ). Thus f is ψ*gα-continuous.
The converse of the above theorem need not be true. It can be seen from the following example.
Example 4.13: Let X = {a,b,c} = Y, τ = {X,ϕ,{b},{c},{b,c}}, σ = {Y,ϕ,{a,b}}.
Define f: (X,τ)→(Y,σ) by f(a) = a, f(b) = c, f(c) = b, ψ*gα-closed sets of (X,τ) = {X,ϕ,{a},{b}, {c},{a,b},{a,c}}, *gα-closed sets of (X,τ) = {X,ϕ,{a},{a,b},{a,c}}.
Here f^{-1}({c}) = {b} is not *gα-closed set in (X,τ). Therefore f is not *gα-continuous. However f is ψ*gα-continuous.
Definition 4.14: A function f: (X,τ)→(Y,σ) is called ψ*gα-irresolute if f^{-1}(V) is ψ*gα-closed in (X,τ) for every ψ*gα-closed set V of (Y,σ).
Theorem 4.15: Every ψ*gα-irresolute function is ψ*gα-continuous.
Proof: Let V be a closed set of (Y,σ). Since every closed set is ψ*gα-closed set. Therefore V is ψ*gα-closed set of (Y,σ). Since f is ψ*gα-irresolute, f^{-1}(V) is ψ*gα-closed set in (X,τ). Hence f is ψ*gα-continuous.
The converse of the above theorem need not be true. It can be seen from the following example.
Example 4.16: Let X = {a,b,c} = Y, τ = {X,ϕ,{a}}, σ = {Y,ϕ,{c}}
Define f: (X,τ) → (Y, σ) by f(a) = b, f(b) = a, f(c) = c, ψ*gα-closed sets of (Y, σ) = {Y,ϕ,{a},{b}, {a,b},{b,c},{a,c}}, ψ*gα-closed sets of (X,τ) = {X,ϕ,{b},{c},{a,b},{b,c},{a,c}}.
Here f is ψ*gα-continuous but f is not ψ*gα-irresolute, since {b} is ψ*gα-closed set in (Y,σ) but f^{-1}({b}) = {a} is not ψ*gα-closed set in (X,τ).
Theorem 4.17: Every ψ*gα-irresolute function is gsp-continuous.
Proof: Let V be a closed set of (Y,σ). Since every closed set is ψ*gα-closed set. Therefore V is ψ*gα-closed set of (Y,σ). Since f is ψ*gα-irresolute, f^{-1}(V) is ψ*gα-closed set in (X,τ). Since every ψ*gα-closed set is gsp-closed set, f^{-1}(V) is gsp-closed set in (X,τ). Hence f is gsp-continuous.
The converse of the above theorem need not be true. It can be seen from the following example.
Example 4.18: Let X = {a,b,c} = Y, τ = {X,ϕ,{a}}, σ = {Y,ϕ,{a,c}}

Define f: $(X,\tau)\to(Y,\sigma)$ by f(a) = a, f(b) = b, f(c) = c, ψ*gα-closed sets of (X,τ) = {X,ϕ,{c},{b,c}, {a,c}}, ψ*gα-closed sets of (Y,σ) = {Y,ϕ,{b},{c},{a,b},{b,c}, {a,c}}, gsp-closed set of (X,τ) = {X,ϕ,{a},{b},{c},{a,b},{b,c}} Here f is gsp-continuous but f is not ψ*gα-irresolute. Since {a,b} is ψ*gα-closed set in (Y,σ) but $f^{-1}(\{a,b\})$ = {a,b} is not ψ*gα-closed set in (X,τ).

Theorem 4.19: Let f: $(X,\tau)\to(Y,\sigma)$ and g: $(Y,\sigma)\to(Z,\eta)$ be any two functions then gof: $(X,\tau)\to(Z,\eta)$ is ψ*gα-continuous if g is continuous and f is ψ*gα-continuous.

Proof: Let V be a closed set in (Z,η). Then $g^{-1}(V)$ is closed in (Y,σ), since g is continuous, ψ*gα-continuity of f implies $f^{-1}(g^{-1}(V)) = (gof)^{-1}(V)$ is ψ*gα-closed set in (X,τ). Hence gof is ψ*gα-continuous

Theorem 4.20: Let f: $(X,\tau)\to(Y,\sigma)$ and g: $(Y,\sigma)\to(Z,\eta)$ be any two functions then gof: $(X,\tau)\to(Z,\eta)$ is ψ*gα-irresolute if both f and g are ψ*gα-irresolute.

Proof: Let V be a ψ*gα-closed set in (Z,η). Since g is ψ*gα-irresolute, $g^{-1}(V)$ is ψ*gα-closed in (Y,σ). As f is ψ*gα-irresolute $f^{-1}(g^{-1}(V)) = (gof)^{-1}(V)$ is ψ*gα-closed in (X,τ). Therefore gof is ψ*gα-irresolute.

Theorem 4.21: Let f: $(X,\tau)\to(Y,\sigma)$ and g : $(Y,\sigma)\to(Z,\eta)$ be any two functions then gof: $(X,\tau)\to(Z,\eta)$ is ψ*gα-continuous if g is ψ*gα-continuous and f is ψ*gα-irresolute.

Proof: Let V be closed set in (Z,η). Since g is ψ*gα-continuous, $g^{-1}(V)$ is ψ*gα-closed in (Y,σ). As f is ψ*gα-irresolute, $f^{-1}(g^{-1}(V)) = (gof)^{-1}(V)$ is ψ*gα-closed in (X,τ). Therefore gof is ψ*gα-continuous.

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Neutrosophic Hesitant Sets and Neutrosophic Hesitant Topological Spaces

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Abstract: The aim of this paper is to introduce the concept of neutrosophic hesitant topological space by defining the basic operations of neutrosophic hesitant set.Suitable examples are provided to prove the claim.

Keywords: Hesitant fuzzy set(HFS),neutrosophic hesitant set (NHS),neutrosophic hesitant topological spaces(NHTS).

I. Introduction

The concept of fuzzy set was introduced by Zadeh[11] in the year 1965 and later Atanassov[1,2,3] generalised this idea to a new class of intuitionistic fuzzy sets using the notions of fuzzy sets.Smarandache[6,7,8] originally gave the definition of a neutrosophic set and neutrosophic logic.Torra and Yasuo Narukawa[9,10] in 2009 introduced the concept of hesitant fuzzy sets.C.L. Chang[4] described the new concept of Fuzzy Topological Spaces and Dogan Coker[5] gave an introduction to intuitionistic fuzzy topological spaces.Later in the year 2010,Torra discussed the relationship between the HFS and other three kinds of fuzzy sets, and showed that the envelope of HFS is an intuitionistic fuzzy set. HFSs can reflect the original information given by the decision makers as much as possible, and more values are obtained from the decision makers or experts, which belong to [0, 1] may be assigned to express hesitant judgments. Therefore, in this paper, the concept of an NHFS is presented on the basis of the combination of NS and HFS.

We can see that the NHS consists of three parts, i.e., the truth-membership hesitancy function,the indeterminacy-membership hesitancy function, and the falsity-membership hesitancy function, supporting a more exemplary and flexible access to assign values for each element in the domain, and can handle three kinds of hesitancy in this situation. Thus, the existing sets, including fuzzy sets, intuitionistic fuzzy sets, NSs, HFSs, and DHFSs, can be regarded as special cases of NHSs.

For our convenience suiting to real life application we consider the real standard interval [0,1] instead of the real non standard interval $]0^-, 1^+[$.

Further the topology for the set is introduced with valid examples.

II. Preliminaries

Definition 2.1[9]

Let X be a fixed set,HFS A on X is defined in terms of a function $h_A(x)$ that when applied to X returns a finite subset of [0, 1], which can be represented as the following mathematical symbol:

$$A=\{<x,h_A(x)>/x\in X\}$$

where $h_A(x)$is a set of some different values in [0, 1], denoting the possible membership degrees of the element $x \in X$ to A.For convenience, we call $h_A(x)$ a hesitant fuzzy element.

Definition2.2[9]

Given a hesitant fuzzy element h, its lower and upper bounds are defined as $h^-(x)$= min$h(x)$ and $h^+(x)$= max $h(x)$, respectively.

Definition2.3[9,10]

Given a hesitant fuzzy element h,$A_{env}(h)$ is called the envelope of h, which is represented by (h^-, 1-h^+), with the lower bound h^- and upper bound h^+.

Definition 2.4[10]
Given three hesitant fuzzy elements h,h_1, h_2some operations in them are as follows:

1.h^c=$\bigcup_{\gamma\in h}(1-\gamma)$
2.$h_1 \cup h_2$=$\bigcup_{\gamma_1\in h_1,\gamma_2\in h_2}\max\{\gamma_1,\gamma_2\}$
3.$h_1 \cap h_2$=$\bigcup_{\gamma_1\in h_1,\gamma_2\in h_2}\min\{\gamma_1,\gamma_2\}$

Definition 2.5[10]
For a hesitant element h,s(h)= $\frac{1}{\#h}\sum_{\gamma\in h}\gamma$ is called the score function of h ,were $\#h$ denotes the number of elements in h.For two hesitant fuzzy elements h_1 and h_2, if s(h_1)>s(h_2) then $h_1 > h_2$, if s(h_1)=s(h_2) then $h_1 = h_2$.

Definition 2.6[6,7,8]
Let X be a non empty fixed set.A neutrosophic set(NS for short) A is an object having the form A={<x,$\mu_A(x),\sigma_A(x),\gamma_A(x)>$} where $\mu_A(x),\sigma_A(x),\gamma_A(x)$ represent the degree of membership function,degree of nonmembership function and degree of inderterminancy function respectively for each element x∈X to the set A.
A neutrosophic set A={<x,$\mu_A(x),\sigma_A(x),\gamma_A(x)>$}can be identified as an ordered triple <μ_A,σ_A,γ_A> in the real non standard interval$]0^-,1^+[$ on X were,

μ_A:X→ $]0^-,1^+[$ σ_A: X→ $]0^-,1^+[$ γ_A: X→ $]0^-,1^+[$

There is no restriction in the NS on T, I, F other than they are subsets of$]0^-,1^+[$
thus,$0^- \le$ inf T + inf I + infF ≤ sup T + sup I + sup F≤3^+.

Definition2.7[6,7,8]
Let T,I,F be the real standard or nonstandard subsets of $]0^-,1^+[$., with
Sup T=t_sup,infT=t_inf
Sup I=i_sup,infI=i_inf
Sup F=f_sup,inf_F=f_inf
n_sup=t_sup+i_sup+f_sup
n_inf=t_inf+i_inf+f_inf
T,I,F are called neutrosophic components.

III. Neutrosophic Hesitant Sets

Definition 3.1

Let Xbe a fixed set, aNHS on X is defined as N={<x, $\tilde{t}_N(x)$, $\tilde{\imath}_N(x),\tilde{f}_N(x)$ >/x ∈ X} where $\tilde{t}_N(x)$, $\tilde{\imath}_N(x),\tilde{f}_N(x)$ denotes the three sets, each one of several values in [0,1] denoting the possible truth,indeterminancy and false membership hesitant degrees of the element x∈X to the set respectively with the conditions,0≤ $\delta,\gamma,\eta \le 1$and 0≤ $\gamma^+ + \delta^+ + \eta^+ \le 3$ where $\gamma \in \tilde{t}_N(x)$,
$\delta \in \tilde{\imath}_N(x), \eta \in \tilde{f}_N(x)$.$\gamma^+ \in \tilde{t}_N{}^+(x)$=$\bigcup_{\gamma\in\tilde{t}_N(x)}\max\{\gamma\}$,$\delta^+ \in \tilde{\imath}_N{}^+(x)$ =$\bigcup_{\delta\in\tilde{\imath}_N(x)}\max\{\delta\}$,
$\eta^+ \in \tilde{f}_N{}^+((x)$=$\bigcup_{\eta\in\tilde{f}_N(x)}\max\{\eta\}$.
The tuple N={<x,$\tilde{t}_N(x),\tilde{\imath}_N(x),\tilde{f}_N(x)$>/x∈Xis called a neutrosophic hesitant element (NHE) or a triple hesitant element, which is denoted by the simplified symbol N={$\tilde{t}_N,\tilde{\imath}_N,\tilde{f}_N$} were $\tilde{t}_N(x),\tilde{\imath}_N(x),\tilde{f}_N(x)$ denotes subsets of [0,1].

Definition 3.2
Let A={<$\tilde{t}_1$,$\tilde{\imath}_1,\tilde{f}_1$>}and B={<$\tilde{t}_2$,$\tilde{\imath}_2,\tilde{f}_2$>}be two NHFSs in a fixed set X then, their union and intersection are defined, respectively, by

(i)A⊆B⇔ $\begin{Bmatrix}\min(\tilde{t}_1)\le\min(\tilde{t}_2),\max(\tilde{t}_1)\le\max(\tilde{t}_2)\\ \min(\tilde{\imath}_1)\ge\min(\tilde{\imath}_2),\max(\tilde{\imath}_1)\ge\max(\tilde{\imath}_2)\\ \min(\tilde{f}_1)\ge\min(\tilde{f}_2),\max(\tilde{f}_1)\ge\max(\tilde{f}_2)\end{Bmatrix}$

(ii)A=B⇔A⊆B and B⊆A

(iii)A^c={<x,{$\tilde{f}(x)$,$\cup_{\delta\in\tilde{\imath}}\{1-\delta\}$,$\tilde{t}(x)$}>}

(iv) $A \cap B$={$\tilde{t}_1 \cap \tilde{t}_2, \tilde{\imath}_1 \cup \tilde{\imath}_2, \tilde{f}_1 \cup \tilde{f}_2$}

(v)$A \cup B$={$\tilde{t}_1 \cup \tilde{t}_2, \tilde{\imath}_1 \cap \tilde{\imath}_2, \tilde{f}_1 \cap \tilde{f}_2$}

(vi)Neutrosophic hesitant empty set(0_N) may be defined as:

$$0_N = \{< x, \{0\}, \{1\}, \{1\}: x \in X\}$$

(vii)Neutrosophic hesitant full set(1_N) may be defined as :
1_N={<x,{1},{0},{0}:x∈ X}
In this section we define neutrosophic hesitant topological spaces and define neutrosophic hesitant interior and neutrosophic hesitant closure along with the properties.

IV. Neutrosophic Hesitant Topological spaces

Definition 4.1

A neutrosophic hesitant topology (NHT for short) on a nonempty set X is a family τ of neutrosophic hesitant subsets in X satisfying the following axioms

Axiom1:0_N,1_N $\in \tau$
Axiom 2:∪ $G_i \in \tau$ for any $\{G_i/i \in J\} \subseteq \tau$
Axiom 3:A∩B $\in \tau$for any $A, B \in \tau$

In this case the pair (X,τ) is called a neutrosophic hesitant topological space(NHTS for short) and any NHS in τ is known as a neutrosophic hesitant open set (NHOS for short) in X.

The complement A^c of a NHOS is called a neutrosophic hesitant closed set.

Definition 4.2
Let (X,τ) be a NHTS and A={<x,$\tilde{t}(x)$,$\tilde{\imath}(x)$, $\tilde{f}(x)$ >} be a NHS in X.Then the neutrosophic hesitant interior and neutrosophic hesitant closure are defined by
NHint(A)=∪{G/G is anNHOS in X and G⊆A}
NHcl(A)=∩{K/K is an NHCS in X and A⊆K }

Definition 4.3
Let A be a subset of a NHTS(X,τ). A point x∈X is said to be a limit point(or accumulation point
Or cluster point)of A if every NH open set, U containing x contains a point of A different from x.

Proposition 4.4
For any NHS set A={<x,$\tilde{t}(x)$,$\tilde{\imath}(x)$, $\tilde{f}(x)$ >} the following results holds
(i)$0_N \subseteq A, 0_N \subseteq 0_N$
(ii)$1_N \subseteq 1_N, A \subseteq 1_N$

Theorem 4.5
Let (X,τ) be a NHTS and if A is an NH subset of X then NHcl(A) is the smallest NH closed set containing A.
Proof:
Let $\{F_\alpha : \alpha \in \Lambda\}$ be the collection of all NH closed sets of X which contain A.Then F=∩$\{F_\alpha : \alpha \in \Lambda\}$ is a NH closed set. Also ∩$\{F_\alpha : \alpha \in \Lambda\}$ is also a NH closed set.Now A is contained in each F_α and hence in their intersection.We know that $F \subset F_\alpha$ for each $\alpha \in \Lambda$ and hence F is the smallest NH closet set containing A.

Properties 4.6
Let (X,τ) be a NHTS and A and B are any two NH subsets of X.Then

(i) A⊂NHcl (A)
(ii) A⊂B⇒ NHcl (A) ⊂NHcl (B)
(iii) A is a NHCS⇔ A=NHcl(A)
(iv) NHcl (NHcl (A)) = NHcl (A)
(v) NHcl (0_N) =0_N ,NHcl (1_N) = 1_N
(vi) NHcl (A∪B) = NHcl (A) ∪ NHcl (B)
(vii) $(NHcl\ (A))^c$ = NH int (A^c)
(viii) (NH cl(A)) $^c \in \tau$

Proof:

(i) NHcl (A) is the smallest NH closed set containing A.
Therefore A⊂ NHcl (A).

(ii) Let A⊂B.We know that A⊂ NHcl (A) and B⊂NHcl (B)(by (i))
Therefore A⊂B⊂ NHcl (B)
⇒A⊂NHcl (B)
Now NHcl (B) also contain A.But the smallest NH closed set which contains A is NHcl (A).Therefore A⊂B⇒ NHcl (A) ⊂ NHcl (B)

(iii) Let A is a NHCS and let x ∈ A.We know that A⊂ NHcl (A)
Now suppose that x ∈NHcl(A) such that x∈A′, where A′ denotes the set of accumulation point of A. Since A is a closed set, A′⊆A, implying that x∈A.Therefore, NHcl(A) ⊂ A . We conclude thatA=NHcl(A) is A closed.

(iv) From (iii),we have A=NHcl(A).But NHcl(A) is also a NH closed set.
Therefore NHcl (NHcl (A))= NHcl (A)

(v) $0_N$and$1_N$are NH closed sets.Therefore NHcl (0_N) =0_N ,NHcl (1_N) = 1_N (by (iii))

(vi) $A \subset A \cup B$ and $B \subset A \cup B$
NHcl (A) ⊂ NHcl ($A \cup B$) and NHcl (B) ⊂ NHcl ($A \cup B$)(by (ii))
Therefore NHcl (A) ∪ NHcl (B) ⊂NHcl ($A \cup B$)-------(1)
NHcl (A) is NH closed set and NHcl (B) is NH closed set.
Therefore NHcl (A) ∪ NHcl (B) is also a NH closed set.
We know that A⊂ NHcl (A) and B⊂ NHcl (B)
Therefore A∪B⊂NHcl (A) ∪ NHcl (B),NHcl (A) ∪ NHcl (B) is a NH closed set containing A∪B.But NHcl (A∪B) is the smallest NH closed set containing A∪B.
Therefore NHcl ($A \cup B$)⊂NHcl (A) ∪ NHcl (B)--------(2)
From (1) and (2) NHcl (A∪B) = NHcl (A) ∪ NHcl (B).

(vii) Obvious

(viii) Obvious

Properties 4.7:

Let (X,τ) be a NHTS and A and B are any two NH subsets of X.Then

(i) NHint(A)⊂A.
(ii) A⊂B⟹ NHint(A)⊂NHint(B)
(iii) A is NHOS ⇔ NHint(A)=A
(iv) NHint(NHint(A)) = NHint(A)
(v) NHint (0_N) =0_N ,NHint (1_N) = 1_N
(vi) NHint (A∩B) = NHint (A) ∩NHint (B)
(vii) $(NHint\ (A))^c$ = NH cl (A^c)
(viii) NHint(A)∈ τ

Proof: obvious

Example 4.8

Let X={a,b} and A={<a,{0.2,0.3},{0.4,0.5},{0.4}>,<b,{0.21,0.29},{0.4,0.6},{0.7}>},
B={<a,{0.3},{0.2,0.4},{0.4}>,<b,{0.5},{0.32,0.4},{0.1}>} be the two NHS then
A∪B={<a,{0.3},{0.2,0.4},{0.4}>,<b,{0.5},{0.32,0.4},{0.1}>}
A∩B={<a,{0.2,0.3},{0.4,0.5},{0.4}>,<b,{0.21,0.29},{0.4,0.6},{0.7}>}
A^c={<a,{0.4 },{0.6,0.5},{0.2,0.3}>,<b,{ 0.7},{0.6,0.4},{0.21,0.29}>}
B^c={<a,{0.4},{0.8,0.6},{0.2}>,<b,{0.1},{0.68,0.6},{0.5}>}
Here A⊆ B.

Example 4.9

Let X={a,b} and A={<a,{0.2,0.3},{0.4,0.5},{0.4}>,<b,{0.21,0.29},{0.4,0.6},{0.7}>}
1_N={<a,{1},{0},{0}>,<b,{ 1},{0},{0}>},0_N={ <a,{0},{1},{0}>,<b,{ 0},{1},{0}>}

Then τ={1_N, 0_N,A} is a neutrosophic hesitant topological space.

Let B={ <a,{0.3},{0.2,0.42},{0.39}>,<b,{ 0.5},{0.32,0.4},{0.1,0.33}>}

Then NHint(B) ={<a,{0.2,0.3},{0.4,0.5},{0.4}>,<b,{0.21,0.29},{0.4,0.6},{0.5}>}=A
NHcl(B)={<a,{1},{0},{0}>,<b,{ 1},{0},{0}>}= 1_N.

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Multi inequality approach to evaluate EOQ and EPQ inventory models with shortages in fuzzy nature

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Abstract*: In this article, A new method by using Arithmetic Geometric Mean (AGM) inequality and Cauchy-Bunyakovsky-Schwarz (CBS) inequality are proposed to compute the Economic Order Quantity (EOQ) and Economic Production Quantity (EPQ) without taking algebraic manipulations or differential calculus. Previously AGM inequality used to compute only EOQ/EPQ without shortages. It does not focus on deriving the shortages level. This paper proposes another simple method to compute both the lot size and the shortage level under fuzzy environment. In this paper EOQ/EPQ with shortages only discussed. The demand rate and the shortage cost are taken as triangular fuzzy number. The effectiveness of the proposed method illustrated by means of numerical examples.*

Keywords*: Inventory, Economic Order Quantity (EOQ), Economic Production Quantity (EPQ), Arithmetic Geometric Mean (AGM) inequality, Cauchy-Bunyakovsky-Schwarz (CBS) inequality, Cost comparison optimization, Triangular fuzzy number, Shortage level*

I. INTRODUCTION

Minner [7] Wee et al. [10] proposed a different approach to obtain the EOQ without differential calculus and algebraic manipulations. He compared the cost co-efficient when a local cost minimum is obtained, took the time horizon to infinity to get the EOQ and then proved it's the global minimum. Later the cost comparisons and the arithmetic geometric mean (AGM) inequality. These methods were developed by Minner [7] and Teng [9] respectively. Minner's method has been modified by Wee et al. [10] . The modified cost comparison method is simpler than Minner's method. Teng's AGM method is also very simple but he is not the first researcher to apply the well known AGM inequality in to optimization functions. It can be argued that Garver (1935) and niven (1981). The arithmetic geometric mean is simple unfortunately the AGM method does not derive the mathematical expression for the shortage level for EOQ/EPQ models. So Cardenas-Barron [2,3] explained how to find the shortage level. . A. Nagoor Gani, U. Mohammed Rafi [8] applied fuzzy techniques in AGM inequality to find the EOQ/EPQ models with and without shortages. This paper is the extension of my previous paper [8].

The research papers by Minner [7], Wee et al. [10] and Teng [9] do not consider the optimization of shortages level. Cardenas-Barron [4] introduces another simple method which uses two well-known inequalities the Arithmetic Geometric Mean (AGM) inequality and Cauchy-Bunyakovsky-Schwarz (CBS) inequality to find the shortages level and the optimal lot size.

A. Assumptions:

Apart from the traditional EOQ/EPQ model some changes are made here as follows

(i) We may assume that the demand rate ($\tilde{D}$) and Shortage cost are uncertain, take it as a triangular fuzzy number
(ii) Replenishment is instantaneous
(iii) Quantity discount is not allowed

(iv) Both the initial and the ending inventory levels are zero

This paper is organized as follows: Section 2 provides some basic idea about the triangular fuzzy number and its arithmetic operations, Section 3 gives methodology to find EOQ and EPQ under fuzzy environment using AGM inequality and CBS inequality. In section 4, the effectiveness of the proposed method is illustrated by means of examples. Finally we conclude the paper.

II PRELIMINARIES

A. Fuzzy Set

A fuzzy set Ã is defined by Ã= {(x, $\mu_A(x)$): x∈A, $\mu_A(x)$∈[0,1] }. In the pair (x, $\mu_A(x)$), the first element x belong to the classical set A, the second element $\mu_A(x)$ belong to the closed interval [0, 1] called Membership function.

B. Fuzzy Number

The notion of fuzzy numbers was introduced by Dubois D. and Prade H [5]. A fuzzy subset $\tilde{A}$ of the real line R with membership function $\mu_{\tilde{A}}: R \to [0,1]$ is called a fuzzy number if

i) A fuzzy set $\tilde{A}$ is normal.

ii) $\tilde{A}$ is fuzzy convex,

(i.e.) $\mu_{\tilde{A}}[\lambda x_1 + (1-\lambda)x_2] \geq \mu_{\tilde{A}}(x_1) \wedge \mu_{\tilde{A}}(x_2), x_1, x_2 \in R, \forall \lambda \in [0,1]$.

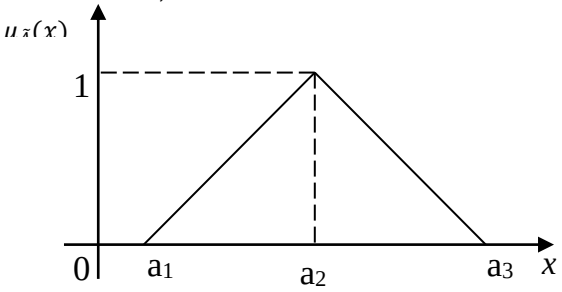
iii) $\mu_{\tilde{A}}$ is upper continuous, and

iv) Supp $\tilde{A}$ is bounded, where supp $\tilde{A} = \{x \in R: \mu_{\tilde{A}}(x) > 0\}$.

C. Triangular Fuzzy Number

A triangular fuzzy number $\tilde{A}$ is denoted as $\tilde{A} = (a_1, a_2, a_3)$ where a_2 is the central value $\mu_{\tilde{A}}(a_2) = 1$, a_1 is the left spread and a_3 is the right spread, and $a_1 < a_2 < a_3$ are defined in R. The membership function is given by,

$$\mu_{\tilde{A}}(x) = \begin{cases} \frac{(x-a_1)}{(a_2-a_1)} \; for \; a_1 \leq x \leq a_2 \\ \frac{(a_3-x)}{(a_3-a_2)} \; for \; a_2 \leq x \leq a_3 \\ 0, \qquad otherwise \end{cases}$$



D. Arithmetic Operations on Triangular Fuzzy Number Using Function Principle

The arithmetic operations on triangular fuzzy number using function principle were given by Chen, S.H [4]. Let $\tilde{A} = (a_1, a_2, a_3)$ and $\tilde{B} = (b_1, b_2, b_3)$ be two triangular fuzzy numbers. Then

i) The addition of $\tilde{A}$ *and* $\tilde{B}$ is
$\tilde{A} + \tilde{B} = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$ where $a_1, a_2, a_3, b_1, b_2, b_3$ are real numbers.

ii) The product of $\tilde{A}$ *and* $\tilde{B}$ is
$\tilde{A} \times \tilde{B} = (c_1, c_2, c_3)$, where T = $\{a_1b_1, a_2b_2, a_3b_3\}$
c_1 = min T, $c_2 = a_2b_2$, c_3 = max T
If $a_1, a_2, a_3, b_1, b_2, b_3$ are all non zero positive real numbers, then
$\tilde{A} \times \tilde{B} = (a_1b_1, a_2b_2, a_3b_3)$

iii) $-\tilde{B} = (-b_3, -b_2, -b_1)$
Then the subtraction of $\tilde{B}$ from $\tilde{A}$ is
$\tilde{A} - \tilde{B} = (a_1 - b_3, a_2 - b_2, a_3 - b_1)$, where $a_1, a_2, a_3, b_1, b_2, b_3$ are real numbers.

iv) $\frac{1}{\tilde{B}} = \tilde{B}^{-1} = \left(\frac{1}{b_3}, \frac{1}{b_2}, \frac{1}{b_1}\right)$, where b_1, b_2, b_3 are all non zero real numbers, then $\frac{\tilde{A}}{\tilde{B}} = \left(\frac{a_1}{b_3}, \frac{a_2}{b_2}, \frac{a_3}{b_1}\right)$.

v) Let $\alpha \in R$, then $\alpha\tilde{A} = (\alpha a_1, \alpha a_2, \alpha a_3)\ if\ \alpha \geq 0$
$= (\alpha a_3, \alpha a_2, \alpha a_1)\ if\ \alpha < 0$

III Derivation of the lot size and the shortages level for the EOQ/EPQ models with shortages

This session provided a method to derive both the optimal lot size and the optimal shortages level for the EOQ models with shortages taking into account the following two well-known inequalities: the Arithmetic Geometric Mean (AGM) inequality and Cauchy-Bunyakovsky-Schwarz (CBS) inequality

The AGM inequality

For any positive real numbers $a_1, a_2, a_3, ...a_n$ such that, $\frac{a_1 + a_2 + a_3 + ... + a_n}{n} \geq \left(a_1.a_2.a_3...a_n\right)^{1/n}$ the equation holds only if $a_1 = a_2 = a_3 = ... = a_n$.

The CBS inequality

Let $a_1, a_2, a_3, ...a_n$ and $b_1, b_2, b_3, ...b_n$ be any two sets of positive real numbers, such that $\left(a_1^{\,2} + a_2^{\,2} + a_3^{\,2} + ... + a_n^{\,2}\right)\left(b_1^{\,2} + b_2^{\,2} + b_3^{\,2} + ... + b_n^{\,2}\right) \geq \left(a_1b_1 + a_2b_2 + a_3b_3 + ... + a_nb_n\right)^2$ the equation holds only if $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3} = ... = \frac{a_n}{b_n}$

Notations

The following notation is used throughout the entire paper.

d	the demand rate per unit time
q	lot size per order (Production run)
C_1	inventory carrying cost (holding cost)
C_2	shortage cost per item
C_3	setup cost per order (ordering cost)
k	the production rate per unit time
q_1	shortage level

A. The EOQ model with shortages

The total inventory cost for the EOQ with shortages is given by

$$TC(q, q_1) = \frac{1}{2}C_1\left(\frac{q_1^{\,2}}{q}\right) + \frac{1}{2}\tilde{C}_2\left[\frac{(q - q_1)^2}{q}\right] + C_3\left(\frac{\tilde{d}}{q}\right) \quad \text{--- (1)}$$

Where $\tilde{d} = (d_1, d_2, d_3)$, $\tilde{C}_2 = (C_{21}, C_{22}, C_{23})$

Eqn. (1) can be written as

$$TC(q, q_1) = \frac{C_3\tilde{d}}{q} + \frac{q}{2}\left[C_1\left(\frac{q_1}{q}\right)^2 + \tilde{C}_2\left(1 - \frac{q_1}{q}\right)^2\right] \quad \text{--- (2)}$$

Now,

$$TC(q, q_1) = \frac{C_3\tilde{d}}{q} + \frac{q}{2}\left[\left(\sqrt{C_1}\left(\frac{q_1}{q}\right)\right)^2 + \left(\sqrt{\tilde{C}_2}\left(1 - \frac{q_1}{q}\right)\right)^2\right]$$

$$= \frac{C_3\tilde{d}}{q} + \frac{q}{2}\left[\left(\sqrt{C_1}\left(\frac{q_1}{q}\right)\right)^2 + \left(\sqrt{\tilde{C}_2}\left(1 - \frac{q_1}{q}\right)\right)^2\right]\left[\left(\frac{\sqrt{\tilde{C}_2}}{\sqrt{C_1 + \tilde{C}_2}}\right)^2 + \left(\frac{\sqrt{C_1}}{\sqrt{C_1 + \tilde{C}_2}}\right)^2\right]$$

--- (3)

Applying CBS inequality in eqn. (3) we get,

$$TC(q,q_1) \geq \frac{C_3\tilde{d}}{q} + \frac{q}{2}\left\{\frac{\sqrt{C_1\tilde{C}_2}}{\sqrt{C_1+\tilde{C}_2}}\left(\frac{q_1}{q}\right) + \frac{\sqrt{C_1\tilde{C}_2}}{\sqrt{C_1+\tilde{C}_2}}\left(1-\frac{q_1}{q}\right)\right\}^2 \quad \text{--- (4)}$$

With equality iff

$$\frac{\sqrt{C_1}\left(\frac{q_1}{q}\right)}{\frac{\sqrt{\tilde{C}_2}}{\sqrt{C_1+\tilde{C}_2}}} = \frac{\sqrt{\tilde{C}_2}\left(1-\frac{q_1}{q}\right)}{\frac{\sqrt{C_1}}{\sqrt{C_1+\tilde{C}_2}}} \quad \text{--- (5)}$$

Iff the two sets of numbers are proportional by CBS inequality. Thus eqn. (4) becomes

$$TC(q) = \frac{C_3\tilde{d}}{q} + \frac{q}{2}\left\{\frac{\sqrt{C_1\tilde{C}_2}}{\sqrt{C_1+\tilde{C}_2}}\left(1-\frac{q_1}{q}+\frac{q_1}{q}\right)\right\}^2$$

$$= \frac{C_3\tilde{d}}{q} + \frac{q}{2}\left\{\frac{\sqrt{C_1\tilde{C}_2}}{\sqrt{C_1+\tilde{C}_2}}\right\}^2$$

$$= \frac{C_3\tilde{d}}{q} + \frac{q}{2}\left\{\frac{C_1\tilde{C}_2}{C_1+\tilde{C}_2}\right\} \quad \text{--- (6)}$$

Here we have two functions $\frac{C_3\tilde{d}}{q}$ & $\frac{q}{2}\left\{\frac{C_1\tilde{C}_2}{C_1+\tilde{C}_2}\right\}$ Thus we have to apply AGM inequality because the product of these two functions is a constant.

That is, $\frac{C_3\tilde{d}}{2}\left\{\frac{C_1\tilde{C}_2}{C_1+\tilde{C}_2}\right\}$ is a constant.

So we have to use AGM inequality in eqn. (6) to solve the minimization problem easily.

$$TC(q) = \frac{\frac{2C_3\tilde{d}}{q} + q\left\{\frac{C_1\tilde{C}_2}{C_1+\tilde{C}_2}\right\}}{2} \geq \sqrt{\frac{2C_3\tilde{d}}{q}q\left\{\frac{C_1\tilde{C}_2}{C_1+\tilde{C}_2}\right\}}$$

$$\geq \sqrt{2C_3\tilde{d}\left(\frac{C_1\tilde{C}_2}{C_1+\tilde{C}_2}\right)} \quad \text{--- (7)}$$

With the inequality iff

$$\frac{C_3\tilde{d}}{q} = \frac{q}{2}\left\{\frac{C_1\tilde{C}_2}{C_1+\tilde{C}_2}\right\} \quad \text{--- (8)}$$

From eqn. (8) iff the product of two functions is a constant. Then simplifying eqn. (5) we get the optimal shortage level is given by

$$q_1^* = \frac{C_1q^*}{C_1+\tilde{C}_2} = \frac{C_1q^*}{C_1+(C_{21},C_{22},C_{23})}$$

On simplifying eqn. (8) we get the optimal lot size is given by

$$q^* = \sqrt{\frac{2C_3\tilde{d}}{C_1}}\sqrt{\frac{C_1+\tilde{C}_2}{\tilde{C}_2}} = \sqrt{\frac{2C_3(d_1,d_2,d_3)}{C_1}}\sqrt{\frac{C_1+(C_{21},C_{22},C_{23})}{(C_{21},C_{22},C_{23})}}$$

From eqn. (7) that the optimal total inventory cost is given by

$$TC^* = \sqrt{2C_1C_3\tilde{d}\left(\frac{\tilde{C}_2}{C_1+\tilde{C}_2}\right)} = \sqrt{2C_1C_3(d_1,d_2,d_3)\left(\frac{(C_{21},C_{22},C_{23})}{C_1+(C_{21},C_{22},C_{23})}\right)}$$

Where $\tilde{d} = (d_1,d_2,d_3),\ \tilde{C}_2 = (C_{21},C_{22},C_{23})$

B. The EPQ model with shortages

The total inventory cost for the EPQ model with shortages is given by

$$TC(q,q_1)=\frac{C_1{q_1}^2}{2q\left(1-\frac{\tilde{d}}{k}\right)}+\frac{\tilde{C}_2\left[q\left(1-\frac{\tilde{d}}{k}\right)-q_1\right]^2}{2q\left(1-\frac{\tilde{d}}{k}\right)}+\frac{C_3\tilde{d}}{q} \quad \text{--- (9)}$$

Where $\tilde{d}=(d_1,d_2,d_3),\ \tilde{C}_2=(C_{21},C_{22},C_{23})$

Eqn. (9) can be written as

$$TC(q,q_1)=\frac{C_3\tilde{d}}{q}+\frac{q\left(1-\tilde{d}/k\right)}{2}\left[C_1\left(\frac{q_1}{q\left(1-\tilde{d}/k\right)}\right)^2+C_2\left(1-\frac{q_1}{q\left(1-\tilde{d}/k\right)}\right)^2\right]$$

$$=\frac{C_3\tilde{d}}{q}+\frac{q\left(1-\tilde{d}/k\right)}{2}\left[\left(\sqrt{C_1}\frac{q_1}{q\left(1-\tilde{d}/k\right)}\right)^2+\left(\sqrt{C_2}\left(1-\frac{q_1}{q\left(1-\tilde{d}/k\right)}\right)\right)^2\right]\left[\left(\frac{\sqrt{\tilde{C}_2}}{\sqrt{C_1+\tilde{C}_2}}\right)^2+\left(\frac{\sqrt{C_1}}{\sqrt{C_1+\tilde{C}_2}}\right)^2\right]$$

--- (10)

The above form is the required form to use the CBS inequality. Now we use CBS inequality in eqn. (10)

$$TC(q,q_1)\geq\frac{C_3\tilde{d}}{q}+\frac{q\left(1-\tilde{d}/k\right)}{2}\left\{\left(\frac{\sqrt{C_1\tilde{C}_2}}{\sqrt{C_1+\tilde{C}_2}}\right)\left(\frac{q_1}{q\left(1-\tilde{d}/k\right)}\right)+\left(\frac{\sqrt{C_1\tilde{C}_2}}{\sqrt{C_1+\tilde{C}_2}}\right)\left(1-\frac{q_1}{q\left(1-\tilde{d}/k\right)}\right)\right\}^2$$

--- (11)

With equality iff

$$\frac{\left(\sqrt{C_1}\frac{q_1}{q\left(1-\tilde{d}/k\right)}\right)}{\frac{\sqrt{\tilde{C}_2}}{\sqrt{C_1+\tilde{C}_2}}}=\frac{\sqrt{C_2}\left(1-\frac{q_1}{q\left(1-\tilde{d}/k\right)}\right)}{\frac{\sqrt{C_1}}{\sqrt{C_1+\tilde{C}_2}}} \quad \text{--- (12)}$$

iff the two sets of numbers are proportional by CBS inequality. Thus eqn. (10) becomes,

$$TC(q)=\frac{C_3\tilde{d}}{q}+\frac{q\left(1-\tilde{d}/k\right)}{2}\left\{\left(\frac{\sqrt{C_1\tilde{C}_2}}{\sqrt{C_1+\tilde{C}_2}}\right)\left(1-\frac{q_1}{q\left(1-\tilde{d}/k\right)}+\frac{q_1}{q\left(1-\tilde{d}/k\right)}\right)\right\}^2$$

$$=\frac{C_3\tilde{d}}{q}+\frac{q\left(1-\tilde{d}/k\right)}{2}\left\{\left(\frac{\sqrt{C_1\tilde{C}_2}}{\sqrt{C_1+\tilde{C}_2}}\right)\right\}^2$$

$$=\frac{C_3\tilde{d}}{q}+\frac{q\left(1-\tilde{d}/k\right)}{2}\left(\frac{C_1\tilde{C}_2}{C_1+\tilde{C}_2}\right) \quad \text{--- (13)}$$

Here we have two functions $\frac{C_3\tilde{d}}{q}$ & $\frac{q\left(1-\tilde{d}/k\right)}{2}\left(\frac{C_1\tilde{C}_2}{C_1+\tilde{C}_2}\right)$. Thus we have to apply AGM inequality because product of these two functions is a constant.

That is, $\frac{C_3\tilde{d}}{2}\left(1-\tilde{d}/k\right)\left(\frac{C_1\tilde{C}_2}{C_1+\tilde{C}_2}\right)$ is a constant.

So we have to use AGM inequality in eqn. (13) to solve the minimization problem easily.

$$TC(q)=\frac{\frac{2C_3\tilde{d}}{q}+q\left(1-\tilde{d}/k\right)\left(\frac{C_1\tilde{C}_2}{C_1+\tilde{C}_2}\right)}{2}\geq\sqrt{\frac{2C_3\tilde{d}}{q}q\left(1-\tilde{d}/k\right)\left(\frac{C_1\tilde{C}_2}{C_1+\tilde{C}_2}\right)}$$

$$\geq \sqrt{2C_3\tilde{d}\left(1-\tilde{d}/k\right)\left(\frac{C_1\tilde{C}_2}{C_1+\tilde{C}_2}\right)} \quad \text{--- (14)}$$

With equality iff

$$\frac{C_3\tilde{d}}{q} = \frac{q\left(1-\tilde{d}/k\right)}{2}\left(\frac{C_1\tilde{C}_2}{C_1+\tilde{C}_2}\right) \quad \text{--- (15)}$$

From eqn. (15) iff the product of two functions is a constant. Then simplifying eqn. (12) we get the optimal shortage level is given by

$$q_1^* = \frac{C_1\left(1-\tilde{d}/k\right)q^*}{C_1+\tilde{C}_2} = \frac{C_1\left(1-\frac{(d_1,d_2,d_3)}{k}\right)}{C_1+(C_{21},C_{22},C_{23})}$$

On simplifying eqn. (15) we get the optimal lot size is given by

$$q^* = \sqrt{\left(\frac{2C_3\tilde{d}}{C_1}\right)\left(\frac{C_1+\tilde{C}_2}{\tilde{C}_2}\right)\left(\frac{k}{k-\tilde{d}}\right)} = \sqrt{\left(\frac{2C_3(d_1,d_2,d_3)}{C_1}\right)\left(\frac{C_1+(C_{21},C_{22},C_{23})}{(C_{21},C_{22},C_{23})}\right)\left(\frac{k}{k-(d_1,d_2,d_3)}\right)}$$

From eqn. (14) that the optimal total inventory cost is given by

$$TC^* = \sqrt{2C_1C_3\tilde{d}\left(\frac{\tilde{C}_2}{C_1+\tilde{C}_2}\right)\left(\frac{k-\tilde{d}}{k}\right)} = \sqrt{2C_1C_3(d_1,d_2,d_3)\left(\frac{(C_{21},C_{22},C_{23})}{C_1+(C_{21},C_{22},C_{23})}\right)\left(\frac{k-(d_1,d_2,d_3)}{k}\right)}$$

Where $\tilde{d}=(d_1,d_2,d_3),\ \tilde{C}_2=(C_{21},C_{22},C_{23})$

IV Illustrate Example:

A. EOQ with Shortages

A particular item has a demand around 9000 units per year. The cost of one procurement is Rs. 100 and the holding cost per unit is Rs. 2.40 per year. The shortage cost is around Rs. 5 per unit per year. Determine the economic lot size, the number of order per year the time between orders and the total cost per year if the cost of one unit is Re.1.

Solution:

Given

Demand $\tilde{d}=(d_1,d_2,d_3)=(8500,9000,9500)$ units per year

Holding cost $C_1 = Rs.2.40$ per year

Shortage cost $C_2=(C_{21},C_{22},C_{23})=Rs.(4.5,\ 5,\ 5.5)$

Setup Cost $C_3 = Rs.100$

1. The economic lot size is given by $q^* = \sqrt{\frac{2C_3(d_1,d_2,d_3)}{C_1}}\sqrt{\frac{C_1+(C_{21},C_{22},C_{23})}{(C_{21},C_{22},C_{23})}}$

   $= \sqrt{\frac{2\times100\times(8500,9000,9500)}{2.40}}\sqrt{\frac{2.40+(4.5,\ 5,\ 5.5)}{(4.5,\ 5,\ 5.5)}}$

   $= (608.4624,\ 680.0478,\ 761)$ units per run

2. The number of order is given by $n=\frac{\tilde{d}}{q^*}=\frac{(8500,9000,9500)}{(608.4624,\ 680.0478,\ 761)}$

   $= (11.1695,\ 13.2344,\ 15.6131)$ orders per year

3. The time between orders $t=\frac{1}{n}=\frac{1}{(11.1695,\ 13.2344,\ 15.6131)}$

   $= (0.0895,\ 0.0756,\ 0.0640)$ years

4. Total cost $TC = 9000\times1+\sqrt{2C_1C_3(d_1,d_2,d_3)\left(\frac{(C_{21},C_{22},C_{23})}{C_1+(C_{21},C_{22},C_{23})}\right)}$

$$= 9000 + \sqrt{2 \times 2.4 \times 100 \times (8500, 9000, 9500)\left(\frac{(4.5,\ 5,\ 5.5)}{2.4 + 4.5,\ 5,\ 5.5)}\right)}$$

$$= Rs.(10{,}524.4193,\ 10{,}708.2871,\ 10906.50)$$

B. EPQ with Shortages

A company has a demand around 12000 units per year for an item and it can be produce 2000 such items per month the cost of one setup is Rs. 400 and the holding cost per unit per month is Rs. 0.15. The shortage cost is around Rs. 20 per unit per year. Find the optimum lot size and the total cost per year assuming the cost of 1 unit is Rs. 4.

Solution:

Given

Demand $\tilde{d} = (d_1, d_2, d_3) = (11000, 12000, 13000)$ units per year

Production rate $k = 2000$ units per month

$= 2000 \times 12 = 24000$ units per year

Holding cost $C_1 = Rs.0.15$ per month

$= 0.15 \times 12 = Rs.1.80$ per year

Shortage cost $C_2 = (C_{21}, C_{22}, C_{23}) = Rs.(18,\ 20,\ 22)$ per units per year

Setup Cost $C_3 = Rs.400$ per setup

1. Optimum lot size $q^* = \sqrt{\left(\frac{2C_3(d_1, d_2, d_3)}{C_1}\right)\left(\frac{C_1 + (C_{21}, C_{22}, C_{23})}{(C_{21}, C_{22}, C_{23})}\right)\left(\frac{k - (d_1, d_2, d_3)}{k}\right)}$

$$= \sqrt{\left(\frac{2 \times 400 \times (11000, 12000, 13000)}{1.8}\right)\left(\frac{1.8 + (18, 20, 22)}{(18, 20, 22)}\right)\left(\frac{24000 - (11000, 12000, 13000)}{24000}\right)}$$

$= (2850.0135,\ 3409.7898,\ 4082.7963)$ units per year

2. Optimum total cost

$$TC = 12000 \times 4 + \sqrt{2C_1C_3(d_1, d_2, d_3)\left(\frac{(C_{21}, C_{22}, C_{23})}{C_1 + (C_{21}, C_{22}, C_{23})}\right)\left(\frac{k - (d_1, d_2, d_3)}{k}\right)}$$

$$= 48000 + \sqrt{2 \times 1.8 \times 400 \times (11000, 12000, 13000)\left(\frac{(18, 20, 22)}{1.8 + (18, 20, 22)}\right)\left(\frac{24000 - (11000, 12000, 13000)}{24000}\right)}$$

$= Rs.(50343.1471, 50815.3749, 51356.6721)$ per year

V Conclusion:

We have two optimization methods to compute the EOQ/EPQ models. These are cost comparisons and AGM inequality. However, neither optimization method attempted to derive the optimal shortage level. Main work of this paper is to introduce an alternative method for deriving EOQ/EPQ models when shortages are allowed in fuzzy environment. Comparing to others method our method is simple and derives both the optimal lot size and the shortage level. This method is also simpler than the algebraic method. Finally this method is very simple and accessible to the learning of inventory models in fuzzy environment for students who lack of knowledge in calculus.

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Criss-Cross Type Algorithm with Symmetric S-Monotone Pivot Rule for Fuzzy Linear Complementarity Problems

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Abstract*: In this paper, criss-cross type algorithm with symmetric s-monotone pivot rule for solving the fuzzy linear complementarity problem(FLCP) is introduced. Here the input fuzzy matrix is non-sufficient and the cost coefficients, constraint coefficients, the right hand side coefficients are represented by an interval numbers. The effectiveness of the proposed method is illustrated by means of a numerical example.*

Keywords*: Fuzzy linear complementarity problem, fuzzy non-sufficient matrix, criss-cross type algorithm with symmetric s-monotone pivot rule.*

I. Introduction

Very few areas of mathematical programming are as studied as the linear complementarity problem. Numerous practical applications and the wide range of unsolved both theoretical and algorithmic-problems make it an stunning field of research.

There are many different pivot algorithms to solve LCP with different matrices.The criss-cross algorithm is one of those which were developed independently-for different optimization problems. Since then, the criss-cross method has become a class of algorithms that differs in the index selection rule.

The criss-cross algorithm for LCP with bisymmetric matrices were developed by Akkeles, Balogh and Illes [1]. They used the LIFO (last-in-first-out) and the MOSV (most-often-selected-variable) pivot rules. These index selection rules are flexible in the sense that they offer a choice of alternative pivots in certain situations, while still preserving finiteness. These index selection rules have been generalized for the linear programming problem in [9]. It is a fascinating question which is the extensive class of matrices for which the criss-cross algorithm with the above mentioned index selection rules can be extended, preserving finiteness.

The class of sufficient matrices was introduced by Cottle, Pang and Venkateswaran [6]. Sufficient matrices can be elucidated as generalizations of P and PSD matrices. Valiaho [20] showed that the class of sufficient matrices is the same as the class of P* matrices, introduced in [15] for interior point methods of LCP. D. den Hertog, Roos and Terlaky [11] proved this method, that the sufficient matrices are exactly those matrices for which the criss-cross algorithm with the minimal index pivot selection rule solves the LCP with any right-hand side vector. There is no familiar effective algorithm to decide whether a matrix is sufficient or not, and Tseng [18] has shown that the classification of column sufficiency of a matrix is co-NP complete. (Valiaho [19] developed a non-polynomial, inductive method to check sufficiency). Most algorithms developed for LCP have the practically unattractive property that they need a priori information that the matrix is adequate or possesses a few other effective properties.

Throughout this paper, vectors by bold, scalars by normal letters. Rows of a matrix are indexed by superscripts and also columns of a matrix are indexed as a subscript. $\widetilde{M}$ denotes the original problem matrix, $\tilde{\boldsymbol{q}}$ the right hand side. Without loss of generality we may assume that rank($\widetilde{M}$) = n. The (ordered) set of all indices is I := $\{1,2,...,n,\bar{1},\bar{2},...,\bar{n}\}\cup\{\tilde{q}\}$ which includes the index of the right hand side, and so |I| = 2n + 1. To denote the complementary pairs, let $\bar{\bar{\alpha}}=\alpha$ for all $\alpha \in I \setminus \{\tilde{q}\}$, so the complementary index pair of $\bar{\alpha}$ is α. For a given basis $\tilde{B}$, we denote the nonbasic part of $\widetilde{M}$ by $\widetilde{N}$; the corresponding set of indices for the basis and nonbasic part are denoted by $I_{\tilde{B}}$ and $I_{\widetilde{N}}$, respectively. The corresponding short pivot tableau for $\tilde{B}$ is denoted by T:=$\tilde{B}^{-1}\widetilde{N}$, while the transformed right hand side is denoted by $\tilde{\tilde{q}}$:=$\tilde{B}^{-1}\tilde{q}$. Individual coefficients of the short pivot tableau will be

denoted by $\bar{\tilde{m}}_{ij}$; The complementary variable pairs ($\tilde{u}_l$, $\tilde{v}_l$) will normally be referred to as variable pairs. When it is advantaged, the element wise Hadamard product will be emphasized by ".".

This paper is organized as follows: In section 2, some basic definitions and lemmas are discussed. Section 3 provides criss-cross type algorithm with symmetric s-monotone pivot rule for fuzzy linear complementarity problems. Section 4 provides a numerical example for determining the solution of FLCP.

II. Preliminaries

2.1. Fuzzy set

A fuzzy set $\tilde{A}$ is defined by $\tilde{A}$ = {(x, μ_A(x)): x∈A, μ_A(x) ∈ [0,1]}. In the pair (x, μ_A(x)), the first element x belong to the classical set A, the second element μ_A(x) belong to the interval [0,1], called membership function.

2.2. Fuzzy number

The notion of fuzzy numbers was introduced by Dubois D. and Prade H. A fuzzy subset $\widetilde{A}$ of the real line R with membership function $\mu_{\widetilde{A}}$: R→[0,1] is called a fuzzy number if

i) A fuzzy set $\tilde{A}$ is normal.

ii) $\tilde{A}$ is fuzzy convex
(i.e.)$\mu_{\tilde{A}}[\lambda x_1 + (1-\lambda)x_2] \geq \mu_{\tilde{A}}(x_1) \wedge \mu_{\tilde{A}}(x_2), x_1, x_2 \in R, \forall \lambda \in [0,1]$.

iii) $\mu_{\tilde{A}}$is upper continuous, and

iv) Supp$\tilde{A}$ is bounded, where supp $\tilde{A}$= {x∈R:$\mu_{\tilde{A}}$ (x)>0}.

2.3. Interval number

If $\widetilde{A}$ is a triangular fuzzy number, we will let $\widetilde{A}_\alpha$ = [A_α^-, A_α^+] be the closed interval which is a α-cut for $\widetilde{A}$ where A_α^- and A_α^+ are its left and right end points respectively. Let I and J be two interval numbers defined by ordered pairs of real numbers with lower and upper bounds. I = [a,b], where a≤b, J = [c,d] where c≤d, when a=b and c=d, these interval numbers degenerate to a scalar real number.

2.4. Fuzzy matrix

A fuzzy matrix $\widetilde{A}$ of order m × n is defined as $\widetilde{A}$ = $[\tilde{a}_{ij}, \tilde{a}_{ij\mu}]_{m\times n}$ where $\tilde{a}_{ij\mu}$ is the membership value of the element $\tilde{a}_{ij}$ in $\widetilde{A}$. For simplicity, we write $\widetilde{A}$ as $\widetilde{A}$ = $[\tilde{a}_{ij\mu}]$.

2.5. Fuzzy sufficient matrix

The fuzzy matrix $\widetilde{A} \in R^{n\times n}$ is called a fuzzy column sufficient if no vector $\tilde{x} \in R^n$ exists, for which

$$\begin{cases} \tilde{x}_i(\widetilde{A}\tilde{x})_i \leq 0 & \text{for all indices } i \in \{1,2,\dots,n\} \\ \tilde{x}_j(\widetilde{A}\tilde{x})_j < 0 & for\ atleast\ one\ index\ j \in \{1,2,\dots,n\} \end{cases}$$

and we call it fuzzy row sufficient if its transpose is fuzzy column sufficient. A fuzzy matrix $\tilde{A}$ is called fuzzy sufficient if it is both fuzzy column and row sufficient at the same time.

2.6. Fuzzy non-sufficient matrix

A fuzzy matrix$\tilde{A}$ is called fuzzy non-sufficient if it is neither fuzzy column nor fuzzy row sufficient.

Lemma 2.7

Let $\tilde{A}$ be a fuzzy sufficient matrix, B a complementary basis and $\underset{\sim}{A}$ = $[\tilde{\tilde{a}}_{ij} | i \in J_B, j \in J_N]$ the corresponding short pivot tableau. Then

(a) $\tilde{\tilde{a}}_{i\bar{\imath}} \geq 0$ for all i∈ J_B ; furthermore

(b) for all i∈ J_B, if $\tilde{\tilde{m}}_{i\bar{\imath}} = 0$ then $\tilde{\tilde{m}}_{i\bar{\jmath}} = \tilde{\tilde{m}}_{j\bar{\imath}} = 0$ or $\tilde{\tilde{m}}_{i\bar{\jmath}}.\tilde{\tilde{m}}_{j\bar{\imath}} < 0$ for all j∈ J_B , j ≠ i.

Lemma 2.8

Let $\tilde{A} \in R^{n\times n}$ be a fuzzy row (column) sufficient matrix. Then

1. any permutation of matrix $\tilde{A}$ is fuzzy row (column) sufficient,
2. the product $\widetilde{D}\tilde{A}\widetilde{D}$ is fuzzy row (column) sufficient, where $\widetilde{D} \in R^{n\times n}$ is a fuzzy positive diagonal matrix,
3. every principal submatrix of $\tilde{A}$ is fuzzy row (column) sufficient.

A fuzzy sufficient matrix $\tilde{A}$ is also fuzzy sufficient after any number of arbitrary principal pivots. The class of fuzzy sufficient matrices is closed under principal block pivot operations [26], and as a consequence their properties are preserved during the criss-cross type algorithms, as the exchange pivot operations carried out by the criss-cross algorithms are equivalent to a block pivot operation of size 2×2.

The decision problem, whether an arbitrary linear complementarity problem has a solution or not, is in NP, and not always in co-NP [5], although for the class of sufficient matrices it belongs to co-NP, and can be stated using the dual of the FLCP problem:

$$(\tilde{x}, \tilde{y}) \in V(\tilde{M}, \tilde{q}) = \left\{(\tilde{x}, \tilde{y}) \middle| \begin{matrix} \tilde{x} + \tilde{M}^T\tilde{y} = 0, \tilde{q}^T\tilde{y} = -1 \\ \tilde{x}\tilde{y} = 0, \tilde{x}, \tilde{y} \geq 0 \end{matrix} \right\}$$

Theorem 2.9

For a fuzzy sufficient matrix $\tilde{A} \in R^{n\times n}$ and a vector $\tilde{q} \in R^n$, exactly one of the following statements holds:

(1) the (P-FLCP) problem has a fuzzy feasible complementary solution ($\tilde{u}$,$\tilde{v}$),
(2) the (D-FLCP) problem has a fuzzy feasible complementary solution ($\tilde{x}$,$\tilde{y}$).

2.10. Possible pivot fuzzy sequence

A fuzzy sequence of index pairs $\tilde{S} = \{\tilde{S}_k = (\tilde{\imath}_k, \tilde{o}_k)$: $\tilde{\imath}_k, \tilde{o}_k \in N$ for some consecutive $k \in N\}$, is called a possible pivot fuzzy sequence, if

(i) $2n = \max\{\max \tilde{\imath}_k , \max \tilde{o}_k \}$,
(ii) There exists a (P-FLCP) in standard form with 2n variables and the rank($\tilde{A}$) = n, and
(iii) (possibly infinite) pivot sequence, where the moving variable pairs of (P-FLCP) correspond to the index pairs of $\tilde{S}$.

The index pairs of a possible pivot fuzzy sequence are thus only required to comply with the basic and nonbasic status. It is now easy to show that

Theorem 2.11

If a possible pivot fuzzy sequence is not finite then there exists a (sub)set of indices, I*, that occurs infinitely many times in $\tilde{S}$.

2.12. Pivot index preference

A fuzzy sequence of fuzzy vectors $\tilde{s}_k \in N^{2n}$ is called a pivot index preference of an index selection rule, if in iteration j, in case of ambiguity according to a pivot selection rule, the index selection rule selects an index with highest value in $\tilde{s}_j$ among the candidates.

2.13. s-monotone index selection rule for fuzzy linear complementarity problems

Let $n \in N$ be given. An index selection rule is called s-monotone, if

1. there exists a pivot index preference $\tilde{s}_k \in N^{2n}$, for which
   (a) the values in the fuzzy vector $\tilde{s}_{j-1}$ after iteration j may only change for $\tilde{\imath}_j$ and $\tilde{o}_j$, where $\tilde{\imath}_j$ and $\tilde{o}_j$ are the indices involved in the pivot operation,
   (b) the values may not decrease.
2. For any infinite possible pivot fuzzy sequence $\tilde{S}$ and for any iteration j there exists iteration $r \geq j$ such that
   (a) The index with minimal value in $\tilde{s}_r$ among $I^* \cap I_{B_r}$ is unique (let it be l), where I_{B_r} is the set of basic indices, in iteration r,
   (b) in iteration $t>r$ when index $l \in I^*$ occurs again in $\tilde{S}$ for the first time, the indices of I* that occurred in $\tilde{S}$ strictly between $\tilde{S}_r$ and $\tilde{S}_t$ have a value in $\tilde{S}_t$ higher than the index l.

An s-monotone pivot rule applied to an FLCP is symmetric, if $\tilde{s}(i) = \tilde{s}(j)$ always hold for any $i \in I$.

The criss-cross algorithm presented in this paper will apply the s-monotone rules in a symmetric way. As examples for s-monotone rules, below is the definition of an s vector for the minimal index.

2.14. Minimal index/ Bland

The variables with the smallest index is selected.Initialize$\tilde{s}$ consisting of constant entries

$$\tilde{s}_i = \text{n-i},\ i \in I$$

where n is the number of the problem columns in the problem.

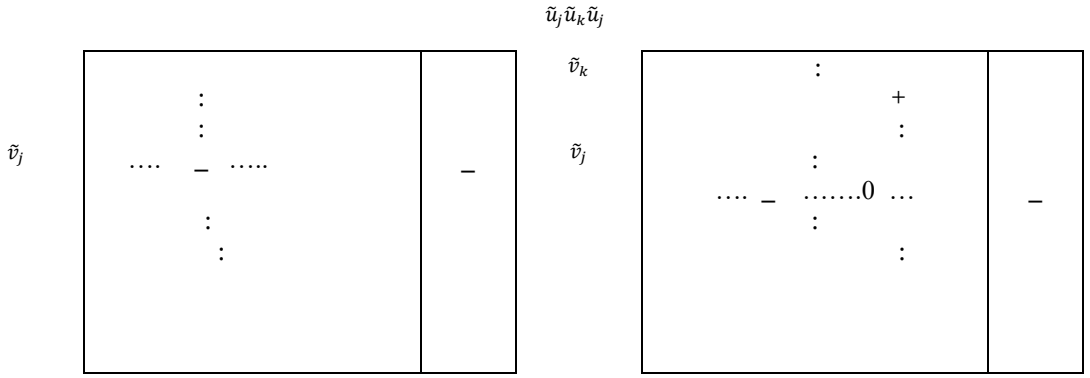
The criss-cross method

This section presents the criss-cross algorithm for fuzzy linear complementarity problems using s-monotone index selection rules.

The criss-cross algorithms for FLCP need a starting fuzzy complementary solution, for which $\tilde{u} = 0$ and $\tilde{v} = \tilde{q}$ is a possible selection using the assumption that rank($\tilde{A}$) = n. Starting from the initial complementarity solution, the criss-cross method performs a sequence of so called diagonal and exchange pivots.

If in any complementary basis during the algorithm $\tilde{v}_j$ is a basic variable and the value of variable $\tilde{v}_j$ is infeasible and $\tilde{a}_{jj} < 0$, then the algorithm, may perform a diagonal pivot where variable $\tilde{u}_j$ enters the basis while variable $\tilde{v}_j$ leaves it.

If $\tilde{a}_{jj} = 0$ and the algorithm select $\tilde{v}_j$ to leave the bases, then it pivots on such an index k for which $\tilde{a}_{jk} < 0$. The solution obtained after a pivot like this is not complementary any more, so to restore the complementarity of the solution the algorithm pivots on the position (k,j) as well. According to lemma 2.2, in case of fuzzy sufficient matrices $\tilde{a}_{kj} > 0$ will hold in this situation. These two pivots together are called an exchange pivot



Diagonal pivotExchange pivot

Duringan exchange pivot, variables $\tilde{u}_j$ and $\tilde{u}_k$ enter the basis, while $\tilde{v}_j$ and $\tilde{v}_k$ leave it.We say that $\tilde{u}_j$and $\tilde{v}_j$are chosen actively, while $\tilde{u}_k$ and $\tilde{v}_k$ are chosen passively. Thus, the terms passively and actively refer to the order in which the indices are selected, irrespective of the variable types (i.e. the same rule applies to an exchange pivot involving different combinations like ($\tilde{v}_j$,$\tilde{u}_k$) exchanged with ($\tilde{u}_j$,$\tilde{v}_k$). The two types of pivot operations are presented in the above figure.

When using s-monotone index selection rules, the role of the $\tilde{s}$-vector is to maintain a history about the movements of the variables. In the FLCP case, these updates are generalized as follows: in case of an exchange pivot, the fuzzy vector $\tilde{s}$ of the s-monotone pivot rule is updated to the passively selected pair of variables first, and only then for the actively selected pair, thus the symmetry of fuzzy vector $\tilde{s}$ (i.e. the value of any variable pair ($\tilde{u}_i$,$\tilde{v}_i$) in $\tilde{s}$ is the same after each pivot) is also maintained. This way, from the $\tilde{s}$ vector's point of view, an exchange pivot is considered as two pivots.

III. Criss-Cross Type Algorithm with Symmetric S-Monotone Pivot Rule for Fuzzy Linear Complementarity Problems

Step 1: Initialization: T = -$\tilde{M}$, $\tilde{\tilde{q}} = \tilde{q}$, r=1 and initialize $\tilde{s}$.
Step 2: If J = {i ∈ I: $\tilde{\tilde{q}}_i < 0$} and J ≠ ∅ then go to next step.
Step 3: J_{max} = {j ∈ J |$\tilde{s}$(j) ≥ $\tilde{s}$(α), for all α ∈ J}, let k ∈ J_{max} arbitrary. Go to next step.
Step 4: Diagonal pivot: If ($\tilde{\tilde{m}}_{kk} < 0$) then diagonal pivot on $\tilde{\tilde{m}}_{kk}$, update vector $\tilde{s}$ for variable pair ($\tilde{u}_k$, $\tilde{v}_k$). Go to next iteration (i.e. r = r + 1).
Step 5: suppose K = {i ∈ I: $\tilde{\tilde{m}}_{ki} < 0$}. If K = ∅ then stop. The FLCP has no feasible solution.
Step 6: Exchange pivot: Suppose K_{max} = {β ∈ K |$\tilde{s}(\beta) \geq \tilde{s}$(α), for all α ∈K }, let l ∈ K_{max} arbitrary. Exchange pivot on $\tilde{\tilde{m}}_{kl}$ and $\tilde{\tilde{m}}_{lk}$.

Step 7: Update fuzzy vector $\tilde{s}$ for variable pair $(\tilde{u}_l,\tilde{v}_l)$ as in iteration r + 1, then for variable pair $(\tilde{u}_k,\tilde{v}_k)$ as in iteration r + 2. Let r = r + 2.
Step 8: Repeat the procedure until we get the fuzzy feasible complementary solution.

IV. Numerical Example

Consider the FLCP$(\tilde{q},\tilde{M})$ with a fuzzy non-sufficient matrix where

$$\tilde{M} = \begin{bmatrix} [1,1] & [-1,-1] & [-1,-1] & [0,0] & [-1,-1] \\ [-1,-1] & [2,2] & [0,0] & [0,0] & [-1,-1] \\ [1,1] & [-2,-2] & [-1,-1] & [-2,-2] & [0,0] \\ [-4,-4] & [-1,-1] & [-1,-1] & [2,2] & [4,4] \\ [-1,-1] & [0,0] & [2,2] & [0,0] & [1,1] \end{bmatrix}$$

and $\tilde{q} = \begin{bmatrix} [-1,-1] \\ [-2,-2] \\ [16,16] \\ [6,6] \\ [4,4] \end{bmatrix}$

Solution:
To demonstrate the criss-cross method, consider the FLCP with the corresponding short pivot tableau where the identity matrix corresponding to $\tilde{v}$ serves as an initial complementary basis.
Solving the problem with the criss-cross method, pivoting first on diagonal elements $(\tilde{u}_1,\tilde{v}_1)$ and $(\tilde{u}_2,\tilde{v}_2)$.

Table 1

	$\tilde{u}_1$	$\tilde{u}_2$	$\tilde{u}_3$	$\tilde{u}_4$	$\tilde{u}_5$	
$\tilde{v}_1$	**[-1,-1]**	[1,1]	[1,1]	[0,0]	[1,1]	[-1,-1]
$\tilde{v}_2$	[1,1]	[-2,-2]	[0,0]	[0,0]	[1,1]	[-2,-2]
$\tilde{v}_3$	[-1,-1]	[2,2]	[1,1]	[2,2]	[0,0]	[16,16]
$\tilde{v}_4$	[4,4]	[1,1]	[1,1]	[-2,-2]	[-4,-4]	[6,6]
$\tilde{v}_5$	[1,1]	[0,0]	[-2,-2]	[0,0]	[-1,-1]	[4,4]

Table 2

	$\tilde{v}_1$	$\tilde{u}_2$	$\tilde{u}_3$	$\tilde{u}_4$	$\tilde{u}_5$	
$\tilde{u}_1$	[-1,-1]	[-1,-1]	[-1,-1]	[0,0]	[-1,-1]	[1,1]
$\tilde{v}_2$	[1,1]	**[-1,-1]**	[1,1]	[0,0]	[2,2]	[-3,-3]
$\tilde{v}_3$	[-1,-1]	[1,1]	[0,0]	[2,2]	[-1,-1]	[17,17]
$\tilde{v}_4$	[4,4]	[5,5]	[5,5]	[-2,-2]	[0,0]	[2,2]
$\tilde{v}_5$	[1,1]	[1,1]	[-1,-1]	[0,0]	[0,0]	[3,3]

then $(\tilde{u}_4, \tilde{v}_4)$:

Table 3

	$\tilde{v}_1$	$\tilde{v}_2$	$\tilde{u}_3$	$\tilde{u}_4$	$\tilde{u}_5$	
$\tilde{u}_1$	[-2,-2]	[-1,-1]	[-2,-2]	[0,0]	[-3,-3]	[4,4]
$\tilde{u}_2$	[-1,-1]	[-1,-1]	[-1,-1]	[0,0]	[-2,-2]	[3,3]
$\tilde{v}_3$	[0,0]	[1,1]	[1,1]	[2,2]	[1,1]	[14,14]
$\tilde{v}_4$	[9,9]	[5,5]	[10,10]	**[-2,-2]**	[10,10]	[-13,-13]
$\tilde{v}_5$	[2,2]	[1,1]	[0,0]	[0,0]	[2,2]	[0,0]

Table 4

	$\tilde{v}_1$	$\tilde{v}_2$	$\tilde{u}_3$	$\tilde{v}_4$	$\tilde{u}_5$	
$\tilde{u}_1$	[-2,-2]	[-1,-1]	[-2,-2]	[0,0]	[-3,-3]	[4,4]
$\tilde{u}_2$	[-1,-1]	[-1,-1]	[-1,-1]	[0,0]	[-2,-2]	[3,3]
$\tilde{v}_3$	[9,9]	[6,6]	[11,11]	[2,2]	[11,11]	[1,1]
$\tilde{u}_4$	[-4.5,-4.5]	[-2.5,-2.5]	[-5,-5]	[-2,-2]	[-5,-5]	[6.5,6.5]
$\tilde{v}_5$	[2,2]	[1,1]	[0,0]	[0,0]	[2,2]	[0,0]

Hence a fuzzy feasible complementary solution for a given problem is $\tilde{u}_1$=[4,4],$\tilde{u}_2$=[3,3], $\tilde{v}_3$=[1,1], $\tilde{u}_4$=[6.5,6.5] and $\tilde{u}_3$=$\tilde{u}_5$=$\tilde{v}_1$=$\tilde{v}_2$=$\tilde{v}_4$=$\tilde{v}_5$=0.

V. Conclusion

In this paper, criss-cross type algorithm with symmetric s-monotone pivot rule for fuzzy linear complementarity problem is suggested. The effectiveness of the proposed method is verified by a numerical example.

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EOQ Models for Non – Deteriorative Items With Constant, Linear and Quadratic Demand Under Inflation - in Third and Fourth Order Equations With a Comparative study

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Abstract: *The classical economic order quantity (EOQ) model is a cost-minimization inventory model with a constant demand rate. It is the most successful model of all inventory theories because it is simple to understand and easy to apply. The effect of inflation and time value of money is very important and it is now included in most of the recent papers concerning inventory. High inflation rates in some countries could determine the entire economic development. Since, money tied up in inventories could decrease their actual value over time, the effect of inflation rate should be reflected in the development of inventory policy. Also, most of the works on inventory models do not take the optimal solution in higher order equation. In this paper, the effect of inflation in inventory models with time dependent demand (constant, linear and quadratic) and the optimum solution is derived in higher order equations. Three models are developed and the optimum time and total cost are derived when the demand is (i) constant, (ii) linear and (iii) quadratic. A comparative study between constant, linear and quadratic demand and it is ascertained that the holding cost and total cost are less in quadratic demand compared to constant and linear demand. Mathematical models are developed for each model and the optimum time and optimal lot size which minimizes the total cost is derived. Numerical examples are given to illustrate the theoretical results and made the sensitivity analysis of parameters on the optimal solutions. The validation of result, this model was coded in Microsoft Visual Basic 6.0*

Key words*: Inventory, Deteriorating, Constant, Linear, Quadratic Demand, Discounted cost.*

I. Introduction

The primary operation strategies and goals of most firms are to seek a high satisfaction to customer's demands and to become a low-cost producer. To achieve these goals, the company must be able to effectively utilize resources and minimize costs. The economic order quantity (EOQ) model was the first mathematical model introduced several decades ago to assist corporations in minimizing total inventory costs. It balances inventory holding and setup costs and derives the optimal order quantity. Regardless of its simplicity, the EOQ model is still applied industry-wide today. The inventory models considers the ideal case where the value of inventory items are unaffected by time and replenishment is done instantaneously. In real life cases, however, the ideal case is not quite as applicable. Inventories are often replenished periodically. Even for purchased items, when the supply arrives at the ware house, it may take days for the receiving department to completely transfer the supply to the storage room. Demand is the major factor in the inventory management. Therefore, decisions of inventory are to be made because of the present and future demands. Take all these points in the mind, the economic order quantity model with constant, linear and quadratic demand is considered in this paper and a comparative study is also carried out. Economic order quantity (EOQ) models have been studied since **Harries (1913)** presented the famous EOQ formulae. **Dave and Patel (1981)** considered an EOQ model in which the demand rate is changing linearly with time and the deteriorating is assumed to be a constant fraction of the onhand inventory. Chund and Tsai (1997) considered the inventory replenishment policy over a fixed planning period for a deteriorating item having a deterministic demand pattern with a linear trend and shortages. **Skouri and Papachristos (2002)** developed an inventory model for the deterioration of items occurs at a fixed rate independent of time. The model allows for partially backlogging and the backlogging rate is an exponentially decreasing, time-dependent function specified by a parameter. **Ghosh and Chaudhuri (2004)** developed an inventory model for a deteriorating item, a quadratic time-vary demand and shortages in inventory. A two-parameter weibull distribution is taken to represent the time of deterioration. **Ghosh and Chaudhuri (2006)** developed an EOQ model over a finite time-horizon for a deteriorating item with a quadratic, time-dependent demand, allowing shortages in inventory and the rate of deterioration is taken to be

time-proportional and it is assumed that shortage occur in every cycle. **Ajanta Roy (2008)** developed a deterministic inventory model when the deterioration rate is time proportional and demand rate is a function of selling price and holding cost is time dependent. **Mingbao Cheng and Guequing Wang (2009)** developed an inventory model for deteriorating items with trapezoidal type demand rate, that is the demand rate is a piecewise linearly function. It is proposed an inventory, replenishment policy for this type of inventory model. The numerical solution of the model is obtained and also examined. **BiswaranjanMandal (2010**) considered in which it is depleted not only by demand but also by deterioration. The Weibull distribution, which is capable of representing constant, increasing and decreasing rate of deterioration, is used to represent the distribution of the time to deterioration. **KripasindhuChauduri al et. (2011)** an economic order quantity inventory problem is discussed over a finite time for deteriorating items with shortages, where the demand rate is of the ramp-type. It is assumed that a constant fraction of the on-hand inventory deteriorates per unit time and time value of money and the effects of inflation are taken into account. **Mingbao Cheng et al. (2011**) considered an inventory model for time-dependent deteriorating items with trapezoidal type demand rate and partial backlogging. **Ghosh et al. (2012)** considered an optimal inventory replenishment policy for a deteriorating items time-quadratic demand and time-dependent partial backlogging which depends on the length of the waiting time for the next replenishment over a finite time horizon and variable replenishment cycle. **Vinod Kumar Mishra et al.(2013)** considered a deterministic inventory model with time-dependent demand and time-varying holding cost where deteriorating is time proportional and the model considered here allows shortages and the demand is partially backlogged. **Sana et al. (2015)** analyzed a single-period newspaper inventory model to define the optimum order quantity where the consumers flinching occur and depend on holding cost. Shortages are allowed and partially backlogged. **Sana and De (2016)** developed an economic order quantity model for fuzzy variables with promotional effort and selling price dependent demand. They observed that demand rate decreases over time during shortage period. **Manna and Manna (2016)** developed an order level inventory system for deteriorating items with demand rate as a ramp type function of time. The finite production rate is proportional to the demand rate and the deterioration rat is independent of time. The unit production cost is inversely proportional to the demand rate. Shortages are not considered in this model. **Mohammad et al. (2016)** develops a time dependent inventory model on the basis of constant production rate and market demands which are exponentially decreasing. It advances in quest of total average optimum cost considering those products which have finite shelf-life. The model also considers the small amount of decay. Without having any sort of backlogs, production starts. Reaching at the desired level of inventories, it stops production. After that due to demands along with the deterioration of the items it initiates its depletion and after certain periods the inventory gets zero. The decay of the products is level dependent.**Kaliraman, et al. (2017)** considered a two warehouse inventory model for deteriorating items with exponential demand rate and permissible delay in payment. Shortages are not considered and deteriorating rate is constant. In this model, one warehouse is rented and the other is owned. The rented warehouse is provided with better facility for the stock than the owned warehouse but it is charged more. The objective of this model is to find the best replenishment policies for minimizing the total appropriate inventory cost. **Tripathi et al. (2017)**established inventory model with exponential time-dependent demand and time- dependent time-dependent deterioration. The purpose of this article is to present a solution procedure for the inventory problem for constant, linear and quadratic trend in demand taking account of time value. Three models are developed. In three models, the optimum time and total cost are derived when the demand is (i) constant, (ii) linear; (iii) quadratic. The remaining of the paper is organized as follows: Section 2 presents the assumptions and notations. Section 3 is for problem formulation and numerical examples. Section 4 is for comparative study. Finally, the paper summarizes and concludes in section 5.

II. Assumptions and Notations

A. Assumptions: i) The demand rate is constant per unit/ year (D), linear ($a+bt$) and quadratic function ($a+bt+ct^2$) where a>0, $b \neq 0, c \neq 0$, at time t and it is continuous function of time. Here, a,b and c are constants and "a" stands for the initial demand "b" and "c" are positive trend in demand. ii) The deteriorating rate is constant, iii) The planning horizon is finite.iv) Lead time is zero. v) There is no repair or replacement of the deteriorated items.vi) Items are produced and added to the inventory and the item is a single product; it does not interact with any other inventory items.

B. Notations: D – Demand rate in units per unit time, Q*-Optimal size of production run , C_0 - ordering cost/order, C_h - holding cost per unit/time, T – Cycle time, r – rate of interest, TC - Total cost.

III. Mathematical Model

The methodology adopted in this paper involves a number of steps. First, the differential inventory equation for the period is developed. Next, these differential equations are solved to formulate the cost model. The details of this methodology are discussed below. In order to develop the differential equation, we need to define the one stage of the inventory cycle shown in figure-1, a simplified representation of the cycle.

A.An EOQ Model for non-Deteriorative items with Constant Demand under Inflation:

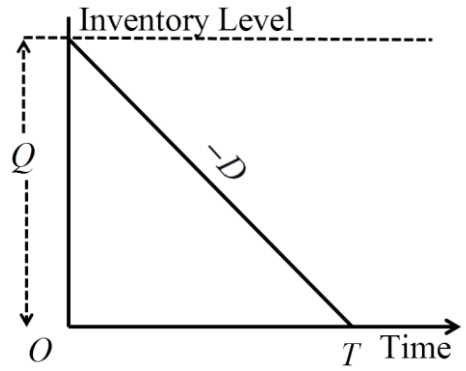


Figure – 1 Purchasing Inventory model (or) Economic Order Quantity Model

The methodology adopted in this paper involves a number of steps. First, the differential inventory equation for all the period is developed. Next, these differential equations is solved to formulate the cost model. The details of this methodology are discussed below. In order to develop the differential equation, we need to define the two stages of the production inventory cycle shown in figure-1

$$\frac{dI(t)}{dt} = -D;\ 0 \le t \le T \qquad (1)$$

With the boundary conditions $I(0) = Q;\ I(T) = 0$ (2)

The solutions of above differential equation is

From (1); $I(t) = D(T-t)$ (3)

From the equation (2), and boundary conditions

$$I(0) = Q \Rightarrow Q = DT \qquad (4)$$

Total cost: The total cost comprises of the sum of the setup cost and holding cost. They are grouped together after evaluating the above cost individually.

Therefore, total cost = Setup cost + holding cost

1. Setup cost = C_0 (5)
2. Holding cost = $C_h \int_0^T I(t)e^{-rT}dt = C_h \int_0^T D(T-t)e^{-rt}dt$

   On simplification,

   $$= DC_h\left[\frac{e^{-rt}}{r^2} - \frac{1}{r^2} + \frac{T}{r}\right] = \frac{DC_h}{r^2}\left[\left(e^{-rT} - 1\right) + rT\right] \qquad (6)$$

Therefore, Total cost (from the equations (5) to (6))

$$TC = C_0 + \frac{DC_h}{r^2}\left[\left(e^{-rT} - 1\right) + rT\right] \qquad (7)$$

According to Figure 2, the present value TC (T), for the first period is repeated at the start of each of the subsequent cycles. The present value of the total cost for N cycles, TC(T) can then be calculated as follows:

$$TC(T) = P\left(1 + e^{-rT} + e^{-2rT} + \ldots\ldots\ldots\ldots + e^{-(N-1)rT}\right)$$

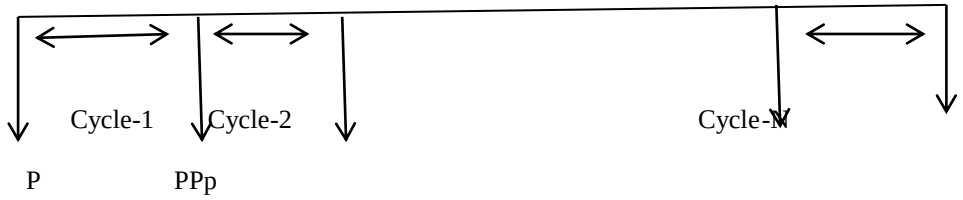


Figure 2 Cash Flow Diagram

For an infinite planning horizon, $N \to \infty$ and TC (T) can be expressed as $TC(T) = P\left(\frac{1}{1-e^{-rT}}\right)$

The present value of the total cost

$$\text{TC(T)} = \frac{C_0}{1-e^{-rT}} + \frac{DC_h}{r^2}\left[-1+\frac{rT}{1-e^{-rT}}\right] \quad (8)$$

Optimality: It can be easily shown that TC (T) is a convex function in T. Hence, an optimal cycle time T can be calculated from

$$\frac{d}{dT}TC(T) = 0 \text{ and } \frac{d^2}{dT^2}TC(T) > 0$$

Differentiate the equation (8) with respect to T,

$$-rC_0e^{-rT} + \frac{DC_h}{r^2}\left[r\left(1-e^{-rT}\right)-r^2Te^{-rT}\right] = 0$$

$$and \quad \frac{d^2(TC)}{dT} > 0$$

For simplification, multiply by e^{-rT}

$$-rC_0 + \frac{DC_h}{r^2}\left[r\left(e^{rT}-1\right)-r^2T\right] = 0$$

$$\frac{T^2}{2} + \frac{rT^3}{6} = \frac{C_0}{DC_h}$$

$$rT^3 + 3T^2 - \frac{6C_0}{DC_h} = 0 \quad (9)$$

which is the optimum solution for T in third order equation.,

Note: When r=0 then T = $\sqrt{\frac{2C_0}{DC_h}}$ and Q = $\sqrt{\frac{2DC_0}{C_h}}$ which is standard inventory models.

For example, D = 4500, C_0=100, C_h =10, and r=0.1

Optimum Solution: The third order equation is

$$0.1T^3 + 3T^2 - 0.01330 = 0$$

(T = 0.0665, -29.999, -0.0666) one positive root and two negative roots. Position root T=0.0665 is consider for this research. The optimum solution is

T = 0.0666, Q = 300, Setup cost = 1500.16,

Holding Cost = 1499.50,Total cost = 2999.67

Sensitivity Analysis

Table 1

Effect of Constant Demand and cost parameters on optimal values

PARAMETERS		OPTIMAL VALUES				
		T	Q	SETUP COST	HOLDING COST	TOTAL COST
C_0	80	0.0596	268.30	1341.77	1341.24	2683.01
	90	0.0632	284.57	1423.17	1422.57	2845.75
	100	0.0666	300.00	1500.16	1499.50	2999.67
	110	0.0699	314.61	1573.40	1572.66	3146.06
	120	0.0730	328.59	1643.37	1642.56	3285.93
C_h	8	0.0745	335.37	1341.81	1341.14	2682.95
	9	0.0903	316.19	1423.19	1422.52	2845.72
	10	0.0666	300.00	1500.16	1499.50	2999.67
	11	0.0635	286.01	1573.38	1572.71	3146.09
	12	0.0608	286.01	1573.38	1572.71	3146.09
r	0.1	0.0666	300.00	1500.16	1499.50	2999.67
	0.2	0.0666	299.93	1500.33	1499.00	2999.33
	0.3	0.0666	299.90	1500.50	1498.50	2999.00
	0.4	0.0666	299.86	1500.66	1498.00	2998.67
	0.5	0.0666	299.83	1500.83	1497.50	2998.33

Observations: From the table 1, it is observed that 1) with the increase in setup cost per unit (C_0), optimum quantity (Q*), cycle time (T), setup cost, holding cost and total cost increases, then there is positive relationship between them, 2) with the increase in holding cost per unit (C_h), optimum quantity (Q*), holding cost (T) decreases, then there is negative relationship between them but set up cost, holding cost and total cost increases then there is positive relationship between them.

BAn EOQ model for Non-Deteriorative items with Linear Demand under inflation:

The instantaneously inventory level I(t) is given by the following differential equation.

$$\frac{d}{dt}I(t) = -(a+bt) \tag{10}$$

with the boundary conditions are I(0) = Q and I(T) =0 (11)

From the equation (10), $I(t) = a(T-t) + \frac{b}{2}\left(T^2 - t^2\right)$ (12)

From the equations (10) and (11),

$$I(0) = Q \Rightarrow Q = aT + \frac{bT^2}{2} \tag{13}$$

Total cost: Total cost comprised of the sum of the setup cost and holding cost. They are grouped together after evaluating the above cost individually.

1. Ordering cost = C_0 (14)
2. Holding cost = $C_h\left[\int_0^T I(t)e^{-rt}dt\right]$

$$= C_h\left[\int_0^T \left(a(T-t) + \frac{b}{2}\left(T^2 - t^2\right)e^{-rt}\right)dt\right]$$

$$= C_h\left[\frac{1}{r^2}(a+bT)e^{-rT} + \frac{b}{r^3}e^{-rT} + \frac{1}{r}\left(aT + \frac{bT^2}{2}\right) - \frac{a}{r^2} - \frac{b}{r^3}\right]$$

$$= C_h\left[\frac{a}{r^2}\left(e^{-rT} - 1\right) + \frac{b}{r^3}\left(e^{-rT} - 1\right) + \frac{bT}{r^2}e^{-rT} + \frac{aT}{r} + \frac{bT^2}{2r}\right] \tag{15}$$

Total cost = Ordering cost + Holding cost

$$\text{TC (T)} = C_0 + C_h\left[\frac{a}{r^2}\left(e^{-rT} - 1\right) + \frac{b}{r^3}\left(e^{-rT} - 1\right) + \frac{bT}{r^2}e^{-rT} + \frac{aT}{r} + \frac{bT^2}{2r}\right] \tag{16}$$

According to Figure 2, the present value of the total cost TC(T)

$$\text{TC (T)} = \frac{C_0}{1-e^{-rT}} + C_h\left[-\frac{a}{r^2} - \frac{b}{r^3} + \frac{bTe^{-rT}}{r^2(1-e^{-rT})} + \frac{aT}{r(1-e^{-rT})} + \frac{bT^2}{2r(1-e^{-rT})}\right] \tag{17}$$

Differentiate the equation (14) with respect to T and simplify, we get,

$$-rC_0e^{-rT} + C_h\left[\begin{array}{l}\frac{1}{r^2}\left\{\left(1-e^{-rT}\right)\left(-rTe^{-rT} + e^{-rT} - rTe^{-2rT}\right)\right\} \\ + \frac{a}{r}\left\{1 - e^{-rT} - rTe^{-rT}\right\} + \frac{b}{2r}\left\{2T - 2Te^{-rT} - rT^2e^{-rT}\right\}\end{array}\right] = 0$$

$$\frac{b}{r^2}\left[\left(e^{rT} - 1\right)\left(-rT + 1\right) - rTe^{-rT}\right)\right] + \frac{a}{r}\left[e^{rT} - 1 - rT\right] + \frac{b}{2r}\left[2Te^{rT} - 2T - rT^2\right] = \frac{rC_0}{C_h}$$

$$\left(\frac{br}{6} + \frac{ar^2}{6} + \frac{br}{2}\right)T^3 + \frac{ar}{2}T^2 = \frac{rC_0}{C_h}$$

$$(ar+4b)T^3+3aT^2-\frac{6C_0}{C_h}=0 \quad (18)$$

which is the solution for optimal cycle time in third order equation.

Note: When a = D and b =0, then T = $\sqrt{\frac{2C_0}{DC_h}}$ and Q = $\sqrt{\frac{2DC_0}{C_h}}$ which is standard inventory models.

For example, $C_0=100$, $C_h=10$, a = 4250, b = 3790

Optimal Solution: The third order equation is

$$15585T^3+12750T^2-60=0$$

T = 0.06599, - 0.81226, - 0.071825 has the three value out of one positive real value and two negative real values. The positive real value T = 0. 06599 is considered for this model.

Therefore, T = 0.0659, Q = 296.96, Setup cost = 1515.39, Holding cost = 1454.02 and Total cost = 2969.41

Sensitivity Analysis:

Table 2
Effect of Linear Demand and cost parameters on optimal values

PARAMETERS		OPTIMAL VALUES				
		T	Q	SETUP COST	HOLDING COST	TOTAL COST
C_0	80	0.0592	266.62	1350.22	1300.77	2651.00
	90	0.0627	280.64	1434.96	1379.54	2814.50
	100	0.0659	296.96	1515.39	1454.02	2969.41
	110	0.0691	310.05	1592.14	1524.86	3117.00
	120	0.0720	325.04	1665.70	1592.54	3258.24
C_h	8	0.0734	330.61	1361.13	1300.25	2661.38
	9	0.0694	314.38	1440.41	1379.28	2819.70
	10	0.0659	296.96	1515.39	1454.02	2969.41
	11	0.0630	284.44	1586.68	1525.12	3111.81
	12	0.0604	271.26	1654.80	1593.06	3247.86
a	3950	0.0682	286.32	1466.88	1401.85	2868.73
	4150	0.0667	294.01	1499.37	1436.84	2936.21
	4250	0.0659	296.96	1515.39	1454.02	2969.41
	4350	0.0653	300.59	1531.25	1471.01	3002.27
	4450	0.0646	303.65	1546.97	1487.81	3034.78
b	3590	0.0661	296.64	1512.58	1453.91	2966.50
	3690	0.0660	296.83	1513.99	1453.97	2967.96
	3790	0.0659	296.96	1515.39	1454.02	2969.41
	3890	0.0659	297.12	1516.78	1454.08	2970.87
	3990	0.0658	297.26	1518.18	1454.14	2972.32
r	0.1	0.0659	296.96	1515.39	1454.02	2969.41
	0.2	0.0659	296.67	1516.87	1449.29	2966.17
	0.3	0.0658	296.36	1518.35	1444.58	2962.93
	0.4	0.0658	296.06	1519.83	1439.88	2959.72
	0.5	0.0657	295.75	1521.30	1435.21	2956.52

Managerial insights: A sensitivity analysis is performed to study the effects of change in the system parameters, ordering cost (C_0), holding cost per unit/year (C_h), r – rate of interest, constant demand (a), and varying demand (b) on optimal values that is optimal cycle time (T), optimal quantity (Q), setup cost, holding cost and total cost. The sensitivity analysis is performed by changing (increasing or decreasing) the parameter taking at a time, keeping the remaining parameters at their original values. The following influences can be obtained from sensitivity analysis based on table 2. 1) with the increase in setup cost per unit (C_0), optimum quantity (Q*), cycle time (T), setup cost, holding cost, and total cost increases then there is positive relationship between them, 2) with the increase in holding cost per unit per unit time (C_h), the setup cost, holding cost and total cost increases then there is positive relationship between them but optimal cycle time (T) and optimal lot size (Q), decreases then there is negative relationship between, 3) Similarly, other parameters initial demand (a), time dependent demand (b), and rate of interest (r) can also be observed from the table 2.

C.An EOQ Model for Non-Deteriorative items with Quadratic Demand under Inflation:

The instantaneously inventory level I(t) is given by the following differential equation

$$\frac{d}{dt}I(t) = -(a + bt + ct^2); 0 < t < T \tag{19}$$

with the boundary conditions are I(0) = Q ; I(T) = 0 (20)

The solution of equation (19) is

$$\text{I(t)} = a(T-t) + \frac{b}{2}\left(T^2 - t^2\right) + \frac{c}{3}\left(T^3 - t^3\right) \tag{21}$$

Total cost: The total cost comprises of the sum of the setup costand holding cost . They are grouped together after evaluating the above cost individually.Therefore, total cost = Setup cost and holding cost + deteriorating cost

1. Setup cost = C_0 (22)
2. Holding cost = $C_h\left[\int_o^T I(t)e^{-rt}dt\right]$

$$= C_h\int_o^T\left[a(T-t) + \frac{b}{2}\left(T^2 - t^2\right) + \frac{c}{3}\left(T^3 - t^3\right)\right]e^{-rt}dt$$

$$= C_h\left[\begin{array}{l}\frac{1}{r^2}\left(a + bT + cT^2\right)e^{-rt} + \frac{1}{r^3}\left(b + 2cT\right)e^{-rt} + \frac{2c}{r^4}e^{-rt} \\ + \frac{1}{r}\left(aT + \frac{bT^2}{2} + \frac{bT^3}{3}\right) - \frac{a}{r^2} - \frac{b}{r^3} - \frac{2c}{r^4}\end{array}\right] \tag{23}$$

Therefore, Total cost = Setup cost + holding cost + deteriorating cost

$$\text{TC (T)} = C_0 + C_h\left[\begin{array}{l}\frac{a}{r^2}\left(e^{-rT} - 1\right) + \frac{b}{r^3}\left(e^{-rT} - 1\right) + \frac{2c}{r^4}\left(e^{-rT} - 1\right) + \frac{bT}{r^2}e^{-rt} \\ + \frac{c}{r^2}T^2e^{-rT} + \frac{2c}{r^3}e^{-rT} + \frac{aT}{r} + \frac{bT^2}{2r} + \frac{cT^3}{3r}\end{array}\right] \tag{24}$$

According to Figure 2, the present value of the total cost TC(T)

$$= \frac{C_0}{1-e^{rT}} + C_h\left[\begin{array}{l}-\frac{a}{r^2} - \frac{b}{r^3} - \frac{2c}{r^4} + \frac{b}{r^2}\left(\frac{Te^{-rT}}{1-e^{-rT}}\right) + \frac{c}{r^2}\left(\frac{T^2e^{-rT}}{1-e^{-rT}}\right) + \frac{2c}{r^3}\left(\frac{Te^{-rT}}{1-e^{-rT}}\right) \\ + \frac{a}{r}\left(\frac{T}{1-e^{-rT}}\right) + \frac{b}{2r}\left(\frac{T^2}{1-e^{-rT}}\right) + \frac{c}{3r}\left(\frac{T^3}{1-e^{-rT}}\right)\end{array}\right] \tag{25}$$

Optimality: It can be easily shown that TC (T) is a convex function in T. Hence, an optimal cycle time T can be calculated from

$$\frac{d}{dT}TC(T) = 0 \text{ and } \frac{d^2}{dT^2}TC(T) > 0$$

Differentiating the total cost equation (25) with respect to T, then

$$= -rC_0e^{-rT} + C_h\left[\begin{array}{l} \frac{b}{r^2}\left\{(1-e^{-rT})\left(e^{-rT} - rTe^{-rT}\right) - rTe^{-rT}\right\} + \frac{c}{r^2}\left\{(1-e^{-rT})\left(\begin{array}{l} 2Te^{-rT} - rT^2e^{-rT} \\ -rT^2e^{-2rT} \end{array}\right)\right\} \\ + \frac{2c}{r^3}\left\{(1-e^{-rT})\left(e^{-rT} - rTe^{-rT}\right) - rTe^{-rT}\right\} + \frac{a}{r}\left\{(1-e^{-rT}) - rTe^{-rT}\right\} \\ + \frac{b}{2r}\left\{2T(1-e^{-rT}) - rT^2e^{-rT}\right\} + \frac{c}{3r}\left\{3T^2(1-e^{-rT}) - rT^3e^{-2rT}\right\} \end{array}\right]$$

and $\frac{d^2}{dT^2}(TC) => 0$

$$= \left[\begin{array}{l} \frac{b}{r^2}(1 - rT - -e^{-rT}) + \frac{c}{r^2}\left(2T - rT^2 - 2Te^{-rT}\right) + \frac{2c}{r^3}\left\{1 - rT - e^{-rT}\right\} \\ + \frac{a}{r}\left\{e^{-rT} - r - rT\right\} + \frac{b}{2r}\left\{2T(e^{rT} - 1) - rT^2\right\} + \frac{c}{3r}\left\{3T^2(e^{rT} - 1) - rT^3\right\} \end{array}\right] = \frac{rC_0}{C_h}$$

On simplification,

$$\left[\frac{brT^3}{6} + \frac{crT^4}{3} + \frac{arT^2}{2} + \frac{ar^2T^3}{6} + \frac{brT^3}{2} + \frac{ar\theta^2T^4}{6} + \frac{br^2T^4}{6} + \frac{crT^4}{2}\right] = \frac{rC_0}{C_h}$$

$$\frac{aT^2}{2} + \frac{5cT^4}{6} + \frac{2bT^3}{3} + \frac{arT^3}{6} + \frac{brT^4}{6} = \frac{C_0}{C_h}$$

$$(br + 5c)T^4 + (ar + 4b)T^3 + 3aT^2 - \frac{6C_0}{C_h} = 0 \qquad (26)$$

which is the equation for optimum solution in fourth order.

Note: When r=0, b =0 , c=0 and a = D then $T = \sqrt{\frac{2C_0}{DC_h}}$ Which is the standard inventory model.

Numerical example,

D = 4500, C_0 =100, C_h =10, θ = 0.01 to 0.10; C_d =100, a = 4250, b = 3600, c=2900,

Optimum Solution: The fourth order is $T^4 + 1.9190T^3 + 2.2112T^2 - 0.10104 = 0$

$T = 0.0629, -0.682, -0.8586 \pm i\,0.8099$ one positive root, one negative root and two complex roots in which positive root is considered for this work. The optimum solution is

T = 0.0666, Q = 299.85, Setup cost = 1500.72,

Holding cost = 1452.62, Total cost = 2953.34

Sensitivity Analysis:

Table 3

Effect of Demand and cost parameters on optimal values with quadratic demand

Parameters		Optimal Values				
		T	Q	Setup cost	Holding cost	Total cost
C_0	80	0.0598	269.01	1338.26	1299.75	2638.01
	90	0.0633	284.88	1421.63	1378.33	2799.96
	100	0.0666	299.85	1500.72	1452.62	2953.34
	110	0.0698	314.06	1576.14	1523.25	3099.40
	120	0.0728	327.59	1648.40	1590.72	3239.12
C_h	8	0.0742	334.13	1346.77	1298.71	2645.48
	9	0.0701	315.59	1425.89	1377.81	2803.70
	10	0.0666	299.85	1500.72	1452.62	2953.34
	11	0.0636	286.28	1571.89	1523.77	3095.66

	12	0.0610	274.41	1639.89	1591.76	3231.65
r	0.1	0.0666	299.85	1500.72	1452.62	2953.34
	0.2	0.0665	299.54	1502.31	1447.77	2950.08
	0.3	0.0664	299.22	1503.89	1442.95	2946.84
	0.4	0.0664	298.91	1505.47	1438.14	2943.61
	0.5	0.0663	298.60	1507.04	1433.36	2940.40
a	4050	0.0687	306.55	1467.94	1417.93	2885.87
	4150	0.0674	303.15	1484.41	1435.38	2919.79
	4250	0.0660	296.80	1516.15	1451.89	2968.05
	4350	0.0659	296.66	1516.86	1469.66	2986.52
	4450	0.0652	293.57	1532.85	1486.51	3019.36
b	2460	0.0667	300.43	1497.83	1452.53	2950.36
	2560	0.0666	300.14	1499.27	1452.58	2951.85
	2660	0.0660	296.80	1516.15	1451.89	2968.05
	2760	0.0665	299.57	1502.16	1452.67	2954.82
	2860	0.0665	299.28	1503.59	1452.71	2956.30
c	900	0.0666	299.90	1500.48	1452.76	2953.24
	1000	0.0666	299.88	1500.59	1452.69	2953.29
	1100	0.0660	296.80	1516.15	1451.89	2968.05
	1200	0.0666	299.83	1500.84	1452.55	2953.39
	1300	0.0666	299.81	1500.95	1452.48	2953.44

Observations: It is observed from the table 3, 1) with the increase in setup cost per unit (C_0), optimum quantity (Q*), setup cost, holding cost, cycle time (T) and total cost increases then there is positive relationship between them, 2) with the increase in holding cost per unit per unit time (C_h), the setup cost, holding cost and total cost increases then there is positive relationship between them but optimal cycle time (T) and optimal lot size (Q) decreases then there is negative relationship between, 3) with increase in the rate of interest (r), the setup cost increases then there is positive relationship between them but the optimal cycle time (T), optimal lot size (Q), holding cost and total cost decreases then there is negative relationship between them. Similarly, other parameters "a", "b", and "c" can also be observed from the table -3.

IV. Comparative Statement: A comparative statement is given in the following table

	HOLDING COST	SETUP COST	TOTAL COST
Constant Demand	1499.50	1500.00	2999.67
Linear Demand	1454.02	1515.39	2969.41
Quadratic Demand	1452.62	1500.72	2954.34

A comparative statement is carried out between constant, linear and quadratic demand and it is observed that the holding cost in linear demand is less with constant demand and the holding cost in quadratic demand is less compare with constant and linear demand. Similarly, the total cost is less in quadratic demand compare with constant and linear demand.

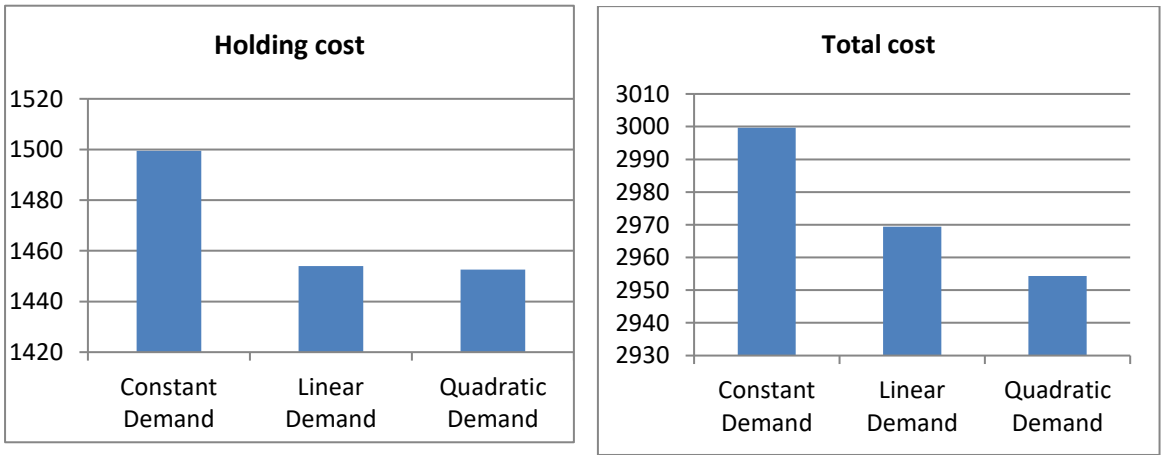


Figure 3 A comparative statement between constant, linear and quadratic Demand with Holding Cost and Total Cost

V. Conclusion:

This research considers inventory systems for purchasing inventory models where the objective is to find the optimal cycle time and optimal lot size which minimizes the total cost. Several aspects such as time value of money, constant, linear and quadratic demand are considered in developing different models. Three models are developed under inflation. It is observed that with the increase in setup cost per unit (C_0) , optimum quantity (Q*), setup cost, holding cost, cycle time (T) and total cost increases then there is positive relationship between them, with the increase in holding cost per unit per unit time (C_h), the setup cost, holding cost and total cost increases then there is positive relationship between them but optimal cycle time (T) and optimal lot size (Q) decreases then there is negative relationship between, with increase in the rate of interest (r), the setup cost increases then there is positive relationship between them but the optimal cycle time (T), optimal lot size (Q), holding cost and total cost decreases then there is negative relationship between them. The proposed model can assist the manufacturer and retailer in accurately determining the optimal quantity, cycle time and inventory total cost. Moreover, the proposed inventory model can be used in inventory control of certain items such as food items, fashionable commodities, stationary stores and others.

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AN M/M/1 QUEUE WITH DIFFERENTIATED WORKING VACATIONS AND IMPATIENCE CUSTOMERS

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Abstract: In this paper, we consider a class of queueing models with differentiated working vacations and impatience customers. Using the probability generating function, we give the joint distribution of the server state and the number of customers in the system in steady state. Finally, some performance measures and numerical examples are presented.

Keywords: Markovian queue, working vacation, steady state analysis, mean system size, expected queue length .

1. Introduction

Consider a system operating as a $M/M/1$ queue. In recent years, the study of vacation queues has a great effect on the queueing theory.Queueing system with server vacations have been extensively studied from $1970's$ onwards by various researchers. This system is applied in various fields such as computer system, communication networks etc..The appication of vacation model is available in the survey paper of Doshi[9] and takagi[8] and Zhang[10].

Waiting for service is usually an unpleasent experience and represents the loss of valuable resources, which translates into psychoogical as well as economic cost of waiting.The concept of waiting server with server vacation was first introduced by Boxma et al [11].There is now growing interest in the analysis of queueing systems with impatience customers.Laxmi and Jyothsna have discussed lot about the impatient customers in [12]. Queueing system with impatience customers reflects in many real life queueing system, particularly when dealing with human behaviour. Customers impatience has been dealt with in the queueing literature mainly in the context of customers abandoning the queue due to either a long wait alread experienced , or a long wait anticipated upon arrival.

Altman and Uri Yechiali [1] have discussed briefly about the impatience customers with single and multiple working vacations.They have also discussed the $M/M/\infty$ queueing model with impatient customers and vacations[6].D.Yue ,G.Xu and W.Yue have etended with impatience customers and working vacations[13].They have derived the probability geneating function of number of customers in the system and calculated values of key performances.Gowsmi[16] and Ammar[15] have discussed about two differentiated vacations. In [14] have discussed impatience customers with vacation and a waiting server in their paper.In [17] D.Yue ,Z.Saffer, and Chen all have consider an multiple vacations with impatience customers.

In this paper we consider the concept of customers impatience during a differentiated vacation policies with a waiting server.

The remaining part of the paper is organised as follows: In section2, we describe the model description and a transition diagram. In section 3, first we develop the probabaility generating functions of the steady state probabilities. After solving the differential equations in 4, we get a closed form expressions of the Mean System size when the srever is in different states in the section 5. At the section 6 the Expected Queue Length was determined. We also obtain expression for other performane measures in section 7 and Some numerical results are presented in section 8.

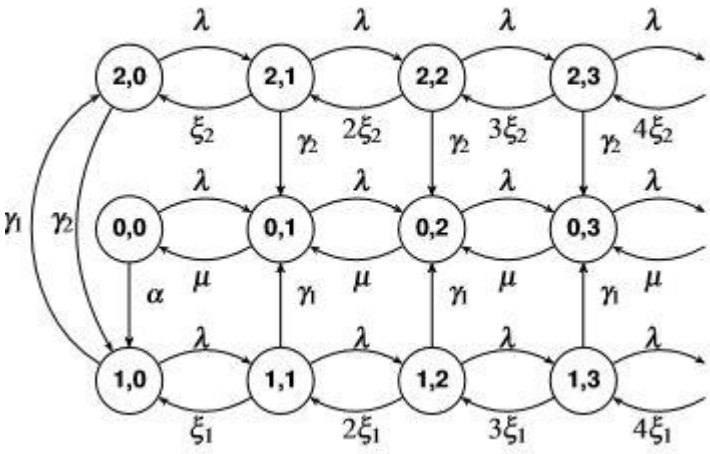
II. Model description and analysis

We consider an $M/M/1$ queueing system with impatient customers and two vacations. Customers arrive according to a Poisson process at rate λ. The service is provided by a single server,who serves the customer on FCFS basis. The service time follows an exponential distribution with a service rate μ.

When the server finishes service and finds the system empty, he does not leave for vacation and he stays ideal for a random period and then he leaves the system with rate α .If the server finds a customer at a vacation completion,the server returns to serve a customer immediately. Otherwise the server will take another vacation with rate γ_1.After the completion of second vacation, if there is no customer the server returns to first vacation with rate γ_2 .If he found customer arrives after completion of second vacation, immediately he start serving the customer.

During the vacations, the customers become impatient.That is whenever a customer arrives at the system it activates an 'imapatience timer' T,which is exponentially distributded with parameters ζ_1, ζ_2 respectively. If the customer's service has not been completed before the customer's timer expire, the customer leaves the queue and never return.

Figure 1 Transition diagram



Let $C(t)$ be the server state at time t. Then

$$C(t)=\begin{cases} 0, \textit{server is in normal busy period} \\ 1, \textit{server is in first vacation} \\ 2, \textit{server is in second vacation} \end{cases}$$

Let $N(t)$ be the number of customers in the system at time t. Then $\{(C(t), N(t)), t \geq 0\}$ is a continuous time Markov chain. Let

$$P_{i,n}(t) = Prob\{C(t) = i, N(t) = n\}, i = 0,1,2 \textit{ and } n \geq 0.$$

III. Steady State Analysis

Let N(t) denote the number of customers in the system at time t.

$$(\lambda + \alpha)P_{0,0} = \mu P_{0,1} \tag{3.1}$$

$$(\lambda + \mu)P_{0,n} = \lambda P_{0,n-1} + \mu P_{0,n+1} + \gamma_1 P_{1,n} + \gamma_2 P_{2,n} \tag{3.2}$$

$$(\lambda + \gamma_1)P_{1,0} = \alpha P_{0,0} + \zeta_1 P_{1,1} + \gamma_2 P_{2,0} \tag{3.3}$$

$$(\lambda + \gamma_1 + n\zeta_1)P_{1,n} = \lambda P_{1,n-1} + (n+1)\zeta_1 P_{1,n+1} \tag{3.4}$$

$$(\lambda + \gamma_2)P_{2,0} = \zeta_2 P_{2,1} + \gamma_1 P_{1,0} \tag{3.5}$$

$$(\lambda + \gamma_2 + n\zeta_2)P_{2,n} = \lambda P_{2,n-1} + (n+1)\zeta_2 P_{2,n+1} \tag{3.6}$$

Probability generating functions

$$P_0(z) = \sum_{n=0}^{\infty} P_{0,n} z^n$$

$$P_1(z) = \sum_{n=0}^{\infty} P_{1,n} z^n$$

$$P_2(z) = \sum_{n=0}^{\infty} P_{2,n} z^n$$

$$P_0^{'}(z) = \frac{d}{dz} P_0 z$$

$$P_1^{'}(z) = \frac{d}{dz} P_1 z$$

$$P_2^{'}(z) = \frac{d}{dz} P_2 z$$

From equation (3.1)and(3.2),we get

$$[\lambda(1-z) + \mu(1-\frac{1}{z})]P_0(z) = \gamma_1 P_1 z + \gamma_2 P_2 z - [\alpha - \mu(1-\frac{1}{z})]P_{0,0} - \gamma_1 P_{1,0} - \gamma_2 P_{2,0}$$

From equations (3.3),(3.4),(3.5) and (3.6),we get

$$(z-1)\zeta_1 P_1^{'}(z) + [\lambda(1-z) + \gamma_1]P_1(z) = \alpha P_{0,0} + \gamma_2 P_{2,0} \tag{3.7}$$

$$(z-1)\zeta_2 P_2^{'}(z) + [\lambda(1-z) + \gamma_2]P_2(z) = \gamma_1 P_{1,0} \tag{3.8}$$

IV. Solution of Differential Equations

Equation (3.8) can be written as

$$P_1^{'}(z) - \left[\frac{\lambda}{\zeta_1} + \frac{\gamma_1}{\zeta_1(1-z)}\right]P_1(z) = -\left[\frac{\alpha P_{0,0} + \gamma_2 P_{2,0}}{\zeta_1(1-z)}\right] \tag{4.9}$$

In order to solve the differential equation(4.9), we multiply both sides by

$$e^{-\frac{\lambda}{\zeta_1}z}(1-z)^{\frac{\gamma_1}{\zeta_1}}$$

Then, we get

$$\frac{d}{dz}\left[e^{-\frac{\lambda}{\zeta_1}z}(1-z)^{\frac{\gamma_1}{\zeta_1}}P_1(z)\right] = -\left[\frac{\alpha P_{0,0} + \gamma_2 P_{2,0}}{\zeta_1}\right]e^{-\frac{\lambda}{\zeta_1}z}(1-z)^{\frac{\gamma_1}{\zeta_1}-1} \tag{4.10}$$

Integrating from 0 to z,we have

$$P_1(z) = \frac{e^{\frac{\lambda}{\zeta_1}z}\left\{P_1(0) - \left[\frac{\alpha P_{0,0} + \gamma_2 P_{2,0}}{\zeta_1}\right]\int_0^z e^{-\frac{\lambda}{\zeta_1}s}(1-s)^{\frac{\gamma_1}{\zeta_1}-1}ds\right\}}{(1-z)^{\frac{\gamma_1}{\zeta_1}}} \tag{4.11}$$

Since $z = 1$ is the root of the denominator of the R.H.S of equation(4.11), we have that $z = 1$ must be the root of numerator of the R.H.S. So we obtain,

$$P_1(0) = \left[\frac{\alpha P_{0,0} + \gamma_2 P_{2,0}}{\zeta_1}\right] C_1 \tag{4.12}$$

$$where C_1 = \int_0^1 e^{-\frac{\lambda}{\zeta_1}s}(1-s)^{\frac{\gamma_1}{\zeta_1}-1} ds$$

Substituting Eqn(4.12) in Eqn(4.11),we obtain

$$P_1(z) = \frac{e^{\frac{\lambda}{\zeta_1}z}}{(1-z)^{\frac{\gamma_1}{\zeta_1}}} \frac{\alpha\zeta_2}{\zeta_1\zeta_2 - \gamma_1\gamma_2 C_1 C_2} C_1 \left[1 - \frac{1}{C_1}\int_0^z e^{-\frac{\lambda}{\zeta_1}s}(1-s)^{\frac{\gamma_1}{\zeta_1}-1} ds\right] P_{0,0} \tag{4.13}$$

Equation (4.3.9) can be written as

$$P_2'(z) - \left[\frac{\lambda}{\zeta_2} + \frac{\gamma_2}{\zeta_2(1-z)}\right] P_2(z) = -\left[\frac{\gamma_1 P_{1,0}}{\zeta_2(1-z)}\right] \tag{4.14}$$

In order to solve the differential equation(4.10), we multiply both sides by

$$e^{-\frac{\lambda}{\zeta_2}z}(1-z)^{\frac{\gamma_2}{\zeta_2}}$$

In a similar manner used for solving Equation (4.3.8),we get

$$P_2(z) = \frac{e^{\frac{\lambda}{\zeta_2}z}\left\{P_2(0) - \left[\frac{\gamma_1 P_{1,0}}{\zeta_2}\right]\int_0^z e^{-\frac{\lambda}{\zeta_2}s}(1-s)^{\frac{\gamma_2}{\zeta_2}-1} ds\right\}}{(1-z)^{\frac{\gamma_2}{\zeta_2}}} \tag{4.15}$$

Since $z = 1$ is the root of the denominator of the R.H.S of equation(4.4.14), we have that $z = 1$ must be the root of numerator of the R.H.S. So we obtain,

$$P_2(0) = \left[\frac{\gamma_1 P_{1,0}}{\zeta_2}\right] C_2 \tag{4.16}$$

$$where C_2 = \int_0^1 e^{-\frac{\lambda}{\zeta_2}s}(1-s)^{\frac{\gamma_2}{\zeta_2}-1} ds$$

substituting (4.16) in (4.15),we get

$$P_2(z) = \frac{e^{\frac{\lambda}{\zeta_2}z}}{(1-z)^{\frac{\gamma_2}{\zeta_2}}} \frac{\alpha\gamma_1}{\zeta_1\zeta_2 - \gamma_1\gamma_2 C_1 C_2} C_1 C_2 \left[1 - \frac{1}{C_2}\int_0^z e^{-\frac{\lambda}{\zeta_2}s}(1-s)^{\frac{\gamma_2}{\zeta_2}-1} ds\right] P_{0,0} \tag{4.17}$$

Equations(4.13) and (4.17) in terms of $P_{0,0}$ .Therefore once $P_{0,0}$ is calculated, the $P_1(z)$ and $P_2(z)$ are determined. In the next section ,we derive the probabilty $P_{0,0}$ and the mean system sizes when the server is in different states.

V. Mean System size

Let L_1 be the system size when the server is in the state 1.Then $E(L_1)$ is the Mean system size when the server is in the state 1,which is defined by

$$E(L_1) = \sum_{n=1}^{\infty} nP_{1,n}$$

From equ (4.9),using $L'Hospital$ rule,we get

$$P_1^{'}(1) = \lim_{z\to 1} \frac{[\lambda(1-z)+\gamma_1]P_1(z) - (\alpha P_{0,0} + \gamma_2 P_{2,0})}{(1-z)\zeta_1}$$

$$= \frac{\gamma_1 P_1^{'}(1) - \lambda P_1(1)}{-\zeta_1} \tag{5.18}$$

Thus,we get

$$E(L_1) = \frac{\lambda}{\zeta_1 + \gamma_1} P_1(1) \tag{5.19}$$

Similarly,Let L_2 be the system size when the server is in the state 2.Then, $E(L_2)$ is the Mean system size when the server is in the state 2,which is defined by

$$E(L_2) = \sum_{n=1}^{\infty} nP_{2,n}$$

.

From equ(4.14),using $L'Hospital$ rule,we get

$$P_2^{'}(1) = \lim_{z\to 1} \frac{[\lambda(1-z)+\gamma_2]P_2(z) - (\gamma_1 P_{1,0})}{(1-z)\zeta_2}$$

$$= \frac{\gamma_2 P_2^{'}(1) - \lambda P_2(1)}{-\zeta_2} \tag{5.20}$$

Thus,we get

$$E(L_2) = \frac{\lambda}{\zeta_2 + \gamma_2} P_2(1) \tag{5.21}$$

Furthermore, the mean system size when the server is on vacation,denoted by $E(L_v)$,is obtained as follows

$$E(L_v) = E(L_1) + E(L_2) \tag{5.23}$$

$$= \frac{P_{0,0}}{\zeta_1\zeta_2 - \gamma_1\gamma_2 C_1 C_2} \left[\frac{\lambda}{\zeta_1 + \gamma_1} \frac{e^{\frac{\lambda}{\zeta_1}} \alpha\zeta_1\zeta_2}{\gamma_1} + \frac{\lambda}{\zeta_2 + \gamma_2} \frac{e^{\frac{\lambda}{\zeta_2}} \alpha\gamma_1\zeta_2}{\gamma_2} C_1 \right] \tag{5.24}$$

Next we derive.

Equation (3.7) can be written as

$$P_0(z) = \frac{\gamma_1 P_1(z) + \gamma_2 P_2(z) - \gamma_1 P_{1,0} - \gamma_2 P_{2,0} - [\alpha - \mu(1-\frac{1}{z})]P_{0,0}}{\lambda(1-z) + \mu(1-\frac{1}{z})} \tag{5.25}$$

Applying $L'Hospital$ rule,we get

$$P_0(1) = \frac{\gamma_1 E(L_1) + \gamma_2 E(L_2) + \mu P_{0,0}}{\mu - \lambda}$$

Substituting equations (4.5.21)and (4.5.19),we get

$$P_0(z) = \frac{P_{0,0}}{\mu - \lambda}\left[\frac{\lambda}{\zeta_1 + \gamma_1}\frac{e^{\frac{\lambda}{\zeta_1}}\alpha\zeta_1\zeta_2}{\gamma_1} + \frac{\lambda}{\zeta_2 + \gamma_2}\frac{e^{\frac{\lambda}{\zeta_2}}\alpha\gamma_1\zeta_2}{\gamma_2}C_1 + \mu\right] \tag{5.26}$$

Using Normalization Condition

$$P_0(1) + P_1(1) + P_2(1) = 1$$

We get,

$$P_{0,0} = [\zeta_1\zeta_2 - \gamma_1\gamma_2 C_1 C_2]\left\{e^{\frac{\lambda}{\zeta_1}}\alpha\zeta_1\zeta_2\left[\frac{1}{\gamma_1} + \frac{1}{\mu-\lambda}\frac{\lambda}{\zeta_1+\gamma_1}\right]\right.$$
$$\left. + e^{\frac{\lambda}{\zeta_2}}\alpha\gamma_1\zeta_2 C_1\left[\frac{1}{\gamma_2} + \frac{1}{\mu-\lambda}\frac{\lambda}{\zeta_2+\gamma_2}\right] + \mu(\zeta_1\zeta_2 - \gamma_1\gamma_2 C_1 C_2)\right\}^{-1} \tag{5.27}$$

Substituting (4.5.27) in (4.5.26),we get

$$E(L_v) = \left[\frac{\lambda}{\zeta_1+\gamma_1}\frac{e^{\frac{\lambda}{\zeta_1}}\alpha\zeta_1\zeta_2}{\gamma_1} + \frac{\lambda}{\zeta_2+\gamma_2}\frac{e^{\frac{\lambda}{\zeta_2}}\alpha\gamma_1\zeta_2}{\gamma_2}C_1\right]\left\{e^{\frac{\lambda}{\zeta_1}}\alpha\zeta_1\zeta_2\left[\frac{1}{\gamma_1} + \frac{1}{\mu-\lambda}\frac{\lambda}{\zeta_1+\gamma_1}\right]\right.$$
$$\left. + e^{\frac{\lambda}{\zeta_2}}\alpha\gamma_1\zeta_2 C_1\left[\frac{1}{\gamma_2} + \frac{1}{\mu-\lambda}\frac{\lambda}{\zeta_2+\gamma_2}\right] + \mu(\zeta_1\zeta_2 - \gamma_1\gamma_2 C_1 C_2)\right\}^{-1} \tag{5.28}$$

Now we derive $E(L_0)$ from the equation (4.5.26), using $L'Hospital$ rule

$$E(L_0) = \frac{\gamma_1 P_1^{''}(1) + \gamma_2 P_2^{''}(1) - 2\mu P_{0,0}}{2(\mu-\lambda)} + \frac{\mu}{(\mu-\lambda)^2}\left[\gamma_1 P_1^{'}(1) + \gamma_2 P_2^{'}(1) + \mu P_{0,0}\right]$$

$$-\frac{3\mu}{(\mu-\lambda)^2}\left[\gamma_1 P_1(1)+\gamma_2 P_2(1)-\gamma_1 P_{1,0}-\gamma_2 P_{2,0}-\alpha P_{0,0}\right] \tag{5.29}$$

Where $P_1^{''}(1)$ and $P_2^{''}(1)$ is obtained by differentiating twice $P_1(z)$ and $P_2(z)$ at z=1.Differentiating twice the equations (4.5.18) and (4.5.20) respectively with z=1,we obtain

$$P_1^{''}(1)=\frac{\lambda P_1^{'}(1)}{\zeta_1} \tag{5.30}$$

$$and P_2^{''}(1)=\frac{\lambda P_2^{'}(1)}{\zeta_2} \tag{5.31}$$

Substituting equation (4.5.31) and (4.5.30) in (4.5.29),we get

$$\begin{aligned}E(L_0)=\frac{1}{\mu-\lambda}\Bigg\{&\gamma_1 P_1(1)\left[\left(\frac{\lambda}{2\zeta_1}+\frac{\mu}{\mu-\lambda}\right)\left(\frac{\lambda}{\zeta_1+\gamma_1}\right)-\frac{3\mu}{\mu-\lambda}\right]\\&+\gamma_2 P_2(1)\left[\left(\frac{\lambda}{2\zeta_2}+\frac{\mu}{\mu-\lambda}\right)\left(\frac{\lambda}{\zeta_2+\gamma_2}\right)-\frac{3\mu}{\mu-\lambda}\right]\Bigg\}\\&+\frac{1}{(\mu-\lambda)^2}\left[3\mu\gamma_1 P_{1,0}+3\mu\gamma_2 P_{2,0}+3\mu\alpha P_{0,0}+\mu P_{0,0}\right]\end{aligned} \tag{5.32}$$

VI. Expected Queue Length

Let L be the number of customers in the system.Then,the mean system size

$$E(L)=E(L_v)+E(L_0) \tag{6.33}$$

can be calaculated from the equations (4.5.32) and (4.5.28)

$$\begin{aligned}E(L)=\frac{P_{0,0}}{\zeta_1\zeta_2-\gamma_1\gamma_2 C_1 C_2}&\left[\frac{\lambda}{\zeta_1+\gamma_1}\frac{e^{\frac{\lambda}{\zeta_1}}\alpha\zeta_1\zeta_2}{\gamma_1}+\frac{\lambda}{\zeta_2+\gamma_2}\frac{e^{\frac{\lambda}{\zeta_2}}\alpha\gamma_1\zeta_2}{\gamma_2}C_1\right]\\&+\frac{1}{\mu-\lambda}\Bigg\{\gamma_1 P_1(1)\left[\left(\frac{\lambda}{2\zeta_1}+\frac{\mu}{\mu-\lambda}\right)\left(\frac{\lambda}{\zeta_1+\gamma_1}\right)-\frac{3\mu}{\mu-\lambda}\right]\\&+\gamma_2 P_2(1)\left[\left(\frac{\lambda}{2\zeta_2}+\frac{\mu}{\mu-\lambda}\right)\left(\frac{\lambda}{\zeta_2+\gamma_2}\right)-\frac{3\mu}{\mu-\lambda}\right]\Bigg\}\\&+\frac{1}{(\mu-\lambda)^2}\left[3\mu\gamma_1 P_{1,0}+3\mu\gamma_2 P_{2,0}+3\mu\alpha P_{0,0}+\mu P_{0,0}\right]\end{aligned}$$

VII. Some Performance Measures

In this subsection,we derive some other performance measures.

1. Probabilty that the server is on vacation

The probability that the server is on vacation is given by

$$P_v = \sum_{n=0}^{\infty} P_{1,n} + \sum_{n=0}^{\infty} P_{2,n}$$
$$= P_1(1) + P_2(1) \tag{7.34}$$

Substituting the values of $P_1(1)$ and $P_2(1)$ in equation (4.7.34)

$$P_v = \frac{P_{0,0}}{\zeta_1\zeta_2 - \gamma_1\gamma_2 C_1 C_2}\left[\frac{e^{\frac{\lambda}{\zeta_1}}\alpha\zeta_1\zeta_2}{\gamma_1} + \frac{e^{\frac{\lambda}{\zeta_2}}\alpha\gamma_1\zeta_2 C_1}{\gamma_2}\right] \tag{7.35}$$

2.Probabilty that the server is busy

The probability that the server is busy is given by

$$P_b = \sum_{n=1}^{\infty} P_{0,n} - P_{0,0} \tag{7.36}$$

Substituting the values of $P_0(1)$,we get

$$P_b = P_{0,0}\left[\frac{1}{\mu-\lambda}\left(\frac{\lambda}{\zeta_1+\gamma_1}\frac{e^{\frac{\lambda}{\zeta_1}}\alpha\zeta_1\zeta_2}{\zeta_1\zeta_2-\gamma_1\gamma_2C_1C_2} + \frac{\lambda}{\zeta_2+\gamma_2}\frac{e^{\frac{\lambda}{\zeta_2}}\alpha\gamma_1\zeta_2C_1}{\zeta_1\zeta_2-\gamma_1\gamma_2C_1C_2} + \mu\right) - 1\right]$$

Where
$$P_{0,0} = [\zeta_1\zeta_2 - \gamma_1\gamma_2 C_1 C_2]\left\{e^{\frac{\lambda}{\zeta_1}}\alpha\zeta_1\zeta_2\left[\frac{1}{\gamma_1} + \frac{1}{\mu-\lambda}\frac{\lambda}{\zeta_1+\gamma_1}\right]\right.$$
$$\left. + e^{\frac{\lambda}{\zeta_2}}\alpha\gamma_1\zeta_2 C_1\left[\frac{1}{\gamma_2} + \frac{1}{\mu-\lambda}\frac{\lambda}{\zeta_2+\gamma_2}\right] + \mu(\zeta_1\zeta_2 - \gamma_1\gamma_2 C_1 C_2)\right\}^{-1}$$

3.Proportion of Customers Served

Clearly,the expected number of customers served per unit of time is μP_b, implying that the proportion of customers served is given by

$$P_s = \frac{\mu}{\lambda}P_b \tag{7.37}$$

where P_b is given by the equation(4.7.36).

4.Average rate of abandonment due to impatience

When the system is in state $(1,n), n \geq 1$, the rate of abandonment of a customer due to impatience is $n\zeta_1$ and in state two the of abdonment of a customer due to impatience is $n\zeta_2$. Thus the average rate of abdonment due to impatience is given by

$$R_a = \sum_{n=1}^{\infty} n\zeta_1 P_{1,n} - P_{1,0} + \sum_{n=1}^{\infty} n\zeta_2 P_{2,n} - P_{2,0} \tag{7.38}$$

VIII. Numerical Examples

In this section, we present examples to illustrate the effect of the different parameters on the system performance measures.

In Figure 2 we have plotted the stationary probability to the server being idle . Figure 2 clearly illustrate the effect of varying μ on $P_{0,0}$ for λ = 4,4.25,4.5 and 4.75. As μ increases the value of $P_{0,0}$ increasing.

Figure 2. $P_{0,0}$ versus μ

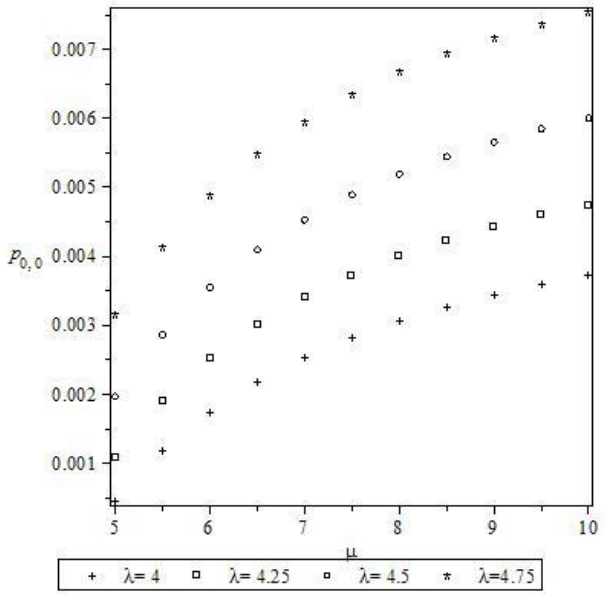
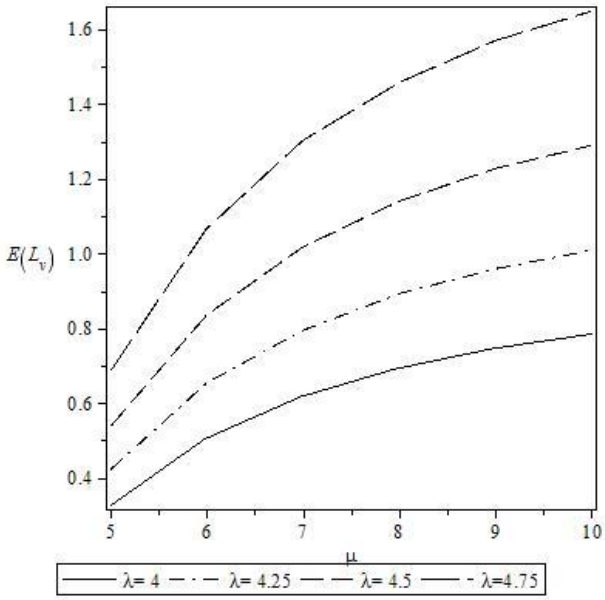


Figure 3, show the influence of μ on the expected queue length due to working vacation for different values of λ respectively. As expected $E(L_v)$ increases. That is the expected queue length increases when arrival increases.

Figure 3 E(Lv) versus μ



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AN M/M/1 QUEUE WITH WORKING VACATIONS, VACATION INTERUPTIONS AND A WAITING SERVER

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Abstract*: In this paper, we study an M/M/1 queue with working vacations, vacation interuptions and a waiting server. We derive the distribution for the number of customers, the virtual time and a stochastic decomposition structures in the system in steady state by employing quasi birth and death process and matrix geometric solution methodology. Furthermore, the formula of expected busy cycle are obtained. Finally, some numerical results considered to illustrate the variation of parameters on the behavior of the System.*

Keywords*: Markovian queue, working vacation, steady state analysis, waiting time distribution, busy period analysis .*

1. Introduction

In many real life congestion situations, the server can be utilized for secondary work and the server provides service with different(lower) rate during the vacation period. Such a queueing models are called queue with working vacations. Servi and Finn [9] first studied the $M/M/1$ queueing system with working vacations, and they have obtained the transform formulae for the distribution of the number of customers in the system and sojourn time in steady state. Wu and Takagi [16], [17] extended [9] to an $M/G/1/WV$ model, where both service periods are generally distributed.In recent years, there are more and more researches analyzing the working vacation queues, the details of which may be found in [14].

Li and Tian[6] introduced vacation interruption policy. During the working vacation period, if at least one customer is present in the system at a service completion epoch, the server will interrupt the vacation and resume regular service. Working vacation interruption has become an important aspect. Li et al. [7] further extended the results of [6] for the queue where the arrivals follows general distribution. Zhang and Hou [19] have analyzed an $M/G/1$ queue with working vacations and vacation interruptions by supplementary variable technique. Tao et al.[12] have discussed an $M/M/1$ retrial queue with collisions and working vacation interruption under N -policy. Tao et al.[13] have studied a retrial queue with working vacation and feedback under N -policy. Sreenivasan et al.[10] have discussed $MAP/PH/1$ queue with working vacations, vacation interruptions and N policy. Cosmika Goswami and Selvaraju [4] sudied a working vacation queue with priority customers and vacation interruptions. Veena Goswami [15] has Analysed impatient customers in queues with Bernoulli schedule working vacations and vacation interruption.

In recent years, interest is growing in queues with delayed vacation. This kind of queue was first inroduced by Boxma et. al. citeboxma Yechiali [18] extended the concept to bulk arrival queues. Altman and Yechiali [1] studied both the multiple-vacation and the single-vacation models of the single-server $M/M/1$ and $M/G/1$ queues, as well as of the multi-server $M/M/c$ systems. Padmavathy et al. [8] studied an $M/M/1$ vacation queue with impatient customers and a waiting server. The time dependent analysis of an $M/M/1$ queue with server vacations and a waiting server, studeid by Kalidass and Ramanath[5]. Sudhesh and Azhagappan [11] have discussed transient analysis of an $M/M/1$ queue with waiting server timer and customer impateince. Recently, Ammar[2] analysed transient solution of a vacation queue with a waiting server and impatient customers.

In this chapter, we consider the working vacations and vacation interruptions queue. Upon finding the system empty, the server waits for a random period of time before proceeding on a working vacation and it can

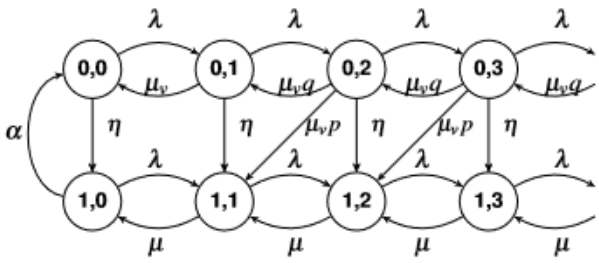
take service at lower rate during the vacation period. If there are customers in the system at the instant of a service completion during the vacation period, the server will come back to the normal working level with probability q $(0 \le q \le 1)$ no matter whether or not the vacation has ended. Otherwise, it continue the vacation with probability $1-q$. Our queuing system can be used to model some practical situations,in modern day communication systems, to keep the server functioning well, virus scan is an important maintenance activity for the servers.It can be performed when the server is idle. This type of maintenance could be programmed to perform on regular basis. Although the operation of virus scan would consume some system resources and reduce the processing speed, the servers could still provide his service with lower processing speed during the period of virus scan and at the instant of a service completion,the process of virus scan is interrupted and the server is resumed to a regular busy period (if there are customers in the queue) no matter whether the virus scan ends,or continues virus scan.We derive an analytic expression for the stationary probability of the number of customers in the system when the server is in a normal busy period and in a working vacation interruption, respectively. The necessary and sufficient condition for the existence of steady state has derived. Some performance measures such as mean system size, the proportion of time the server being normal and in a working vacation have been discussed. We demonstrate the stochastic decomposition structures of the stationary queue length and virtual time. Busy period of the system also studeied. Numerical results have been illustrated in the form of graphs.

The organization of the chapter is as follows. We described the model and derived the necessary and sufficient condition for the existence of steady state in the immediate section. In section 3, queue length distribution is presented. The stochastic decomposition on the queue lenght distribution is studied in section 4. Virtual time analysis and the busy period of our model is presented in section 5 and 6 respectively. Finally, Some numerical examples to illustrate the effect of the parameters on the system performance measures are presented in Section 7.

II. Model description and analysis

We consider the multiple working vacations $M/M/1$ queueing system with Bernoulli schedule vacation interruptions. Customers arrive according to a Poisson process with arrival rate λ. The service times during a normal service period, the service times during a working vacation period, and the vacation times are exponentially distributed with rates μ, μ_v and η, respectively. Whenever the server free, the server activates a 'timer' T, exponentially distributed with parameter α. If a customer arrives before the time T expires, the server stays in the system and provides service. However, if T expires goes for a working vacation. During the working vacation period, a customer is serviced at a lower rate, and, at the instants of the service completion, the vacation is interrupted and the server resumes a regular busy period with probability $1-p$ (if there are customers in the queue) or remains in the vacation with probability p. The interarrival times, service times and the vacation duration times all are taken to be mutually independent. The customers are served on a firstcome first-served (FCFS) queue discipline. Let $\{N(t), t \ge 0\}$ denote the number of customer in the system at time t. Let $C(t)$ be the state of the server at time t.

Transition Diagram



Then

$$C(t) = \begin{cases} 0, & \text{the server is in working vacation state,} \\ 1, & \text{server is in normal state.} \end{cases}$$

Then $\{(N(t),C(t)),t \geq 0\}$ is a continuous time Markov chain. Let

$$\pi_{n,i}(t) = Prob\{N(t) = n, C(t) = i\},\ i = 0,1 \text{ and } n \geq 0$$

Using the lexicographical ordering for the states, the infinitesimal generator of the process can be written as

$$Q = \begin{pmatrix} A_{00} & C & & & & \\ B_{10} & A & C & & & \\ & B & A & C & & \\ & & B & A & C & \\ & & & & \vdots & \vdots \end{pmatrix}$$

$$where\ A_{00} = \begin{pmatrix} -(\lambda+\eta) & \eta \\ \alpha & -(\alpha+\lambda) \end{pmatrix}$$

$$B_{10} = \begin{pmatrix} \mu_v & 0 \\ 0 & \mu \end{pmatrix}$$

$$A = \begin{pmatrix} -(\lambda+\eta+\mu_v) & \eta \\ 0 & -(\lambda+\mu) \end{pmatrix}$$

$$C = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

$$B = \begin{pmatrix} \mu_v q & \mu_v(1-q) \\ 0 & \mu \end{pmatrix}$$

To analyze this QBD process, it is necessary to solve for the minimal non-negative solution of the matrix quadratic equation

$$R^2B + RA + C = 0 \tag{1.2}$$

and this solution is called the rate matrix and denoted by R.

Theorem 2.1 :

If $\rho = \frac{\lambda}{\mu} < 1$, the matrix equation (1.2) has the minimal non-negative solution.

Proof:

Because the matrices A,B,C of (2.1) are all upper triangular, we can assume that R has the same structure as

$$R = \begin{pmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{pmatrix}$$

Substituting $R^2 and R$ into (3.2) yields the following set of equations:

$$\mu_v q r_{11}^2 - (\lambda+\eta+\mu_v)r_{11} + \lambda = 0, \tag{2.1}$$

$$\mu r_{22}^2 - (\lambda+\mu)r_{22} + \lambda = 0, \tag{2.2}$$

$$\mu_v(1-q)r_{11}^2 + \mu(r_{11}r_{12} + r_{12}r_{22}) + \eta r_{11} - (\lambda+\mu)r_{12} = 0. \tag{2.3}$$

To obtain the minimal non-negative solution of (2.1) taking $r_{22} = \rho$ (the other root is greater than 1) in the second equation and $r_{11} = r$ in the first equation. Using elementory method ,we can prove that $0 < r < 1$. Substituting $\rho \; and \; r$ in third equation we get,

$$r_{12} = \frac{r\eta + r^2\mu_v(1-q)}{\mu(1-r)}$$

Because r satisfies the following equation

$$r^2\mu_v q - (\lambda + \eta + \mu_v)r + \lambda = 0,$$

Divide both sides of this equation by r, we get

$$\lambda + \eta + \mu_v - r\mu_v q = \frac{\lambda}{r}$$

$$\frac{\eta}{1-r} + \frac{\mu_v rp}{1-r} = \frac{\lambda}{r} - \mu_v$$

Theorem 2.2:

The QBD process $\{N(t), C(t)\}$ is positive recurrent if and only if $\rho < 1$.

Proof:

Based on Theorem 3.1.1 of Neuts [12], the QBD process $\{N(t), C(t)\}$ is positive recurrent if and only if the spectral radius $SP(R)$ of the rate matrix R is less than 1, and set of equations $(x_0, x_1, x_2, x_3, x_4, x_5)B[R] = 0$ has a positive solution, where

$$B[R] = \begin{bmatrix} A_{00} & c \\ B_{10} & RB + A \end{bmatrix}$$

$$= \begin{bmatrix} -(\lambda+\eta) & \eta & \lambda & 0 \\ \alpha & -(\alpha+\lambda) & 0 & \lambda \\ \mu_v & 0 & -\frac{\lambda}{r} & \frac{\lambda}{r} - \mu_v \\ 0 & \mu & 0 & -\mu \end{bmatrix}$$

$B[R]$ is an irreducible and aperiodic generator with finite state. Therefore,

$$(x_0, x_1, x_2, x_3, x_4, x_5)B[R] = 0$$

has a positive solution. Thus, process $\{N(t), C(t)\}$ is positive recurrent if and only if

$$SP(R) = \max(r, \rho) < 1$$

Since $\rho < 1$, we must have $r < 1$.

III. Queue Length Distribution

If $\rho < 1$, let (N, C) be the stationary limit of the QBD process $\{N(t), C(t)\}$.Let

$$\pi_0 = \pi_{0,0},$$

$$\pi_k = (\pi_{k,0}, \pi_{k,1}), k \geq 1$$

$$\pi_{k,j} = P(N = k, C = j), (k \geq 0, \text{ and } j = 0)$$

Theorem 3.1

If $\rho < 1$, the joint probabiility distribution of (N, C) is

$$\begin{aligned} \pi_{0,0} &= \mathsf{K} \\ \pi_{k,0} &= \mathsf{K} r^k \\ \pi_{k,1} &= \mathsf{K}\left[\frac{\lambda - r\mu_v}{\mu}\sum_{m=0}^{k-1} r^m \rho^{k-1-m} + \frac{\rho^{k-1}}{\mu\alpha}[\lambda(\lambda + \eta - r\mu_v)]\right] \end{aligned} \tag{3.4}$$

where

$$\mathsf{K} = \left[\frac{(1-r)(1-\rho)}{1-\rho + \frac{\lambda + \eta - r\mu_v}{\alpha}(1-r) + \frac{\lambda - r\mu_v}{\mu}}\right]$$

Proof:

With the matrix-geometric solution method see Neut[12],we have

$$\begin{aligned} \pi_k &= (\pi_{k,0}, \pi_{k,1}) \\ &= (\pi_{1,0}, \pi_{1,1}) R^{k-1}, \ k \geq 1 \end{aligned}$$

and

$$(\pi_{0,0}, \pi_{0,1}, \pi_{1,0}, \pi_{1,1})$$

satiesfies

$$(\pi_{0,0}, \pi_{0,1}, \pi_{1,0}, \pi_{1,1}) B(R) = 0 \tag{3.5}$$

Substituting B(R) into the above equation we obtain the set of equations

$$\pi_{0,0}(-\lambda + \eta) + \alpha\pi_{0,1} + \mu_v\pi_{1,0} = 0 \tag{3.6}$$

$$\eta\pi_{0,0} + (-\lambda - \alpha)\pi_{0,1} + \mu\pi_{1,1} = 0 \tag{3.7}$$

$$\lambda\pi_{0,0} - \frac{\lambda}{r}\pi_{1,0} = 0 \tag{3.8}$$

$$\lambda\pi_{0,1} + (\frac{\lambda}{r} - \mu_v)\pi_{1,0} - \mu\pi_{1,1} = 0 \tag{3.9}$$

Let

$\pi_{0,0} = \mathsf{K}$. Then (3.8) becomes

$$\pi_{1,0} = \mathsf{K}r \tag{3.10}$$

Substitute value of π_{10} in (3.6), we have

$$\pi_{0,1} = \frac{\mathsf{K}}{\alpha}\left[\lambda + \eta - r\mu_v\right] \tag{3.11}$$

From (3.7), we have

$$\pi_{1,1} = \frac{\mathrm{K}}{\mu}\left[\frac{\lambda}{\alpha}(\lambda+\eta-r\mu_v)+\lambda-r\mu_v\right] \tag{3.12}$$

Now

$$R^{k-1} = \begin{bmatrix} r^{k-1} & (\frac{\lambda-r\mu_v}{\rho\mu})\left[-r^{k-1}+\sum_{m=0}^{k-1}\rho^{k-1-m}r^m\right] \\ 0 & \rho^{k-1} \end{bmatrix}$$

$$(\pi_{k,0}, \pi_{k,1}) = (\pi_{1,0}, \pi_{1,1})R^{k-1}$$

$$\pi_{k,0} = \mathrm{K}r \tag{3.13}$$

$$\pi_{k,1} = \mathrm{K}\left[\frac{\lambda-r\mu_v}{\mu}\sum_{m=0}^{k-1}r^m\rho^{k-1-m}+\frac{\rho^{k-1}}{\mu\alpha}[\lambda(\lambda+\eta-r\mu_v)]\right] \tag{3.14}$$

By using normalization condition, we get the above equations.

With (3.13 ,3.14) the probabilities that the system in a normal period and a working vacation period are as follows,respectively:

$$P(j=0) = \pi_{0,0}\frac{1}{1-r}$$

$$P(j=1) = \pi_{0,0}\left[\frac{\lambda-r\mu_v}{\mu}[\frac{1}{(1-r)(1-\rho)}]+\frac{\lambda+\eta-r\mu_v}{\alpha}\left[\frac{\rho}{1-\rho}\right]\right]$$

The distribution of the number of customers N in system in steady-state is as follows

$$P(N=0) = \pi_{0,0}$$

$$P(N=k) = (\pi_{k,0}+\pi_{k,1}),\ k \geq 1$$

IV. Stochastic decomposition

Theorem 4.1 :

If $\rho < 1$ and $\mu > \mu_v$, the number of customers Q in the system can be decomposed into the sum of two independent random variables: $Q = Q_o + Q_d$, where Q_o is the number of customers of a classical $M/M/1$ queue in steady state and follows a geometric distribution with parameter $1-\rho$ .Additional number of customers Q_d has a modified geometric distribution

$$P[Q_d = k] = \begin{cases} \mathrm{K}\left[\frac{1}{1-rz}\right], & k=0, \\ \mathrm{K}\left[\frac{\lambda-r\mu_v}{\mu}\left(\frac{z}{(1-rz)(1-\rho z)}\right)+\frac{\lambda+\eta-r\mu_v}{\alpha}\left(\frac{1}{1-\rho z}\right)\right], & k \geq 1. \end{cases}$$

where

$$\mathsf{K}^{-1} = \frac{1-\rho+\dfrac{\lambda-r\mu_v}{\mu}+\dfrac{\lambda+\eta-r\mu_v}{\alpha}(1-r)}{(1-r)(1-\rho)}$$

Proof:

With probability generating function of Q can be written as

$$Q(z) = \sum_{k=0}^{\infty} z^k \pi_{k,0} + \sum_{k=0}^{\infty} z^k \pi_{k,1}$$

$$= \pi_{0,0}\left[\frac{1}{1-rz}+\frac{\lambda-r\mu_v}{\mu}\left(\frac{z}{(1-rz)(1-\rho z)}\right)+\frac{\lambda+\eta-r\mu_v}{\alpha}\left(\frac{1}{1-\rho z}\right)\right]$$

Q(z) can be rewritten as

$$Q(z) = \frac{1-\rho}{1-\rho z}\mathsf{K}\left[\frac{1-\rho z}{(1-\rho)(1-rz)}+\frac{\lambda-r\mu_v}{\mu}\left(\frac{z}{(1-rz)(1-\rho)}\right)+\frac{\lambda+\eta-r\mu_v}{\alpha}\left(\frac{1}{1-\rho z}\right)\right]$$

$$= \frac{1-\rho}{1-\rho z}Q_d(z) \qquad (4.15)$$

The above equation indicates

$$Q_d(z) = \mathsf{K}\left[\frac{1-\rho z}{(1-\rho)(1-rz)}+\frac{\lambda-r\mu_v}{\mu}\left(\frac{z}{(1-rz)(1-\rho)}\right)+\frac{\lambda+\eta-r\mu_v}{\alpha}\left(\frac{1}{1-\rho z}\right)\right]$$

is a PGF.

Expanding $Q_d(z)$ into power series of z,we get the distribution of additional numbers of customers Q_d .

With the stochastic decomposition structure in theorem (4.1),we can easily get

$$E(Q) = \frac{\rho}{1-\rho} + E(Q_d) \qquad (4.16)$$

$$where E(Q_d) = \mathsf{K}\left[\frac{r-\rho}{(1-\rho)(1-r)^2}+\frac{\lambda-r\mu_v}{\mu(1-\rho)}\frac{1}{(1-r)^2}\right]$$

V. Virtual time Distribution

Denote the virtual time of a customer in the system by W ,we have the following decomposition results.

Theorem 5.1:

If $\rho < 1$ and $\mu > \mu_v$, the virtual time W can be decomposed into the sum of two independent random variables: $W = W_o + W_d$, where W_o is the virtual time of a costumer in a corresponding classical $M/M/1$ queue and has an exponential distribution with parameter $\mu(1-\rho_1)$;Additional delay W_d has the LST

$$W_d^*(s) = \mathsf{K}\left[\frac{\lambda(1-\rho)+\rho s}{(\lambda(1-r)+rs)(1-\rho)} + \frac{\lambda - r\mu_v}{\mu}\left(\frac{\lambda - s}{(\lambda(1-r)+rs)(1-\rho)}\right) + \frac{\lambda+\eta-r\mu_v}{\alpha}\left(\frac{1}{1-\rho}\right)\right]$$

Proof:

The classical relaionship between the PGF of Q and the LST of virtual time W is

$$Q(z) = W^*(\lambda(1-z)).$$

From theorem (4.1),the PGF of the number of customers Q can be written as

$$\begin{aligned} Q(z) &= \frac{1-\rho}{1-\rho z}\mathsf{K}\left[\frac{1-\rho z}{(1-\rho)(1-rz)} + \frac{\lambda - r\mu_v}{\mu}\left(\frac{z}{(1-rz)(1-\rho)}\right) + \frac{\lambda+\eta-r\mu_v}{\alpha}\left(\frac{1}{1-\rho z}\right)\right] \\ &= \frac{1-\rho}{1-\rho z}Q_d(z) \end{aligned} \tag{5.17}$$

Taking $z = 1 - \frac{s}{\lambda}$ in (5.17)

now

$$\frac{1-\rho}{1-\rho z} = \frac{\mu(1-\rho)}{\mu(1-\rho)+s}$$
$$\frac{1-\rho z}{1-rz} = \frac{\lambda(1-\rho)+\rho s}{\lambda(1-r)+rs}$$

substituting the above results into (5.17), we have

$$W^*(s) = \frac{\mu(1-\rho)}{\mu(1-\rho)+s}\mathsf{K}\left[\frac{\lambda(1-\rho)+\rho s}{(\lambda(1-r)+rs)(1-\rho)} + \frac{\lambda - r\mu_v}{\mu}\left(\frac{\lambda - s}{(\lambda(1-r)+rs)(1-\rho)}\right) + \frac{\lambda+\eta-r\mu_v}{\alpha}\left(\frac{1}{1-\rho}\right)\right]$$

Note that (4.16) and the proof of theorem (4.1), give the equivalent expession of K

$$\mathsf{K}^{-1} = \left[\frac{\lambda(1-\rho)}{[\lambda(1-r)(1-\rho)]} + \frac{\lambda - r\mu_v}{\mu}\left(\frac{\lambda}{[\lambda(1-r)(1-\rho)]}\right) + \frac{\lambda+\eta-r\mu_v}{\alpha}\left(\frac{1}{1-\rho}\right)\right]$$

This verifies that

$$W_d^*(s) = \mathsf{K}\left[\frac{\lambda(1-\rho)+\rho s}{(\lambda(1-r)+rs)(1-\rho)} + \frac{\lambda - r\mu_v}{\mu}\left(\frac{\lambda - s}{(\lambda(1-r)+rs)(1-\rho)}\right) + \frac{\lambda+\eta-r\mu_v}{\alpha}\left(\frac{1}{1-\rho}\right)\right]$$

is a LST.

It is easy to obtain

$$\begin{aligned} E[W_d] &= \mathsf{K}\left[\frac{1}{1-\rho}\left(\frac{r-\rho}{\lambda(1-r)^2}\right) + \frac{\lambda - r\mu_v}{\lambda\mu(1-\rho)}\left(\frac{1}{(1-r)^2}\right)\right] \\ E[W] &= \frac{1}{\mu(1-\rho)} + E[W_d] \end{aligned} \tag{5.18}$$

VI. Busy Period Analysis

The duration in which the server works at a rate of μ continuously is called a regular busy period,denoted by B. A regular busy period can start at the instant when a working vacation or an idle period finishes.

A busy cycle can be composed of a working vacation V, idle period after this V finishes(if exists) and subsequent regular busy period B.

Note that there is just a working vacation V in a busy cycle C, and $E(V) = \frac{1}{\eta}$.

Using the limiting theorem of alternative renewal process,we have

$$P(J = 0) = \frac{E(V)}{E(C)}$$
$$= \pi_{0,0} \frac{1}{1-r}$$

Thus,we have expected busy cycle

$$E(C) = \frac{(1-r)}{\eta \pi_{0,0}} \tag{6.19}$$

Similarly, using alternative renewal theorem, we have

$$P(j = 1) = \frac{E(B)}{E(C)}$$
$$= \sum_{k=1}^{\infty} \pi_{k,1} \tag{6.20}$$

Substituting $E(C)$ of (6.19) into the above expression, we can conclude the expected busy period B

$$E(B) = E(C) \sum_{k=1}^{\infty} \pi_{k,1} \tag{6.21}$$
$$= \frac{1}{\eta(1-\rho)} \left[\frac{\lambda - r\mu_v}{\mu} + \frac{\lambda + \eta - r\mu_v}{\alpha} \right] \tag{6.22}$$

We can conclude the expected idle period in a busy cycle,

$$\pi_{0,1} = \frac{E(I)}{E(C)} \tag{6.23}$$

Substituting $E(C)$ of (6.19) into the above expression, we can conclude the expected idle period in a busy cycle

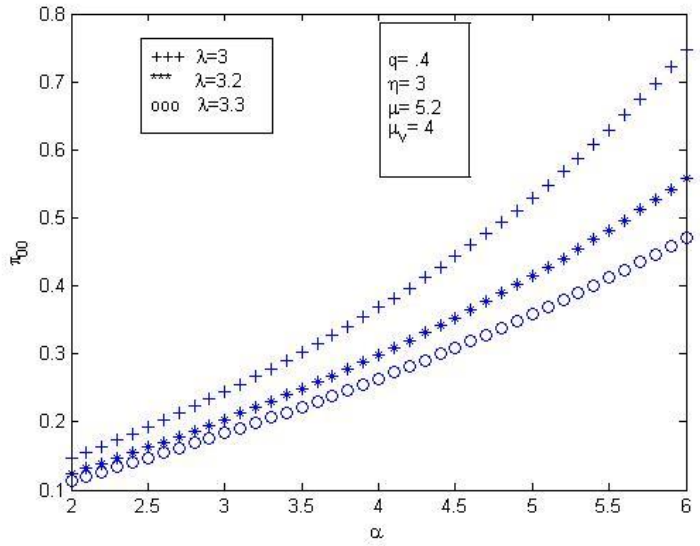
$$E(I) = \frac{1-r}{\eta \alpha} (\lambda + \eta - r\mu_v) \tag{6.24}$$

VII. Numerical Examples

In this section, we present examples to illustrate the effect of the different parameters on the system performance measures.

In Figure 1 to 5 we have plotted the stationary probability to the server being idle for $q = 0.4$. Figure 1 clearly illustrate the effect of varying α on $\pi_{0,0}$ for $\lambda = 3.3.2$ and 3.3. As α increases the value of $\pi_{0,0}$ increasing.

Figure 1: $\pi_{0,0}$ versus α



From figure 2, we observe that $\pi_{0,0}$ decreases with increasing values of λ for different values of η=3,3.3 and 3.5. Figure 3 shows that the probability $\pi_{0,0}$ decreases for incresing values of λ with respect to the working vacation service rate q=.3,.2 and.1. From fig 2 and 3, we find that $\pi_{0,0}$ decreasing. However, the later decrease fastly.

Figure 2: $\pi_{0,0}$ versus λ

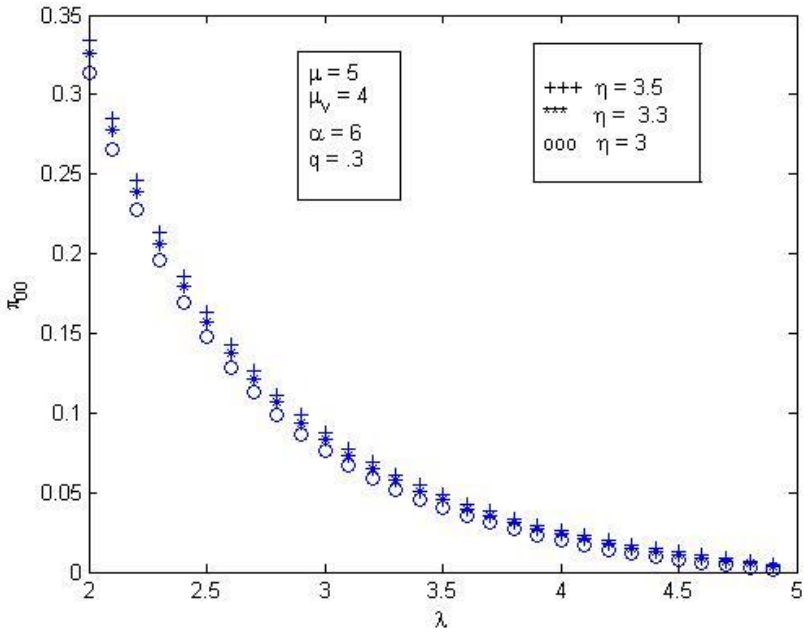


Figure 3: $\pi_{0,0}$ versus λ

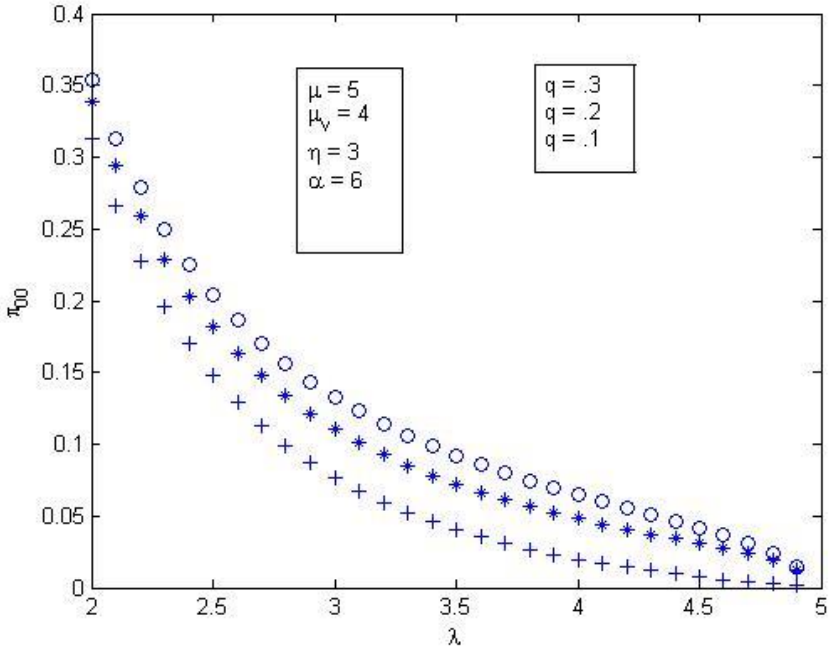


Figure 4: $\pi_{0,0}$ versus μ

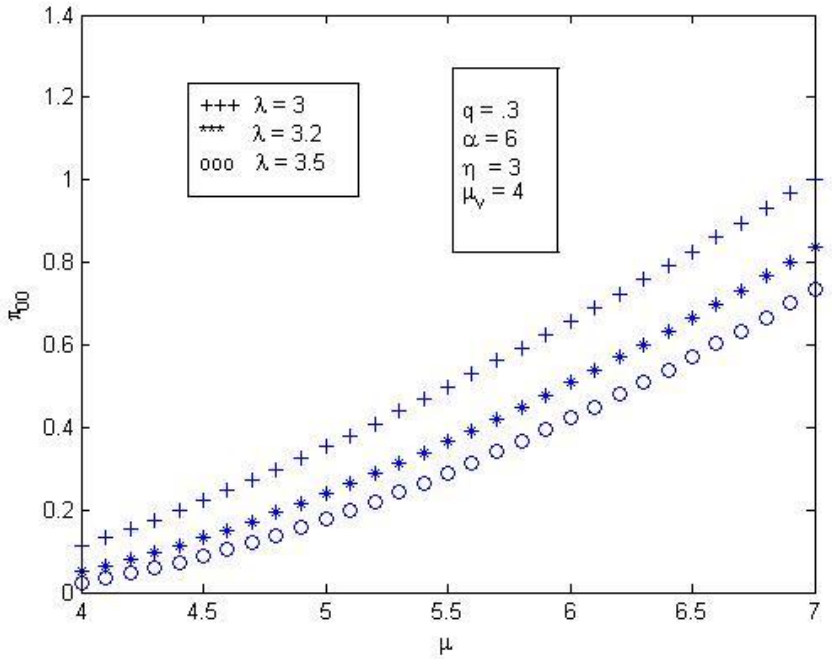


Figure 4 dipicts the influence of the normal service rate μ on the probability $\pi_{0,0}$. The probability of the server being idle decreases with the increasing values of μ for different values of the arrival rate λ.

The effect of λ on $\pi_{0,0}$ is presented in Figure 5; as expected, $\pi_{0,0}$ decreases with increasing values of λ for different values of q.

Figure 5: $\pi_{0,0}$ versus λ

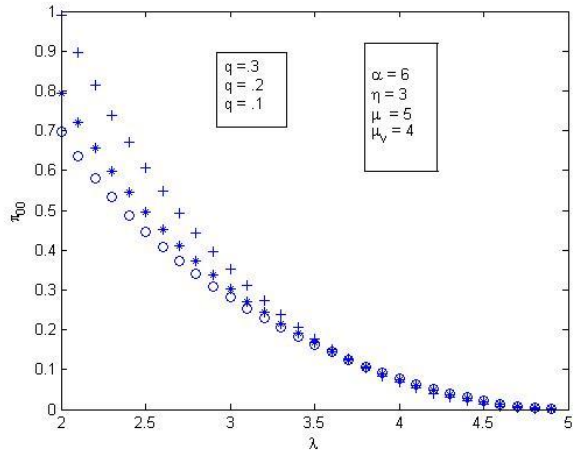


Figure 6 and 7 indicate that the effect of α and the working vacation service rate μ_v on the expected waiting time of customer due to working vacation, respectively.

Figure 6: $E[W_d]$ versus α

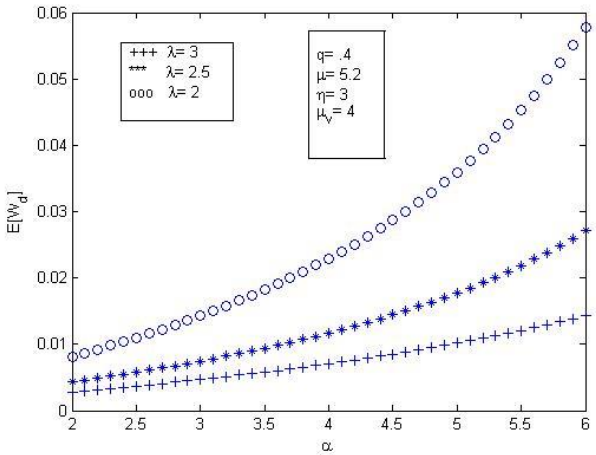


Figure 7: $E[W_d]$ versus μ_v

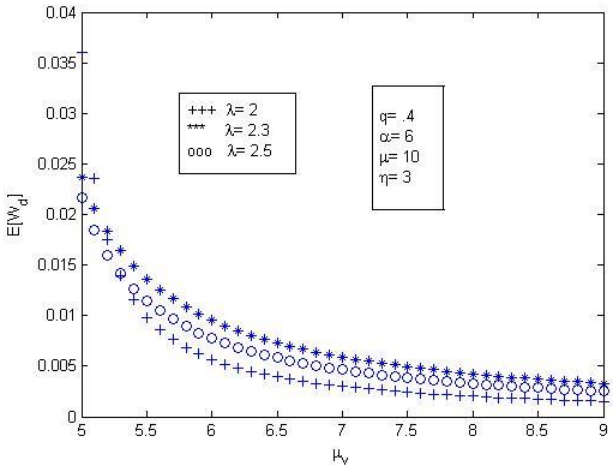


Figure 8 and 9, show the influence of μ_v on the expected queue length due to working vacation for different values of η and q respectively. As expected $E[Q_d]$ decreases.

Figure 8: $E[Q_d]$ versus μ_v

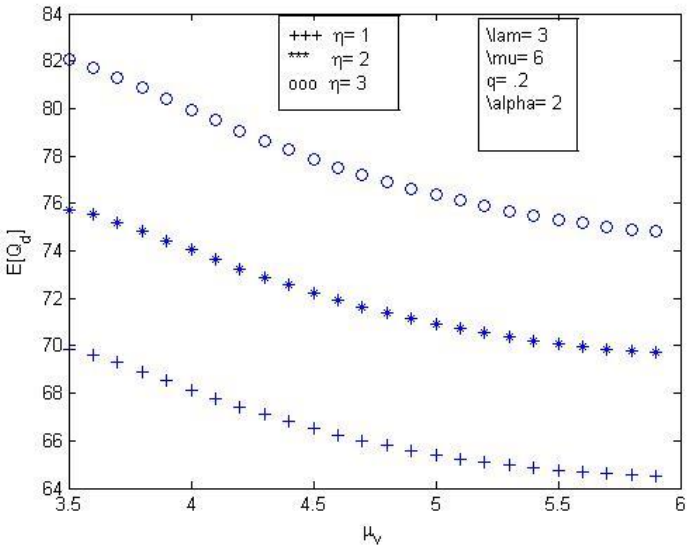


Figure 9: $E[Q_d]$ versus μ_v

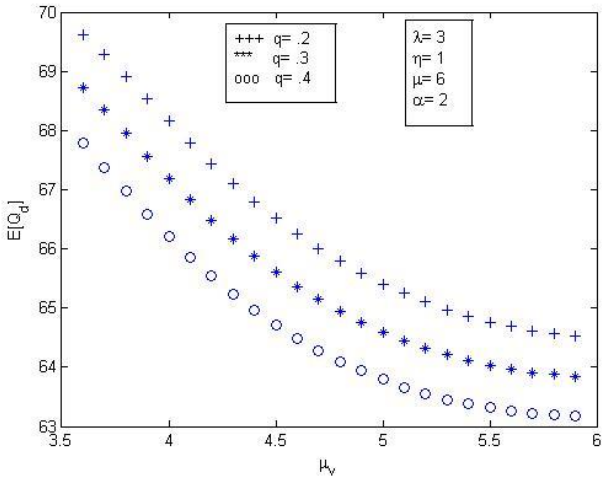


Figure 10: Π_0 versus μ

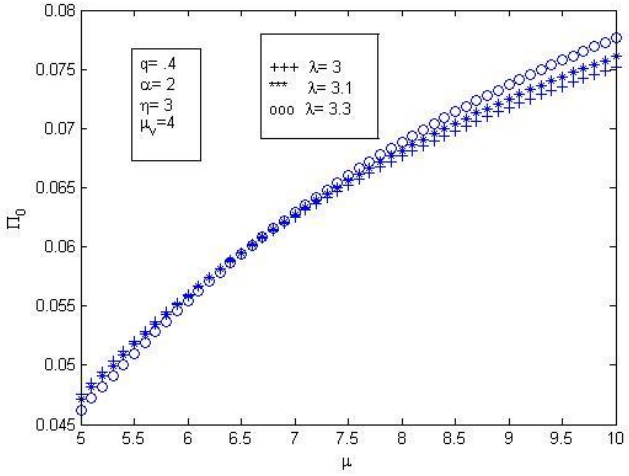


Figure 11: Π_0 versus λ

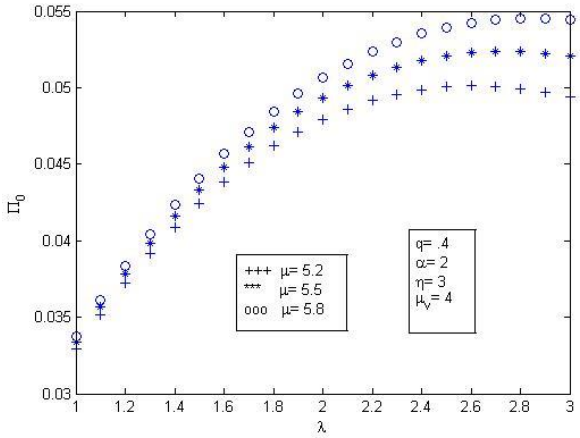
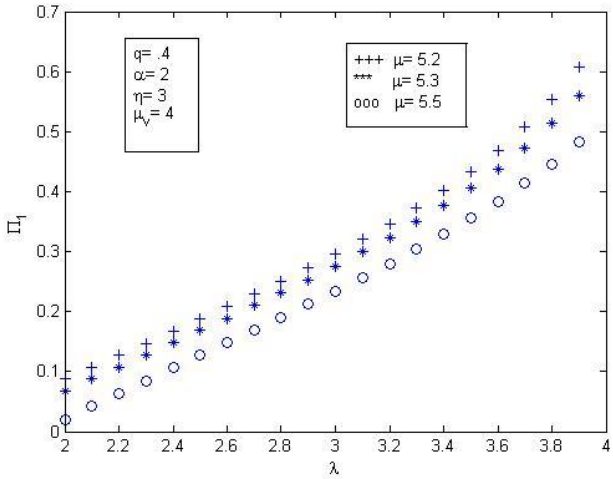


Figure 12: Π_1 versus λ



Figures 10,11 and 12 show the influence of parameters λ and μ on the server state probability. We find that the probability of the server in working vacation and busy Π_0, the probability of the server in regular busy period Π_1 decrease with the increasing λ and μ , respectively. However, the decrease of later fastly. Figure 11, shows that the server in working vacation and busy Π_0 increases for increasing values of λ.

Figure 13,14,15 and 16 show the influence of the expected queue length and expected waiting time with the parameters of λ, $\mu_v$and μ.

Fig 13: E(Q) versus μ and μ_v

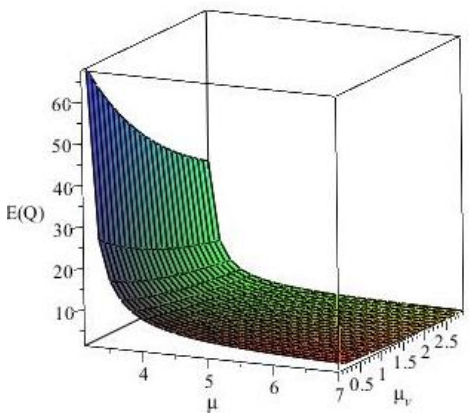


Fig 14: E(W) versus μ and λ

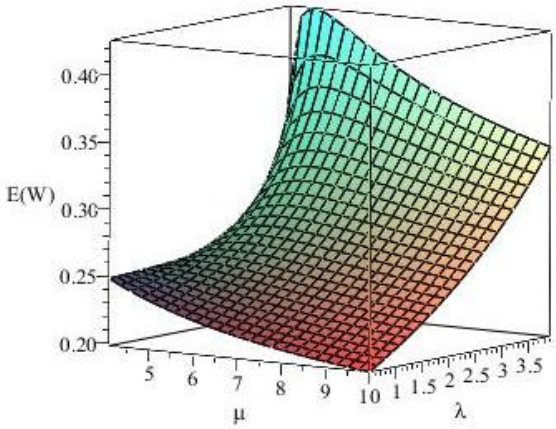


Fig 15: E(W) versus μ and μ_v

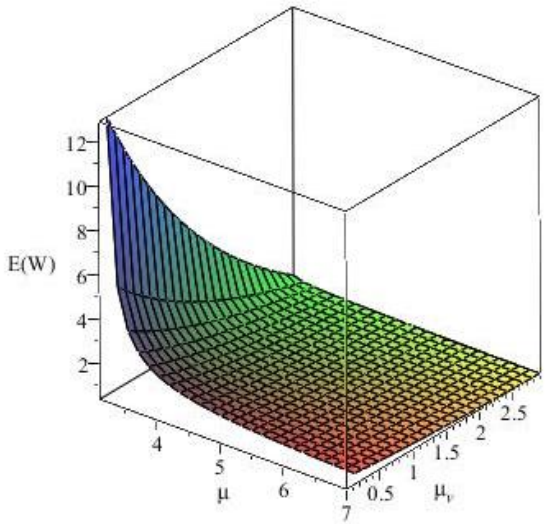
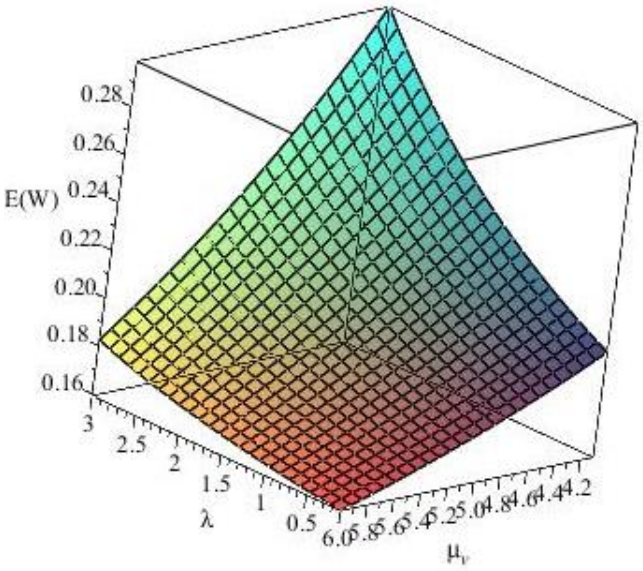


Fig 16: E(W) versus λ and μ_1



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On Interval Arithmetic Operations With Intuitionistic Fuzzy Numbers For Solving An Intuitionistic Fuzzy Multi–Objective Linear Programming Problem

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Abstract*: In the real world circumstances, whenever haziness exists in the modeling of intuitionistic fuzzy numbers(IFNS), interval valued intuitionistic fuzzy numbers(IVIFNS) are often used, in order to represent a range of IFNS, unstable from the most pessimistic evaluation to most the optimistic. IVIFNS are a structure which helps to elude such an preventable complexity. This paper is addressed on the two types of arithmetic operations on interval valued intuitionistic fuzzy numbers(IVIFNS) to solve the interval valued intuitionistic fuzzy multi-objective linear programming problem with hexagonal intuitionistic fuzzy numbers(HIFNS) by assuming different α and β cut values in a comparative way. The objective functions involved in the problem are ranked by ratio ranking method and the problem is solved by preemptive optimization method. An illustrative example with MATLAB outputs is presented in order to clarify the potential approach.*

Keywords*: Hexagonal intuitionistic fuzzy number, Interval valued intuitionistic fuzzy number, Interval valued intuitionistic fuzzy arithmetic, Modified interval valued intuitionistic fuzzy arithmetic, Intuitionistic fuzzy multi-objective linear programming problem, Interval valued intuitionistic fuzzy multi-objective linear programming problem.*

I. Introduction

Fuzzy set theory has been used for managing fuzzy decision-making problems for a long extent of period, but many researchers have shown concern in the intuitionistic fuzzy set theory (IFS) and applied it to the field of decision making. The notion of an intuitionistic fuzzy set can be viewed as an alternative approach to identify a fuzzy set in cases where accessible information is not ample for the definition of an inaccurate concept by means of a conventional fuzzy set. In the year 1989, Atanassov introduced Interval-valued intuitionistic fuzzy sets and many researchers have exposed interest in the interval valued intuitionistic fuzzy set theory (IVIFS)

Linear programming(LP] is measured as one of the most pertinent optimization techniques. It is based on the optimizations of a linear function while fulfilling a set of linear equality and/or inequality constraints or restrictions and it involved a lot of parameters whose values are assigned by decision makers. However, the value of those parameters in most of the cases are not acknowledged by both experts and decision makers. In the literature, Interval Arithmetic was first suggested by Dwyer(1951). The same was developed by Moore(1966,1979). This paper is focused on the interval arithmetic operations of interval valued intuitionistic fuzzy numbers in mathematical optimization problems. In the field of interval linear programming, Ishibuchi *et al.*(1990), Inuiguchi *et al.*(1995), Chanas *et al.*(1996), Chinneck *et al.*(2000), Sengupta *et al.*(2001) developed different procedures to deal with these problems. Some frameworks are proposed to solve multi-objective problems with interval parameters by Das *et al.*(1999), Ida(1999) and Wang *et al.*(2001). Oliveira *et al.*(2007) presented an overview on some current procedures of interval valued multi-objective linear programming problem. With the progress of intuitionistic fuzzy set theory, ranking has become a subject that has been studied by many researchers. Deng-Feng Li(2010) proposed a ratio ranking between triangular intuitionistic fuzzy numbers. We also extended the ratio ranking method to interval valued intuitionistic fuzzy numbers (IVIFNS). Various operations on fuzzy numbers were also available in the past decades which included the new arithmetic operations on interval valued fuzzy numbers by Ganesan *et al.*(2005) and Nagoor Gani *et al.*(2007). Here in this work, we modified the same operations to interval valued intuitionistic fuzzy numbers(IVIFNS) to get the preferred maximum conclusion. This paper is focused on the extension of the existing interval valued fuzzy arithmetic operations(Timothy,2010) and the modified interval valued

intuitionistic fuzzy arithmetic operations based on Ganesan *et al.*(2005) with the proposed division operation in a comparative approach.

This paper is focused on hexagonal intuitionistic fuzzy numbers (HIFNS) and interval valued intuitionistic fuzzy numbers by assuming various α and β cut values from them. The hexagonal intuitionistic fuzzy Number is represented by the six parameters such as a, b, c, d, e and f, where a and b are denoted by the smallest possible values (decreasing order), c, d the most promising value and d, e the largest possible value (increasing order) respectively. Each number in the pair wise comparison represented the vague judgments.

The paper is organized as follows: Section 2 introduces the preliminaries of hexagonal intuitionistic fuzzy numbers(HIFNS), interval valued intuitionistic fuzzy numbers, extension of interval arithmetic, extension of modified arithmetical operations for interval valued intuitionistic numbers (IVIFNS) with proposed division operation and proposed ratio ranking between interval valued intuitionistic numbers (IVIFNS). Section 3 deals with the formulation of interval valued intuitionistic fuzzy multi objective linear programming problem (IVIFMOLPP). In section 4, some important theorems and results on interval valued intuitionistic fuzzy multi objective linear programming problem are given. An application of these new operations is discussed by a numerical illustration in section 5 and some concluding remarks are given in section 6.

II. Preliminaries

This section introduces the preliminary notations of the area of intuitionistic fuzzy set theory .

2.1.Definition. (*Atanassov. 1986***)** Given a fixed set X={$x_1,x_2,\ldots\ldots,x_n$} , an intuitionistic fuzzy set(IFS) is defined as $\tilde{A}^I = \{\langle x_i, \mu_{\tilde{A}^I}(x_i), \vartheta_{\tilde{A}^I}(x_i)\rangle / x_i \epsilon X\}$ which assigns to each element x_i, a membership degree $\mu_{\tilde{A}^I}(x_i)$ and a non-membership degree $\vartheta_{\tilde{A}^I}(x_i)$ under the condition $0 \le \mu_{\tilde{A}^I}(x_i) + \vartheta_{\tilde{A}^I}(x_i) \le 1$, for all $x_i \in$ X .

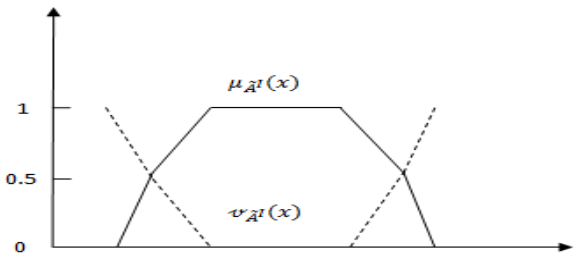
2.2. Definition. (*Atanassov. 1986***)** Let D[0, 1] be the set of all closed subintervals of the interval [0, 1] and X(= Ø) be a given set. An interval valued intuitionistic fuzzy set[IVIFS] $\tilde{A}^I$ in X is defined as $\tilde{A}^I = \{\langle x_i, \mu_{\tilde{A}^I}(x_i), \vartheta_{\tilde{A}^I}(x_i)\rangle / x_i \epsilon X\}$, where $\mu_{\tilde{A}^I}(x_i)$: X → D[0, 1], $\vartheta_{\tilde{A}^I}(x_i)$: X → D[0, 1] with the condition 0 ≤ sup$(\mu_{\tilde{A}^I}(x_i))$ + sup $\vartheta_{\tilde{A}^I}(x_i)$ ≤1 for any x ∈ X.

2.3. Definition. A hexagonal intuitionistic fuzzy number(HITN) $\tilde{A}^I$ is an intuitionistic fuzzy set in R with the following membership function $\mu_{\tilde{A}^I}[x]$ and non-membership function $\nu_{\tilde{A}^I}[x]$

$$\mu_{\tilde{A}^I}[x] = \begin{cases} 0 & ,x < a \\ \frac{x-a}{2b-2a} & ,a \le x \le b \\ \frac{1}{2} + (\frac{x-b}{2c-2b}) & ,b \le x \le c \\ 1 & , c \le x \le d \\ 1 - (\frac{x-d}{2e-2d}) & ,d \le x \le e \\ \frac{f-x}{2f-2e} & ,e \le x \le f \\ 0 & ,x > f \end{cases} \text{ and } \nu_{\tilde{A}^I}[x] = \begin{cases} 1 & ,x < a \\ \frac{2b-a-x}{2b-2a} & ,a \le x \le b \\ \frac{1}{2} - (\frac{x-b}{2c-2b}) & ,b \le x \le c \\ 0 & ,c \le x \le d \\ \frac{x-d}{2e-2d} & ,d \le x \le e \\ \frac{x-2e+f}{2f-2e} & , e \le x \le f \\ 1 & ,x > f \end{cases}$$

where $a \le b \le c \le d \le e \le f$ and $\mu_{\tilde{A}^I}(x) + \nu_{\tilde{A}^I}(x) \le 1$ or $\mu_{\tilde{A}^I}(x) = \nu_{\tilde{A}^I}(x)$, for all $x \in$ R . This HITN is denoted by $\tilde{A}^I = \{(a, b, c, d, e, f), \mu_{\tilde{A}^I}(x), \nu_{\tilde{A}^I}(x)\}$. Throughout this paper, HITN is denoted by $\tilde{A}^I = \{(a, b, c, d, e, f), 1, 0\}$.

Figure 1 Hexagonal intuitionistic fuzzy number(HITN)



2.4. Definition. An (α, β) - cut set of a intuitionistic fuzzy number is defined as $\tilde{A}^I_{\alpha,\beta}$ = $\{x / \mu_{\tilde{A}^I}(x) \ge \alpha, \nu_{\tilde{A}^I}(x) \le \beta\}$, where $0 \le \alpha \le 1; 0 \le \beta \le 1$ and $0 \le \alpha + \beta \le 1$.

2.5. Definition. If $\tilde{A}^I$=$\{(a,b,c,d,e,f),\mu_{\tilde{A}^I}(x),\nu_{\tilde{A}^I}(x)\}$ is a hexagonal intuitionistic fuzzy number, we will let $\tilde{A}^I{}_{[\alpha,\beta]}$ = $\{[A_\alpha^-, A_\alpha^+];[A_\beta^-, A_\beta^+]\}$ where $[A_\alpha^-, A_\alpha^+]$=$[a+2\alpha(b-a), f-2\alpha(f-e)]$ be the closed interval which is a α-cut for $\tilde{A}^I$ in $0\leq\alpha\leq\frac{1}{2}$and where $[A_\beta^-, A_\beta^+]$=$[(2b-a)$-$2\beta(b-a),(2e-f)+2\beta(f-e)]$ be the closed interval which is a β-cut for $\tilde{A}^I$ in $\frac{1}{2}\leq\beta\leq 1$ and also we will let $\tilde{A}^I{}_{[\alpha,\beta]}$ = $\{[A_\alpha^-, A_\alpha^+];[A_\beta^-, A_\beta^+]\}$ where $[A_\alpha^-, A_\alpha^+]$=$[(2b-c)+2\alpha(c-b),(2e-d)-2\alpha(e-d)]$ be the closed interval which is a α-cut for $\tilde{A}^I$ in $\frac{1}{2}\leq\alpha\leq 1$ and where $[A_\beta^-, A_\beta^+]$=$[c-2\beta(c-b), d+2\beta(e-d)]$ be the closed interval which is a β-cut for $\tilde{A}^I$ in $0\leq\beta\leq\frac{1}{2}$. A positive interval valued intuitionistic fuzzy number is denoted as $\{[a_1,a_2];[a_1',a_2']\}$ where all a_i's anda_i''s > 0 for all i=1, 2. A negative interval valued intuitionistic fuzzy number is denoted as $\{[a_1,a_2];[a_1',a_2']\}$ where all a_i's and a_i''s < 0 for all i=1, 2.

2.6. Interval arithmetic operations on Intuitionistic fuzzy numbers. The existing interval arithmetic operations on fuzzy numbers(*Timothy, 2010*) are extended to intuitionistic fuzzy numbers and are given below:

Let $\tilde{A}^I=\{[a_1,a_2];[a_1',a_2']\}$ and $\tilde{B}^I=\{[b_1,b_2];[b_1',b_2']\}$ be two interval valued intuitionistic fuzzy numbers. Then the various arithmetic operations are

(i) Addition:$\tilde{A}^I+\tilde{B}^I=\{[a_1+b_1,a_2+b_2];(a_1'+b_1',a_2'+b_2')\}$

(ii) Subtraction:$\tilde{A}^I-\tilde{B}^I=\{[a_1-b_2,a_2-b_1];[a_1'-b_2',a_2'-b_1']\}$

[or]

$\{[a_1-b_1,a_2-b_2];[a_1'-b_1',a_2'-b_2']\}$,

provided that, $D[\tilde{A}^I]\geq D[\tilde{B}^I]$ and $D[\tilde{A}^{I'}]\geq D[\tilde{B}^{I'}]$, where $D[\tilde{A}^I]=\frac{a_2-a_1}{2}$, $D[\tilde{B}^I]=\frac{b_2-b_1}{2}$, $D[\tilde{A}^{I'}]=\frac{a_2'-a_1'}{2}$ and $D[\tilde{B}^{I'}]=\frac{b_2'-b_1'}{2}$. Here D denotes difference point of an interval valued intuitionistic fuzzy number.

(iii) Multiplication: $\tilde{A}^I X\tilde{B}^I$= $\{[\min[a_1b_1,a_1b_2,a_2b_1,a_2b_2],\max[a_1b_1,a_1b_2,a_2b_1,a_2b_2]];$
$[\min[a_1'b_1',a_1'b_2',a_2'b_1',a_2'b_2'],\quad \max[a_1'b_1',a_1'b_2',a_2'b_1',a_2'b_2']]\}$

(iv) Division: $\tilde{A}^I/\tilde{B}^I$ = $\{[\min[\frac{a_1}{b_1},\frac{a_1}{b_2},\frac{a_2}{b_1},\frac{a_2}{b_2}],\max[\frac{a_1}{b_1},\frac{a_1}{b_2},\frac{a_2}{b_1},\frac{a_2}{b_2}]];\ [\min[\frac{a_1'}{b_1'},\frac{a_1'}{b_2'},\frac{a_2'}{b_1'},\frac{a_2'}{b_2'}],\max[\frac{a_1'}{b_1'},\frac{a_1'}{b_2'},\frac{a_2'}{b_1'},\frac{a_2'}{b_2'}]]\}$

[or]

$\{[\frac{a_1}{b_1},\frac{a_2}{b_2}];[\frac{a_1'}{b_1'},\frac{a_2'}{b_2'}]\}$, provided that, $\left|\frac{D[\tilde{A}^I]}{M[\tilde{A}^I]}\right|\geq\left|\frac{D[\tilde{B}^I]}{M[\tilde{B}^I]}\right|$; $\left|\frac{D[\tilde{A}^{I'}]}{M[\tilde{A}^{I'}]}\right|\geq\left|\frac{D[\tilde{B}^{I'}]}{M[\tilde{B}^{I'}]}\right|$,

where $M[\tilde{A}^I]=\frac{a_2+a_1}{2}$,$M[\tilde{B}^I]=\frac{b_2+b_1}{2}$, $M[\tilde{A}^{I'}]=\frac{a_2'+a_1'}{2}$ and $M[\tilde{B}^{I'}]=\frac{b_2'+b_1'}{2}$. Here M denotes mid value of an interval valued intuitionistic fuzzy number.

(v) Scalar multiplication: Let $\lambda\in$ R. Then $\lambda\tilde{A}^I=\begin{cases}\{[\lambda a_1,\lambda a_2];[\lambda a_1',\lambda a_2']\}, \text{for } \lambda\geq 0\\ \{[\lambda a_2,\lambda a_1];[\lambda a_2',\lambda a_1']\}, \text{for } \lambda<0\end{cases}$

2.7. Modified interval arithmetic operations on interval valued intuitionistic fuzzy numbers. K.Ganesan *et al.*(2005) developed the modified arithmetic operations on interval fuzzy numbers. These operations are extended here for interval valued intuitionistic fuzzy numbers.

Any interval valued intuitionistic fuzzy number $\tilde{A}^I=\{[a_1,a_2];[a_1',a_2']\}$ is alternatively represented as $\tilde{A}^I$=$\{[m_1(\tilde{A}^I),w_1(\tilde{A}^I)];[m_1'(\tilde{A}^I),w_1'(\tilde{A}^I)]\}$, where $m_1(\tilde{A}^I)$, $m_1'(\tilde{A}^I)$ are the mid-points and $w_1(\tilde{A}^I)$, $w_1'(\tilde{A}^I)$ are the half-widths of an interval valued intuitionistic fuzzy number $\tilde{A}^I$. i.e., $m_1(\tilde{A}^I)=\frac{a_1+a_2}{2}$, $w_1(\tilde{A}^I)=\frac{a_2-a_1}{2}$, $m_1'(\tilde{A}^I)=\frac{a_1'+a_2'}{2}$ and $w_1'(\tilde{A}^I)=\frac{a_2'-a_1'}{2}$.

Let $\tilde{A}^I=\{[a_1,a_2];[a_1',a_2']\}$ and $\tilde{B}^I=\{[b_1,b_2];[b_1',b_2']\}$ be two interval valued intuitionistic fuzzy numbers. Then $m_1=\frac{a_1+a_2}{2}$, $m_2=\frac{b_1+b_2}{2}$, $m_1'=\frac{a_1'+a_2'}{2}$ and $m_2'=\frac{b_1'+b'_2}{2}$.

(i) Addition: $\tilde{A}^I+\tilde{B}^I=\{[(m_1+m_2-k);(m_1+m_2+k)];[(m_1'+m_2'-k'),(m_1'+m_2'+k')]\}$,

where $k=\frac{[b_2+a_2]-[b_1+a_1]}{2}$, $k'=\frac{[b'_2+a_2']-[b_1'+a_1']}{2}$.

(ii) Subtraction: $\tilde{A}^I-\tilde{B}^I=\{[(m_1-m_2-k);(m_1-m_2+k)];[(m_1'-m_2'-k'),(m_1'-m_2'+k')]\}$,

where $k=\frac{[b_2+a_2]-[b_1+a_1]}{2}$, $k'=\frac{[b'_2+a_2']-[b_1'+a_1']}{2}$.

(iii) Multiplication: $\tilde{A}^I X\tilde{B}^I$= $\{[(m_1m_2-k);(m_1m_2+k)];[(m_1'm_2'-k'),(m_1'm_2'+k')]\}$,

where $k=\min\{[m_1m_2-\alpha,\beta-m_1m_2]\}$, $\alpha=\min[a_1b_1,a_1b_2,a_2b_1,a_2b_2]$, $\beta=\max[a_1b_1,a_1b_2,a_2b_1,a_2b_2]$, $k'=\min\{[m_1'm_2'-\alpha,\beta-m_1'm_2']\}$, $\alpha'=\min[a_1'b_1',a_1'b_2',a_2'b_1',a_2'b_2']$, $\beta'=\max[a_1'b_1',a_1'b_2',a_2'b_1',a_2'b_2']$

(iv) Inverse: $1/\tilde{A}^I=\{[a_1,a_2];[a_1',a_2']\}^{-1}=\{[\frac{1}{m_1}-k,\frac{1}{m_1}+k];[\frac{1}{m_1'}-k',\frac{1}{m_1'}+k']\}$,

Where $k=\min\{\frac{1}{a_2}\left(\frac{a_2-a_1}{a_1+a_2}\right),\frac{1}{a_1}\left(\frac{a_2-a_1}{a_1+a_2}\right)\}$, $k'=\min\{\frac{1}{a_2'}\left(\frac{a_2'-a_1'}{a_1'+a_2'}\right),\frac{1}{a_1'}\left(\frac{a_2'-a_1'}{a_1'+a_2'}\right)\}$ for all positive real numbers a_1,a_2,a_1',a_2' and $0\notin\{[a_1,a_2];[a_1',a_2']\}$.

If $a_1,a_1'<0$, $a_1=a_1'$, $|a_1|<|a_2|$ and $|a_1'|<|a_2'|$ the above operation is not working. So in that case, the quantities k and k' are proposed as $k=\min\{\frac{1}{|a_2|}\left(\frac{|a_2|-|a_1|}{|a_1|+|a_2|}\right),\frac{1}{|a_1|}\left(\frac{|a_2|-|a_1|}{|a_1|+|a_2|}\right)\}$, $k'=\min\{\frac{1}{|a_2'|}\left(\frac{|a_2'|-|a_1'|}{|a_1'|+|a_2'|}\right),\frac{1}{|a_1'|}\left(\frac{|a_2'|-|a_1'|}{|a_1'|+|a_2'|}\right)\}$

(v) Scalar multiplication: Let $\lambda\in R$.

Then $\lambda\tilde{A}^I=\begin{cases}\{[\lambda a_1,\lambda a_2];[\lambda a_1',\lambda a_2']\}, \text{for } \lambda\geq 0\\ \{[\lambda a_2,\lambda a_1];[\lambda a_2',\lambda a_1']\}, \text{for } \lambda<0\end{cases}$

2.8. Ranking algorithm for interval valued intuitionistic fuzzy numbers. *Deng-Feng Li(2010)* developed a ratio ranking method on triangular intuitionistic fuzzy numbers to compare them. Here, an algorithm was developed to rank the interval valued intuitionistic fuzzy numbers(IVIFNS).

Step.1 Enter the interval numbers, which are to be ranked.

i.e., $\tilde{A}^I=\{[a_1,a_2];[a_1',a_2']\}$ and $\tilde{B}^I=[\{b_1,b_2];[b_1',b_2']\}$

Step.2 For $\tilde{A}^I$ and $\tilde{B}^I$, compute $v_\mu(\tilde{A}^I)=\frac{a_1+a_2}{2}$;

$v_\gamma(\tilde{A}^I)=\frac{a_1'+a_2'}{2}$; $\mathcal{A}_\mu(\tilde{A}^I)=\frac{a_2-a_1}{3}$, $\mathcal{A}_\gamma(\tilde{A}^I)=\frac{a_2'-a_1'}{3}$, $v_\mu(\tilde{B}^I)=\frac{b_1+b_2}{2}$; $v_\gamma(\tilde{B}^I)=\frac{b_1'+b_2'}{2}$; $\mathcal{A}_\mu(\tilde{B}^I)=\frac{b_2-b_1}{3}$ and $\mathcal{A}_\gamma(\tilde{B}^I)=\frac{b_2'-b_1'}{3}$.

Step.3 Calculate the value index $v(\tilde{A}^I,\lambda)=v_\mu(\tilde{A}^I)+\lambda(v_\gamma(\tilde{A}^I)-v_\mu(\tilde{A}^I))$ and the ambiguity index $\mathcal{A}(\tilde{A}^I,\lambda)=\mathcal{A}_\gamma(\tilde{A}^I)-\lambda(\mathcal{A}_\gamma(\tilde{A}^I)-\mathcal{A}_\mu(\tilde{A}^I))$ for $\tilde{A}^I$ and calculate the same for $\tilde{B}^I$, where $\lambda\in[\frac{1}{2},1]$ shows that the decision maker prefers certainty or positive feeling; $\lambda=\frac{1}{2}$ shows that the decision maker is indifferent between positive feeling and negative feeling.

Step.4 Calculate the ratios $\mathcal{R}[\tilde{A}^I,\lambda)=\frac{v(\tilde{A}^I,\lambda)}{1+\mathcal{A}(\tilde{A}^I,\lambda)}$ and $\mathcal{R}(\tilde{B}^I,\lambda)=\frac{v(\tilde{B}^I,\lambda)}{1+\mathcal{A}(\tilde{B}^I,\lambda)}$.

Step.5 Compare the ratios $\mathcal{R}(\tilde{A}^I,\lambda)$ and $\mathcal{R}(\tilde{B}^I,\lambda)$

(i) If $\mathcal{R}(\tilde{A}^I,\lambda)<R(\tilde{B}^I,\lambda)$, then $\tilde{A}^I<\tilde{B}^I$.

(ii) If $\mathcal{R}(\tilde{A}^I,\lambda)>R(\tilde{B}^I,\lambda)$, then $\tilde{A}^I>\tilde{B}^I$.

(iii) If $\mathcal{R}(\tilde{A}^I,\lambda)=\mathcal{R}(\tilde{B}^I,\lambda)$, then $\tilde{A}^I=\tilde{B}^I$.

III. Formulation of the problem

In this section, it is intended to solve the conventional multi-objective linear programming problem with intuitionistic fuzzy attributes according to the appearance of the vagueness in the real world situation.

3.1. Formulation of the interval valued intuitionistic fuzzy multi objective linear programming problem[IVIFMOLPP]. The general form of multi-objective optimization problem with 'k' intuitionistic fuzzy objective functions $\tilde{Z}_1^I,\tilde{Z}_2^I,\ldots\ldots,\tilde{Z}_k^I$ and 'm' intuitionistic fuzzy constraints is given by

$$\left.\begin{array}{l}\text{Maximize or Minimize: } \tilde{Z}_l^I=\sum_{j=1}^n \tilde{c}_j^I\,\tilde{x}_j^I\\ \text{Subject to } \sum_{j=1}^n \tilde{a}_{ij}^I\,\tilde{x}_j^I \preccurlyeq or \succcurlyeq or \approx \tilde{b}_i^I, \tilde{x}_j^I\succcurlyeq\tilde{0}^I\\ \quad\text{(or) } \tilde{A}^I\tilde{X}^I\approx\tilde{b}^I\end{array}\right\}\ (1)$$

where l=1,2,…..,k, i = 1,2,…….,m, j = 1,2,……,n where $\tilde{A}^I=[\tilde{a}_{ij}^I]$, $\tilde{C}_j^I$, $\tilde{b}^I$ and $\tilde{x}_j^I$ whose states are also given by hexagonal intuitionistic fuzzy numbers(HIFNS).

By assuming prescribed values of α and β, the above problem (a) can be restated as

$$\left.\begin{array}{l}\text{Maximize or Minimize:}\\ (\tilde{Z}_l^I)_{(\alpha,\beta)}=\sum_{j=1}^n(\tilde{c}_j^I)_{(\alpha,\beta)}\,(\tilde{x}_j^I)_{(\alpha,\beta)}\\ \text{Subject to}\\ \sum_{j=1}^n(\tilde{a}_{ij}^I)_{(\alpha,\beta)}\,(\tilde{x}_j^I)_{(\alpha,\beta)}\preccurlyeq or\succcurlyeq or\approx(\tilde{b}_i^I)_{(\alpha,\beta)},\tilde{x}_j^I)_{(\alpha,\beta)}\succcurlyeq\tilde{0}^I\\ \text{where } l=1,2,\ldots..,k;\ i=1,2,\ldots\ldots,m\ ;\ j=1,2,\ldots\ldots,n \text{ where } (\tilde{a}_{ij}^I)_{(\alpha,\beta)},(\tilde{c}_j^I)_{(\alpha,\beta)},\end{array}\right\}\ (2)$$

$(\tilde{b}_i^I)_{(\alpha,\beta)}$, are interval valued intuitionistic fuzzy numbers(IVIFNS) and $(\tilde{x}_j^I)_{(\alpha,\beta)}$ whose states are also given by interval valued intuitionistic fuzzy numbers(IVIFNS).

Any interval valued intuitionistic fuzzy multi objective linear programming problem can be converted to its standard form as:

maximize or minimize
$(\tilde{Z}_l^{\,I})_{(\alpha,\beta)}=\sum_{j=1}^{n}(\tilde{c}_j^{\,I})_{(\alpha,\beta)}\,(\tilde{x}_j^{\,I})_{(\alpha,\beta)}$,
l=1,2,.....,k
subject to
$\sum_{j=1}^{n}(\tilde{a}_{ij}^{\,I})_{[\alpha,\beta]}\,(\tilde{x}_j^{\,I})_{[\alpha,\beta]} \pm (\tilde{s}_i^{\,I})_{[\alpha,\beta]} \approx (\tilde{b}_i^{\,I})_{[\alpha,\beta]}$,
$i = 1,2,\ldots\ldots,m$, $(\tilde{x}_j^{\,I})_{(\alpha,\beta)} \succcurlyeq \tilde{0}^I$ } (3)

where $(\tilde{s}_i^{\,I})_{(\alpha,\beta)}$ are called interval valued slack or surplus variables. $[\tilde{1}^I]_{(\alpha,\beta)}$ and $[\tilde{0}^I]_{(\alpha,\beta)}$ are conveniently taken as $[\tilde{1}^I]_{(\alpha,\beta)}$= {[1,1];[1,1]} and $[\tilde{0}^I]_{(\alpha,\beta]}$={[0,0];[0,0]} throughout this paper.

IV. Main results

Now we are interested to prove interval valued intuitionistic fuzzy analogues of some important theorems of multi objective linear programming.

4.1. Definition. Any $\tilde{Z}_1^{\,I}$ $\tilde{X}^I = (\tilde{x}_1^{\,I}, \tilde{x}_2^{\,I}, \ldots, \tilde{x}_n^{\,I}) \in \mathbb{N}^n(S)$, where each $\tilde{x}_j^{\,I} \in \mathbb{N}(S)$, which satisfies all the constraints and non-negativity restrictions of (3) is said to be an interval valued intuitionistic fuzzy feasible solution of (3).

4.2. Definition. Let $\tilde{X}^I = (\tilde{x}_1^{\,I}, \tilde{x}_2^{\,I}, \ldots, \tilde{x}_n^{\,I})$. Suppose $\tilde{X}^I$ solves $\tilde{A}^I\tilde{X}^I \approx \tilde{b}^I$. If all $\tilde{x}_j^{\,I} \approx \tilde{0}^I$ for some j, then $\tilde{x}_j^{\,I}$ is said to be an interval valued intuitionistic fuzzy basic solution. If all $\tilde{x}_j^{\,I} \not\approx \tilde{0}^I$ for some j, then $\tilde{X}^I$ has some non-zero components, say $\tilde{x}_1^{\,I}, \tilde{x}_2^{\,I}, \ldots, \tilde{x}_t^{\,I}$, $1 \le t \le n$. then $\tilde{a}_1^{\,I}\tilde{x}^I{}_1 + \tilde{a}_2^{\,I}\tilde{x}_2^{\,I} + \ldots + \tilde{a}_t^{\,I}\tilde{0}^I + \tilde{a}_{t+1}^{\,I}\tilde{0}^I + \tilde{a}_{t+2}^{\,I}\tilde{0}^I + \cdots + \tilde{a}_n^{\,I}\tilde{0}^I \approx \tilde{b}^I$. If the columns $\tilde{a}_1^{\,I}, \tilde{a}_2^{\,I}, \tilde{a}_3^{\,I}, \ldots \tilde{a}_t^{\,I}$ corresponding to these non-zero components $\tilde{x}_1^{\,I}, \tilde{x}_2^{\,I}, \ldots, \tilde{x}_t^{\,I}$ are linearly independent, then $\tilde{X}^I$ is said to be an interval valued intuitionistic fuzzy basic solution.

4.3 Definition: Let $\tilde{B}^I = [\widetilde{b_1}^{\,I}, \widetilde{b_2}^{\,I}, \ldots, \widetilde{b_m}^{\,I})$ form a basis for the columns of $\tilde{A}^I$. Let $\tilde{X}^I{}_{\tilde{B}^I} = \tilde{B}^{-1^I}\tilde{b}^I$ be an interval valued intuitionistic fuzzy basic feasible solution and the value of the objective function $\tilde{z}_l^{\,I}$ [l=1,2,.....,k) is given by $\tilde{z}_{0l}{}^I \approx \tilde{c}^I{}_{\tilde{B}^I}\,\tilde{X}^I{}_{\tilde{B}^I}$, where $\tilde{c}^I{}_{\tilde{B}^I} = [\tilde{c}^I{}_{\tilde{B}1^I}, \tilde{c}^I{}_{\tilde{B}2^I}, \tilde{c}^I{}_{\tilde{B}3^I}, \ldots, \tilde{c}^I{}_{\tilde{B}m^I})$ be the cost vector corresponding to $\tilde{X}^I{}_{\tilde{B}^I}$. Assume that $\widetilde{a_j}^{\,I} = \sum_{i=1}^{m}\tilde{y}_{ij}{}^I\tilde{b}_i^{\,I} = \tilde{y}_j{}^I\widetilde{B}^I$ and the interval valued intuitionistic fuzzy numbers $\tilde{z}_j^{\,I} = \sum_{i=1}^{m}\tilde{c}^I{}_{\tilde{B}i^I}\,\tilde{y}_{ij}{}^I = \tilde{c}^I{}_{\tilde{B}^I}\tilde{y}_j{}^I$ are known for every column vector $\widetilde{a_j}^{\,I}$ in $\tilde{A}^I$, which is not in $\tilde{B}^I$. Now we intended to examine the possibility of finding another interval valued intuitionistic fuzzy basic feasible solution with an improved interval valued intuitionistic fuzzy value of $\tilde{z}_l^{\,I}$(l=1,2,.....,k) by replacing one of the columns of $\tilde{B}^I$ by $\widetilde{a_j}^{\,I}$.

Theorem 4.1. Let $\tilde{X}^I{}_{\tilde{B}^I} = \tilde{B}^{-1^I}\tilde{b}^I$ be an interval valued intuitionistic fuzzy basic feasible solution of (3). If for any column $\widetilde{a_j}^{\,I}$ in $\tilde{A}^I$ which is not in $\tilde{B}^I$, the condition $(\tilde{z}_j^{\,I} - \tilde{c}_j^{\,I}) \prec \tilde{0}^I$ hold and $\tilde{y}_{ij}{}^I \succ \tilde{0}^I$ for some i , $i \in \{1, 2, 3, \ldots, m\}$ then it is possible to obtain a new interval valued intuitionistic fuzzy basic feasible solution by replacing one of the columns in $\tilde{B}^I$ by $\widetilde{a_j}^{\,I}$.

Proof: Suppose that $\tilde{X}^I{}_{\tilde{B}^I} = (\tilde{x}^I{}_{\tilde{B}1^I}, \tilde{x}^I{}_{\tilde{B}2^I}, \tilde{x}^I{}_{\tilde{B}3^I}, \ldots, \tilde{x}^I{}_{\tilde{B}m^I}]$ be an interval valued intuitionistic fuzzy basic feasible solution with k positive components such that $\tilde{B}^I\tilde{X}^I{}_{\tilde{B}^I} \approx \tilde{b}$ or $\tilde{X}^I{}_{\tilde{B}^I} \approx \tilde{B}^{-1^I}\tilde{b}^I$, where $\tilde{x}^I{}_{\tilde{B}i^I} \in \mathbb{N}(S)$, for i=1,2,...m and $\tilde{x}^I{}_{\tilde{B}i^I} \succ \tilde{0}^I$ for i=1,2,....k, $k < m$

Now equation $\tilde{B}^I\tilde{X}^I{}_{\tilde{B}^I} \approx \tilde{b}^I$ becomes

$\sum_{i=1}^{k}\tilde{x}^I{}_{\tilde{B}i^I}\tilde{b}_i^{\,I} + \tilde{0}^I\tilde{b}_{k+1}{}^I + \tilde{0}^I\tilde{b}_{k+2}{}^I + \ldots + \tilde{0}^I\tilde{b}_m{}^I \approx \tilde{b}^I$,

where $\tilde{0}^I \approx$ {(0,0,0,0];1,0,0} or {(0,0,0];(1,0,0]} without loss of generality.

That is $\sum_{i=1}^{k}\tilde{x}^I{}_{\tilde{B}i^I}\tilde{b}_i^{\,I} + \sum_{i=k+1}^{m}\tilde{0}^I\tilde{b}_i^{\,I} \approx \tilde{b}^I$ (4)

Then for any column $\widetilde{a_j}^{\,I}$ in $\tilde{A}^I$ which is not in $\tilde{B}^I$, we write

$\widetilde{a_j}^{\,I} = \sum_{i=1}^{m}\tilde{y}_{ij}{}^I\tilde{b}_i^{\,I} = \tilde{y}_j{}^I\tilde{B}^I$.We know that if the basis vector $\tilde{b}_r^{\,I}$ for which $\tilde{y}_{rj}{}^I \not\approx \tilde{0}^I$ is replaced by $\widetilde{a_j}^{\,I}$in $\tilde{A}^I$, then the new set of vectors $\tilde{b}_i^{\,I}$ $(i = 1,2,\ldots m)$ form a basis.

Now for $\tilde{y}_{rj}{}^I \not\approx \tilde{0}^I$ and $r \le k$, we can write $\tilde{b}_r^{\,I} = \frac{\widetilde{a_j}^I}{\tilde{y}_{rj}{}^I} - \sum_{\substack{i=1\\ i\neq r}}^{m}\frac{\tilde{y}_{ij}{}^I}{\tilde{y}_{rj}{}^I}\,\tilde{b}_i^{\,I} = \frac{\widetilde{a_j}^I}{\tilde{y}_{rj}{}^I} - \sum_{\substack{i=1\\ i\neq r}}^{m}\frac{\tilde{y}_{ij}{}^I}{\tilde{y}_{rj}{}^I}\,\tilde{b}_i^{\,I} - \sum_{i=k+1}^{m}\frac{\tilde{y}_{ij}{}^I}{\tilde{y}_{rj}{}^I}\,\tilde{b}_i^{\,I}$

Equation (4) becomes

$\sum_{\substack{i=1\\ i\neq r}}^{m}\tilde{x}^I{}_{\tilde{B}i^I}\,\tilde{b}_i^{\,I} + \tilde{x}^I{}_{\tilde{B}r^I}\tilde{b}_r^{\,I} + \sum_{i=k+1}^{m}\tilde{0}^I\,\tilde{b}_i^{\,I} \approx \tilde{b}^I$.

$\Longrightarrow \sum_{\substack{i=1\\ i\neq r}}^{m}\tilde{x}^I{}_{\tilde{B}i^I}\,\tilde{b}_i^{\,I} + \frac{\tilde{x}^I{}_{\tilde{B}r^I}}{\tilde{y}_{rj}{}^I}\widetilde{a_j}^{\,I} - \frac{\tilde{x}^I{}_{\tilde{B}r^I}}{\tilde{y}_{rj}{}^I}\sum_{\substack{i=1\\ i\neq r}}^{k}\tilde{y}_{ij}{}^I\,\tilde{b}_i^{\,I} - \frac{\tilde{x}^I{}_{\tilde{B}r^I}}{\tilde{y}_{rj}{}^I}\sum_{i=k+1}^{m}\tilde{y}_{ij}{}^I\,\tilde{b}_i^{\,I} + \sum_{i=k+1}^{m}\tilde{0}^I\,\tilde{b}_i^{\,I} \approx \tilde{b}^I$.

$\Longrightarrow \sum_{\substack{i=1\\ i\neq r}}^{k}(\tilde{x}^I{}_{\tilde{B}i^I} - \frac{\tilde{x}^I{}_{\tilde{B}r^I}}{\tilde{y}_{rj}{}^I}\tilde{y}_{ij}{}^I)\,\tilde{b}_i^{\,I} + \frac{\tilde{x}^I{}_{\tilde{B}r^I}}{\tilde{y}_{rj}{}^I}\,\widetilde{a_j}^{\,I} + \sum_{i=k+1}^{m}(\tilde{0}^I - \frac{\tilde{x}^I{}_{\tilde{B}r^I}}{\tilde{y}_{rj}{}^I}\,\tilde{y}_{ij}{}^I)\,\tilde{b}_i^{\,I} \approx \tilde{b}^I$.

Since $\tilde{x}^I{}_{\tilde{B}i^I} \approx \tilde{0}^I$, for $i = k+1, k+2, \ldots, m$, we have

$\sum_{\substack{i=1 \\ i\neq r}}^{k}[\tilde{x}^I{}_{\tilde{B}i^I} - \frac{\tilde{x}^I{}_{\tilde{B}r^I}}{\tilde{y}_{rj}{}^I}\tilde{y}_{ij}{}^I)\,\tilde{b}_i{}^I + \frac{\tilde{x}^I{}_{\tilde{B}r^I}}{\tilde{y}_{rj}{}^I}\,\tilde{a}_j{}^I \approx \tilde{b}^I$

$\Longrightarrow \sum_{\substack{i=1 \\ i\neq r}}^{m}\hat{\tilde{x}}^I{}_{\tilde{B}i^I}\,\tilde{b}_i{}^I - \hat{\tilde{x}}^I{}_{\tilde{B}r^I}\,\tilde{a}_j{}^I \approx \tilde{b}^I,$

where $\hat{\tilde{x}}^I{}_{\tilde{B}i^I} = (\tilde{x}^I{}_{\tilde{B}i^I} - \frac{\tilde{x}^I{}_{\tilde{B}r^I}}{\tilde{y}_{rj}{}^I}\tilde{y}_{ij}{}^I], i \neq r$ and $\hat{\tilde{x}}^I{}_{\tilde{B}r^I} = \frac{\tilde{x}^I{}_{\tilde{B}r^I}}{\tilde{y}_{rj}{}^I}$

which gives a new interval valued intuitionistic fuzzy basic solution to $\tilde{A}^I\tilde{X}^I \approx \tilde{b}^I$.

We shall prove that this new interval valued intuitionistic fuzzy basic solution is also feasible.

Then choose $\tilde{y}_{rj}{}^I \succ \tilde{0}^I$ such that $\frac{\tilde{x}^I{}_{\tilde{B}r^I}}{\tilde{y}_{rj}{}^I} \approx \min_i\left\{\frac{\tilde{x}^I{}_{\tilde{B}i^I}}{\tilde{y}_{ij}{}^I} : \tilde{y}^I{}_{ij} \succ \tilde{0}^I\right\}$. Then $\frac{\tilde{x}^I{}_{\tilde{B}r^I}}{\tilde{y}_{rj}{}^I} \preccurlyeq \frac{\tilde{x}^I{}_{\tilde{B}i^I}}{\tilde{y}_{ij}{}^I}$.

That implies that $\frac{\tilde{x}^I{}_{\tilde{B}i^I}}{\tilde{y}_{ij}{}^I} - \frac{\tilde{x}^I{}_{\tilde{B}r^I}}{\tilde{y}_{rj}{}^I} \succcurlyeq \tilde{0}^I$. (5)

Hence the new interval valued intuitionistic fuzzy basic solution is an interval valued intuitionistic fuzzy basic feasible solution.

Theorem 4.2. If $\tilde{X}^I{}_{\tilde{B}^I} = \tilde{B}^{-1^I}\tilde{b}^I$ is a new interval valued intuitionistic fuzzy basic feasible solution of (3) with $\tilde{z}_0{}^I \approx \tilde{c}^I{}_{\tilde{B}^I}\tilde{X}^I{}_{\tilde{B}^I}$ as the interval valued intuitionistic fuzzy value of the objective function and if $\hat{\tilde{x}}^I{}_{\tilde{B}i^I}$ is another interval valued intuitionistic fuzzy basic feasible solution with $\widehat{\tilde{z}_0}^I \approx \widehat{\tilde{c}^I{}_{\tilde{B}^I}}\widehat{\tilde{X}^I{}_{\tilde{B}^I}}$ obtained by admitting a non-basic column vector $\tilde{a}_j{}^I$ in the basis for which $(\tilde{z}_j{}^I - \tilde{c}_j{}^I) \prec \tilde{0}^I$ and $\tilde{y}_{ij}{}^I \succ \tilde{0}^I$ for some $i, i \in \{1,2,3,\ldots,m\}$, then $\hat{\tilde{z}}^I \succcurlyeq \tilde{z}_0{}^I$.

Proof: Let $\tilde{X}^I{}_{\tilde{B}^I}$ be an interval valued intuitionistic fuzzy basic feasible solution and $\tilde{z}_0{}^I \approx \tilde{c}^I{}_{\tilde{B}^I}\tilde{X}^I{}_{\tilde{B}^I}$. Let $\tilde{a}_j{}^I$ be the column vector introduced in the basis for which $(\tilde{z}_j{}^I - \tilde{c}_j{}^I) \prec \tilde{0}^I$. Let $\tilde{b}_r{}^I$ be the column vector removed from the basis and $\widehat{\tilde{X}^I{}_{\tilde{B}^I}}$ be a new interval valued intuitionistic fuzzy basic feasible solution, then $\hat{\tilde{x}}^I{}_{\tilde{B}i^I} = (\tilde{x}^I{}_{\tilde{B}i^I} - \frac{\tilde{x}^I{}_{\tilde{B}r^I}}{\tilde{y}_{rj}{}^I}\tilde{y}_{ij}{}^I), i \neq r$ and

$\hat{\tilde{x}}^I{}_{\tilde{B}r^I} = \frac{\tilde{x}^I{}_{\tilde{B}r^I}}{\tilde{y}_{rj}{}^I}$. Since $\widehat{\tilde{c}^I{}_{\tilde{B}i^I}} \approx \tilde{c}^I{}_{\tilde{B}i^I}, i \neq r$ and $\widehat{\tilde{c}^I{}_{\tilde{B}r^I}} \approx \tilde{c}_j{}^I$, the new interval valued intuitionistic fuzzy value of the objective function is

$\hat{\tilde{z}}^I \approx \widehat{\tilde{c}^I{}_{\tilde{B}^I}}\widehat{\tilde{X}^I{}_{\tilde{B}^I}} \approx \sum_{i=1}^{m}\widehat{\tilde{c}^I{}_{\tilde{B}i^I}}\,\hat{\tilde{x}}^I{}_{\tilde{B}i^I}$

$\approx \sum_{\substack{i=1 \\ i\neq r}}^{m}\widehat{\tilde{c}^I{}_{\tilde{B}i^I}}\,\hat{\tilde{x}}^I{}_{\tilde{B}i^I} + \widehat{\tilde{c}^I{}_{\tilde{B}r^I}}\hat{\tilde{x}}^I{}_{\tilde{B}r^I}$

$\approx \sum_{\substack{i=1 \\ i\neq r}}^{m}\tilde{c}^I{}_{\tilde{B}i^I}(\tilde{x}^I{}_{\tilde{B}i^I} - \frac{\tilde{x}^I{}_{\tilde{B}r^I}}{\tilde{y}_{rj}{}^I}\tilde{y}_{ij}{}^I) + \tilde{c}^I{}_{\tilde{B}r^I}(\tilde{x}^I{}_{\tilde{B}r^I} - \frac{\tilde{x}^I{}_{\tilde{B}r^I}}{\tilde{y}_{rj}{}^I}\tilde{y}_{rj}{}^I) + \tilde{c}_j{}^I\frac{\tilde{x}^I{}_{\tilde{B}r^I}}{\tilde{y}_{rj}{}^I}$

$\approx \sum_{i=1}^{m}\tilde{c}^I{}_{\tilde{B}i^I}(\tilde{x}^I{}_{\tilde{B}i^I} - \frac{\tilde{x}^I{}_{\tilde{B}r^I}}{\tilde{y}_{rj}{}^I}\tilde{y}_{ij}{}^I) + \tilde{c}_j{}^I\frac{\tilde{x}^I{}_{\tilde{B}r^I}}{\tilde{y}_{rj}{}^I}$

$\approx \sum_{i=1}^{m}\tilde{c}^I{}_{\tilde{B}i^I}(\tilde{x}^I{}_{\tilde{B}i^I} - \frac{\tilde{x}^I{}_{\tilde{B}r^I}}{\tilde{y}_{rj}{}^I}\sum_{i=1}^{m}\tilde{c}^I{}_{\tilde{B}i^I}\tilde{y}_{ij}{}^I) + \tilde{c}_j{}^I\frac{\tilde{x}^I{}_{\tilde{B}r^I}}{\tilde{y}_{rj}{}^I}$

$\approx \tilde{z}_0{}^I - \frac{\tilde{x}^I{}_{\tilde{B}r^I}}{\tilde{y}_{rj}{}^I}\tilde{z}_j{}^I + \tilde{c}_j{}^I\frac{\tilde{x}^I{}_{\tilde{B}r^I}}{\tilde{y}_{rj}{}^I} \approx \tilde{z}_0{}^I - \frac{\tilde{x}^I{}_{\tilde{B}r^I}}{\tilde{y}_{rj}{}^I}(\tilde{z}_j{}^I - \tilde{c}_j{}^I)$

Since $\tilde{y}_{rj}{}^I \succ \tilde{0}^I$, $(\tilde{z}_j{}^I - \tilde{c}_j{}^I) \prec \tilde{0}^I$ and $\frac{\tilde{x}^I{}_{\tilde{B}r^I}}{\tilde{y}_{rj}{}^I} \succcurlyeq \tilde{0}^I$, then $\hat{\tilde{z}}^I \succcurlyeq \tilde{z}_0{}^I$. (6)

Hence the new interval valued intuitionistic fuzzy basic feasible solution gives the improved interval valued intuitionistic fuzzy value of the objective function.

4.4 Definition: In the classical linear programming problems, the process of inserting and removing vectors from the basis matrix has been done from any one of the following situations:

(i) There exist *j* such that $[\tilde{z}_j{}^I - \tilde{c}_j{}^I) \prec \tilde{0}^I$, $\tilde{y}_{ij}{}^I \preccurlyeq \tilde{0}^I$, $i =$ 1,2,3,...,m (or) (ii) For all *j*, $(\tilde{z}_j{}^I - \tilde{c}_j{}^I) \succcurlyeq \tilde{0}^I$.

In the first case, we obtained an unbounded solution and if the second case occurred , it made easy to show that the interval valued intuitionistic fuzzy linear programming problem has an interval valued intuitionistic fuzzy optimal solution.

Theorem 4.4. If $\tilde{X}^I{}_{\tilde{B}^I} = \tilde{B}^{-1^I}\tilde{b}^I$ is an interval valued intuitionistic fuzzy basic feasible solution of (2) and if $(\tilde{z}_j{}^I - \tilde{c}_j{}^I) \succcurlyeq \tilde{0}^I$ for every column $\tilde{a}_j{}^I$ of $\tilde{A}^I$, then $\tilde{X}^I{}_{\tilde{B}^I}$ is an interval valued intuitionistic fuzzy optimal solution.

4.5. Definition. Preemptive *optimization (Rardin,2003)* treated the multi-objective optimization by considering objectives one at a time. The most important one is optimized, then the second most important one is optimized subject to a requirement that the first achieve its optimal value in the preemptive optimization and so on. If each stage of preemptive optimization yields a single objective optimum, the final solution is concluded as an efficient point of the full multi-objective model.

V. Illustrative example

Consider the following intuitionistic fuzzy multi objective linear programming problem(IFMOLPP)

$$\left.\begin{array}{l}\text{Maximize } \tilde{z}_1^I = \{(-3.5,-0.5,2.5,5.5,8.5,11.5);1,0\}\ \tilde{x}_1^I + \{(3,6,9,11,14,17);1,0\}\tilde{x}_2^I \\ \text{Maximize } \tilde{z}_2^I = \{(-4.5,-1.5,1.5,4.5,7.5,10.5);1,0\}\ \tilde{x}_1^I + \{(2,5,8,10,13,16);1,0\}\tilde{x}_2^I \\ \text{Subject to the constraints} \\ \{(-5,-2,1,3,6,9);1,0\}\ \tilde{x}_1^I + \{(-7.5,-4.5,-1.5,3.5,6.5,9.5);1,0\}\tilde{x}_2^I \le \{(-2,1,4,6,9,12);1,0\} \\ \{(-5,-2,1,3,6,9);1,0\}\tilde{x}_1^I + \{(-2,1,4,6,9,12);1,0\}\tilde{x}_2^I \le \{(3,6,9,11,14,17);1,0\} \,\&\, \tilde{x}_1^I, \tilde{x}_2^I \ge \tilde{0}^I.\end{array}\right\} \quad (7)$$

For Solving the above intuitionistic fuzzy multi objective linear programming problem(IFMOLPP), it is reduced into the following interval valued intuitionistic fuzzy linear programming problem(IVIFLPP) by taking different $\alpha$and β -cuts.

When α = 1, β=0, the model (7) for the first objective function takes the form,

$$\left.\begin{array}{l}\text{Maximize } \widetilde{Z}_1^I = \{[2.5,5.5];[2.5,5.5]\}\ \tilde{x}_1^I + \{[9,11];[9,11]\}\ \tilde{x}_2^I \\ \text{subject to the constraints} \\ \{[1,3];[1,3]\}\ \tilde{x}_1^I + \{[-1.5,3.5];[-1.5,3.5]\}\ \tilde{x}_2^I \le \{[4,6];[4,6]\} \\ \{[1,3];[1,3]\}\ \tilde{x}_1^I + \{[4,6];[4,6]\}\ \tilde{x}_2^I \le \{[9,11];[9,11]\}\ \ \&\ \ \tilde{x}_1^I, \tilde{x}_2^I \ge \tilde{0}^I\end{array}\right\} \quad (8)$$

The above model [8] can be rewritten as

$$\left.\begin{array}{l}\text{Maximize } \widetilde{Z}_1^I = \{[2.5,5.5];[2.5,5.5]\}\ \tilde{x}_1^I + \{[9,11];[9,11]\}\ \tilde{x}_2^I + \{[0,0];[0,0]\}\ \tilde{s}_1^I + \{[0,0];[0,0]\}\ \tilde{s}_2^I \\ \text{subject to the constraints} \\ \{[1,3];[1,3]\}\ \tilde{x}_1^I + \{[-1.5,3.5];[-1.5,3.5]\}\ \tilde{x}_2^I + \{[1,1];[1,1]\}\ \tilde{s}_1^I + \{[0,0];[0,0]\}\ \tilde{s}_2^I = \{[4,6];[4,6]\} \\ \{[1,3];[1,3]\}\ \tilde{x}_1^I + \{[4,6];[4,6]\}\ \tilde{x}_2^I + \{[0,0];[0,0]\}\ \tilde{s}_1^I + \{[1,1];[1,1]\}\ \tilde{s}_2^I = \{[9,11];[9,11]\}\ \ \&\ \ \tilde{x}_1^I, \tilde{x}_2^I \ge \tilde{0}^I\end{array}\right\} \quad (9)$$

By the theorems 4.1 and 4.2, it is observed that all $\tilde{z}_j^I - \tilde{c}_j^I \ge 0$. Then the current solution is optimal and the optimal solution is as $\tilde{x}_1^I$ ={[0,0];[0,0]}, $\tilde{x}_2^I$ = {[1.53,2.47];[1.53,2.47]} with Maximize $\tilde{z}_1^I$ = {[1.26,38.74];[1.26,38.74]}.

The optimal solutions of $\widetilde{Z}_1^I$ for using different α, β cut values by interval arithmetic operations and modified interval arithmetic operations and also the respective MATLAB outputs(Figure 2 and Figure 3) are given below.

Table 1 Optimum $\widetilde{Z}_1^I$ -values

α-values	β-values	Optimum $\widetilde{Z}_1^I$ -values by interval arithmetic operations	Optimum $\widetilde{Z}_1^I$ -values by modified interval arithmetic operations
α=1	β=0	$\tilde{Z}^I$ = {[-144.50,241.40];[-144.50,241.40]}	$\tilde{Z}^I$ = {[1.26,38.74];[1.26,38.74]}
α =0.9	β=0.1	$\tilde{Z}^I$ = {[-192.04,23.62];[-192.04,23.62]}	$\tilde{Z}^I$ = {[1.79,38.81];[1.79,38.81]}
α =0.8	β=0.2	$\tilde{Z}^I$ = {[-312.05,23.07];[-312.05,23.07]}	$\tilde{Z}^I$ = {[2.1,37.9];[2.1,37.9]}
α =0.7	β=0.3	$\tilde{Z}^I$ = {[-1155.20,22.65];[-1155.20,22.65]}	$\tilde{Z}^I$ = {[2.54,37.46];[2.54,37.46]}
α =0.6	β=0.4	$\tilde{Z}^I$ = {[3.02,532.90];[3.02,532.90]}	$\tilde{Z}^I$ = {[2.92,37.08];[2.92,37.08]}
α =0.5	β=0.5	$\tilde{Z}^I$ = {[4.02,196];[4.02,196]}	$\tilde{Z}^I$ = { [3.96,36.04];[3.96,36.04]}
α =0.4	β=0.6	$\tilde{Z}^I$ = {[5.21,112.29];[5.21,112.29]}	$\tilde{Z}^I$ = {[5.21,34.79];[5.21,34.79]}
α =0.3	β=0.7	$\tilde{Z}^I$ = {[6.62,74.50];[6.62,74.50]}	$\tilde{Z}^I$ = {[6.67,33.23];[6.67,33.23]}
α =0.2	β=0.8	$\tilde{Z}^I$ = {[8.42,53.19];[8.42,53.19]}	$\tilde{Z}^I$ = {[8.50,31.5];[8.50,31.5]}
α =0.1	β=0.9	$\tilde{Z}^I$ = {[10.67,39.56];[10.67,39.56]}	$\tilde{Z}^I$ = {[10.58,29.42];[10.58,29.42]}
α =0	β=1	$\tilde{Z}^I$ = {[13.5,24.75];[13.5,24.75]}	$\tilde{Z}^I$ = {[13.77,26.23];[13.77,26.23]}

Figure 2: Optimum values $\widetilde{z}_1^I$---->>

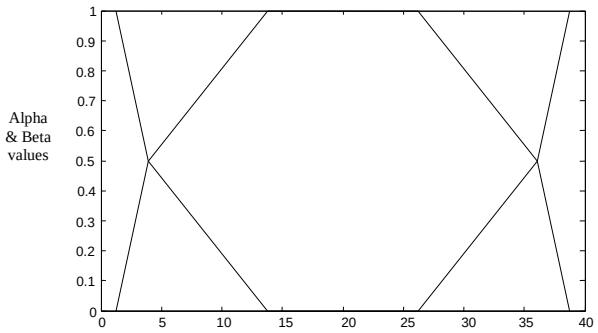
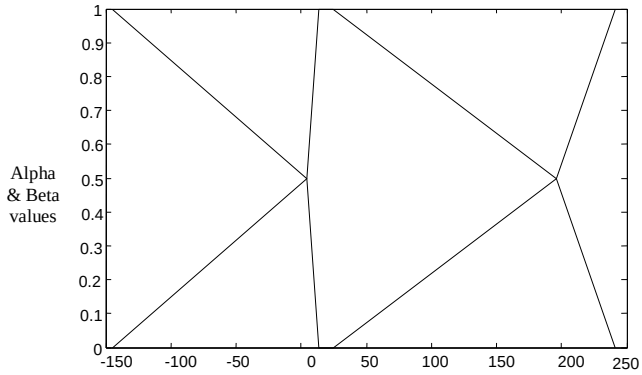


Figure 3 Optimum values $\widetilde{z_1}^I$



The next constraint is considered as

{[2.5,5.5]; [2.5,5.5]} $\tilde{x}_1^I$+{[9,11]; [9,11]} $\tilde{x}_2^I \geq$ {[1.26,38.74];[1.26,38.74]} to optimize the second objective.

The second objective is written as

Maximize $\widetilde{z_2}^I$ = {[1.5,4.5];[1.5,4.5]} $\tilde{x}_1^I$ +{[8,10];[8,10]}$\tilde{x}_2^I$
Subject to the constraints
{[1,3];[1,3]} $\tilde{x}_1^I$ +{[-1.5,3.5];[-1.5,3.5]} $\tilde{x}_2^I \leq$ {[4,6];[4,6]}
{[1,3];[1,3]} $\tilde{x}_1^I$ +{[4,6];[4,6]} $\tilde{x}_2^I \leq$ {[9,11];[9,11]}
{[-5.5,-2.5];[-5.5,-2.5]} $\tilde{x}_1^I$+{[-11,-9];[-11,-9]} $\tilde{x}_2^I \leq$ {[-38.74,-1.26];[-38.74,-1.26]} & $\tilde{x}_1^I, \tilde{x}_2^I \geq 0$ (10)

The above model [10] can be rewritten as

Maximize $\widetilde{z_2}^I$ = {[1.5,4.5];[1.5,4.5]} $\tilde{x}_1^I$ +{[8,10];[8,10]}$\tilde{x}_2^I$
+ {[0,0]; [0,0]} $\tilde{s}_1^I$+{[0,0]; [0,0]} $\tilde{s}_2^I$ + {[0,0]; [0,0]} $\tilde{s}_3^I$

Subject to the constraints
{[1,3];[1,3]} $\tilde{x}_1^I$ +{[-1.5,3.5];[-1.5,3.5]} $\tilde{x}_2^I$ + {[1,1]; [1,1]} $\tilde{s}_1^I$+{[0,0]; [0,0]} $\tilde{s}_2^I$ + {[0,0]; [0,0]} $\tilde{s}_3^I$
={[4,6];[4,6]}
{[1,3];[1,3]} $\tilde{x}_1^I$ +{[4,6];[4,6]} $\tilde{x}_2^I$ {[0,0]; [0,0]} $\tilde{s}_1^I$ +{[1,1]; [1,1]} $\tilde{s}_2^I$ + {[0,0]; [0,0]} $\tilde{s}_3^I$ = {[9,11];[9,11]}
{[-5.5,-2.5];[-5.5,-2.5]} $\tilde{x}_1^I$+{[-11,-9];[-11,-9]} $\tilde{x}_2^I$ {[0,0]; [0,0]} $\tilde{s}_1^I$ +{[0,0]; [0,0]} $\tilde{s}_2^I$ + {[1,1]; [1,1]} $\tilde{s}_3^I$
= {[-38.74,-1.26];[-38.74,-1.26]} & $\tilde{x}_1^I, \tilde{x}_2^I \geq 0$ (11)

By the theorems 4.1 and 4.2, it is observed that all $\tilde{z}_j^I - \tilde{c}_j^I \geq 0$. Then the current solution is optimal and the optimal solution is as $\tilde{x}_1^I$={[0,0];[0,0]}, $\tilde{x}_2^I$= {[1.53,2.47];[1.53,2.47]} with

Maximize $\widetilde{z_2}^I$ = {[12.24,23.76];[12.24,23.76]}. The optimal solutions of $\widetilde{Z_2}^I$ for using different α, β cut values by interval arithmetic operations and modified interval arithmetic operations and also the respective MATLAB outputs(Figure 4 and Figure 5) are given below.

Table 2 Optimum $\widetilde{Z_1}^I$ -values

α-values	β-values	Optimum $\widetilde{Z_2}^I$ -values by interval arithmetic operations	Optimum $\widetilde{Z_2}^I$ -values by modified interval arithmetic operations
α=1	β=0	$\widetilde{Z_2}^I$ = {[12,22.5];[12,22.5]}	$\widetilde{Z_2}^I$ = {[12.24,23.76];[12.24,23.76]}
α =0.9	β=0.1	$\widetilde{Z_2}^I$ = {[9.4,36.15];[9.4,36.15]}	$\widetilde{Z_2}^I$ = {[9.32,26.68];[9.32,26.68]}
α =0.8	β=0.2	$\widetilde{Z_2}^I$ = {[7.34,48.83];[7.34,48.83]}	$\widetilde{Z_2}^I$ = {[7.41,28.59];[7.41,28.59]}
α =0.7	β=0.3	$\widetilde{Z_2}^I$ = {[5.7,68.62];[5.7,68.62]}	$\widetilde{Z_2}^I$ = {[5.83,30.17];[5.83,30.17]}
α =0.6	β=0.4	$\widetilde{Z_2}^I$ = {[4.42,103.91];[4.42,103.91]}	$\widetilde{Z_2}^I$ = {[4.42,31.58];[4.42,31.58]}
α =0.5	β=0.5	$\widetilde{Z_2}^I$ = {[3.35,182];[3.35,182]}	$\widetilde{Z_2}^I$ = {[3.3,32.70];[3.3,32.70]}
α =0.4	β=0.6	$\widetilde{Z_2}^I$ = {[2.46,496.40];[2.46,496.40]}	$\widetilde{Z_2}^I$ = {[2.38,33.62];[2.38,33.62]}
α =0.3	β=0.7	$\widetilde{Z_2}^I$ = {[-1079.20,21.16];[-1079.20,21.16]}	$\widetilde{Z_2}^I$ = {[2.01,33.99];[2.01,33.99]}
α =0.2	β=0.8	$\widetilde{Z_2}^I$ = {[-292.30,21.61];[-292.30,21.61]}	$\widetilde{Z_2}^I$ = {[1.60,34.40];[1.60,34.40]}
α =0.1	β=0.9	$\widetilde{Z_2}^I$ = {[-180.33,22.18];[-180.33,22.18]}	$\widetilde{Z_2}^I$ = {[1.22,34.78];[1.22,34.78]}
α =0	β=1	$\widetilde{Z_2}^I$ = {[-136,22.72];[-136,22.72]}	$\widetilde{Z_2}^I$ = {[0.84,35.16];[0.84,35.16]}

Figure 4 Optimum values $\widetilde{z_2}^I$

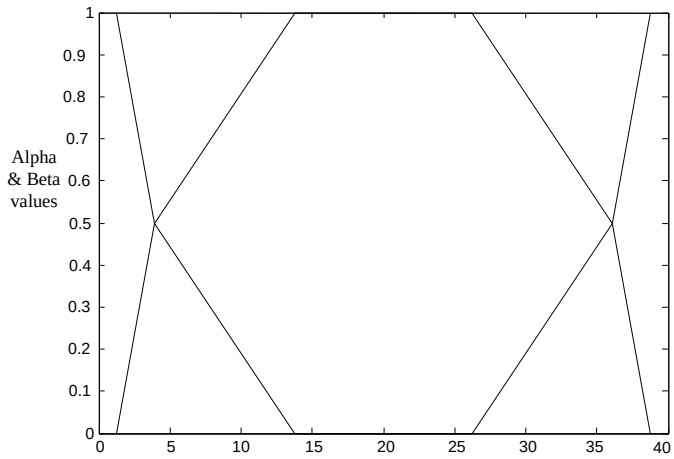
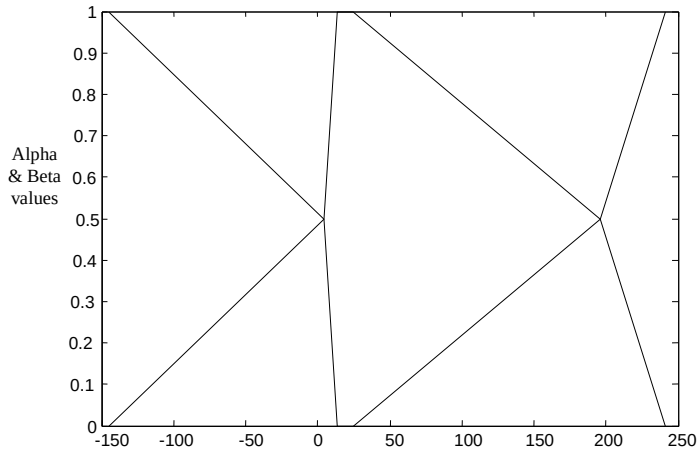


Figure 5 Optimum values $\widetilde{z_2}^I$



VI. Conclusion

This paper offered that how the interval arithmetic operations(*Timothy, 2010*) and the modified interval arithmetic operations(*Ganesan et al. 2005*) can be powerfully used for solving multi-objective linear programming problem with hexagonal intuitionistic fuzzy numbers in a comparative approach. Based on the MATLAB outputs, the modified interval arithmetic operations with proposed division operation are better than the interval arithmetic operations. Although here the linear case is considered, these operations into multi-objective non-linear programming with interval valued intuitionistic fuzzy coefficients and the other intuitionistic fuzzy numbers like triangular, trapezoidal, exponential, piece-wise quadratic, etc., with real life applications can be extended.

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A Mathematical model for Product Life Cycle

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Abstract: *A product life cycle is the life span of a product which the period begins with the initial product specification and ends with the withdrawal from the market of both the product and its support. A product life cycle can be divided into several stages characterized by revenue generated by the product. This study investigates a mathematical model for product life cycle which consists of four stages that is introduction, growth, maturity and decline stages. In this paper,* ***triangular inequalities view is considered for developing the mathematical models****. Two mathematical models are developed for two operational policies. The first policy covers the case that the mathematical model for product life cycle with maturity stage and in the second case a mathematical model for product life cycle with growth stage. The objective is to derive the cycle time and optimal production lot size that minimizes total costs of the cycle. The relevant model is built, solved. Illustrative examples are provided and numerically verified. Sensitivity analysis is performed to show how the optimal values of the policy variables in the model change as various model parameters are changed. The validation of result in this model was coded in Microsoft Visual Basic 6.0*

Key words: *Mathematical models, Product Life Cycle, Maturity stage, Growth stage, Demand and Production*.

I. Introduction

A product life cycle is the life span of a product which the period begins with the initial product specification and ends with the withdrawal from the market of both the product and its support. A new product is first developed and then introduced to the market. Once the introduction gets successful, a growth period follows with wider awareness of the product and increases sales. The product enters maturity when sales stop growing and demand stabilizes. Eventually, sales may decline until the product is finally with drawn from the market or re-developed. A product life cycle can be divided into several stages characterized by the revenue generated by the product. The product life cycle concept may apply to brand or to a category of the product. Its duration may be too short to few months for a faded item or a century or more for product categories. When the product is introduced, sales will be low until customers become aware of the product and its benefits. The concept of a Product Life Cycle (PLC) has occupied a prominent position in the marketing literature as both a forecasting instrument by **Kovac et al. (1972)** and a guideline for corporate marketing strategy by **Levitt (1965)** and it has been discussed widely in research (see the over view by **Kotler, 2003**). In the theory, atleast two conflicting definitions about the PLC can be derived. The first refers to the progress of a product from raw material, through which the production and use, to its final disposal. The second definition of the PLC describes the evolution of a product measured by its sales over time as seen in figure. **Kotler(1967 and 2003)** present the product life cycle concept as a marketing management tool for consumer branded products, i.e. (i) Introduction – the product is introduced in the market, and its awareness and acceptance are minimal. (ii) Growth- the product begins to make rapid sales and gains because of the cumulative effects of introductory promotion, distribution and word-of mouth influence. (iii) Maturity-growth of sales continuous. Sales reach and remain on a plateau marked by the level of replacement demand. (iv) Decline – Sales begin to diminish absolutely as the product is gradually edged out by better products or by its substitutes.

Figure – 1 Product Life Cycle

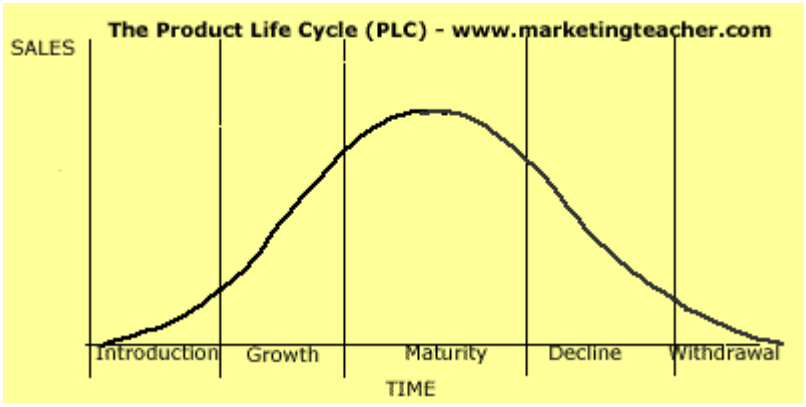
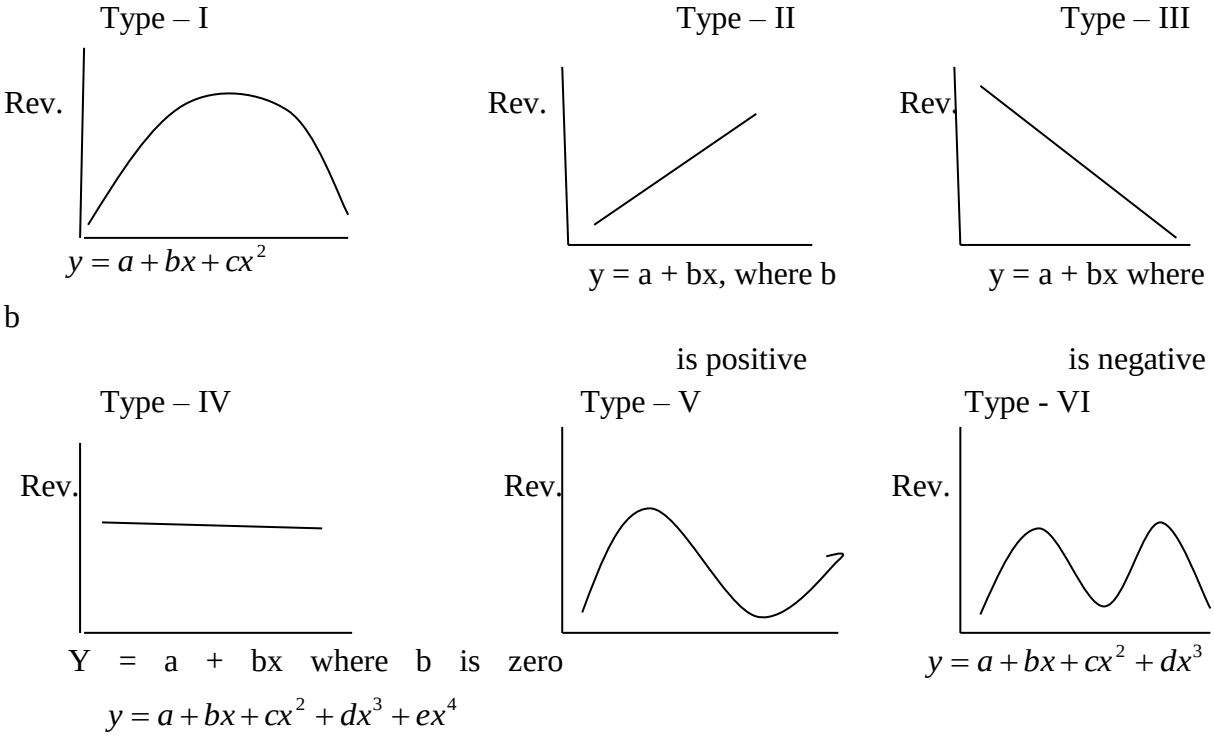


Figure – 2 Six Types of Product Life Cycle for Ethical Drugs



Source: W.E. Cox, "Product Life Cycles as Marketing Models", The Journal of Business 40, Oct., 1967, pp. 382

Cox (1967) presents every product passes through a series of phases in the course of its life, referred to as the product life cycle. The phases that a product goes through during it life cycle is the introduction, growth, maturity and decline stages. The most common product life cycle pattern is the S – shaped logistic curve with four major stages – introduction, growth, maturity and decline. For example, Cox in a study of 258 ethical-drug products introduced during 1955-59, found that six types of life cycle curves were needed to describe the sales patterns of these products. These six product life cycle curves are presented in figure -2. It is interesting to note down that only 28.3% of the products studied by Cox were described by the traditional S – shaped curve. The sixth life cycle pattern-polynomials of the fourth degree – is found to be the most appropriate for the doing products with 39.1% of the products following this pattern. **Robert D. Buzzell (1972)** – the introductory period is characterized by heavy promotion aimed to build up primary demands; price is relatively unimportant. During this growth phase, more competition appears and there is an increasing pressure on price. Promotional expenditures decline in relation to sales; there is a shift to competition on the basis of brands and specific features. As the product enters maturity, there is increasing product brand competition, promotional expenditures and prices tend to stabilize, manufacturers begin efforts to extend life cycles and new brands may appear. Finally, in the decline phase, further declines in price and promotional expenditures can be expected.

Tellis and Crawford (1981) presents the product life cycle as modeled on the fixed cycle of birth-growth-maturity-death through which higher living organisms pass. The PLC can be analyzed on different levels from the main product type (product class) down to different product models. Steven and Kleppper (1996) developed a model and use it to analyse this whole process in the industry and highlighted two types of innovation, the product innovation and process innovation. The characteristics of the life cycle and its effects on the reversed supply chain have been discussed by **Tibben-Lembke (2002)** presents although it lacks a discussion on its effects on remanufacturing operation. When the historical sales data is known, this data can be used as a basis for forecasting when these products are likely to be returned. **Laan and Salomon (1997)** presents the robustness of the control parameters of PUSH and PULL disposal strategy over the different stages of a Product Life Cycle. **Umeda el. al (2005)** present a model based on empirical data from return rates for remanufacturing of a single use camera and a photo copier. In this model, a simple normal distribution function has been shown to sufficient results in predicting returns when using average life as an indicator for timing of returns. The distribution of disposed products S(t) is calculated as the historical sales Data D(t) over a limited time frame $D(\tau)\Delta\tau$, distributed as a normal distribution function with a standard deviation (σ) after an average usage time μ. **Jha et al. (2005)** have concluded the concept of market segmentation in diffusion model for to advertise a new product considering the external as well as internal influences and study the optimal advertising effectiveness rate in a segment. **Seo et al. (2007**) have studied an approximate method of providing the preliminary life cycle cost. Learning algorithms trained to use the known characteristics of existing products can perhaps allow the life cycle cost of new products to be approximated quickly during the conceptual design phase without the overheads of defining new Product life cycle cost models. **Huang and Treng (2008)**, the authors proposed a forecast methodology for predicting both product life time and non-line product life cycle based upon a two-stage, fuzzy, piecewise regression analysis model. In different to traditional time-based forecast methodology, a generation-based approach was applied, which predicts product life cycle by deriving the annual fuzzy regression lines, based upon the annual shipments of earlier generation products. **Alexandru and Voda (2008)** considered a model regarding the product life cycle from a reliability theory view point and modified the transfer curve in a probability density function which allows the application of statistical inferential procedures. **Che-Fu Hsuch (2010)** investigates inventory control policies in a manufacturing system during the product life cycle, the closed-form formulas of optimal production in lot size, reorder point and safety stock in each phase of product life cycle are derived. **Ostlin et al. (2009)** have studied strategies to balance supply and demand for its remanufactured product life cycle; that does not present a clear inventory control policy. Li **Guoquing and Chen Zhongliang (2011) designed** a mechanism for the competition and simulate to coincide with the past regularities of product life cycle. This model has a generous structure with discrete periods that can be applied to specific industry. **C. Krishnamoorthi (2012)** developed an inventory model for product life cycle with defective items single manufacturing system which consists of introduction, growth, maturity and decline stages and the defective rate is considered as a variable of known proportions. **C. Krishnamoorthi (2012)** developed an inventory model for product life cycle with defective items single manufacturing system which consists of introduction, growth, maturity and decline stages and the defective rate is considered as a variable of known proportions and also considered as shortages in this paper.

The objective of this paper is to derive the cycle time and optimal production lot size that minimizes total costs for the product life cycle with growth and maturity stages. The remainder of the paper is organized as follows. Section 2 presents the assumptions and notations. Section 3 is for problem formulation and numerical examples. Finally, the paper summarizes and concludes in section 4.

II. ASSUMPTIONS AND NOTATIONS

A. Assumptions: 1) The demand rate is known, constant and continuous, 2**)** Items are produced and added to theinventory, 3**)** The item is a single product; it does not interact with any other inventory items, 4**)** The production rate is always greater than or equal to the sum of the demand rate, 5) The introduction time (T_1), the time for growth stage (T_2) and the time for maturity stage (T_3) are calculated based on triangular inequality, 6**)** The other assumptions are in classical Production Inventory model.

B. Notations: **1)** P – Production rate in units per unit time, **2)** D – Demand rate in units per unit time, 3) T – optimal cycle time, 4) T_1 - time during introduction of the product, 5) T_2 - time during growth stage, 6) T_3 - time during maturity stage period, 7) optimal quantity, 8**)** Q_1 – on hand inventory during introduction time T_1 , 9**)** Q_2 -on hand inventory level at time T_2, 10) Q_3 - on hand inventory level at time T_3, 11**)** C_p – Production Cost per unit, 12**)** C_h -Holding cost per unit/ per unit time, 13**)** C_0 – Setup cost per setup , 14**)** TC - Total cost.

III. MATHEMATICAL MODELS

A. A Mathematical model for Product Life Cycle with Maturity Stage

In **introduction stage**, products have to be carefully monitored to ensure that they start to grow. Substantial research and development costs may be high in order to test the market, undergo launch promotion and setup distribution channels. The cycle start at t =0. In this stage, inventory is increasing at the rate of P and simultaneously decreasing at the rate of D. Thus inventory accumulates at the rate of P - D units. Therefore, the maximum inventory level shall be equal to $(P-D)t_1$. In **growth stage**, more customers become aware of the product and its benefits and additional market segments are targeted. The growth stage is characterized by rapid growth in sales and profits. Profits arise due to an increase in output and possibly better prices. When the product enters growth stage at T_1, Production and Demand increases at the rate of "m" time of P-D i.e. m(P-D) where "m" is a constant. In maturity stage, sales growth continuous and a company has achieved its market share goals enjoys that most profitable period. Production and 2Demand increases at the rate of "n" time of P-D i.e. n(P-D) where "n" is a constant. In **decline stage**, the market is shrinking, reducing overall amount of profit that can be shared amongst the remaining competitors. The product becomes technically obsolete or customer taste changes. Care should be taken to control the amount of stocks of the product. The inventory level starts to decrease due to demand at a rate D up to time T_3. Time T needed to consume all units Q at demand rate. The process is repeated. The variation of the underlying inventory system for one cycle is shown in the figure 2.

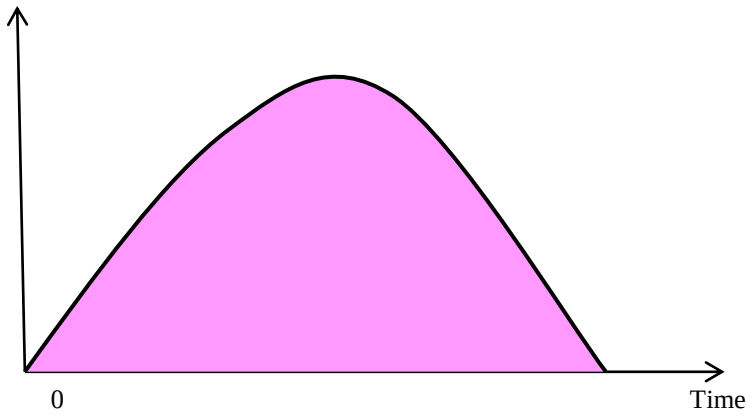


Figure- 1 Product life cycle with Maturity Stage)

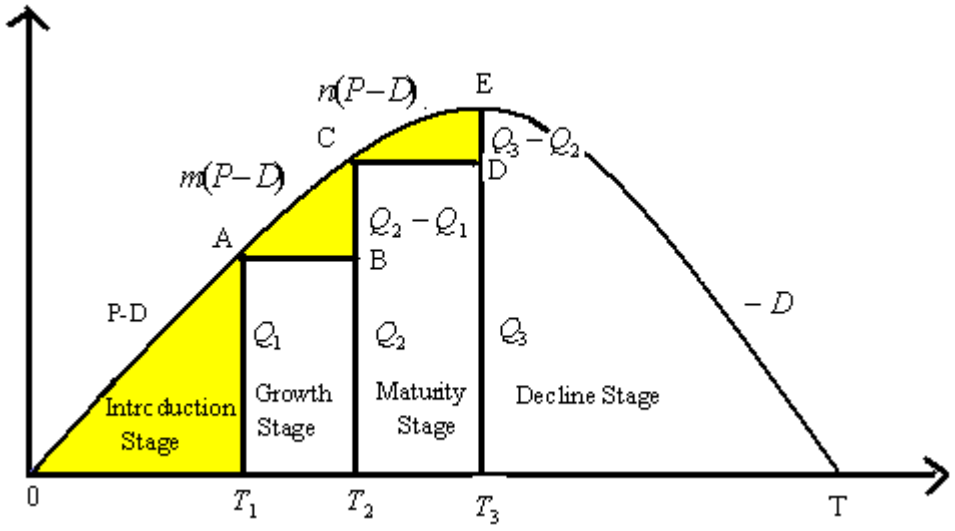


Figure- 2 On hand inventory in product life cycle with Maturity Stage

The production rate of good items is always greater than or equal to the demand rate. So, we must have P ≥ D. Let I(t) denote the inventory level of the system at time T. The differential equation describing the system in the interval (0,T) are as follows

$$\frac{dI(t)}{dt} = P - D;\ 0 \le t \le T_1 \quad (1)$$

$$\frac{dI(t)}{dt} = m(P - D);\ T_1 \le t \le T_2 \quad (2)$$

$$\frac{dI(t)}{dt} = n(P - D);\ T_2 \le t \le T_3 \quad (3)$$

$$\frac{dI(t)}{dt} = -D;\ T_3 \le t \le T \quad (4)$$

The boundary conditions are

$$I(0) = 0;\ I(T_1) = Q_1;\ I(T_2) = Q_2,\ I(T_3) = Q_3,\ I(T) = 0 \quad (5)$$

The solutions of the above equations are

From the equation (1), $I(t) = (P - D)t;\ 0 \le t \le T_1$ (6)

From the equation (2), $I(t) = m(P - D)t$ (7)

From the equation (3), $I(t) = n(P - D)t$ (8)

From the equation (4), $I(t) = D(T - t)$ (9)

From the triangular inequalities OA T_1 , ABC and CDE

$$T_1 = \frac{T_3}{m+n+1} \text{ and } T_2 = \frac{(m+1)T_3}{m+n+1} \quad (10)$$

Maximum inventory Q_1 : The maximum inventory during introduction time (T_1) is calculated as follows. Therefore, $Q_1 = (P-D)T_1$ (11)

Maximum inventory Q_2 : The maximum inventory during growth stage time (T_2) is calculated as follows. Therefore, $Q_2 = m(P-D)T_2$ (12)

Maximum inventory Q_3 : The maximum inventory during maturity stage time T_3 is calculated as follows. Therefore, $Q_3 = n(P-D)T_3$ (13)

Total Cost: The total cost comprises of the sum of the Production cost, ordering cost and holding cost. They are grouped together after evaluating the above cost individually.

(i)Production Cost per unit time = DC_P (14)

(ii)Ordering Cost per unit time = $\frac{C_0}{T}$ (15)

(iii) Holding Cost per unit time: =

$$= \frac{C_h}{T}\left[\int_0^{T_1} I(t)dt + \int_{T_1}^{T_2} I(t)dt + \int_{T_2}^{T_3} I(t)dt + \int_{T_3}^{T} I(t)dt\right]$$

$$= \frac{C_h}{T}\left(\int_0^{T_1} (P-D)tdt + \int_{T_1}^{T_2} m(P-D)tdt + \int_{T_2}^{T_3} n(P-D)dt + \int_{T_3}^{T} D(T-t)dt\right)$$

$$= \frac{C_h}{T}\left[(P-D)\left(\frac{T_1^2}{2}\right) + m(P-D)\frac{\left(T_2^2 - T_1^2\right)}{2} + n(P-D)\frac{\left(T_3^2 - T_2^2\right)}{2} + D\frac{(T-T_3)^2}{2}\right]$$

$$= \frac{C_h}{2T}\left[(P-D)T_1^2 + m(P-D)(T_2^2 - T_1^2) + n(P-D)(T_3^2 - T_2^2) + D(T-T_3)^2\right]$$

Using equation (10), substitute the value of T_1 and T_2 and simplify

$$= \frac{C_h}{2T}\left[\begin{array}{l} \dfrac{(P-D)T_3^2}{(m+n+1)^2} + m(P-D)\left(\dfrac{(1+m)^2T_3^2}{(m+n+1)^2} - \dfrac{T_3^2}{(m+n+1)^2}\right) \\ + n(P-D)\left(T_3^2 - \dfrac{(1+m)^2T_3^2}{(m+n+1)^2}\right) + D(T-T_3)^2 \end{array}\right]$$

$$= \frac{C_h}{2(m+n+1)^2T}\left[\begin{array}{l} (P-D)\left(m^3+n^3+2m^2+2n^2+2mn^2+1\right)T_3^2 \\ + D(m+n+1)^2(T-T_3)^2 \end{array}\right] \quad (16)$$

TC = Purchase Cost + Ordering Cost + Holding Cost

$$= DC_P + \frac{C_0}{T} + \frac{C_h}{2(m+n+1)^2T}\left[\begin{array}{l} (P-D)\left(m^3+n^3+2m^2+2n^2+2mn^2+1\right)T_3^2 \\ + D(m+n+1)^2(T-T_3)^2 \end{array}\right] \quad (17)$$

For optimality conditions

$$\frac{\partial}{\partial T_3}(TC) = 0 \quad and \quad \frac{\partial^2}{\partial T_3^2}(TC) < 0$$

$$\frac{\partial}{\partial T}(TC) = 0 \quad and \quad \frac{\partial^2}{\partial T^2}(TC) < 0$$

Partially differentiate the total cost (17) with respect to T_3,

$$\frac{\partial}{\partial T_3}(TC) = \left[(P-D)\left(m^3+n^3+2m^2+2n^2+2mn^2+1\right)T_3 - D(m+n+1)^2(T-T_3)\right] = 0$$

Therefore, $T_3 = \dfrac{D(m+n+1)^2 T}{(P-D)\left(m^3+2m^2+n^3+2mn^2+2n^2+1\right)+D\left(m+n+1\right)^2}$

Partially differentiate the total cost (17) with respect to T

$$\begin{bmatrix} -(P-D)\left(m^3+n^3+2m^2+2n^2+2mn^2+1\right)T_3^2 \\ +D(m+n+1)^2T^2 - D(m+n+1)^2T_3^2 \end{bmatrix} = \frac{2C_0(m+n+1)^2}{C_h}$$

$$\left[(P-D)\left(m^3+n^3+2m^2+2n^2+2mn^2+1\right)+D(m+n+1)^2\right]T_3^2 + D(m+n+1)^2T^2 = 0$$

Substituting the value of T_3 and simplify, therefore

$$\frac{-D^2(m+n+1)^2T^2}{(P-D)\left(m^3+n^32m^2+2n^2+2mn^2+1\right)+D(m+n+1)^2} + DT^2 = \frac{2C_0}{C_h}$$

$$T = \sqrt{\frac{2C_0\left[(P-D)\left(m^3+2m^2+n^3+2mn^2+2n^2+1\right)+D\left(m+n+1\right)^2\right]}{DC_h(P-D)\left(m^3+2m^2+n^3+2mn^2+2n^2+1\right)}} \qquad (18)$$

When $T = \dfrac{Q}{D}$ then $Q = \sqrt{\dfrac{2DC_0\left[(P-D)\left(m^3+2m^2+n^3+2mn^2+2n^2+1\right)+D\left(m+n+1\right)^2\right]}{C_h(P-D)\left(m^3+2m^2+n^3+2mn^2+2n^2+1\right)}}$

Note : when m = n =0, then $T = \sqrt{\dfrac{2PC_0}{DC_h(P-D)}}$

Numerical Example, Let us consider the cost parameters

P = 5000 units, D = 4500 units, C_h =10, C_p = 100, C_0 =100, m = 2, n= 3

Optimum solution:

Cycle Times: T = 0.1383; T_1 = 0.0177; T_2 = 0.0531; T_3 = 0.1062; Optimum Quantity Q* = 622.54, Q_1 = 8.85; Q_2 = 53.11; Q_3 = 159.32; Production cost =450,000,

Setup cost = 722.85, Holding cost = 722.85, Total cost = 451445.70

Sensitivity Analysis:

The total cost functions are the real solution in which the model parameters are assumed to be a static values. It is reasonable to study the sensitivity i.e. the effect of making chances in the model parameters over a given optimum solution. It is important to find the effects on different system performance measures, such as cost function, inventory system, etc. For this purpose, sensitivity analysis of various system parameters for the models of this research are required to observe whether the current solutions remain unchanged, the current solutions become infeasible, etc.

Table – 1
Effect of Demand and Cost parameters on optimal policies

PARAMETERS		OPTIMUM VALUES										
		T	T_1	T_2	T_3	Q	Q_1	Q_2	Q_3	SETUP COST	HOLDING COST	TOTAL COST
C_0	80	0.1237	0.0158	0.0475	0.0950	556.81	7.91	47.50	142.50	646.54	646.54	451293.07
	90	0.1312	0.0168	0.0503	0.1007	590.59	8.40	50.38	151.15	685.76	685.76	451371.51
	100	0.1383	0.0177	0.0531	0.1062	622.54	8.85	53.11	159.32	722.85	722.85	451445.70
	110	0.1451	0.0186	0.0557	0.1114	652.92	9.28	55.70	167.09	758.13	758.13	451516.26
	120	0.1515	0.0194	0.0582	0.1163	681.95	9.70	58.17	174.53	791.84	791.84	451583.68
C_h	8	0.1547	0.0198	0.0594	0.1187	696.02	9.89	59.37	178.12	646.53	646.53	451293.07
	9	0.1458	0.0186	0.0559	0.1119	656.21	9.33	55.98	167.94	685.75	685.75	451371.51
	10	0.1383	0.0177	0.0531	0.1062	622.54	8.85	53.11	159.32	722.85	722.85	451445.70
	11	0.1319	0.0168	0.0506	0.1013	593.56	8.44	50.63	151.91	758.13	758.13	451516.26
	12	0.1263	0.0161	0.0484	0.0970	568.29	8.08	48.48	145.44	791.84	791.84	451583.68
C_P	80	0.1237	0.0158	0.0475	0.0950	556.81	7.91	47.50	142.50	646.54	646.54	361249.49
	90	0.1312	0.0168	0.0503	0.1007	590.59	8.40	50.38	151.15	685.76	685.76	406249.90
	100	0.1383	0.0177	0.0531	0.1062	622.54	8.85	53.11	159.32	722.85	722.85	451445.70
	110	0.1451	0.0186	0.0557	0.1114	652.92	9.28	55.70	167.09	758.13	758.13	496249.49
	120	0.1515	0.0194	0.0582	0.1163	681.95	9.70	58.17	174.53	791.84	791.84	541249.49
m	1	0.1392	0.0214	0.0429	0.1072	626.29	10.72	21.45	718.52	718.52	718.52	451437.03
	2	0.1383	0.0177	0.0531	0.1062	622.54	8.85	53.11	159.32	722.85	722.85	451445.70
	3	0.1340	0.0144	0.0576	0.1008	603.09	7.20	86.45	151.29	746.15	746.15	451492.30
	4	0.1281	0.0167	0.0584	0.0934	576.40	5.84	116.74	140.09	780.70	780.70	451561.40
	5	0.1219	0.0094	0.0570	0.0854	548.60	4.74	142.45	128.18	820.27	820.27	451640.53
n	1	0.1764	0.0378	0.1134	0.1518	793.77	18.89	113.38	75.59	566.94	566.94	451133.89
	2	0.1576	0.0258	0.0776	0.1294	709.41	12.94	77.67	129.45	634.33	634.33	451268.65
	3	0.1383	0.0177	0.0531	0.1062	622.54	8.85	53.11	159.32	722.85	722.85	451445.70
	4	0.1245	0.0127	0.0381	0.0889	560.56	6.35	38.08	177.78	802.76	802.76	451605.51
	5	0.1149	0.0095	0.0286	0.0762	517.24	4.76	28.60	196.96	190.68	190.68	451740.01

Note: Production cost = 450000, Production cost in C_P **are 360000, 405000, 450000, 495000, 540000**

Managerial insights: A sensitivity analysis is performed to study the effects of change in the system parameters, ordering cost per order (C_0), holding cost per unit per year (C_h), production cost per unit (C_P), m and n constants rate during growth and maturity period on optimal values that is optimal cycle time (T), optimal quantity (Q), introduction time (T_1), time during growth stage (T_2), time during maturity stage (T_3), maximum inventory during introduction stage (Q_1), maximum inventory during growth stage (Q_2), maximum inventory during maturity stage (Q_3), setup cost, holding cost, and total cost. The sensitivity analysis is performed by changing (increasing or decreasing) the parameter taking at a time, keeping the remaining parameters at their original values. The following influences can be obtained from sensitivity analysis based on table 1. 1) with the increase in setup cost per unit (C_0), optimal values that is optimal cycle time (T), optimal quantity (Q), introduction time (T_1), time during growth stage (T_2), time during maturity stage (T_3), maximum inventory during introduction stage (Q_1), maximum inventory during growth stage (Q_2), maximum inventory during maturity stage (Q_3), setup cost, holding cost, and total cost increases then there is positive relation between them. 2) with the increase in holding cost per unit per unit time (C_h), the setup cost, holding cost and total cost increases then there is positive relationship between them and optimal cycle time (T), optimal quantity (Q), introduction time (T_1), time during growth stage (T_2), time during maturity stage (T_3), maximum inventory during introduction stage (Q_1), maximum inventory during growth stage (Q_2), maximum inventory during maturity stage (Q_3) are decreases then there is negative relationship

between them. 3) Similarly, other parameters production cost per unit (C_P), the constant rate " m " and " n " can also be observed from the table 1.

B. A Mathematical model for Product Life Cycle with Growth Stage

In **introduction stage**, products have to be carefully monitored to ensure that they start to grow. Substantial research and development costs may be high in order to test the market, undergo launch promotion and setup distribution channels. The cycle start at t =0. In this stage, inventory is increasing at the rate of P and simultaneously decreasing at the rate of D. Thus inventory accumulates at the rate of P - D units. Therefore, the maximum inventory level shall be equal to $(P-D)t_1$. In **growth stage**, more customers become aware of the product and its benefits and additional market segments are targeted. The growth stage is characterized by rapid growth in sales and profits. Profits arise due to an increase in output and possibly better prices. When the product enters growth stage at T_1, Production and Demand increases at the rate of "a" time of P-D i.e. a(P-D) where "a" is a constant. In maturity stage, sales growth continuous and a company has achieved its market share goals enjoys that most profitable period. Production and 2Demand increases at the rate of "b" time of P-D i.e. b(P-D) where "b" is a constant. In **decline stage**, the market is shrinking, reducing overall amount of profit that can be shared amongst the remaining competitors. The product becomes technically obsolete or customer taste changes. Care should be taken to control the amount of stocks of the product. The inventory level starts to decrease due to demand at a rate D up to time T_3. Time T needed to consume all units Q at demand rate. The process is repeated. The variation of the underlying inventory system for one cycle is shown in the figure 2.

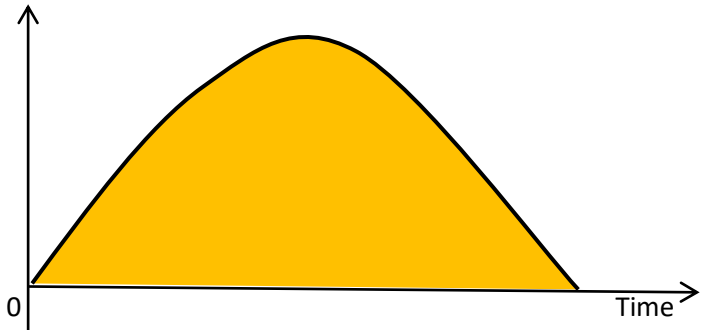


Figure- 3 Product Life Cycle with only Growth Stage

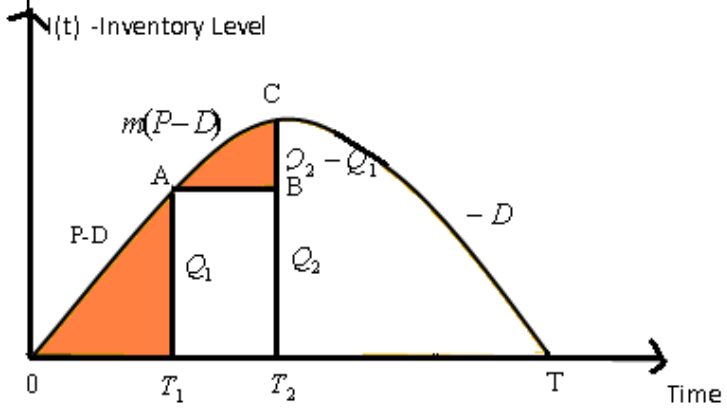


Figure 4 On hand inventory in product life cycle with only Growth Stage

$$\frac{dI(t)}{dt} = P - D;\ 0 \le t \le T_1 \tag{19}$$

$$\frac{dI(t)}{dt} = m(P-D);\ T_1 \le t \le T_2 \tag{20}$$

$$\frac{dI(t)}{dt} = -D;\ T_2 \le t \le T \tag{21}$$

The boundary conditions are

I(0) =0; I(T_1) $= Q_1$; I(T_2) $= Q_2$, $I(T) = 0$ (22)

The solutions of the above equations are

From the equation (19), I(t) = $(P-D)\,t$ (23)

From the equation (20), $I(t) = m(P-D)t$ (24)

From the equation (21), $I(t) = D(T-t)$ (25)

From the triangular inequalities OAT_1, ABC, $T_1 = \frac{T_2}{1+m}$ (26)

Maximum inventory Q_1 : The maximum inventory during introduction time T_1 is calculated as follows. From equations (22) and (23), $I(T_1) = Q_1 \Rightarrow Q_1 = (P-D)T_1$ (27)

Maximum inventory Q_2 : The maximum inventory during growth stage time T_2 is calculated as follows. From the equations (22) and (24), $I(T_2) = Q_2 \Rightarrow Q_2 = m(P-D)T_2$ (28)

Total Cost: The total cost comprises of the sum of the Production cost, ordering cost, holding cost and deteriorating cost. They are grouped together after evaluating the above cost individually.

(i)Production Cost per unit time = DC_P (29)

(ii)Setup cost per unit time = $\frac{C_0}{T}$ (30)

(iii)Holding Cost per unit time

$$= \frac{C_h}{T}\left[\int_0^{T_1} I(t)dt + \int_{T_1}^{T_2} I(t)dt + \int_{T_2}^{T} I(t)dt\right]$$

$$= \frac{C_h}{T}\left(\int_0^{T_1} (P-D)t\,dt + \int_{T_1}^{T_2} m(P-D)\,t\,dt + \int_{T_2}^{T} D(T-t)dt\right)$$

$$= \frac{C_h}{2T}\left[(P-D)T_1^2 + m(P-D)\left(T_2^2 - T_1^2\right) + D(T-T_2)^2\right]$$

Substitute the value of T_1 and simplify

$$= \frac{C_h}{2T}\left[(P-D)\frac{T_2^2}{(1+m)^2} + m(P-D)\left(T_2^2 - \frac{T_2^2}{(1+m)^2}\right) + D(T-T_2)^2\right]$$

$$= \frac{C_h}{2(1+m)^2 T}\left[(P-D)T_2^2 + m(P-D)(m^2+2m)T_2^2 + D(1+m)^2(T-T_2)^2\right] \tag{31}$$

$$= \frac{C_h}{2(1+m)^2 T}\left[(P-D)T_2^2 + (P-D)(m^3+2m^2)T_2^2 + D(1+m)^2(T-T_2)^2\right]$$

$$= \frac{C_h}{2(1+m)^2 T}\left[(P-D)\left(m^3+2m^2+1\right)T_2^2 + D(1+m)^2(T-T_2)^2\right]$$

Total cost = setup cost + production cost + holding cost

$$\text{TC} = \frac{C_0}{T} + DC_P + \frac{C_h}{2(1+m)^2 T}\left[(P-D)\left(m^3+2m^2+1\right)T_2^2 + D(1+m)^2(T-T_2)^2\right] \tag{32}$$

For optimality conditions

$$\frac{\partial}{\partial T_2}(TC) = 0 \quad and \quad \frac{\partial^2}{\partial T_2^2}(TC) < 0$$

$$\frac{\partial}{\partial T}(TC) = 0 \quad and \quad \frac{\partial^2}{\partial T^2}(TC) < 0$$

Partially differentiate the total cost (32) with respect to T_2,

$$\frac{\partial}{\partial T_2}(TC) = (P-D)T_2(m^3+2m^2+1)T_2 - D(1+m)^2(T-T_2) = 0$$

$$T_2 = \frac{D(1+m)^2 T}{(P-D)(m^3+2m^2+1)+D(1+m)^2} \tag{33}$$

Partially differentiate the total cost (13) with respect to T

$$-\left[(P-D)(m^3+2m^2+1)+D(1+m)^2\right]T_2^2 + D(1+m)^2T^2 = \frac{2C_0(1+m)^2}{C_h}$$

Substitute the value of T_2 and simplify

$$\frac{-D^2(1+m)^2T^2}{(P-D)\left(m^3+2m^2+1\right)+D(1+m)^2} + DT^2 = \frac{2C_0}{C_h}$$

$$D(P-D)\left(m^3+2m^2+1\right)T^2 = \frac{2C_0\left[(P-D)\left(m^3+2m^2+1\right)+D(1+m)^2\right]}{C_h}$$

Therefore, $T = \sqrt{\dfrac{2C_0\left[(P-D)(m^3+2m^2+1)+D(1+m)^2\right]}{DC_h(P-D)(m^3+2m^2+1)}}$ (34)

When $T = \dfrac{Q}{D}$ then $Q = \sqrt{\dfrac{2DC_0\left[(P-D)(m^3+2m^2+1)+D(1+m)^2\right]}{C_h(P-D)(m^3+2m^2+1)}}$

Note: When m =0, then $T = \sqrt{\dfrac{2PC_0}{D(P-D)C_h}}$ which is standard inventory model

Numerical Example, Let us consider the cost parameters

P = 5000 units, D = 4500 units, C_h =10, C_p = 100, C_0 =100, m = 2

Optimum solution:

Cycle Times: T = 0.1600; T_1 = 0.0441; T_2 = 0.1323; Optimum Quantity Q* = 720.29,

Q_1 = 22.05; Q_2 = 132.29; Production cost = 450,000, Setup cost = 624.74, Holding cost = 624.74,

Total cost = 451249.49

Sensitivity Analysis:

The total cost functions are the real solution in which the model parameters are assumed to be a static values. It is reasonable to study the sensitivity i.e. the effect of making chances in the model parameters over a given optimum solution. It is important to find the effects on different system performance measures, such as cost function, inventory system, etc. For this purpose, sensitivity analysis of various system parameters for the models of this research are required to observe whether the current solutions remain unchanged, the current solutions become infeasible, etc.

Table – 2

Effect of Demand and Cost parameters on optimal policies

Parameters		Optimum values								
		T	Q	T_1	T_2	Q_1	Q_2	Setup cost	Holding cost	Total cost
C_0	80	0.1432	644.25	0.0394	0.1183	19.72	118.33	558.79	558.79	451117.58
	90	0.1518	683.33	0.0418	0.1255	20.91	125.51	592.68	592.68	451185.37
	100	0.1600	720.29	0.0441	0.1323	22.05	132.29	624.74	624.74	451249.49
	110	0.1679	755.45	0.0462	0.1387	23.12	138.75	655.24	655.24	451310.47
	120	0.1753	759.04	0.0483	0.1449	24.15	144.92	684.37	684.37	451368.75
C_h	8	0.1789	805.31	0.0493	0.1479	24.65	147.95	558.79	558.79	451117.58
	9	0.1687	759.25	0.0464	0.1394	23.24	139.45	592.68	592.68	451185.37
	10	0.1600	720.29	0.0441	0.1323	22.05	132.29	624.74	624.74	451249.49
	11	0.1526	686.77	0.0420	0.1261	21.02	126.14	655.24	655.24	451310.47
	12	0.1461	657.53	0.0402	0.1207	20.13	120.77	684.37	684.37	451368.75
C_P	80	0.1432	644.25	0.0394	0.1183	19.72	118.33	558.79	558.79	361249.49
	90	0.1518	683.33	0.0418	0.1255	20.91	125.51	592.68	592.68	406249.49
	100	0.1600	720.29	0.0441	0.1323	22.05	132.29	624.74	624.74	451249.49
	110	0.1679	755.45	0.0462	0.1387	23.12	138.75	655.24	655.24	496249.49
	120	0.1753	759.04	0.0483	0.1449	24.15	144.92	684.37	684.37	541249.49
m	1	0.2108	948.68	0.0949	0.1897	47.43	94.86	474.34	474.34	450948.68
	2	0.1600	720.29	0.0441	0.1323	22.05	132.29	624.74	624.74	451249.49
	3	0.1355	609.70	0.0257	0.1027	12.83	154.03	738.06	738.06	451476.12
	4	0.1215	546.59	0.0170	0.0848	8.49	169.75	823.28	823.28	451646.57
	5	0.1124	505.65	0.0121	0.0728	6.07	182.03	889.94	889.94	451779.89

Note: Production cost in all cases 450000 but in parameter C_P, production costs are 360000, 405000, 495000, 450000, and 540000 for the production cost per unit 80, 90, 100, 110 and 120 respectively.

Managerial insights: A sensitivity analysis is performed to study the effects of change in the system parameters, ordering cost per order (C_0), holding cost per unit per year (C_h), production cost per unit (C_P), m constant rate during growth period on optimal values that is optimal cycle time (T), optimal quantity (Q), introduction time (T_1), time during growth stage (T_2), maximum inventory during introduction stage (Q_1), maximum inventory during growth stage (Q_2), setup cost, holding cost, and total cost. The sensitivity analysis is performed by changing (increasing or decreasing) the parameter taking at a time, keeping the remaining parameters at their original values. The following influences can be obtained from sensitivity analysis based on table 2.

1) with the increase in setup cost per unit (C_0), optimal values that is optimal cycle time (T), optimal quantity (Q), introduction time (T_1), time during growth stage (T_2), maximum inventory during introduction stage (Q_1), maximum inventory during growth stage (Q_2), setup cost, holding cost, and total cost increases then there is positive relation between them.

2) with the increase in holding cost per unit per unit time (C_h), the setup cost, holding cost and total cost increases then there is positive relationship between them and optimal cycle time (T), optimal quantity (Q), introduction time (T_1), time during growth stage (T_2), maximum inventory during introduction stage (Q_1), maximum inventory during growth stage (Q_2) are decreases then there is negative relationship between them.

3) Similarly, other parameters production cost per unit (C_P), the constant rate " m " can also be observed from the table 2.

IV Conclusions

A product life cycle is the life span of a product which the period begins with the initial product specification and ends with the withdrawal from the market of both the product and its support. A product life cycle can be divided into several stages characterized by revenue generated by the product. This study investigates a mathematical model for product life cycle which consists of four stages that is introduction, growth, maturity and decline stages. The objective is to cycle time and optimal production lot size that minimizes total costs of the product life cycle with maturity and growth stages. The relevant model is built, solved and some main results about the uniqueness of this solution with the use of rigorous mathematical methods are obtained. Illustrative examples are provided and numerical verified. The validation of result in this model was coded in Microsoft Visual Basic 6.0 The following conclusions are obtained in this paper. 1) with the increase in setup cost per unit (C_0) , optimal values that is optimal cycle time (T), optimal quantity (Q), introduction time (T_1), time during growth stage (T_2), time during maturity stage (T_3), maximum inventory during introduction stage (Q_1), maximum inventory during growth stage (Q_2), maximum inventory during maturity stage (Q_3), setup cost, holding cost, and total cost increases then there is positive relation between them. 2) with the increase in holding cost per unit per unit time (C_h), the setup cost, holding cost and total cost increases then there is positive relationship between them and optimal cycle time (T), optimal quantity (Q), introduction time (T_1), time during growth stage (T_2), time during maturity stage (T_3), maximum inventory during introduction stage (Q_1), maximum inventory during growth stage (Q_2), maximum inventory during maturity stage (Q_3) are decreases then there is negative relationship between them

This research can be extended as follows:

(1) Most of the production systems today are multi-stage system and in a multi-stage system the defective items and scrap can be produced in each stage. Again, the defectives and scrap proportion for a multi-stage system can be different in different stages. Taking these factors into consideration this research can be extended for a multi-stage production process.

(2)Traditionally, inspection procedures incurring cost is an important factor to identify the defectives and scrap and remove them for the finished goods inventory. For better production, the placement and effectiveness of inspection procedures are required which is ignored for this research, so inspection cost can be included in developing the future models.

(3) The demand of a product may decrease with time owing to the introduction of a new product which is either technically superior or more attractive and cheaper than the old one. On the other hand the demand of new product will increase. Thus, demand rate can be varied with time, so variable demand rate can be used to develop the model. The proposed model can assist the manufacturer and retailer in accurately determining the optimal quantity, cycle time and inventory total cost. Moreover, the proposed inventory model can be used in inventory control of certain items such as food items, fashionable commodities, stationary stores and others.

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Dynamic analysis of a discrete fractional order 3D chaotic system

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Abstract: *In this paper, we investigate the dynamic analysis of a fractional order 3D chaotic system with discrete version. Stability of the fixed points is analyzed. Time series and phase plane diagrams are presented for different sets of parameter values. Bifurcation diagrams and chaotic attractors are presented for different sets values of the discrete fractional order 3D chaotic system.*

Keywords: *fractional order, discretization process, chaotic system, bifurcation, chaotic attractor*

I. Introduction

Fractional derivatives have a long mathematical history. Fractional calculus is a generalization of integration and differentiation to non-integer order. It is useful in studies in biology, physics, electrochemistry, viscoelasticity, dynamic systems and etc. In recent years fractional order differential equations and fractional order difference equations have attracted many researchers because of their applications in many areas of science and engineering [3].

II. Fractional order 3D chaotic system with discretization

In 1963, Lorenz system was first formulated and studied by Edward Lorenz. For certain parameter values and initial conditions in the Lorenz system having chaotic solutions are notable [1, 2]. In 2011 Ihsan Pehlivan and Zhouchao Wei [5, 6] introduced a 3D chaotic system with 8 terms including 4 quadratic nonlinearity terms. The system is defined by nonlinear ordinary differential equations

$$x' = y(1-z),\ y' = y(1+z) - ax\ ,\ z' = a - xy - y^2.$$

These equations contain three state variables $x\ , y\ , z$ and one real parameter $'a'$.

We consider the fractional order 3D chaotic system as follows:

$$\begin{aligned} D^{\alpha}x(t) &= y(t)(1-z(t)) \\ D^{\alpha}y(t) &= y(t)\big(1+z(t)\big) - ax(t) \\ D^{\alpha}z(t) &= a - x(t)y(t) - y^2(t) \end{aligned} \qquad (1)$$

where α is the fractional order. Applying a discretization process for a fractional order 3D chaotic system described in [4, 7, 9], we obtain the discrete fractional order 3D chaotic system as follow:

$$\begin{aligned} D^{\alpha}x(t) &= y\left(r\left[\frac{t}{r}\right]\right)\left(1 - z\left(r\left[\frac{t}{r}\right]\right)\right) \\ D^{\alpha}y(t) &= y\left(r\left[\frac{t}{r}\right]\right)\left(1 + z\left(r\left[\frac{t}{r}\right]\right)\right) - a\,x\left(r\left[\frac{t}{r}\right]\right) \end{aligned} \qquad (2)$$

$$D^{\alpha}z(t) = a - x\left(r\left[\frac{t}{r}\right]\right)y\left(r\left[\frac{t}{r}\right]\right) - \left(y\left(r\left[\frac{t}{r}\right]\right)\right)^2$$

with initial condition $x(0) = x_0\,, y(0) = y_0\,, z(0) = z_0$.

Let $t \in [0, r)$, then $\frac{t}{r} \in [0,1)$. We get

$$D^{\alpha}x(t) = y_0(1 - z_0)$$
$$D^{\alpha}y(t) = y_0(1 + z_0) - ax_0$$
$$D^{\alpha}z(t) = a - x_0y_0 - y_0^2$$

and the solution of (2) is given by

$$x_1(t) = x_0 + I^{\alpha}(y_0(1 - z_0)) = x_0 + \frac{t^{\alpha}}{\Gamma(1+\alpha)}(y_0(1 - z_0))$$
$$y_1(t) = y_0 + I^{\alpha}(y_0(1 + z_0) - ax_0) = y_0 + \frac{t^{\alpha}}{\Gamma(1+\alpha)}(y_0(1 + z_0) - ax_0)$$
$$z_1(t) = z_0 + I^{\alpha}(a - x_0y_0 - y_0^2) = z_0 + \frac{t^{\alpha}}{\Gamma(1+\alpha)}(a - x_0y_0 - y_0^2)$$

Let $t \in [r, 2r)$, then $\frac{t}{r} \in [1,2)$ and we get

$$D^{\alpha}x(t) = y_1(1 - z_1)$$
$$D^{\alpha}y(t) = y_1(1 + z_1) - ax_1$$
$$D^{\alpha}z(t) = a - x_1y_1 - y_1^2$$

and the solution of (2) is given by

$$x_2(t) = x_1 + I^{\alpha}(y_1(1 - z_1)) = x_1 + \frac{t^{\alpha}}{\Gamma(1+\alpha)}(y_1(1 - z_1))$$
$$y_2(t) = y_1 + I^{\alpha}(y_1(1 + z_1) - ax_1) = y_1 + \frac{t^{\alpha}}{\Gamma(1+\alpha)}(y_1(1 + z_1) - ax_1)$$
$$z_2(t) = z_1 + I^{\alpha}(a - x_1y_1 - y_1^2) = z_1 + \frac{t^{\alpha}}{\Gamma(1+\alpha)}(a - x_1y_1 - y_1^2)$$

Repeating the process for $n + 1$, we get

$$x_{n+1}(t) = x_n(nr) + \frac{(t - nr)^{\alpha}}{\Gamma(1+\alpha)}\left(y_n(nr)\left(1 - z_n(nr)\right)\right)$$
$$y_{n+1}(t) = y_n(nr) + \frac{(t - nr)^{\alpha}}{\Gamma(1+\alpha)}\left(y_n(nr)\left(1 + z_n(nr)\right) - a\,x_n(nr)\right)$$
$$z_{n+1}(t) = z_n(nr) + \frac{(t - nr)^{\alpha}}{\Gamma(1+\alpha)}(a - x_n(nr)y_n(nr) - y_n^2(nr))$$

Let $t \to (n + 1)r$, we obtain the following discretization

$$x_{n+1}((n + 1)r) = x_n(nr) + \frac{(r)^{\alpha}}{\Gamma(1+\alpha)}\left(y_n(nr)\left(1 - z_n(nr)\right)\right)$$
$$y_{n+1}((n + 1)r) = y_n(nr) + \frac{(r)^{\alpha}}{\Gamma(1+\alpha)}\left(y_n(nr)\left(1 + z_n(nr)\right) - a\,x_n(nr)\right)$$
$$z_{n+1}((n + 1)r) = z_n(nr) + \frac{(r)^{\alpha}}{\Gamma(1+\alpha)}(a - x_n(nr)y_n(nr) - y_n^2(nr))$$

which can be expressed as

$$x_{n+1} = x_n + \frac{r^{\alpha}}{\Gamma(1+\alpha)}\left(y_n(1 - z_n)\right)$$

$$y_{n+1} = y_n + \frac{r^\alpha}{\Gamma(1+\alpha)}(y_n(1+z_n) - a\,x_n) \qquad (3)$$

$$z_{n+1} = z_n + \frac{r^\alpha}{\Gamma(1+\alpha)}(a - x_n y_n - y_n^2).$$

III. Fixed points and Stability

System (5) has two fixed points,

(i) $F_1 = \left(2\sqrt{\frac{1}{a+2}}, a\sqrt{\frac{1}{a+2}}, 1\right)$

(ii) $F_2 = \left(-2\sqrt{\frac{1}{a+2}}, -a\sqrt{\frac{1}{a+2}}, 1\right)$

Linearization of the system (3) about F_1 or F_2 yield the characteristic equation

$$P(\lambda) = \lambda^3 + \lambda^2[-3-2s] + \lambda[3+4s+s^2a(1+2b^2)+s^2a^2b^2] + [s^3a^2b^2(3-2a) - 2s^2ab(1+b+ab) - sab - 2s - 1] = 0$$

where $s = \frac{r^\alpha}{\Gamma(1+\alpha)}$ and $b = \sqrt{\frac{1}{a+2}}$. Let $a_1 = -3-2s$, $a_2 = 3+4s+s^2a(1+2b^2)+s^2a^2b^2$, $a_3 = s^3a^2b^2(3-2a) - 2s^2ab(1+b+ab) - sab - 2s - 1$.

From the Jury test, if $P(1) > 0$, $P(-1) < 0$ and $P(0) = a_3 < 0, |b_3| > b_1, c_2 > |c_3|$, where $b_3 = 1 - a_3^2$, $b_2 = a_1 - a_3a_2$, $b_1 = a_2 - a_3a_1$, $c_3 = b_3^2 - b_1^2$ and $c_2 = b_3b_2 - b_1b_2$ then the roots of $P(\lambda)$ satisfy $|\lambda_j| < 1$ and thus F_1 or F_2 is asymptotically stable. Suppose $P(1) < 0$, then F_1 or F_2 is unstable.

IV. Numerical Simulations

In this section, we present the time series for $x(t), y(t), z(t)$ and phase plane diagrams for the fractional order discrete dynamical system (3).

Example 1. Let us consider the values $a = 0.5$, $r = 0.1$ and $\alpha = 0.9$ with initial values $(x_0, y_0, z_0) = (0.1, 0.1, 0.1)$. Applying Jury test, we get $P(-1) = -9.1264 < 0$ and $a_3 = -1.3239 < 0$, thus $P(1) = -0.0449 < 0$ fixed point F_1 or F_2 is unstable. See figure-1 & 2.

Figure 1. Time series for the discrete fractional order 3D chaotic system (3)

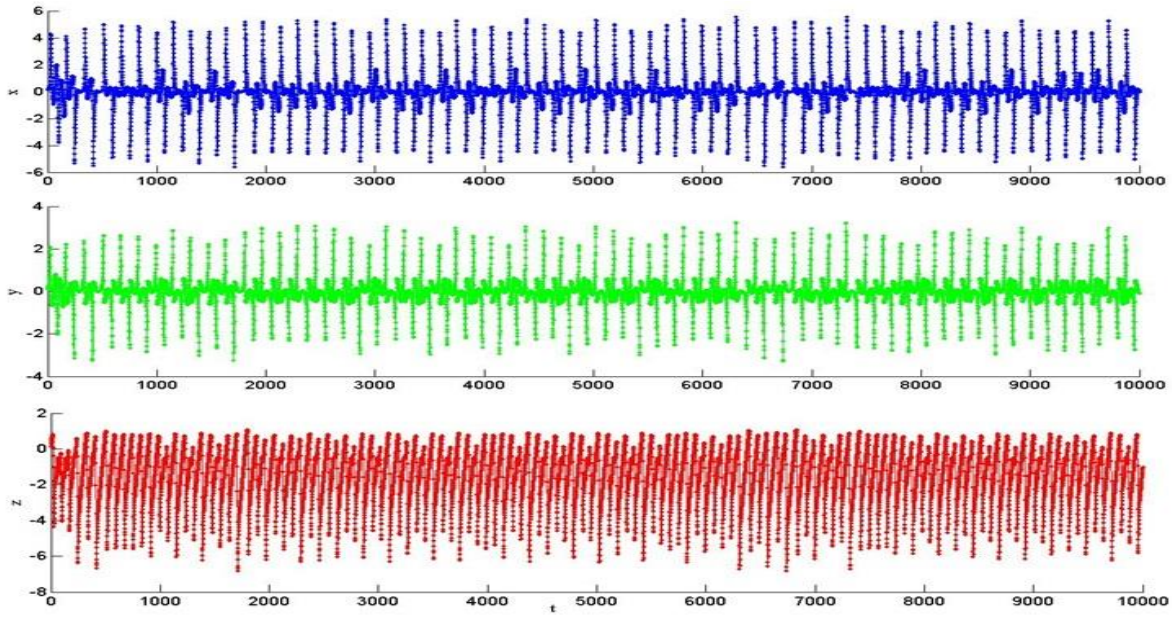
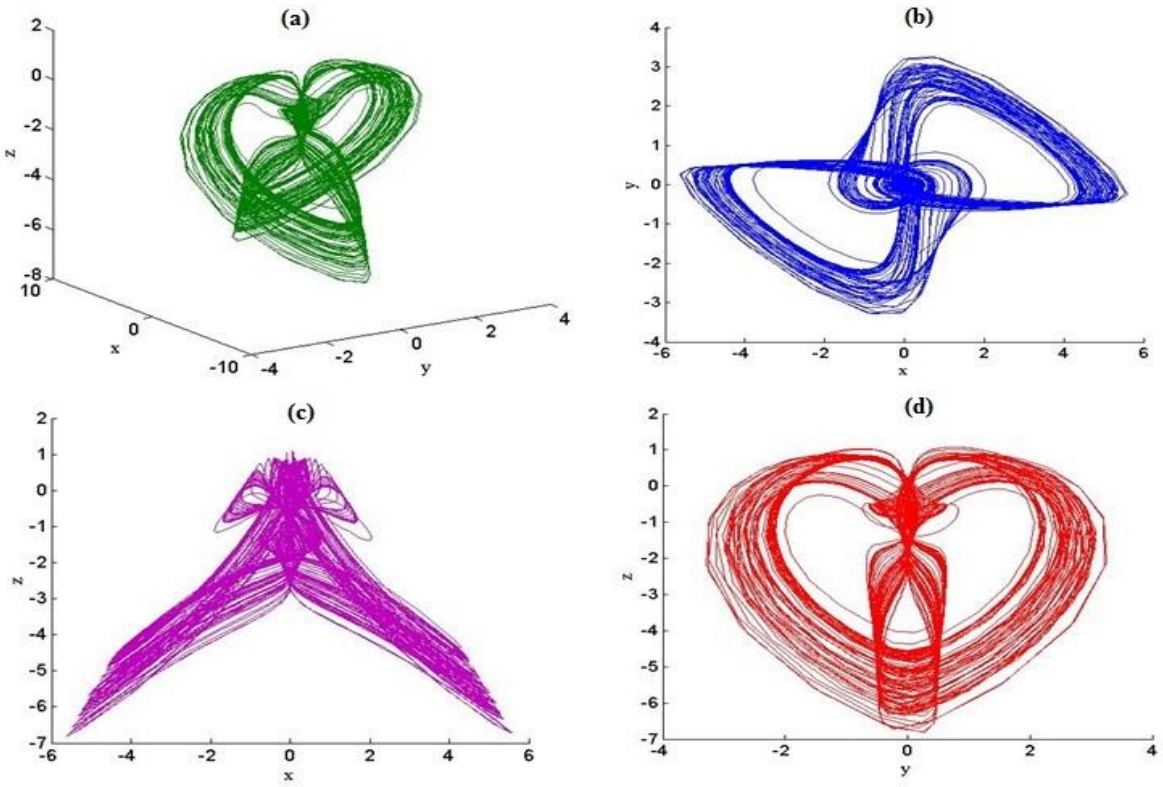


Figure 2. Various phase plane diagrams of the system (3): a) 3D phase plane, b) x-y phase plane, c) x-z phase plane, d) y-z phase plane



Example 2. Let us consider the values $a = 0.55$, $r = 0.1$ and $\alpha = 0.9$ with initial values $(x_0, y_0, z_0) = (0.1, 0.1, 0.1)$ Applying Jury test, we get $P(-1) = -9.1339 < 0$ and $a_3 = -1.3296 < 0$, thus $P(1) = -0.0490 < 0$ fixed point F_1 or F_2 is unstable. See figure 3&4.

Figure 3. Time series for the discrete fractional order 3D chaotic system (3)

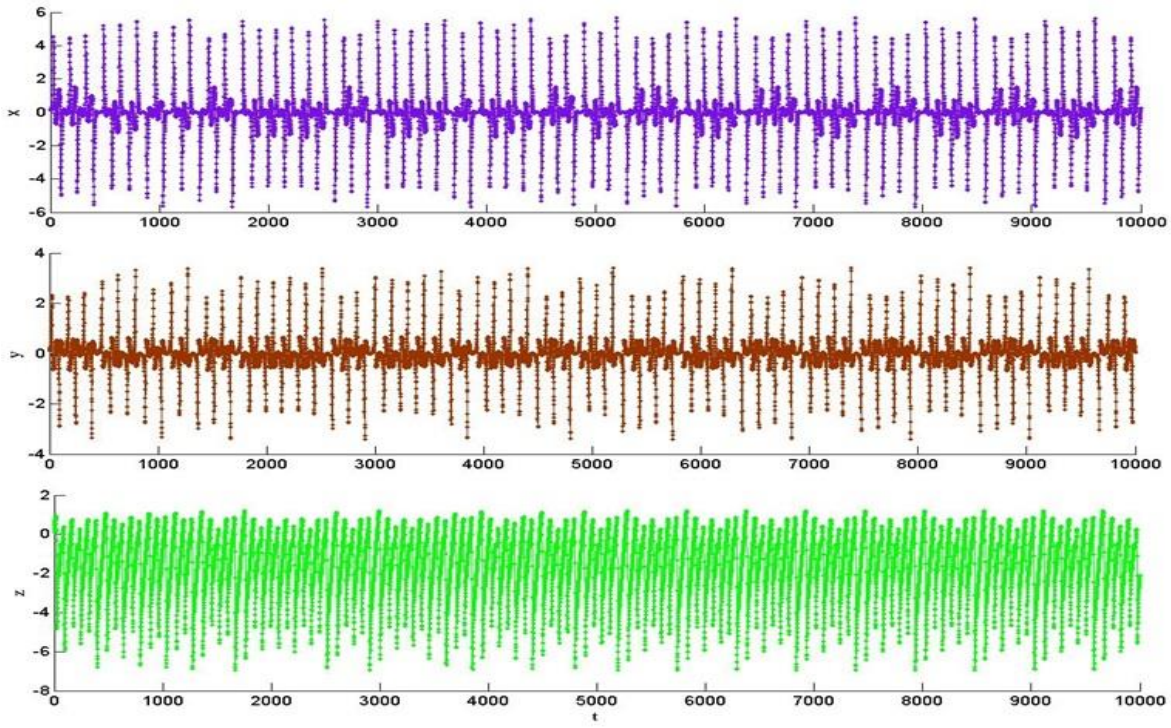
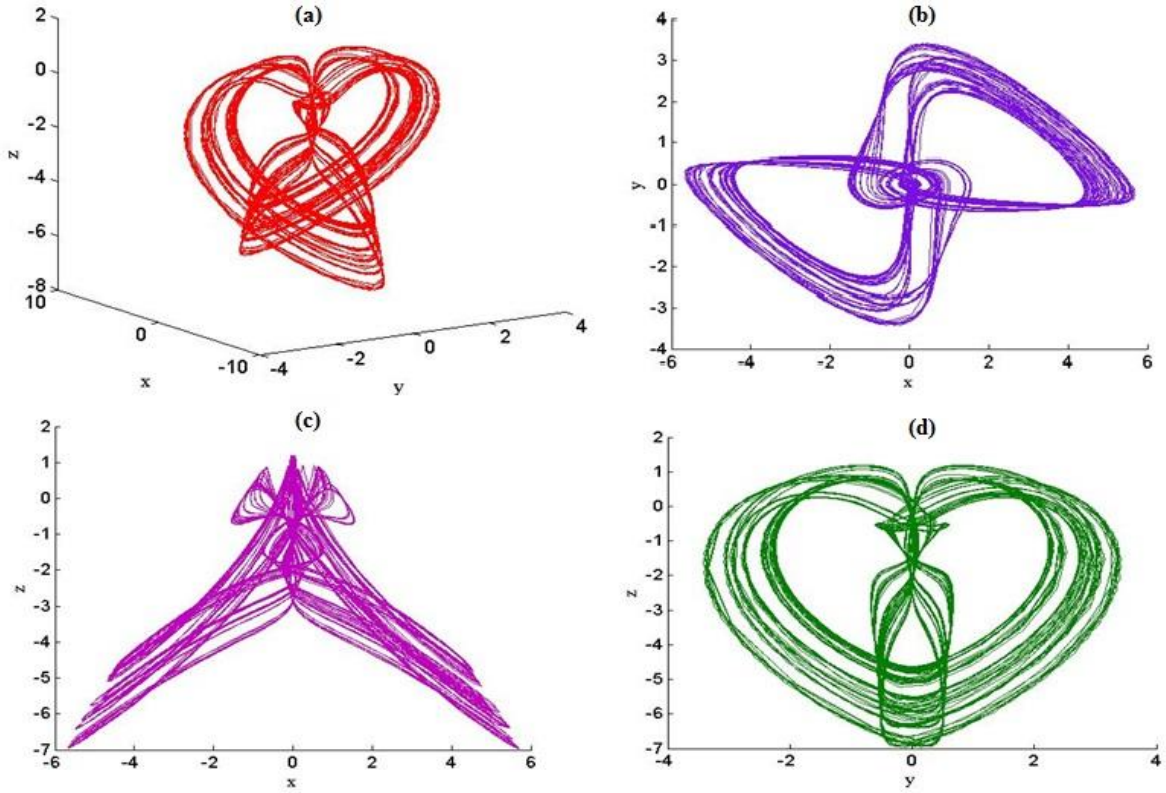


Figure 4. Various phase plane diagrams of the system (3): a) 3D phase plane, b) x-y phase plane, c) x-z phase plane, d) y-z phase plane



Example 3. Let us consider the values $a = 0.6$, $r = 0.1$ and $\alpha = 0.9$ with initial values $(x_0, y_0, z_0) = (0.1, 0.1, 0.1)$ Applying Jury test, we get $P(-1) = -9.1413 < 0$ and $a_3 = -1.3353 < 0$, thus $P(1) = -0.0530 < 0$ fixed point F_1 or F_2 is unstable. See figure - 5 & 6.

Figure 5. Time series for the discrete fractional order 3D chaotic system (3)

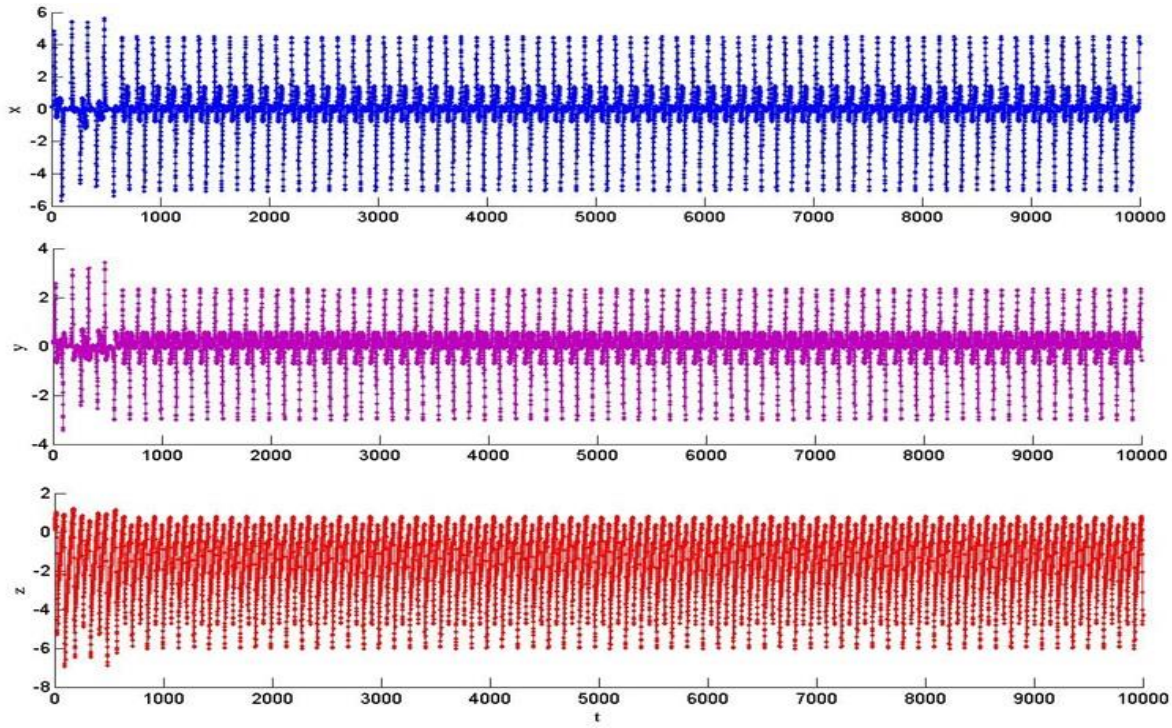
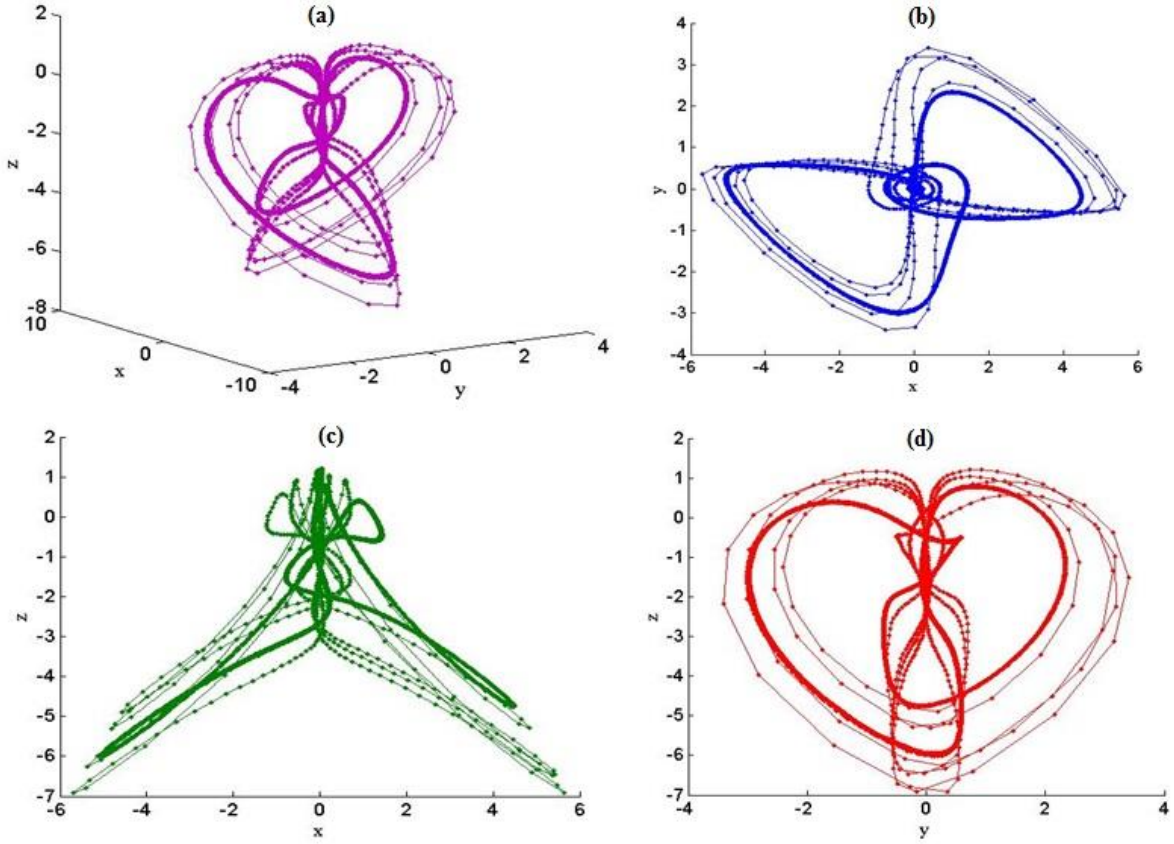


Figure 6. Various phase plane diagrams of the system (3): a) 3D phase plane, b) x-y phase plane, c) x-z phase plane, d) y-z phase plane



V. Bifurcation and Chaotic attractor

Bifurcation diagrams provide information about abrupt changes in the qualitative behavior in the dynamics of the system. The parameter values at which these changes occur are called bifurcation points [4]. We present the bifurcation diagrams, phase plane diagrams and chaotic attractor of the system (3), see Figure 7, 8 & 9. We take the initial values $(x_0, y_0, z_0) = (0.1, 0.1, 0.1)$.

Figure 7. Bifurcation diagrams for x, y, z of the system (3) and $a \in (0, 0.35)$, $r = 0.1, \alpha = 0.6$

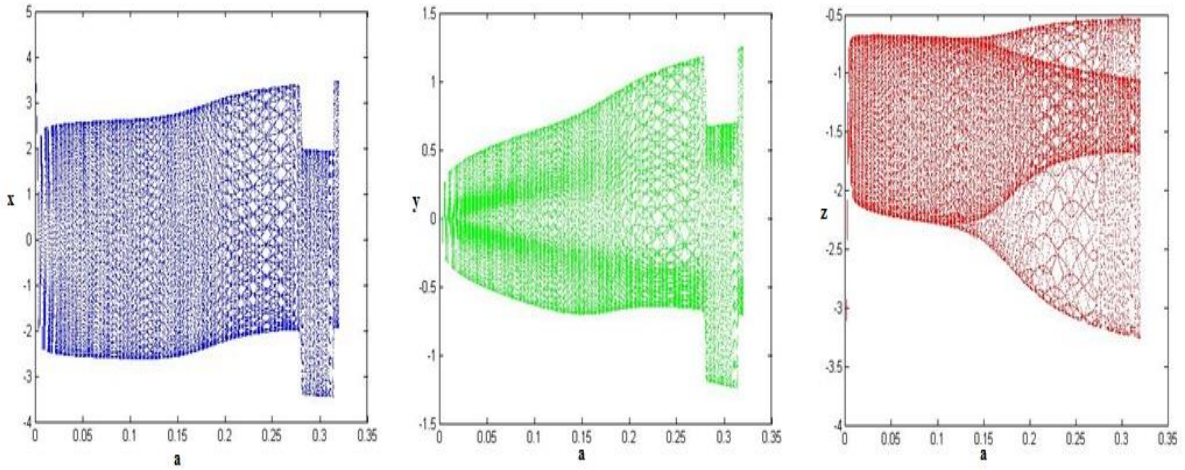


Figure 8. Bifurcation diagrams for the system (3), $a \in (0, 0.35), r = 0.1, \alpha = 0.7$ and various phase portraits (a) $a = 0.25$, $r = 0.1, \alpha = 0.7$ (b) $a = 0.35$, $r = 0.1, \alpha = 0.7$ i) x-y phase plane, ii) x-z phase plane, iii) y-z phase plane.

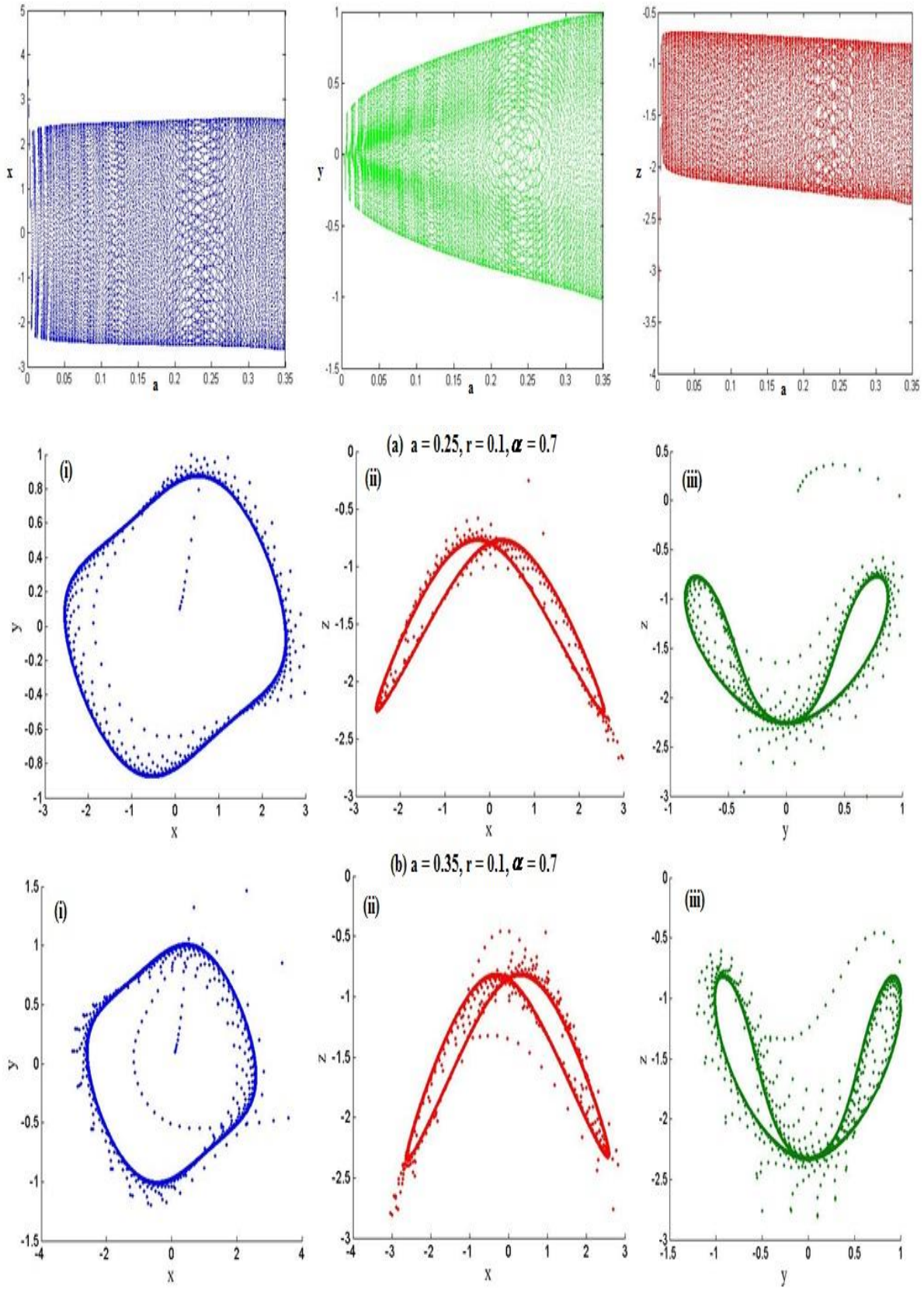
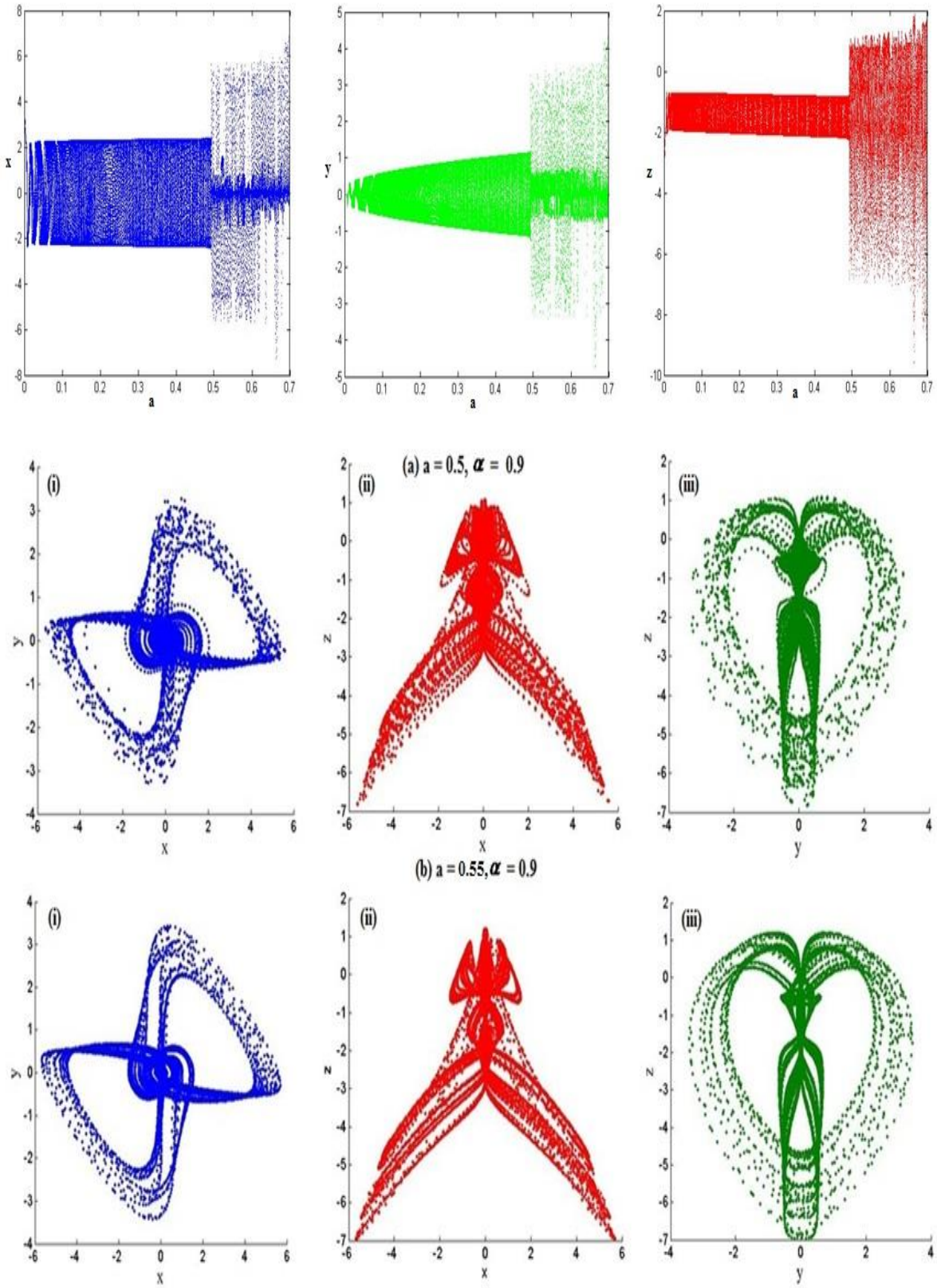


Figure 9. Bifurcation diagrams for the system (3), $a \in (0, 0.7), r = 0.1, \alpha = 0.9$ and various chaotic attractors (a) $a = 0.5,\ r = 0.1, \alpha = 0.9$ (b) $a = 0.55,\ r = 0.1, \alpha = 0.9$ i) x-y chaotic attractor, ii) x-z chaotic attractor, iii) y-z chaotic attractor



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Elecrohydrodynamic convection of aerosols in a vertical channel

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Abstract *The electrohydrodynamic convection of atmospheric aerosols consisting of couple stress fluid through a vertical channel when the walls are held at different temperatures is investigated.Using regular perturbation technique an analytical solution for the nonlinear coupled equation is obtained with buoyancy number N as the perturbation parameter. The effect of electric field on the velocity,temperature, skin friction and rate of heat transfer are computed.The results are represented graphically.*

Keywords: couple stress fluid, heat transfer, skin friction, temperature profiles, electrodes

I. Introduction

Nowadays, aerospace engineering has many types of environmental problems. The problem of convection in fluids is of importance in geophysics, soil sciences, ground water hydrology and astrophysics. In geophysical situations, more often, the fluid is not pure but may instead be permeated with aerosols. Recently spacecraft observations have confirmed that the dust particle play an important role in the dynamics of atmosphere. The pollution caused by space craft depends not only on the total quantity of the fuels used, but on their chemical composition when the space shuttle lifts off. The shuttle takes off vertically and it takes a lot of fuel just to lift the fuel. The vertical transport of properties depends on three processes namely convection, turbulent diffusion and fall by gravity.Convection is the vertical transport of warmer air through the environment and ultimately dependent on the vertical stability of the air. Turbulent diffusion is the random motion of air parcels of different sizes.Several authors have studied the problem of vertical distribution of aerosols. Yonghang Chen et al., [20] have studied vertical distribution of optical and micro physical properties of ambient aerosols. Meilu He and Suresh Dhaniyala[16] have investigated vertical and horizontal concentration distributions of ultrafine particles near a highway. Laurent et al.,[9] have studied vertical spray droplet dispersion in the marine boundary layer.Srinivasacharya and Kaladhar[12] have investigated the Soret and Dufour effects on the convection along a semi-infinite vertical plate embedded in a couple stress fluid. Gnaneswara Reddy and Bhaskar Reddy [7] reported the steady MHD free convection flow past a semi-infinite moving vertical plate in a porous medium with viscous dissipation. Lakshmi Narayana and Murthy [10] studied free convection heat and mass transfers from a vertical surface in a doubly stratified Darcian porous medium. Alam and Rahman [1] have investigated the mixed convection flow past a vertical porous flat plate with variable suction. Postelnicu[11] studied numerically the influence of a magnetic field on heat transfer by natural convection from a vertical surface in porous media. Umavathi and Malashetty[19] have investigated oberbeck convection flow of a couple stress fluid through a vertical porous stratum. Kafoussias and Williams [8] presented the mixed convective and mass transfer steady laminar boundary layer flow over a vertical flat plate with temperature dependent viscosity. From the above references, we note that it is of great importance to study the vertical distribution of aerosols in the presence of electric field and couplestress.In the present study, the electrohydrodynamic convection of atmospheric aerosols consisting of a couple stress fluid through a vertical channel, when the walls are held at different temperatures is studied.

II. Mathematical Formulation

Consider a two dimensional geometry as shown in Figure 1. It consists of flow through a vertical channel extended to infinity on both directions. The x-axis is taken along the walls and the y-axis perpendicular to it. The walls are placed at a distance b apart and maintained at constant different temperatures,T_0 and T_1. The couple stress fluid is filled in the channel with embedded electrodes of different potentials at y=0 and y=b.

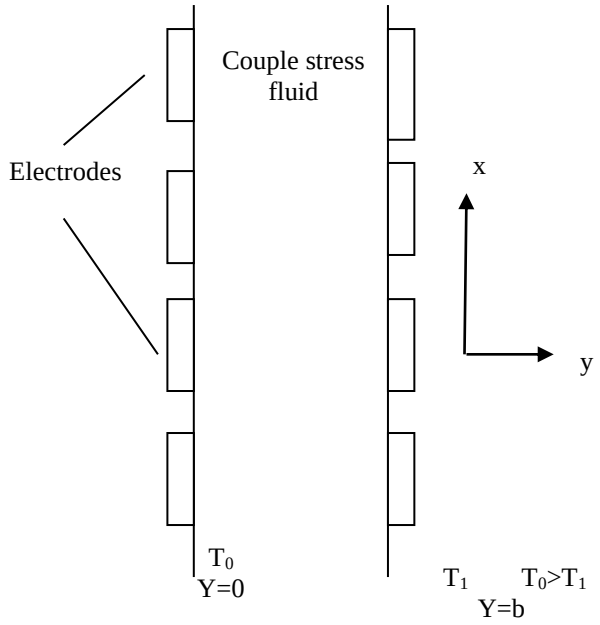


Figure 1.physical configuration

In this paper,we make the assumption that the electrical conductivity ( ) is negligibly small and hence the magnetic field is negligible. This assumption makes the electric field $\vec{E}$, to be conservative.

ie $\vec{E} = -\nabla\emptyset$ (1)

The basic equations are,

The Conversation of mass for an incompressible flow

$\nabla.\vec{q} = 0$ (2)

The Conservation of momentum

$$\rho\left(\frac{\partial\vec{q}}{\partial t} + (\vec{q}.\nabla)\vec{q}\right) = -\nabla p + \mu\nabla^2\vec{q} - \lambda\nabla^4\vec{q} + \rho g + \rho_e\overrightarrow{E} \quad (3)$$

where λ is a couple stress parameter.

The energy equation, in the presence of viscous dissipation is,

$$(pC_p)\left(\frac{\partial T}{\partial t} + (\vec{q}\cdot\nabla)T\right) = K\nabla^2 T + \psi \quad (4)$$

The Equation of state,

$$\rho = \rho_0\,(1-\beta\,(T-T_0)) \quad (5)$$

The Conservation of species

$$\frac{\partial C}{\partial t} + (\vec{q}\cdot\nabla)C = D\nabla^2 C \quad (6)$$

The Conservation of charges

$$\frac{\partial\rho_e}{\partial t} + (\vec{q}\cdot\nabla)\rho_e + \nabla\cdot\vec{J} = 0 \quad (7)$$

Maxwell's equation

$$\nabla\cdot\vec{E} = \frac{\rho e}{\in_0}\,(Gauss\,law) \quad (8)$$

$$\nabla\times\vec{E} = 0\,(Faraday's\,law) \quad (9)$$

$$\vec{J} = \sigma \vec{E} \ (Ohm's\ law) \tag{10}$$

where $\vec{q}$ is the velocity, p the pressure, ρ the density, E the electric field, ρ_e the electric charge density, J the current density, $\in_0$ the dielectric constant for the free space, ΔT the temperature gradient, K the thermal conductivity, β the volumetric expansion for density, C_p the specific heat at a constant pressure and ψ the viscous dissipation.

The above equations are solved using the following boundary conditions on velocity and potential are,

$$\left.\begin{aligned} u = 0 \ and \ T = T_1 \ at \ y = b \\ u = 0 \ and \ T = T_0 \ at \ y = 0 \end{aligned}\right\} \tag{11}$$

$$\text{The couple stress condition, } \frac{d^2u}{dy^2} = 0 \text{ at } y = 0 \text{ and } y = b \tag{12}$$

$$\left.\begin{aligned} \phi = \frac{V}{h} x \ at \quad y = 0 \\ \phi = \frac{V}{h}(x - x_0) \ at \quad y = b \end{aligned}\right\} \tag{13}$$

Boundary conditions on velocity represents the no- slip conditions of the couple stresses at the solid boundaries and that on the temperature points that the plates are isothermally maintained at different temperatures T_1 and T_0 ($T_0 > T_1$).

In cartesian form, using the above approximation equation (3) and (4) becomes

$$0 = \rho_0 \beta (T - T_0) g + \mu \frac{d^2u}{dy^2} - \lambda \frac{d^4u}{dy^4} + \rho_e E_x$$

$$0 = K \frac{d^2T}{dy^2} + \mu \left(\frac{du}{dy}\right)^2 - \sigma \vec{\mathbf{E}}_x{}^2$$

In a poorly conducting fluid, the electrical conductivity is assumed to vary linearly with temperature and increases with temperature in the form

$$\sigma = \sigma_0 \left[1 + \alpha_h (T_b - T_0)\right] \tag{14}$$

where α_h is the volumetric coefficient of expansion and σ is the electrical conductivity.

We assume the flow is fully developed and unidirectional in the x – direction. This means the velocity is independent of time and all physical quantities except pressure and concentration are independent of x, so that the velocity and temperature will be functions of y only. Using the following dimensionless quantities,

$y* = \frac{y}{b}; \theta = \frac{T - T_0}{T_1 - T_0}; \ u* = \frac{\nu}{g\beta b^2 (T_1 - T_0)} u; \ E_x^* = \frac{E_x}{\frac{V}{b}}; \rho_e^* = \frac{\rho_e}{\frac{\varepsilon_0 V}{b^2}}$; where V is electric potential, we get electric potential through electrodes. Equations (3) to (13) becomes,

$$\frac{d^4u}{dy^4} - a^2 \frac{d^2u}{dy^2} - a^2\theta = a^2 \ W_e \rho_e E_x \tag{15}$$

where $W_e = \frac{\varepsilon_0 V^2}{\mu}, l = \sqrt{\frac{\lambda}{\mu}}, a = \frac{b}{l}$ is the couple stress parameter.

$$\frac{d^2\theta}{dy^2} + N \left(\frac{du}{dy}\right)^2 - \frac{\sigma}{\sigma_0} R_e E_x{}^2 = 0 \tag{16}$$

Where $N = \frac{\rho_0 g^2 \beta^2 b^4 \Delta T}{k\nu}$ is the buoyancy parameter, ie. $N = \frac{\mu b g \beta}{k} g_r$, $g_r = \frac{\beta g b^3 \Delta T}{\nu^2}$, the grashof number, $R_e = \frac{\sigma_0 V^2}{K\Delta T}$, is the thermal electric number.

equation (7) becomes, $\nabla . \quad = 0$

Using equation (1) we get,

$$\sigma(\nabla^2\phi) + \nabla\phi \cdot \nabla\sigma = 0 \tag{17}$$

The boundary conditions on velocity, couple stress, temperature and electric potential after dimensionless are

u = 0 at y = 0,1 (18)

$$\frac{d^2u}{dy^2} = 0 \; at \; y = 0,1 \tag{19}$$

$$\left.\begin{aligned} \theta = 0 \; at \; y = 0 \\ \theta = 0 \;\; at \; y = 1 \end{aligned}\right\} \tag{20}$$

$$\left.\begin{aligned} \phi = x \; at \qquad y = 0 \\ \phi = x - x_0 \; at \qquad y = 1 \end{aligned}\right\} \tag{21}$$

The solution for ϕ, according to (17) depends on σ which in turn depends on the temperature T_b as in (14). In a poorly conducting fluid, σ<<1 and hence any perturbation on it is negligible and hence any perturbation on it is negligible and hence it depends on the conduction temperature T_b namely,

$$\frac{d^2T_b}{dy^2} = 0 \tag{22}$$

with the boundary conditions

$$\left.\begin{aligned} T_b = T_0 \; at \; y = 0 \\ T_b = T_1 \; at \; y = b \end{aligned}\right\} \tag{23}$$

is $T_b - T_0 = \Delta T y$ (24)

Therefore equation (14) becomes $\sigma_0[1+\alpha_h \Delta T y] = \sigma_0(1+\alpha y) = \sigma_0 e^{\alpha y}$

$$\sigma \approx e^{\alpha y} \tag{25}$$

where $\alpha = \alpha_h \Delta T$.

Then (17) using (25) we get

$$\frac{d^2\phi}{dy^2} + \alpha \frac{d\phi}{dy} = 0 \tag{26}$$

Its solution satisfying the boundary condition (21) is

$$\phi = x - \frac{x_0}{1-e^{-\alpha}}\left[1 - e^{-\alpha y}\right] \tag{27}$$

Using the dimensionless quantities and equation (27), equations (8), (9) and (10) reduce to

$$\rho_e = \nabla . \vec{E} = -\nabla^2\phi = -\frac{x_0\alpha^2 e^{-\alpha y}}{1-e^{-\alpha}} \; ; \; E_x = -1$$

Therefore,

$$\rho_e E_x = \frac{x_0\alpha^2 e^{-\alpha y}}{1-e^{-\alpha}} \tag{28}$$

III. Analytical solutions

Using a regular perturbation technique with the buoyancy parameter N as the perturbation parameter, which is very small,to find the analytical solutions.

A.velocity and temperature distributions

To obtain analytical solutions for velocity and temperature, the solutions of (15) and (16) in the form

$$\left.\begin{aligned} u &= u_0 + Nu_1 + N^2u_2 + ... \\ \theta &= \theta_0 + N\theta_1 + N^2\theta_2 + ... \end{aligned}\right\} \tag{29}$$

Here the subscript 0 refers to the solutions for the case in which N = 0, which represents physically the solutions in the absence of viscous dissipation and u_1, u_2, …,θ_1, θ_2, …., are the perturbation quantities which are assumed to be small compared to u_0 and θ_0. Equations (15) and (16), using (28) and (29) and equating the coefficients of the like powers of N to zero, we get the following set of equations:

Zeroth order equations

$$\frac{d^4u_0}{dy^4} - a^2\frac{d^2u_0}{dy^2} - a^2\theta_0 - a_0a^2e^{-\alpha y} = 0 \tag{30}$$

$$\frac{d^4\theta_0}{dy^2} - R_e e^{\alpha y} = 0 \tag{31}$$

First order equations

$$\frac{d^4u_1}{dy^4} - a^2\frac{d^2u_1}{dy^2} - a^2\theta_1 = 0 \tag{32}$$

$$\frac{d^2\theta_1}{dy^2} + \left(\frac{du_0}{dy}\right)^2 = 0 \tag{33}$$

Where $a_0 = \dfrac{w_e x_0 \alpha^2}{1-e^{-\alpha}}$

The boundary conditions are

$$u_0 = \frac{d^2u_0}{dy^2} = 0 \; at \; y = 0,1 \tag{34}$$

$$\left.\begin{aligned} \theta_0 &= 0 \; at \; y = 0 \\ \theta_0 &= 0 \;\; at \; y = 1 \end{aligned}\right\} \tag{35}$$

$$u_1 = \frac{d^2u_1}{dy^2} = 0 \; at \; y = 0,1 \tag{36}$$

$$\left.\begin{aligned} \theta_1 &= 0 \; at \; y = 0 \\ \theta_1 &= 1 \;\; at \; y = 1 \end{aligned}\right\} \tag{37}$$

The solutions of equation (30) and (31), satisfying the conditions (34) and (35) are

$$\theta_0 = \frac{R_e}{2}y^2 + \frac{R_e}{6}y^3\,\alpha + A_3 y \tag{38}$$

$$u_0 = (c_1y + c_2)e^{ay} + c_3e^{ay} + c_4e^{-ay} - \frac{y^2}{2}a_0 + (a_0\alpha - A_3)\frac{y^3}{6} - \frac{R_e y^4}{24} - \frac{R_e y^2}{2a^2} - \frac{R_e \alpha y^5}{120} - \frac{R_e \alpha y^3}{6a^2}$$

(39)

$$\theta_1 = \frac{-c_4^2}{4}e^{-2ay} - \frac{-c_3^2}{4}e^{2ay} - b_1e^{-ay} + b_2e^{ay} - b_3ye^{-ay} - b_4y^2e^{-ay}$$
$$-\frac{R_e c_4}{3a}y^3e^{-ay} - \frac{R_e\alpha}{12a}c_4\ y^4e^{-ay} + b_5ye^{ay} + b_6y^2e^{ay} + b_7y^3e^{ay}$$
$$+\frac{R_e\alpha}{12a}c_3y^4e^{ay} + a_8\frac{y^4}{4} + a_9\frac{y^5}{5} + a_{10}\frac{y^6}{6} + a_{11}\frac{y^7}{7} + a_{12}\frac{y^8}{8} \quad (40)$$
$$-\frac{R_e^2\alpha}{5184}\ y^9 - \frac{R_e^2\alpha^2}{51840}y^{10} + m_{14}$$

$$u_1 = -\frac{c_4^2}{48a^2}e^{-2ay} - \frac{c_3^2}{48a^2}e^{2ay} - b_8e^{-2a-ay} + b_9e^{-2a+ay} + b_{10}ye^{-2a-ay}$$
$$+b_{11}ye^{-2a+ay} + b_{12}y + b_{13}y^2e^{-2a-ay} + b_{14}y^2e^{-2a+ay} + b_{15}y^2 + g_2y^3e^{-2a-ay}$$
$$+g_3y^3e^{-2a+ay} + g_4y^3 + g_5y^4e^{-2a-ay} + g_6y^4e^{-2a+ay} + g_7y^4 \quad (41)$$
$$+ g_{14}y^9 + g_{15}y^{10} + g_{16}y^{11} + g_{17}y^{12}$$

$$\theta = \theta_0 + N\theta_1 \quad (42)$$

$$u = u_0 + Nu_1 \quad (43)$$

$$= (c_1 + Nb_{12})y + (c_2 + Ng_1) + (c_3 + Ne^{-2a}b_9)\,e^{-ay} + (C_4 + Ne^{-2a}b_8)e^{-ay}$$
$$+\left(-\frac{a_0}{2} - \frac{R_e}{2a^2} + Nb_{15}\right)y^2 + \left(\left(\frac{a_0\alpha - A_3}{6}\right) + Ng_4 - \frac{R_e\alpha}{6a^2}\right)y^3$$
$$+\left(\frac{-R_e}{24} + Ng_7\right)y^4 + \left(\frac{-R_e\alpha}{120} + Ng_{10}\right)y^5 + \left(\frac{-Nc_4^2}{48a^2}\right)e^{-2ay}$$
$$+\left(\frac{-Nc_3^2}{48a^2}\right)e^{2ay} + b_{10}Ne^{-2a-ay}\ y + b_{11}Nye^{-2a+y} + b_{13}Ny^2\cdot e^{-2a-ay}$$
$$+ b_{14}Ny^2\ e^{-2a+ay} + g_2Ny^3e^{-2a-ay} + g_3Ny^3e^{-2a+ay} + g_5Ny^4e^{-2a-ay} + g_6Ny^4e^{-2a+ay} + g_8Ny^5e^{-2a-ay}$$
$$+g_9Ny^5e^{-2a+ay} + g_{11}y^6 + g_{12}y^7 + g_{13}y^8 + g_{14}y^9 + g_{15}y^{10} + g_{16}y^{11} + g_{17}y^{12}$$

B. Skin friction and Rate of heat transfer

In many practical applications it is advantageous to know the skin friction and the rate of heat transfer at the boundaries. These can be determined once we know the velocity and temperature distributions. The skin friction τ at the walls is defined as

$$\tau = \mu\left(\frac{du}{dy}\right)$$

making this dimensionless using the scale $\rho g b \Delta T$ for τ and using the scale for u used earlier we get,

$$\tau = \frac{du}{dy}$$

This equation ,using (27) takes the form

$$\tau = \frac{du_0}{dy} + N\left(\frac{du_1}{dy}\right)$$

Here $\frac{du_0}{dy}$ and $\frac{du_1}{dy}$ can be obtained using (37) and (39). Similarly the rate of heat transfer, between the fluid and the plate is given by the heat flux.

$$H' = \frac{\partial T}{\partial t}$$

This using the scale-K Δ T/b for the heat flux H' takes the form ,

$$H = \frac{d\theta}{dy}$$

This using (27), takes the form,

$$H = \frac{d\theta_0}{dy} + N\left(\frac{d\theta_1}{dy}\right)$$

where $\frac{d\theta_0}{dy}$ and $\frac{d\theta_1}{dy}$ are obtained using (36) and (38)

IV. Results and Discussions

The heat transfer mechanisms of atmospheric aerosols is discussed.The effect of electric field and couple stress parameter on the velocity u,temperature θ ,skin friction τ and rate of heat transfer H are obtained analytically using regular perturbation technique for small values of buoyancy parameter N.The analytical results are reported and are represented graphically through figures.

In figure 2 the effect of electric number on the magnitude of velocity profiles of aerosols is plotted. It is observed that the velocity profile is parabolic and it increases with an increase in electric number We.Figures 3 and 4 represent the effect of thermal electric number Re and couple stress parameter a on the temperature. From these figures, it shows that the temperature θ decreases with an increase in Re and We.In figures 5 the effect of thermal electric number Re and couple stress parameter a on the skin friction τ is plotted. It reveals that skin friction increases with increase in thermal electric number.

Figure 6 represents the effect of electric number weon the skin friction τ . It is seen that the skin friction increases with an increase in electric numberWe. The effect of thermal electric numberRe and couple stress parameter a on the heat transfer is plotted in Figures 7. It concludes that heat transfer H increases with increase in thermal electric number.Figure 8 represents the effect of electric number on the magnitude of heat transfer of aerosols. It shows that the heat transfer H decreases with an increase in electric number We.

V. Conclusion

The convection of atmospheric aerosols in a vertical channel bounded by electrodes is studied in the presence of couple stress.It concludes that the enhancement of external constraints like electric field and couple stress reduces the heat transfer mechanisms of atmospheric aerosols. The problem considered in this paper is a representative of electrohydrodynamic convection of atmospheric aerosols.

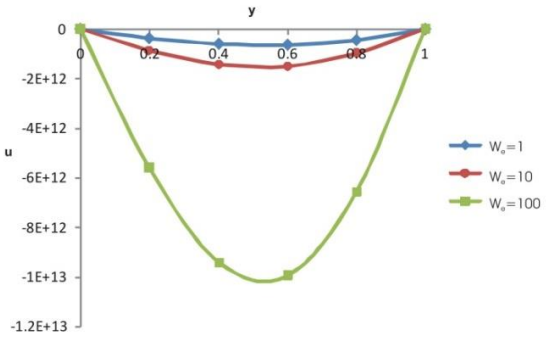


Figure 2.Velocity profiles for different values of We With α =0.1, Re=1, x0=2.1, a=3 and N=0.5

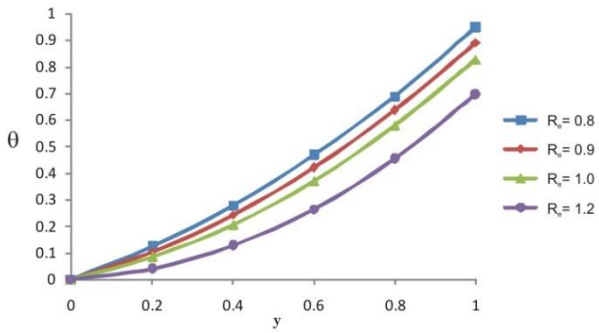


Figure 3. Temperature profiles for different values of Re with α =0.1, x0=2.1, N=0.1, a=1

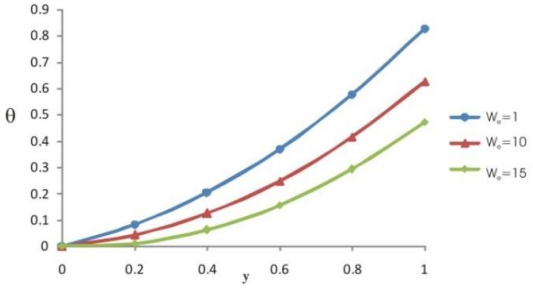


Figure 4. The effect of We on temperature distribution with α =0.1, Re=1, x0=2.1, N=0.1 and a=1

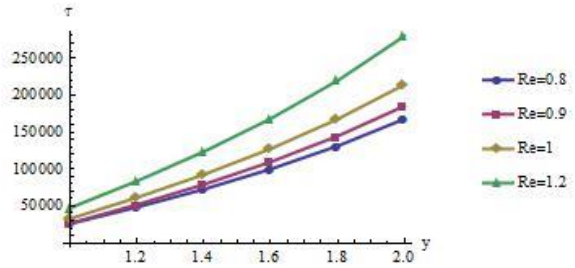


Figure 5.skin friction for different values of Re with α =0.1, X0=2.1, N=0.1, a=1 and We=1

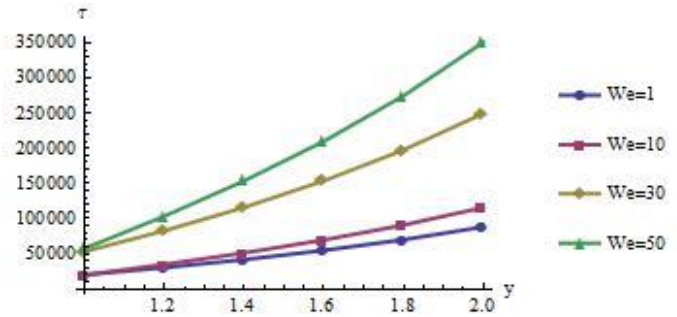


Figure 6.The effect of We on skin friction with α =0.1, Re=1, x0=2.1, N=0.1 and a=1.

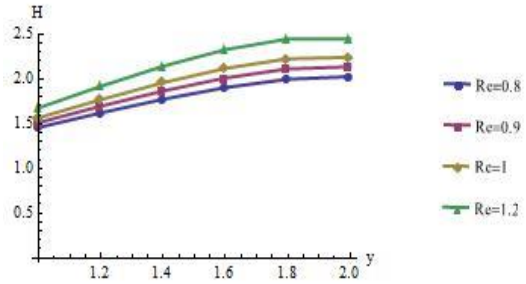


Figure 7.Heat transfer for different values of Re with α =0.1, x0=2.1, N=0.1, a=1 and We=1.

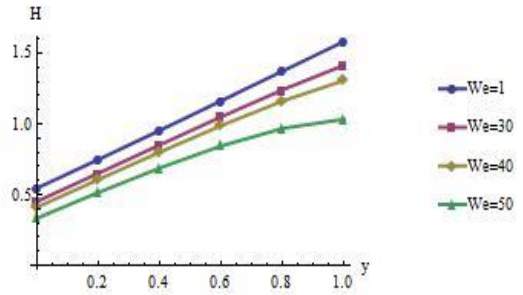


Figure 8.The effect ofWe on heat transfer with α =0.1, Re=1, x0=2.1, N=0.1 and a=1.

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Appendix

$$A_3=\left(1-\frac{R_e}{2}-\frac{R_e\alpha}{2}\right);a_0=\left(\frac{w_e x_0\alpha^2}{1-e^{-\alpha}}\right);a_1=\left(\frac{a_0\alpha-A_3}{2}\right);a_2=\frac{R_e}{6};a_3=\frac{R_e\alpha}{24};$$

$$a_4=\frac{R_e}{a^2};a_5=\frac{R_e\alpha}{2a^2};$$

$$c_1=\left((a_0\alpha-A_3)\left(\frac{1}{a^2}-\frac{1}{6}\right)+\frac{a_0}{2}+R_e\alpha\left(\frac{1}{120}-\frac{1}{a^4}\right)+\frac{R_e}{24}\right);c_2=\left(\frac{1}{a^2}\left(-a_0-\frac{R_e}{a^2}\right)\right);$$

$$c_3=\left(\left(\frac{1}{a^2(e^a-e^{-a})}\right)\left(a_0\left(e^a-1\right)+\frac{R_e}{a^2}e^a+\left(a_0\alpha-A_3\right)-\frac{R_e}{2}-\frac{R_e}{a^2}-\frac{R_e\alpha}{6}-\frac{R_e\alpha}{a^2}\right)+\frac{a_0}{a^2}+\frac{R_e}{a^4}\right);$$

$$c_4=\left(\left(\frac{1}{a^2(e^a-e^{-a})}\right)\left(a_0\left(e^a-1\right)+\frac{R_e}{a^2}e^a+\left(a_0\alpha-A_3\right)-\frac{R_e}{2}-\frac{R_e}{a^2}-\frac{R_e\alpha}{6}-\frac{R_e\alpha}{a^2}\right)\right);$$

$$a_6=\left(-c_1^2+2a^2c_3c_4\right);a_7=\left(a_0c_1+a_4c_1\right);a_8=\frac{1}{3}\left(-a_0^2-2a_0a_4-a_4^2-2a_1c_1+2a_5c_1\right);$$

$$a_9=\frac{1}{2}\left(a_0a_1+a_1a_4-a_0a_5-a_4a_5+a_2c_1\right);a_{10}=\frac{1}{5}\left(-a_1^2-2a_0a_2-2a_2a_4+2a_1a_5-a_5^2+2a_3c_1\right);$$

$$a_{11}=\frac{1}{3}\left(a_1a_2-a_0a_3-a_3a_4-a_2a_5\right);a_{12}=\frac{1}{7}\left(-a_2^2+2a_1a_3-2a_3a_5\right);$$

$$a_{13}=\frac{1}{4}a_2a_3;a_{14}=\frac{a_3^2}{9};m_1=\left(\frac{-2a_0c_3}{a}-\frac{4a_1c_3}{a^2}-\frac{12a_2c_3}{a^3}+\frac{48a_3c_3}{a^4}-\frac{2a_4c_3}{a}+\frac{4a_5c_3}{a^2}-2c_1c_3\right);$$

$$m_2=\left(2a_0c_3+\frac{4a_1c_3}{a^2}-\frac{12a_2c_3}{a^2}-\frac{48a_3c_3}{a^3}-2a_4c_3-\frac{4a_5c_3}{a}\right);m_3=\left(-2a_1c_3-\frac{6a_2c_3}{a}+\frac{24a_3c_3}{a^2}+2a_5c_3\right);$$

$$m_4=\left(2a_2c_3-\frac{8a_3c_3}{a}\right);m_5=2a_3c_3;m_6=\left(\frac{2a_0c_4}{a}-\frac{4a_1c_4}{a^2}+\frac{12a^2c_4}{a^3}+\frac{48a_3c_4}{a^4}+\frac{2a_4c_4}{a}+\frac{4a_5c_4}{a^2}-2c_1c_4\right);$$

$$m_7=\left(2a_0c_4-\frac{4a_1c_4}{a^2}+\frac{12a^2c_4}{a^3}+\frac{48a_3c_4}{a^4}+\frac{2a_4c_4}{a}+\frac{4a_5c_4}{a^2}\right);m_8=\left(-2a_1c_4+\frac{6a_2c_4}{a}+\frac{24a_3c_4}{a^2}+2a_5c_4\right);$$

$$m_9=\left(2a_2c_4+\frac{8a_3c_4}{a}\right);m_{10}=2a_3c_4;m_{11}=\frac{1}{2}ac_4^2;m_{12}=\frac{1}{2}ac_3^2;$$

$$m_{13}=\left(\frac{m_{11}}{2a}\left(e^{-2a}-1\right)+\frac{m_6}{a}\left(e^{-a}-1\right)+\frac{m_1}{a}\left(1-e^a\right)+\frac{m_{12}}{2a}\left(e^{2a}-1\right)\right.$$

$$-\frac{a_6}{2}-\frac{a_7}{3}-\frac{a_8}{4}-\frac{a_9}{5}-\frac{a_{10}}{6}-\frac{a_{11}}{7}-\frac{a_{12}}{8}+\frac{a_{13}}{9}+\frac{a_{14}}{10}+\frac{m_7}{a^2}\left(e^{-a}+ae^{-a}-1\right)+\frac{m_2}{a^2}\left(e^a-1-ae^a\right)$$

$$\frac{2m_8}{a^3}\left(-1+e^{-a}+ae^{-a}+\frac{a^2e^{-a}}{2}\right)+\frac{2m_3}{a^3}\left(1-e^a+ae^a+\frac{a^2e^a}{2}\right)$$

$$+\frac{6m_9}{a^4}\left(-1+e^{-a}+ae^{-a}+\frac{a^2e^{-a}}{2}+\frac{a^3e^{-a}}{6}\right)+\frac{6m_4}{a^4}\left(-1+e^{-a}-ae^{-a}+\frac{a^2e^a}{2}-\frac{a^3e^a}{6}\right)$$

$$+\frac{24m_{10}}{a^5}\left(-1+e^{-a}+ae^{-a}+\frac{a^2e^{-a}}{2}+\frac{a^3e^{-a}}{6}+\frac{a^4e^{-a}}{24}\right)$$

$$\left.+\frac{24m_5}{a^5}\left(-1+e^a+ae^a+\frac{a^2e^a}{2}+\frac{a^3e^a}{6}-\frac{a^4e^a}{24}\right)\right);$$

$$m_{14}=\left(\frac{m_{11}}{2a}+\frac{m_6}{a}-\frac{m_1}{a}+\frac{m_{12}}{2a}+\frac{m_7}{a^2}+\frac{m_2}{a^2}+\frac{2m_8}{a^3}-\frac{2m_3}{a^3}+\frac{6m_9}{a^4}+\frac{6m_4}{a^4}+\frac{24m_{10}}{a^5}-\frac{24m_5}{a^5}\right);$$

$$b_1=\left(\frac{m_6}{a}+\frac{m_7}{a^2}+\frac{2m_8}{a^3}+\frac{6m_9}{a^4}+\frac{24m_{10}}{a^5}\right);b_2=\left(\frac{m_1}{a}-\frac{m_2}{a^2}+\frac{2m_3}{a^3}-\frac{6m_4}{a^4}+\frac{24m_5}{a^5}\right);$$

$$b_3=\left(\frac{m_7}{a}+\frac{2m_8}{a^2}+\frac{6m_9}{a^3}+\frac{24m_{10}}{a^4}\right);b_4=\left(\frac{m_8}{a}-\frac{3m_9}{a^2}+\frac{12m_{10}}{a^3}\right);$$

$$s_1=\left(\frac{m_9}{a}-\frac{4m_{10}}{a^2}\right);s_2=\frac{m_{10}}{a};b_5=\left(\frac{m_2}{a}-\frac{2m_3}{a^2}+\frac{6m_4}{a^3}-\frac{24m_5}{a^4}\right);b_6=\left(\frac{m_3}{a}-\frac{3m_4}{a^2}+\frac{12m_5}{a^3}\right);$$

$$b_7=\left(\frac{m_4}{a}-\frac{4m_5}{a^2}\right);s_3=\left(\frac{m_5}{a}\right);s_4=\left(\frac{m_{11}}{2a}\right);s_5=\left(\frac{m_{12}}{2a}\right);s_6=\left(\frac{R_e}{2}+N\frac{a_6}{2}\right);s_7=\left(\frac{R_e\alpha}{6}+N\frac{a_7}{3}\right);$$

$$s_8=\left(A_3+Nm_{13}\right);s_9=-\frac{c_4^2e^{2a}}{48a^2};s_{10}=-\frac{c_3^2e^{2a}}{48a^2};$$

$$g_1=\left(-\frac{1}{4a^{12}}\left(480a^4a_{10}+20160a^2a_{12}-141520a_{14}+8a^9\left(b_3+b_5\right)+96a^7\left(s_1+b_7\right);\right.\right.$$

$$\left.\left.+a^{10}\left(4m_{14}+4b_1-4b_2+s_4+s_5\right)+4a^8\left(6b_4-6b_6+a_6\right)+24a^6\left(20as_2-20s_3+a_8\right)\right)\right);$$

$$b_8=\left(\frac{1}{2520a^{12}\left(-1+e^{2a}\right)}e^a\left(-914457600a_{14}e^{2a}\left(-1+e^a\right)+21a^{11}e^a\left(60b^1+30b^3\right)+20b_4+15s_1\right)\right.$$

$$+12s_2+60b_2+e^{2a}+30b_5e^{2a}+20b_6e^{2a}+15b_7e^{2a}+12s_3e^{2a}\big)+302400\,a^4e^{2a}\big(-6a^{11}-21a_{12}$$
$$+56a_{13}+126a_{14}+a_{10}\big(-1+e^a\big)\big)+12700800a^2e^{2a}\big(8a_{13}+36a_{14}+a_{12}\big(-1+e^a\big)\big)$$
$$+315a^8e^a\big(3s_1+6s_2-40a_{10}e^a-48a_{11}e^a-56a_{12}e^a+64a_{13}e^a+72a_{14}e^a-8a_6e^a$$
$$-16a_7e^a-24a_8e^a-32a_9e^a-3b_7e^{2a}-6s_3e^{24}+8a_6e^{2a}+48b_4\big(-1+e^{2a}\big)\big)$$
$$+1890a^7e^a\big(s_2+s_3e^{2a}+32s_1\big(-1+e^{2a}\big)\big)+15120\,a^6e^a\big(e^a-10a_{10}-20a_{11}-35a_{12}+56a_{13}+$$
$$84a_{14}-a_8-4a_9+a_8e^a\big)+20s_2\big(-1+e^{2a}\big)\big)+a^{10}\big(840s^4\big(-1+e^{3a}\big)-e^a\big(-630b_3$$
$$-630b_4-630s_1-630s_2+420a_{10}e^a+360a_{11}e^a+315a_{12}e^a-280a_{13}e^a-252a_{14}e^a$$
$$+2520m_{13}e^a+2520m^{14}e^a+1260a_6e^a+840a_7e^a+630a_8e^a+504a_9e^a-2520m_{14}e^{2a}$$
$$+630b_5e^{2a}+630b_6e^{2a}+630b_7e^{2a}+630s_3e^{2a}-840s_5e^{2a}+840s_5e^{3a}-2520b_1\big(-1+e^{2a}\big)\big)\big)\big)\big);$$

$$b_9=\bigg(\frac{1}{2520a^{12}\big(-1+e^{2a}\big)}e^a\big(-914457600a_{14}e^{2a}\big(-1+e^a\big)-315a^8e^a\big(3s^1+6s_2-48b_6+8a_6$$
$$-40a_{10}e^a-48a_{11}e^a-56a_{12}e^a+64a_{13}e^a+72a_{14}e^a-8a_6e^a-16a_7e^a-24a_8e^a$$
$$-32a_9e^a+48b_6e^{2a}-3b_7e^{2a}-6s_3e^{2a}\big)+315a^9e^a\big(-2b_4-3s_1-4s_2-16b_5+16b_5e^{2a}$$
$$-2b_6e^{2a}-3b_7e^{2a}-4s_3e^{2a}\big)-21a^{11}e^a\big(60b_1+30b_3+20b_4+15s_1+12s_2+60b_2e^{2a}$$
$$+30b_5e^{2a}+20b_6e^{2a}+15b_7e^{2a}+12s_3e^{2a}\big)+302400a^4e^a\big(\big(6a_{11}+7\big(3a_{12}-8a_{13}-18a_{14}\big)\big)e^a$$
$$+a_{10}\big(-1+e^a\big)\big)+12700800a^2e^a\big(-4\big(2a_{13}+9a_{14}\big)e^a+a_{12}\big(-1+e^a\big)\big)-1890a^7e^a\big(s_2$$
$$+s_3e^{2a}-32b_7\big(-1+e^{2a}\big)\big)-15120a^6e^a\big(a_8-a_8e^a-\big(10a_{10}+20a_{11}+35a_{12}-56a_{13}\big)$$
$$-84a_{14}+4a_9\big)+20s_3\big(-1+e^{2a}\big)\big)+a^{10}\big(-840s_4\big(-1+e^a\big)+e^a-630b_3-630b_4$$
$$-630s_1-630s_2-840s_5+420a_{10}e^a+360a_{11}e^a+315a_{12}e^a-280a_{13}e^a-252a_{14}e^a$$
$$+2520m_{13}e^a+1260a_6e^a+840a_7e^a+630a_8e^a+504a_9e^a+630b_5e^{2a}+630b_6e^{2a}+630b_7e^{2a}$$
$$+630s_3e^{2a}+840s_5e^{3a}+2520m_{14}\big(-1+e^a\big)-2520b_2\big(-1+e^{2a}\big)\big)\big)\big);$$

$$b_{10}=\frac{1}{4a^6}\big(-\big(-627aa_2-3438a_3-18a^3\big(a_0+a_4\big)+98a^2\big(a_1-a_5\big)+4a^4c_1\big)c_4e^{2a}\big);$$

$$b_{11}=\frac{1}{4a^6}\big(-\big(-627aa_2-4614a_3-18a^3\big(a_0+a_4\big)+98a^2\big(a_1-a_5\big)+4a^4c_1\big)c_3e^{2a}\big);$$

$$b_{12}=\bigg(\frac{1}{5544a^{10}}e^{-2a}\big(-39916800a^2a_{11}e^{2a}+2235340800a_{13}e^{2a}+a^{10}\big(165a_{10}+110a_{11}+77a_{12}$$
$$-56a_{13}-42a_{14}+9240m_{13}+27720m_{14}+2310a_6+924a_7+462a_8+264a_9\big)e^{2a}$$
$$-110880\,a^7e^a\big(b_3+2b_4+3s_1+4s_2-b_3e^a-b_5e^a+b_5e^{2a}+2b_6e^{2a}+3b_7e^{2a}+4s_3e^{2a}\big)$$
$$+110880a^6\big(-9s_1-18s_2-3b_6e^a-a_7e^a+3b_6e^{2a}+9b_7e^{2a}+18s_3e^{2a}+3b_4\big(-1+e^a\big)\big)$$
$$+13860\,a^8\big(s_4\big(-1+e^{2a}\big)+e^a\big(-4b_3-4b_4-4s_1-4s_2-4m_{13}e^a-4b_2e^a+s_5e^a+4b_2e^{2a}+4b_5e^{2a}$$
$$+4b_6e^{2a}+4b_7e^{2a}+4s_3e^{2a}-s_5e^{3a}+b_1\big(-1+e^a\big)\big)\big)-1330560a^5e^a\big(s_1+4s_2$$
$$-s_1e^a+e^a\big(4s_3e^a+b_7\big(-1+e^a\big)\big)\big)+1330560a^4e^a\big(5s_2\big(-1+e^a\big)+e^a\big(-a_9+5s_3\big(-1+e^a\big)\big)\big)\big);$$

$$b_{13}=\frac{1}{4a^5}\left(\left(147aa_2+846a_3+2a^3(a_0+a_4)-18a^2(a_1-a_5)\right)c_4e^{2a}\right);$$

$$b_{14}=\frac{1}{4a^5}\left(\left(147aa_2-1254a_3+2a^3(a_0+a_4)-18a^2(a_1-a_5)\right)c_3e^{2a}\right);$$

$$b_{15}=\frac{1}{2a^{10}}\left(-120a^4a_{10}+5040a^2a_{12}-362880a_{14}+a^{10}m_{14}+a^8a_6+6a^6a_8\right);$$

$$g_2=\frac{1}{12a^3}\left(\left(2a^2b_4+15as_1+102s_2\right)e^{2a}\right);\ g_3=\frac{1}{12a^3}\left(\left(2a^2b_6-15ab_7+102s_3\right)e^{2a}\right);$$

$$g_4=\frac{1}{6a^8}\left(-720a^2a_{11}-40320a_{13}+a^8m_{13}+2a^6a_7+24aa_9\right);\ g_5=\left(\frac{(as_1+10s_2)e^{2a}}{8a^2}\right);$$

$$g_6=\left(\frac{\left(ab_7-10s_3e^{2a}\right)}{8a^2}\right);\ g_7=\left(-\frac{5a_{10}}{a^4}-\frac{210a_{12}}{a^6}+\frac{15120a_{14}}{a^8}-\frac{a_6}{24}-\frac{a_8}{4a^2}\right);$$

$$g_8=\left(\frac{s_2e^{2a}}{10a}\right);\ g_9=\left(\frac{s_3e^{2a}}{10a}\right);\ g_{10}=\left(-\frac{6a_{11}}{a^4}+\frac{336a^{13}}{a^6}-\frac{a_7}{60}-\frac{a_9}{8a^2}\right);\ g_{11}=\left(-\frac{a_{10}}{6a^2}-\frac{7a_{12}}{a^4}+\frac{504a_{14}}{a^6}-\frac{a_8}{120}\right);$$

$$g_{12}=\left(-\frac{a_{11}}{7a^2}+\frac{8a_{13}}{a_4}-\frac{a_9}{210}\right);\ g_{13}=\left(-\frac{a_{10}}{336}-\frac{a_{12}}{8a^2}+\frac{9a_{14}}{a^4}\right);\ g_{14}=\left(-\frac{a_{11}}{504}+\frac{a_{13}}{9a^2}\right);$$

$$g_{15}=\left(-\frac{a_{12}}{720}+\frac{a_{14}}{10a^2}\right);\ g_{16}=\left(\frac{a_{13}}{990}\right);\ g_{17}=\left(\frac{a_{14}}{1320}\right);$$

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Cartesian Product Over Antagonistic Intuitionistic Fuzzy H-Ideals In Division BG-Algebras

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Abstract*: The purpose of this paper is to define the notion of a Cartesian product of a antagonistic intuitionistic fuzzy H-ideals in division BG-algebras, and to study some related properties of it. Eventually we found that an antagonistic intuitionistic fuzzy H-ideal of BG-algebras is a Cartesian product over antagonistic intuitionistic fuzzy H-ideal in division BG-algebras.*

Keywords*: BG-Division algebra, intuitionistic fuzzy ideal, intuitionistic fuzzy H-ideal, antagonistic fuzzy H-ideal, antagonistic intuitionistic fuzzy H-ideal, Cartesian product over antagonistic intuitionistic fuzzy H-ideal.*

I. Introduction

In 1965, Zadeh introduced the notion of a fuzzy subset of a set [12]. The concept of intuitionistic fuzzy subsets was introduced by Atanassov [1] in1986, which is a generalization of the notion of fuzzy sets. In 1966, Imai and Iseki [6] introduced two classes of abstract algebra: *BCK*-algebras and *BCI*-algebras. It is known that the class of *BCK*-algebra is a proper subclass of the class of *BCI*-algebras. Neggers and Kim [9] introduced a new notion, called a *B*-algebras which is related to several classes of algebras such as *BCI/BCK*-algebras. Kim and Kim [10] introduced the notion of *BG*-algebras, which is a generalization of *B*-algebras. Zarandi and Saeid [13] presented intuitionistic fuzzy ideals of *BG*-algebras. In 2012 Senapati, Bhowmik and Pal [11] presented the concept and basic properties of intuitionistic fuzzy ideals in *BG*- algebras. Jin Bai Kim, Young Hee Kim, Chang Bum Kim[7] presented Resolution of the Cartesian Product of fuzzy sets. K.A.Dib Nabil L.Youssef [4] presented Fuzzy Cartesian product and fuzzy relations and fuzzy functions.

In this paper, some extended result of Cartesian product over intuitionistic fuzzy ideal of *BG*-algebras is presented based on intuitionistic fuzzy sets (IFSs), and obtain some results on them. At the same time, the notion of equivalence relations on the family of all intuitionistic fuzzy ideals of a *BG*-algebra are presented and investigated some related properties. The Cartesian product of intuitionistic fuzzy *BG*-algebra has been introduced and some important properties are of it are also studied.

The paper is organized as follows. The following section briefly reviews some basic definitions related to *BG*-algebra, *BG*-sub algebra, division BG-algebra, intuitionistic fuzzy set, intuitionistic fuzzy H-ideals, antagonistic intuitionistic fuzzy set, antagonistic intuitionistic fuzzy H-ideal. In Section 3, we propose the concepts and operations of Cartesian product over antagonistic intuitionistic fuzzy ideal in division BG-algebra and discuss their properties in detail. Finally, in Section 4, we draw the conclusion and present some topics for future research.

II. Preliminaries

Definition2.1: A division BG-algebra is a non-empty set X with a constant 1 and a binary operation $\div$ satisfying the following axioms:.

(i) $x \div x = 1$,

(ii) $x \div 1 = x$,

(iii) $(x \div y) \div (1 \div y) = x \; \forall x, y \in X$.

Example : 1.The set of all Complex numbers is a division BG-algebras.

Example: Let X={0,1,2,3,4} with the following Cayley table:

Table 1 : Cayley table for division BG-algebra.

$\div$	0	1	2	3	4
0	1	0	0	0	0
1	1	1	1	1	1
2	2	2	1	2	2

3	3	3	3	1	3
4	4	4	4	4	1

Then (X, ÷, 1) is a division BG- algebra.

Definition 2.2 : If a division BG-algebra satisfies $(x \div y) \div z = x \div (y \div z)$ for all $x, y, z \in X$ then it is called associative.
Throughout this paper, X always means a division BG-algebra without any specification.

Definition 2.3: A non-empty subset I of a division BG-algebra X is called an ideal of X if
(i) $1 \in I$
(ii) $x \div y \in I$ and $y \in I$ then $x \in I$, for all $x, y \in X$.

Definition 2.4 : A non-empty subset I of a division BG-algebra X is said to be a H-ideal of X if
(i) $1 \in I$
(ii) $x \div (y \div z) \in I$ and $y \in I$ then $x \div z \in I$, for all $x, y, z \in X$.

Definition 2.5 : A fuzzy set $A = \{< x, \sigma_A(x)> : x \in X\}$ in X is called a fuzzy H-ideal of X if
(i) $\sigma_A(1) \geq \sigma_A(x)$
(ii) $\sigma_A(x \div z) \geq \max \{\sigma_A\{(x \div (y \div z)), \sigma_A(y)\}$ for all $x, y, z \in X$.

Definition 2.6 : An intuitionistic fuzzy set $A = (\sigma_A, \tau_A)$ in X is called an intuitionistic fuzzy ideal of X, if it satisfies the following axioms:
(i) $\sigma_A(1) \geq \sigma_A(x), \tau_A(1) \leq \tau_A(x)$,
(ii) $\sigma_A(x) \geq \min \{\sigma_A(x \div y), \sigma_A(y)\}$
(iii) $\tau(x) \leq \max \{\tau(x \div y\}, \tau_A(y)\}$ for all $x, y \in X$.

Definition 2.7 : An intuitionistic fuzzy set $A = (\sigma_A, \tau_A)$ in X is called an intuitionistic fuzzy H-ideal of X, if it satisfies the following axioms:
(i) $\sigma_A(1) \geq \sigma_A(x), \tau_A(1) \leq \tau_A(x)$,
(ii) $\sigma_A(x \div z) \geq \min\{\sigma_A(x \div (y \div z)), \sigma_A(y)\}$,
(iii) $\tau_A(x \div z) \leq \max(\tau_A(x \div (y \div z)), \tau_A(y))$, for all $x, y, z \in X$.

Definition 2.8: A fuzzy set $A = \{<x, \sigma_A(x)>: x \in X\}$ in X is called a antagonistic fuzzy sub algebra of X if
$\sigma_A(x \div y) \leq \max \{\{\sigma_A(x), \sigma_A(y)\}$, for all $x, y \in X$.

Definition 2.9 : A fuzzy set $A = \{<x, \sigma_A(x)> : x \in X\}$ in X is called a antagonistic fuzzy ideal of X if
(i) $\sigma_A(1) \leq \sigma_A(x)$
(ii) $\sigma_A(x) \leq \max\{\sigma_A(x \div y), \sigma_A(y)\}$, for all $x, y \in X$.

Definition 2.10: An intuitionistic fuzzy set $A = (\sigma_A, \tau_A)$ in a division BG-algebra X is called a antagonistic intuitionistic fuzzy ideal if
(i) $\sigma_A(1) \leq \sigma_A(x); \tau_A(1) \geq \tau_A(x)$
(ii) $\sigma_A(x) \leq \max\{\sigma_A(x \div y), \sigma_A(y)\}$
(iii) $\tau_A(x) \geq \min \{\tau_A(x \div y), \tau_A(y)\}$, for all $x, y \in X$.

III. Cartesian products over Antagonistic intuitionistic fuzzy H-ideals:

Definition 3.1. If A and B be two antagonistic intuitionstic fuzzy sets in X be a division BG-algebras X. Then the Cartesian products of two antagonistic intuitionstic fuzzy sets A and B are defined as follows:

1. The Cartesian product $\times_1$
   $A \times_1 B = \{<<x, y>, \sigma_A(x). \sigma_B(y), \tau_A(x).\tau_B(y)>/x, y \in X\}$
2. The Cartesian product $\times_2$

$A \times_2 B = \{<<x, y>, \sigma_A(x)+\sigma_B(y) -\sigma_A(x)\ \sigma_B(y), \tau_A(x).\tau_B(y)>/x, y \in X\}$

3. The Cartesian product $\times_3$

   $A \times_3 B = \{<<x, y>, \sigma_A(x) + \sigma_B(y) , \tau_A(x)+ \tau_B(y)- \tau_A(x).\ \tau_B(y)>/x, y \in X\}$

4. The Cartesian product $\times_4$

   $A \times_4 B = \{<<x, y>, \min (\sigma_A(x), \sigma_B(y)),\max(\tau_A(x), \tau_B(y))>/x, y \in X\}$

5. The Cartesian product $\times_5$

   $A \times_5 B = \{<<x, y>, \max (\sigma_A(x), \sigma_B(y)),\min(\tau_A(x), \tau_B(y)>/x, y \in X\}$

Thoerem3.1 : If A and B are antagonistic- intuitionistic fuzzy H-ideals in division BG-algebras X, then the Cartesian product of $A \times_1 B$ is also an antagonistic- intuitionistic fuzzy H-ideals in division BG-algebras X.

Proof: Let A and B are antagonistic- intuitionistic fuzzy H-ideals in division BG-algebras X.

Then

(i) $\sigma_A(1) \le \sigma_A(x)$, $\tau_A(1) \ge \tau_A(x)$ and $\sigma_B(1) \le \sigma_B(x)$, $\tau_B(1) \ge \tau_B(x)$

Now, $\sigma_A(1)\cdot\ \sigma_B(1) \le \sigma_A(x)\cdot\ \sigma_B(x)$

$\sigma_{A \times_1 B}(1) \le \sigma_{A \times_1 B}(x)$, for all $x \in X$. And

$\tau_A(1)\cdot\ \tau_B(1) \le \tau_A(x)\cdot\ \tau_B(x)$

$\tau_{A \times_1 B}(1) \le \tau_{A \times_1 B}(x)$, for all $x \in X$.

(ii) $\sigma_A(x \div z) \le \max\{\sigma_A(x \div (y \div z)), \sigma_A(y)\}$

$\sigma_B(x \div z) \le \max\{\sigma_B(x \div (y \div z)), \sigma_B(y)\}$

$\sigma_A(x \div z)\cdot\ \sigma_B(x \div z) \le \max\{\sigma_A(x \div (y \div z)), \sigma_A(y)\}\cdot \max\{\sigma_B(x \div (y \div z)), \sigma_B(y)\}$

$\sigma_{A \times_1 B}(x \div z) \le \max\{\sigma_A(x \div (y \div z))\cdot \sigma_B(x \div (y \div z)\}, \max\{\sigma_A(y)\cdot\ \sigma_B(y)\}$

$\le \max\{\sigma_{A \times_1 B}((x \div (y \div z)), \sigma_{A \times_1 B}(y)\}$

(iii) $\tau_A(x \div z) \ge \text{mini}\{\tau_A(x \div (y \div z)), \tau_A(y)\}$

$\tau_B(x \div z) \ge \text{mini}\{\tau_B(x \div (y \div z)), \tau_B(y)\}$

$\tau_A(x \div z)\cdot\ \tau_B(x \div z) \ge \text{mini}\ \tau_A(x \div (y \div z)), \tau_A(y)\}\cdot \text{mini}\{\tau_B(x \div (y \div z)), \tau_B(y)\}$

$\tau_{A \times_1 B}(x \div z) \ge \text{mini}\{\tau_A(x \div (y \div z))\cdot\ \tau_B(x \div (y \div z)\}, \text{mini}\{\tau_A(y)\cdot\ \tau_B(y)\}$

$\ge \text{mini}\{\tau_{A \times_1 B}((x \div (y \div z)), \tau_{A \times_1 B}(y)\}$

Example: Let X={0,1,2,3,4} with the following Cayley table:

Table 1 : Cayley table for division BG-algebra.

÷	0	1	2	3	4
0	1	0	0	0	0
1	0	1	1	1	1
2	2	2	1	2	2
3	3	3	3	1	3
4	4	4	4	4	1

We define an antagonistic- intuitionistic fuzzy set A, then $A = < x, \sigma_A, \tau_A >$ by routine calculation A is antagonistic intuitionstic fuzzy H-ideal in division BG-algebras X.

X	0	1	2	3	4
σ_A	.73	.40	.66	.60	.50
τ_A	.36	.58	.33	.38	.42

We define an antagonistic- intuitionistic fuzzy set A, then $A = < x, \sigma_A, \tau_A >$ by routine calculation B is antagonistic intuitionstic fuzzy H-ideal in division BG-algebras X.

X	0	1	2	3	4
σ_B	.63	.35	.62	.52	.42
τ_B	.35	.54	.32	.36	.40

If Aand B are antagonistic intuitionstic fuzzy H-ideal of division BG-algebras X. Obvisouly,the Cartesian product $A \times_1 B$. Since x=0, y=1, z=3

(i) $\sigma_A(1) \le \sigma_A(0)$, $\tau_A(1) \ge \tau_A(0)$ and $\sigma_B(1) \le \sigma_B(0)$, $\tau_B(1) \ge \tau_B(0)$
Now, $\sigma_A(1)\cdot \sigma_B(1) \le \sigma_A(0)\cdot \sigma_B(0)$
.40•.35≤ .73•.63
.14 ≤ .4599
$\tau_A(1)\cdot \tau_B(1) \ge \tau_A(0)\cdot \tau_B(0)$
.58•.54 ≥ .36•.35
.3132 ≥ .126
(ii)$\sigma_A(0 \div 3) \le \max\{\sigma_A(0 \div (1 \div 3)), \sigma_A(1)\}$
$\sigma_B(0 \div 3) \le \max\{\sigma_B(0 \div (1 \div 3)), \sigma_B(1)\}$
$\sigma_A(0 \div 3)\cdot \sigma_B(0 \div 3) \le \max\{\sigma_A(0 \div (1 \div 3)), \sigma_A(1)\}\cdot \max\{\sigma_B(0 \div (1 \div 3)), \sigma_B(1)\}$
$\sigma_A(0)\cdot \sigma_B(0) \le \max\{\sigma_A(0 \div 1), \sigma_A(1)\}\cdot \max\{\sigma_B(0 \div 1), \sigma_B(1)\}$
$\le \max\{\{\sigma_A(0) \bullet \sigma_B(0)\}, \{\sigma_A(1) \bullet \sigma_B(1)\}\}$
.73•.63 ≤ max {.73• 63,40•.35}
.4599 ≤ .max {.4599, .14}
.4599 ≤ .4599
(iii) $\tau_A(0 \div 3)\cdot \tau_B(0 \div 3) \ge \text{mini}\{\tau_A(0 \div (1 \div 3)), \tau_A(1)\}\cdot \text{mini}\{\tau_B(0 \div (1 \div 3)), \tau_B(2)\}$
$\tau_A(0)\cdot \tau_B(0) \ge \text{mini}\{\tau_A(0 \div 1), \tau_A(1)\}\cdot \text{mini}\{\tau_B(0 \div 1), \tau_B(1)\}$
$\ge \text{mini}\{\tau_A(0), \tau_A(1)\}\cdot \text{mini}\{\tau_B(0), \tau_B(1)\}$
$\ge \text{mini}\{\tau_A(0) \bullet \tau_B(0)\}, \{\tau_A(0) \bullet \tau_B(1)\}\}$
.36•.35 ≥ mini {.36•.35}, {.36•.54}
.126 ≥ mini {.126 .1944}
.126 ≥ .126

If A and B are antagonistic- intuitionistic fuzzy H-ideals in division BG-algebras X, the Cartesian product of A $\times_1$ B is also an antagonistic- intuitionistic fuzzy H-ideals in division BG-algebras X.

Thoerem3.2 : If A and B are antagonistic- intuitionistic fuzzy H-ideals in division BG-algebras X, then the Cartesian product of A $\times_2$ B is also an antagonistic- intuitionistic fuzzy H-ideals in division BG-algebras X.
Proof: Let A and B are antagonistic- intuitionistic fuzzy H-ideals in division BG-algebras X.
Then i
(i) $\sigma_A(1) \le \sigma_A(x)$, $\tau_A(1) \ge \tau_A(x)$ and $\sigma_B(1) \le \sigma_B(x)$, $\tau_B(1) \ge \tau_B(x)$
Now, $\sigma_A(1) + \sigma_B(1) - \sigma_A(1)\cdot \sigma_B(1) \le \sigma_A(x) + \sigma_B(x) - \sigma_A(x)\cdot \sigma_B(x)$
$\sigma_{A \times_2 B}(1) \le \sigma_{A \times_2 B}(x)$, for all x ∈ X. And
$\tau_A(1)\cdot \tau_B(1) \le \tau_A(x)\cdot \tau_B(x)$
$\tau_{A \times_2 B}(1) \le \tau_{A \times_2 B}(x)$, for all x∈ X.
(ii) $\sigma_A(x \div z) \le \max\{\sigma_A(x \div (y \div z)), \sigma_A(y)\}$
$\sigma_B(x \div z) \le \max\{\sigma_B(x \div (y \div z)), \sigma_B(y)\}$
$\sigma_A(x \div z) + \sigma_B(x \div z) - \sigma_A(x \div z)\cdot \sigma_B(x \div z) \le \max\{\sigma_A(x \div (y \div z)), \sigma_A(y)\} + \max\{\sigma_B(x \div (y \div z)), \sigma_B(y)\} - \max\{\sigma_A(x \div (y \div z)), \sigma_A(y)\}\cdot \max\{\sigma_B(x \div (y \div z)), \sigma_B(y)\}$
$\sigma_{A \times_2 B}(x \div z) \le \max\{\sigma_A(x \div (y \div z)) + \sigma_B(x \div (y \div z) - \sigma_A(x \div (y \div z)) . \sigma_B(x \div (y \div z)\}$, $\max\{\sigma_A(y) + \sigma_B(y) - \sigma_A(y)\cdot \sigma_B(y)\}$
$\sigma_{A \times_2 B}(x \div z) \le \max\{\{\sigma_A(x \div (y \div z)) + \sigma_B(x \div (y \div z) - \sigma_A(x \div (y \div z)) . \sigma_B(x \div (y \div z)\}, \{\sigma_A(y) + \sigma_B(y) - \sigma_A(y)\cdot \sigma_B(y)\}\}$
$\sigma_{A \times_2 B}(x \div z) \le \max\{\sigma_{A \times_2 B}((x \div (y \div z)), \sigma_{A \times_2 B}(y)\}$
(iii) $\tau_A(x \div z) \ge \text{mini}\{\tau_A(x \div (y \div z)), \tau_A(y)\}$
$\tau_B(x \div z) \ge \text{mini}\{\tau_B(x \div (y \div z)), \tau_B(y)\}$
$\tau_A(x \div z)\cdot \tau_B(x \div z) \ge \text{mini}\ \tau_A(x \div (y \div z)), \tau_A(y)\}\cdot \text{mini}\{\tau_B(x \div (y \div z)), \tau_B(y)\}$

$\sigma_{A \times_2 B}(x \div z) \ge \text{mini}\{\tau_A(x \div (y \div z))\cdot \tau_B(x \div (y \div z)\}, \text{mini}\{\tau_A(y)\cdot \tau_B(y)\}$
$\ge \text{mini}\{\{\tau_A(x \div (y \div z))\cdot \tau_B(x \div (y \div z)\}, \{\tau_A(y)\cdot \tau_B(y)\}\}$
$\ge \text{mini}\{\tau_{A \times_2 B}((x \div (y \div z)), \tau_{A \times_2 B}(y)\}$

Example: Let X={0,1,2,3,4} with the following Cayley table:
Table 1 : Cayley table for division BG-algebra.

÷	0	1	2	3	4
0	1	0	0	0	0

1	0	1	1	1	1
2	2	2	1	2	2
3	3	3	3	1	3
4	4	4	4	4	1

We define an antagonistic- intuitionistic fuzzy set A, then $A = < x, \sigma_A, \tau_A >$ by routine calculation A is antagonistic intuitionstic fuzzy H-ideal in division BG-algebras X.

X	0	1	2	3	4
σ_A	.73	.40	.66	.60	.50
τ_A	.36	.58	.33	.38	.42

We define an antagonistic- intuitionistic fuzzy set B, then $B = < x, \sigma_B, \tau_B >$ by routine calculation B is antagonistic intuitionstic fuzzy H-ideal in division BG-algebras X.

X	0	1	2	3	4
σ_B	.63	.35	.62	.52	.42
τ_B	.35	.54	.32	.36	.40

If A and B are antagonistic intuitionstic fuzzy H-ideal of division BG-algebras X. Obvisouly, the Cartesian product $A \times_2 B$. Since x = 2,y = 3,z = 1

Now $\sigma_A(1) \le \sigma_A(2)$, $\tau_A(1) \ge \tau_A(2)$ and $\sigma_B(1) \le \sigma_B(2)$, $\tau_B(1) \ge \tau_B(2)$

(i), $\sigma_A(1) + \sigma_B(1) - \sigma_A(1) \cdot \sigma_B(1) \le \sigma_A(2) + \sigma_B(2) - \sigma_A(2) \cdot \sigma_B(2)$

$.40 + .35 - .40 \bullet .35 \le .66 + .62 - .66 \bullet .62$

$.75 - .14 \le 1.28 - .4092$

$.61 \le .8708$

$\tau_A(1) \cdot \tau_B(1) \ge \tau_A(2) \cdot \tau_B(2)$

$.58 \bullet .54 \ge .33 \bullet .32$

$.3132 \ge .1056$

(ii) $\sigma_A(2 \div 1) + \sigma_B(2 \div 1) - \sigma_A(2 \div 1) \cdot \sigma_B(2 \div 1) \le \text{Max}\{\sigma_A(2 \div (3 \div 1)), \sigma_A(3)\} + \text{Max}\{\sigma_B(2 \div (3 \div 1)), \sigma_B(3)\} - \text{Max}\{\sigma_A(2 \div (3 \div 1)), \sigma_A(3)\} + \text{Max}\{\sigma_B(2 \div (3 \div 1)), \sigma_B(3)\}$

$\sigma_A(2) + \sigma_B(2) - \sigma_A(2) \cdot \sigma_B(2) \le \text{Max}\{\sigma_A(2) + \sigma_B(2) - \sigma_A(3) \cdot \sigma_B(3)\}, \text{Max}\{\sigma_B(3) + \sigma_B(3) - \sigma_A(3) \cdot \sigma_B(3)\}$

$.66 + .62 - .66 \bullet .62 \le \max\{.66 + .62 - .66 \bullet .62, .60 + .52 - .60 \bullet .52\}$

$0.8708 \le \max\{0.8708, 0.808\}$

$0.8708 \le 0.8708$

(iii) $\tau_A(2 \div 1) \cdot \tau_B(2 \div 1) \ge \text{mini}\{\{\tau_A(2 \div (3 \div 1)), (3)\} \cdot \text{mini}\{\tau_B(2 \div (3 \div 1))\}$

$.66 + .62 - .66 \bullet .62 \ge \text{mini}\{\tau_A(2 \div (3 \div 1)), \{\tau_A(2 \div (3 \div 1)), \text{mini}\{\tau_A(3), \tau_B(3)\}$

$0.8708 \ge \text{mini}\{0.8708, 0.808\}$

$0.8708 \ge 0.808$

If A and B are antagonistic- intuitionistic fuzzy H-ideals in division BG-algebras X, the Cartesian product of $A \times_2 B$ is also an antagonistic- intuitionistic fuzzy H-ideals in division BG-algebras X.

Thoerem3.3 : If A and B are antagonistic- intuitionistic fuzzy H-ideals in division BG-algebras X, then the Cartesian product of $A \times_3 B$ is also an antagonistic- intuitionistic fuzzy H-ideals in division BG-algebras X.

Proof: Let A and B are antagonistic- intuitionistic fuzzy H-ideals in division BG-algebras X.

Then

(i) $\sigma_A(1) \le \sigma_A(x)$, $\tau_A(1) \ge \tau_A(x)$ and $\sigma_B(1) \le \sigma_B(x)$, $\tau_B(1) \ge \tau_B(x)$

Now, $\sigma_A(1) \cdot \sigma_B(1) \le \sigma_A(x) \cdot \sigma_B(x)$

$\sigma_{A \times_1 B}(1) \le \sigma_{A \times_3 B}(x)$, for all $x \in X$. And

$\tau_A(1) + \tau_B(1) - \tau_A(1) \cdot \tau_B(1) \le \tau_A(x) + \tau_B(x) - \tau_A(x) \cdot \tau_B(x)$

$\tau_{A \times_3 B}(1) \le \tau_{A \times_3 B}(x)$, for all $x \in X$.

(ii) $\sigma_A(x \div z) \le \text{Max}\{\sigma_A(x \div (y \div z)), \sigma_A(y)\}$

$\sigma_B(x \div z) \le \text{Max}\{\sigma_B(x \div (y \div z)), \sigma_B(y)\}$

$\sigma_A(x \div z) \cdot \sigma_B(x \div z) \le \text{Max}\{\sigma_A(x \div (y \div z)), \sigma_A(y)\} \cdot \text{Max}\{\sigma_B(x \div (y \div z)), \sigma_B(y)\}$

$\sigma_{A \times_3 B}(x \div z) \le \text{Max}\{\sigma_A(x \div (y \div z)) \cdot \sigma_B(x \div (y \div z)\}, \text{Max}\{\sigma_A(y) \cdot \sigma_B(y)\}$

$\le \text{Max}\{\sigma_{A \times_3 B}((x \div (y \div z)), \sigma_{A \times_3 B}(y)\}$

(iv) $\tau_A(x \div z) \ge \text{mini}\{\tau_A(x \div (y \div z)), \tau_A(y)\}$

$\tau_B(x \div z) \ge \text{mini}\{\tau_B(x \div (y \div z)), \tau_B(y)\}$

$\tau_A(x \div z) + \tau_B(x \div z) - \tau_A(x \div z) \cdot \tau_B(x \div z) \geq \text{mini}(\tau_A(x \div (y \div z)), \tau_A(y)\} + \text{mini}\{\tau_B(x \div (y \div z)), \tau_B(y)\} - \text{mini}(\tau_A(x \div (y \div z), \tau_A(y)\}) \cdot \text{mini}(\tau_B(x \div (y \div z)), \tau_B(y)\}$

$\sigma_{A \times_3 B}(x \div z) \geq \text{mini}\{\tau_A(x \div (y \div z)) + \tau_B(x \div (y \div z) - \tau_A(x \div (y \div z)) \cdot \tau_B(x \div (y \div z)\},$

$\text{mini}\{\tau_A(y) + \tau_B(y) - \tau_A(y) \cdot \tau_B(y)\}$

$\sigma_{A \times_3 B}(x \div z) \geq \text{mini}\{\{\tau_A(x \div (y \div z)) + \tau_B(x \div (y \div z) - \tau_A(x \div (y \div z)) \cdot \tau_B(x \div (y \div z)\}, \{\tau_A(y) + \tau_B(y) - \tau_A(y) \cdot \tau_B(y)\}\}$

$\sigma_{A \times_3 B}(x \div z) \geq \text{mini}\{\tau_{A \times_3 B}((x \div (y \div z)), \tau_{A \times_3 B}(y)\}$

Example: Let X={0,1,2,3,4} with the following Cayley table:

Table 1 : Cayley table for division BG-algebra.

÷	0	1	2	3	4
0	1	0	0	0	0
1	0	1	1	1	1
2	2	2	1	2	2
3	3	3	3	1	3
4	4	4	4	4	1

We define an antagonistic- intuitionistic fuzzy set A, then $A = < x, \sigma_A, \tau_A >$ by routine calculation A is antagonistic intuitionstic fuzzy H-ideal in division BG-algebras X.

X	0	1	2	3	4
σ_A	.73	.40	.66	.60	.50
τ_A	.36	.58	.33	.38	.42

We define an antagonistic- intuitionistic fuzzy set B, then $B = < x, \sigma_B, \tau_B >$ by routine calculation B is antagonistic intuitionstic fuzzy H-ideal in division BG-algebras X.

X	0	1	2	3	4
σ_B	.63	.35	.62	.52	.42
τ_B	.35	.54	.32	.36	.40

If A and B are antagonistic intuitionstic fuzzy H-ideal of division BG-algebras X. Obvisouly, the Cartesian product $A \times_3 B$. Since x =1,y = 2,z = 4

Now i) $\sigma_A(1) \leq \sigma_A(1)$, $\tau_A(1) \geq \tau_A(1)$ and $\sigma_B(1) \leq \sigma_B(1)$, $\tau_B(1) \geq \tau_B(1)$

(i) $\sigma_A(1) \cdot \sigma_B(1) \leq \sigma_A(1) \cdot \sigma_B(1)$

$.40 \bullet .35 \leq .40 \bullet .35$

$0.14 \leq 0.14$

$\tau_A(1) + \tau_B(1) - \tau_A(1) \cdot \tau_B(1) \geq \tau_A(1) + \tau_B(1) - \tau_A(1) \cdot \tau_B(1)$

$.58 + .54 - .58 \bullet .54 \geq .58 + .54 - .58 \bullet .54$

$0.8068 \geq 0.8068$

(ii) $\sigma_A(1 \div 4) \cdot \sigma_B(1 \div 4) \leq \text{Max}\{\sigma_A(1 \div (2 \div 4)), \sigma_A(2)\} \cdot \text{Max}\{\sigma_B(1 \div (2 \div 4)), \sigma_B(2)\}$

$.40 \bullet .35 \leq \max\{.40 \bullet .35, .66 \bullet .62\}$

$0.14 \leq \max\{0.14, 0.4092\}$

$0.14 \leq 0.4092$

(iii) $\tau_A(1 \div 4) + \tau_B(1 \div 4) - \tau_A(1 \div 4) \cdot \tau_B(1 \div 4) \geq \text{mini}\{\{\tau_A(1 \div (2 \div 4)), \tau_A(2)\} + \text{mini}\{\tau_B(1 \div (2 \div 4)), \tau_B(2)\} - \text{mini}\{\{\tau_A(1 \div (2 \div 4)), \tau_A(2)\} \cdot \text{mini}\{\tau_B(1 \div (2 \div 4)), \tau_B(2)\}$

$\tau_A(1) + \tau_B(1) - \tau_A(1) \cdot \tau_B(1) \geq \text{mini}\{(\tau_A(1) + \tau_B(1) - \{\tau_A(1) \cdot \tau_B(1)), (\tau_A(2) + \tau_B(2) - \tau_A(2) \cdot \tau_B(2))\}$

$0.8068 \geq \text{mini}\{0.8068, 0.5444\}$

$0.8068 \geq 0.5444$

If A and B are antagonistic- intuitionistic fuzzy H-ideals in division BG-algebras X, the Cartesian product of $A \times_3 B$ is also an antagonistic- intuitionistic fuzzy H-ideals in division BG-algebras X.

Thoerem3.4 : If A and B are antagonistic- intuitionistic fuzzy H-ideals in division BG-algebras X, then the Cartesian product of $A \times_4 B$ is also an antagonistic- intuitionistic fuzzy H-ideals in division BG-algebras X.

Proof: Let A and B are antagonistic- intuitionistic fuzzy H-ideals in division BG-algebras X.

Then

(i) $\sigma_A(1) \leq \sigma_A(x)$, $\tau_A(1) \geq \tau_A(x)$ and $\sigma_B(1) \leq \sigma_B(x)$, $\tau_B(1) \geq \tau_B(x)$

Now, $\text{mini}\{\sigma_A(1), \sigma_B(1)\} \leq \text{mini}\{\sigma_A(x), \sigma_B(x)\}$

$\sigma_{A \times_4 B}(1) \leq \sigma_{A \times_4 B}(x)$, for all $x \in X$. And

$\max\{\tau_A(1), \tau_B(1)\} \leq \max\{\tau_A(x) \cdot \tau_B(x)\}$

$\tau_{A \times_4 B}(1) \leq \tau_{A \times_4 B}(x)$, for all $x \in X$.

(ii) $\sigma_A(x \div z) \leq \max\{\sigma_A(x \div (y \div z)), \sigma_A(y)\}$

$\sigma_B(x \div z) \leq \max\{\sigma_B(x \div (y \div z)), \sigma_B(y)\}$

$\min\{\sigma_A(x \div z), \sigma_B(x \div z)\} \leq \min\{\max\{\sigma_A(x \div (y \div z)), \sigma_A(y)\} \cdot \max\{\sigma_B(x \div (y \div z)), \sigma_B(y)\}$

$\sigma_{A \times_4 B}(x \div z) \leq \min\{\max\{\sigma_A(x \div (y \div z)), \sigma_B(x \div (y \div z)\}, \max\{\sigma_A(y), \sigma_B(y)\}\}$

$\sigma_{A \times_4 B}(x \div z) \leq \min\{\max\{\{\sigma_A(x \div (y \div z)), \sigma_B(x \div (y \div z)\}, \{\sigma_A(y), \sigma_B(y)\}\}\}$

If $A \subset B$ or $B \subset A$

$\sigma_{A \times_4 B}(x \div z) \leq \max\{\min\{\{\sigma_A(x \div (y \div z)), \sigma_B(x \div (y \div z)\}, \{\sigma_A(y), \sigma_B(y)\}\}\}$

$\sigma_{A \times_4 B}(x \div z) \leq \max\{\sigma_{A \times_4 B}((x \div (y \div z)), \sigma_{A \times_4 B}(y)\}$

(iii) $\tau_A(x \div z) \geq \min\{\tau_A(x \div (y \div z)), \tau_A(y)\}$

$\tau_B(x \div z) \geq \min\{\tau_B(x \div (y \div z)), \tau_B(y)\}$

$\max(\tau_A(x \div z), \tau_B(x \div z)) \geq \max\{\min\{\tau_A(x \div (y \div z)), \tau_A(y)\}, \min\{\tau_B(x \div (y \div z)), \tau_B(y)\}\}$

$\tau_{A \times_4 B}(x \div z) \geq \max\{\min\{\tau_A(x \div (y \div z)), \tau_B(x \div (y \div z)\}, \min\{\tau_A(y), \tau_B(y)\}\}$

$\tau_{A \times_4 B}(x \div z) \geq \max\{\min\{\{\tau_A(x \div (y \div z)), \tau_B(x \div (y \div z)\}, \{\tau_A(y), \tau_B(y)\}\}\}$

If $A \subset B$ or $B \subset A$

$\tau_{A \times_4 B}(x \div z) \geq \min\{\max\{\{\tau_A(x \div (y \div z)), \tau_B(x \div (y \div z)\}, \{\tau_A(y), \tau_B(y)\}\}\}$

$\tau_{A \times_4 B}(x \div z) \geq \min\{\tau_{A \times_4 B}((x \div (y \div z)), \tau_{A \times_4 B}(y)\}$

Example: Let X={0,1,2,3,4} with the following Cayley table:

Table 1 : Cayley table for division BG-algebra.

÷	0	1	2	3	4
0	1	0	0	0	0
1	0	1	1	1	1
2	2	2	1	2	2
3	3	3	3	1	3
4	4	4	4	4	1

We define an antagonistic- intuitionistic fuzzy set A, then $A = < x, \sigma_A, \tau_A >$ by routine calculation A is antagonistic intuitionstic fuzzy H-ideal in division BG-algebras X.

X	0	1	2	3	4
σ_A	.73	.40	.66	.60	.50
τ_A	.36	.58	.33	.38	.42

We define an antagonistic- intuitionistic fuzzy set B, then $B = < x, \sigma_B, \tau_B >$ by routine calculation B is antagonistic intuitionstic fuzzy H-ideal in division BG-algebras X.

X	0	1	2	3	4
σ_B	.63	.35	.62	.52	.42
τ_B	.35	.54	.32	.36	.40

If A and B are antagonistic intuitionstic fuzzy H-ideal of division BG-algebras X. Obvisouly, the Cartesian product $A \times_4 B$. Since x =0, y = 1, z = 3

Now $\sigma_A(1) \leq \sigma_A(0)$, $\tau_A(1) \geq \tau_A(0)$ and $\sigma_B(1) \leq \sigma_B(0)$, $\tau_B(1) \geq \tau_B(0)$

(i) $\min\{\sigma_A(1), \sigma_B(1)\} \leq \min\{\sigma_A(0), \sigma_B(0)\}$

$\min(.40, .35) \leq \min(.73,.63)$

$0.35 \leq 0.63$

$\max(\tau_A(1), \tau_B(1)) \geq \max(\tau_A(0), \tau_B(0))$

$\max(.58, .54) \geq .\max(.36,.35)$

$0.58 \geq 0.36$

(ii) $\min\{\sigma_A(0 \div 3), \sigma_B(0 \div 3)\} \leq \min\{\max\{\sigma_A(0 \div (1 \div 3)), \sigma_A(1)\} \cdot \max\{\sigma_B(0 \div (1 \div 3)), \sigma_B(1)\}$

$\min(\sigma_A(0), \sigma_B(0)) \leq \min\{\{\{\max\{\sigma_A(0), \sigma_B(0)\}, \max\{\sigma_A(0), \sigma_B(0)\}\}$

$\min(.73, .63) \leq \max\{\min\{\{\sigma_A(0), \sigma_B(0)\}, \min\{\sigma_A(1), \sigma_B(1)\}\}$

$0.63 \leq \max\{\min(.73,.63), \min(.40,.35)\}$

$0.63 \leq \max(.63, .35)$

$0.63 \leq 0.63$

(iii) $\max(\tau_A(0 \div 3), \tau_B(0 \div 3)) \geq \max\{\min\{\tau_A(0 \div (1 \div 3)), \tau_A(1)\}, \min\{\tau_B(0 \div (1 \div 3)), \tau_B(1)\}\}$

$\max(\tau_A(0), \tau_B(0)) \geq \max\{\min(\tau_A(0), \tau_B(0)), \min(\tau_A(1), \tau_B(1))\}$

$\max(.36,.35) \geq \max\{\min(.36,.35), \min(.58,.54)\}$

$\max(.36,.35) \geq \text{mini}\{\max(.36,.35), \max(.58,.54)$
$0.36 \geq \text{mini}(.36, .58)$
$0.36 \geq 0.36$

If A and B are antagonistic- intuitionistic fuzzy H-ideals in division BG-algebras X, the Cartesian product of A $\times_4$ B is also an antagonistic- intuitionistic fuzzy H-ideals in division BG-algebras X.

Thoerem3.5: If A and B are antagonistic- intuitionistic fuzzy H-ideals in division BG-algebras X, then the Cartesian product of A $\times_5$ B is also an antagonistic- intuitionistic fuzzy H-ideals in division BG-algebras X.

Proof: Let A and B are antagonistic- intuitionistic fuzzy H-ideals in division BG-algebras X.

Then

(i) $\sigma_A(1) \leq \sigma_A(x)$, $\tau_A(1) \geq \tau_A(x)$ and $\sigma_B(1) \leq \sigma_B(x)$, $\tau_B(1) \geq \tau_B(x)$
Now, $\max\{\sigma_A(1), \sigma_B(1)\} \leq \max\{\sigma_A(x), \sigma_B(x)\}$
$\sigma_{A \times_4 B}(1) \leq \sigma_{A \times_4 B}(x)$, for all $x \in X$. And
$\text{mini}\{\tau_A(1), \tau_B(1)\} \leq \text{mini}\{\tau_A(x) \cdot \tau_B(x)\}$
$\tau_{A \times_4 B}(1) \leq \tau_{A \times_4 B}(x)$, for all $x \in X$.

(ii) $\sigma_A(x \div z) \leq \text{mmx}\{\sigma_A(x \div (y \div z)), \sigma_A(y)\}$
$\sigma_B(x \div z) \leq \text{Max}\{\sigma_B(x \div (y \div z)), \sigma_B(y)\}$
$\max\{\sigma_A(x \div z), \sigma_B(x \div z)\} \leq \max\{\max\{\sigma_A(x \div (y \div z)), \sigma_A(y)\} \cdot \max\{\sigma_B(x \div (y \div z)), \sigma_B(y)\}$
$\sigma_{A \times_4 B}(x \div z) \leq \max\{\max\{\sigma_A(x \div (y \div z)), \sigma_B(x \div (y \div z)\}, \max\{\sigma_A(y), \sigma_B(y)\}\}$
$\sigma_{A \times_4 B}(x \div z) \leq \max\{\max\{\{\sigma_A(x \div (y \div z)), \sigma_B(x \div (y \div z)\}, \{\sigma_A(y), \sigma_B(y)\}\}\}$
$\sigma_{A \times_4 B}(x \div z) \leq \max\{\max\{\{\sigma_A(x \div (y \div z)), \sigma_B(x \div (y \div z)\}, \{\sigma_A(y), \sigma_B(y)\}\}\}$
$\sigma_{A \times_4 B}(x \div z) \leq \max\{\sigma_{A \times_4 B}((x \div (y \div z)), \sigma_{A \times_4 B}(y)\}$

(iii) $\tau_A(x \div z) \geq \text{mini}\{\tau_A(x \div (y \div z)), \tau_A(y)\}$
$\tau_B(x \div z) \geq \text{mini}\{\tau_B(x \div (y \div z)), \tau_B(y)\}$
$\text{mini}(\tau_A(x \div z), \tau_B(x \div z)) \geq \text{mini}\{\text{mini}\{\tau_A(x \div (y \div z)), \tau_A(y)\}, \text{mini}\{\tau_B(x \div (y \div z)), \tau_B(y)\}\}$
$\tau_{A \times_4 B}(x \div z) \geq \text{mini}\{\text{mini}\{\tau_A(x \div (y \div z)), \tau_B(x \div (y \div z)\}, \text{mini}\{\tau_A(y), \tau_B(y)\}\}$
$\tau_{A \times_4 B}(x \div z) \geq \text{mini}\{\text{mini}\{\{\tau_A(x \div (y \div z)), \tau_B(x \div (y \div z)\}, \{\tau_A(y), \tau_B(y)\}\}\}$
$\tau_{A \times_4 B}(x \div z) \geq \text{mini}\{\tau_{A \times_4 B}((x \div (y \div z)), \tau_{A \times_4 B}(y)\}$

Example: Let X={0,1,2,3,4} with the following Cayley table:

Table 1 : Cayley table for division BG-algebra.

÷	0	1	2	3	4
0	1	0	0	0	0
1	0	1	1	1	1
2	2	2	1	2	2
3	3	3	3	1	3
4	4	4	4	4	1

We define an antagonistic- intuitionistic fuzzy set A, then $A = < x, \sigma_A, \tau_A >$ by routine calculation A is antagonistic intuitionstic fuzzy H-ideal in division BG-algebras X.

X	0	1	2	3	4
σ_A	.73	.40	.66	.60	.50
τ_A	.36	.58	.33	.38	.42

We define an antagonistic- intuitionistic fuzzy set B, then $B = < x, \sigma_B, \tau_B >$ by routine calculation B is antagonistic intuitionstic fuzzy H-ideal in division BG-algebras X.

X	0	1	2	3	4
σ_B	.63	.35	.62	.52	.42
τ_B	.35	.54	.32	.36	.40

If A and B are antagonistic intuitionstic fuzzy H-ideal of division BG-algebras X. Obvisouly, the Cartesian product A $\times_5$B . Since x = 1, y = 2, z = 3

Now $\sigma_A(1) \leq \sigma_A(1)$, $\tau_A(1) \geq \tau_A(1)$ and $\sigma_B(1) \leq \sigma_B(1)$, $\tau_B(1) \geq \tau_B(1)$

i) $\max\{\sigma_A(1), \sigma_B(1)\} \leq \max\{\sigma_A(1), \sigma_B(1)\}$
$\max(.40, .35) \leq \max(.40,.35)$
$0.40 \leq 0.40$

$\text{mini}(\tau_A(1), \tau_B(1)) \geq \text{mini}(\tau_A(1), \tau_B(1))$
$\text{mini}(.58, .54) \geq .\text{mini}(.58,.54)$

$0.54 \geq 0.54$

(ii) $\max\{\sigma_A(1 \div 3), \sigma_B(1 \div 3)\} \leq \max\{\max\{\sigma_A(1 \div (2 \div 3)), \sigma_A(2)\} \cdot \max\{\sigma_B(1 \div (2 \div 3)), \sigma_B(2)\}$

$\max(\sigma_A(1), \sigma_B(1)\} \leq \max\{\{\{\max\{\sigma_A(1), \sigma_B(1)\}, \max\{\sigma_A(2), \sigma_B(2)\}\}$

$\max(.40, .35) \leq \max\{\max\{\{\sigma_A(1), \sigma_B(1)\}, \max\{\sigma_A(2), \sigma_B(2)\}\}$

$0.40 \leq \max\{\max(.40, .35), \max(.66, .62)\}$

$0.40 \leq \max(.40, .66)$

$0.40 \leq 0.66$

(iii) $\text{mini}(\tau_A(1 \div 3), \tau_B(1 \div 3)) \geq \text{mini}\{\text{mini}\{\tau_A(1 \div (2 \div 3)), \tau_A(2)\}, \text{mini}\{\tau_B(1 \div (2 \div 3)), \tau_B(2)\}\}$

$\text{mini}(\tau_A(1), \tau_B(1) \geq \text{mini}\{\text{mini}(\tau_A(1), \tau_B(1)), \text{mini}(\tau_A(2), \tau_B(2))\}$

$\text{mini}(.58, .54) \geq \text{mini}\{\text{mini}(.58, .54), \text{mini}(.33, .32)$

$0.54 \geq \text{mini}(.54, .32)$

$0.54 \geq 0.32$

If A and B are antagonistic- intuitionistic fuzzy H-ideals in division BG-algebras X, the Cartesian product of A $\times_5$ B is also an antagonistic- intuitionistic fuzzy H-ideals in division BG-algebras X.

Thoerem 3.6: If A, B and C are antagonistic- intuitionistic fuzzy H-ideals in division BG-algebras X, then the Cartesian product of (A + B) $\times_1$ C is also an antagonistic- intuitionistic fuzzy H-ideals in division BG-algebras X.

Proof: Let A , B and C are antagonistic- intuitionistic fuzzy H-ideals in division BG-algebras X.Then

(i) $\sigma_A(1) \leq \sigma_A(x)$, $\tau_A(1) \geq \tau_A(x)$, $\sigma_B(1) \leq \sigma_B(x)$, $\tau_B(1) \geq \tau_B(x)$ and

$\sigma_C(1) \leq \sigma_C(x)$, $\tau_C(1) \geq \tau_C(x)$

(ii) $\sigma_A(x \div z) \leq \text{Max}\{\sigma_A(x \div (y \div z)), \sigma_A(y)\}$

$\sigma_B(x \div z) \leq \text{Max}\{\sigma_B(x \div (y \div z)), \sigma_B(y)\}$

$\sigma_C(x \div z) \leq \text{Max}\{\sigma_C(x \div (y \div z)), \sigma_C(y)\}$

(iii) $\tau_A(x \div z) \geq \text{mini}\{\tau_A(x \div (y \div z)), \tau_A(y)\}$

$\tau_B(x \div z) \geq \text{mini}\{\tau_B(x \div (y \div z)), \tau_B(y)\}$

$\tau_C(x \div z) \geq \text{mini}\{\tau_C(x \div (y \div z)), \tau_C(y)\}$

Now (i) $[\{\sigma_A(1) + \sigma_B(1) - \sigma_A(1)\ \sigma_B(1)\} \bullet \sigma_C(1)] \leq [\{\sigma_A(x) + \sigma_B(x) - \sigma_A(x)\ \sigma_B(x)\} \bullet \sigma_C(x)]$

$\sigma_{(A+B)\times_1 C}(1) \leq \sigma_{(A+B)\times_1 C}(x)$ and

$[\{\tau_A(1) + \tau_B(1) - \tau_A(1)\ \tau_B(1)\} \bullet \tau_C(1)] \geq [\{\tau_A(x) + \tau_B(x) - \tau_A(x)\ \tau_B(x)\} \bullet \tau_C(x)]$

$\tau_{(A+B)\times_1 C}(1) \geq \tau_{(A+B)\times_1 C}(x)$

(ii) $\sigma_A(x \div z) + \sigma_B(x \div z) - \sigma_A(x \div z)\ \sigma_B(x \div z)\} \bullet \sigma_C(x \div z) \leq \{\text{Max}\{\sigma_A(x \div (y \div z)), \sigma_A(y)\} + \text{Max}\{\sigma_B(x \div (y \div z)), \sigma_B(y)\} - \{\text{Max}\{\sigma_A(x \div (y \div z)), \sigma_A(y)\}\ \text{Max}\{\sigma_B(x \div (y \div z)), \sigma_B(y)\}\} \bullet \text{Max}\{\sigma_C(x \div (y \div z)), \sigma_C(y)\}$.

$\sigma_A(x \div z) \bullet \sigma_C(x \div z) + \sigma_B(x \div z) \bullet \sigma_C(x \div z) - \sigma_A(x \div z)\ \sigma_B(x \div z)\} \bullet \sigma_C(x \div z) \leq \text{Max}\{\sigma_A(x \div (y \div z))\}, \bullet\text{Max}\{\{\sigma_C(x \div (y \div z))\}, \text{Max}\{\sigma_A(y), \{\sigma_C(y)\} + \text{Max}\{\sigma_B(x \div (y \div z))\},$

$\bullet\text{Max}\{\sigma_C(x \div (y \div z))\}, \text{Max}\{\sigma_B(y), \{\sigma_C(y)\} - \{\text{Max}\{\sigma_A(x \div (y \div z) \bullet \{\sigma_B(x \div (y \div z)), \sigma_B(y))\}\} \bullet \text{Max}\{\{\sigma_C(x \div (y \div z))\}, \text{Max}\{\sigma_A(y), \sigma_A(x,)\ \sigma_C(y)\}$

$\leq \text{Max}\{\sigma_A(x \div (y \div z))\} \bullet \{\{\sigma_C(x \div (y \div z))\}, (\sigma_A(y) \bullet \sigma_C(y)) + \text{Max}\{\sigma_B(x \div (y \div z)) \bullet \sigma_C(x \div (y \div z))\}, (\sigma_B(y) \bullet \sigma_C(y)) - \text{Max}\{\sigma_A(x \div (y \div z)) \bullet \sigma_B(x \div (y \div z)) \bullet \sigma_C(x \div (y \div z)), (\sigma_A(y), \sigma_A(x,)\ \sigma_C(y))$

$\sigma_{(A+B)\times_1 C}(x \div z) \leq \text{Max}\{\sigma_{(A+B)\times_1 C}(x \div (y \div z)), \sigma_{(A+B)\times_1 C}(y)\}$

(iii) $\tau_A(x \div z) \bullet \tau_B(x \div z) \bullet \tau_C(x \div z) \leq \{\text{mini}\{\tau_A(x \div (y \div z)), \tau_A(y)\} \bullet \text{mini}\{\tau_B(x \div (y \div z)), \tau_B(y)\} \bullet \text{mini}\{\tau_C(x \div (y \div z)), \tau_C(y)\}$.

$\tau_A(x \div z) \bullet \tau_C(x \div z) \bullet \tau_C(x \div z) \leq \text{mini}\{\tau_A(x \div (y \div z)) \bullet \tau_B(x \div (y \div z)) \bullet \tau_C(x \div (y \div z))\}, \text{mini}\{\tau_A(y) \bullet \tau_B(y) \bullet \tau_C(y)\}$

$\tau_{(A+B)\times_1 C}(x \div z) \leq \text{mini}\{\tau_{(A+B)\times_1 C}(x \div (y \div z))\}, \text{mini}\{\sigma_{(A+B)\times_1 C}(y)\}$

$\tau_{(A+B)\times_1 C}(x \div z) \leq \text{mini}\{\tau_{(A+B)\times_1 C}(x \div (y \div z)), \sigma_{(A+B)\times_1 C}(y)\}$

Example: Let X={0,1,a,b} with the following Cayley table:

Table 1 : Cayley table for division BG-algebra.

÷	0	1	a	b

0	1	0	0	0
1	1	1	1	1
a	a	a	1	a
b	b	b	b	1

We define an antagonistic- intuitionistic fuzzy set A, then $A = < x, \sigma_A, \tau_A >$ by routine calculation A is antagonistic intuitionstic fuzzy H-ideal in division BG-algebras X.

X	0	1	a	B
σ_A	.64	.53	.66	.68
τ_A	.36	.46	.33	.32

We define an antagonistic- intuitionistic fuzzy set B, then $B = < x, \sigma_B, \tau_B >$ by routine calculation B is antagonistic intuitionstic fuzzy H-ideal in division BG-algebras X.

X	0	1	a	b
σ_B	.62	.51	.57	.58
τ_B	.30	.47	.42	.40

We define an antagonistic- intuitionistic fuzzy set C, then $C = < x, \sigma_C, \tau_C >$ by routine calculation C is antagonistic intuitionstic fuzzy H-ideal in division BG-algebras X.

X	0	1	a	b
σ_C	.66	..44	.55	.53
τ_C	.35	.50	.42	.48

If A ,Band C are antagonistic intuitionstic fuzzy H-ideal of division BG-algebras X. Obviously, the Cartesian product $(A + B) \times_1 C$. Since x = a, y = b, z = 1.

Now $\sigma_A(1) \le \sigma_A(a)$, $\tau_A(1) \ge \tau_A(a)$ and $\sigma_B(1) \le \sigma_B(a)$, $\tau_B(1) \ge \tau_B(a)$

(i) $[\{\sigma_A(1) + \sigma_B(1) - \sigma_A(1).\sigma_B(1)\} \bullet \sigma_C(1)] \le [\{\sigma_A(a) + \sigma_B(a) - \sigma_A(a)\ \sigma_B(a)\} \bullet \sigma_C(a)]$

$\{[.53+.51-(.53\bullet.51)]\bullet.44\} \le \{[.66+.57-(.66 \bullet.57)]\bullet.55\}$

$0.576532 \le 0.88341$

$\tau_A(1) \bullet \tau_B(1) \bullet \tau_C(1) \ge \tau_A(a) \bullet \tau_B(a) \bullet \tau_C(a)$

$.46\bullet.47\bullet.64 \ge .33\bullet.42\bullet.42$

$0.138368 \le 0.058212$

(ii) $\sigma_A(a\div1) + \sigma_{\mathbf{B}}(a\div1) - \sigma_A(a\div1).\sigma_{\mathbf{B}}(a \div1)\} \bullet \sigma_C(a \div 1) \le \{\max\{\sigma_A(a\div(b\div1)), \sigma_A(b)\} + \max\{\sigma_B(a\div(b\div1)), \sigma_B(b)\} - \{\max\{\sigma_A(a\div(b\div1)), \sigma_A(b)\}.\max\{\sigma_B(a\div(b\div1)), \sigma_B(b)\}\} \bullet \max\{\sigma_C(a\div(b\div1)), \sigma_C(b)\}$.

$\{[.66+.57-(.66\bullet.57)]\bullet.55\} \le \{[\max(.66,.68) + \max(.57,.58) - \max(.66,.68).\max(.57,.58)] \bullet \max(.55,.51)\}$

$0.88341 \le \max(.88341, .84374)$

$0.88341 \le 0.88341$

(iii) $\tau_A(a\div1) \bullet \tau_{\mathbf{B}}(a\div1) \bullet \tau_C(a\div 1) \ge \{\min\{\tau_A(a\div(b\div1)), \tau_A(b)\} \bullet \min\{\tau_B(a\div(b\div1)), \tau_B(b)\} \bullet \min\{\tau_C(a\div(b\div1)), \tau_C(b)\}$

$.33\bullet.42\bullet.42 \ge \min(.33,.32)\bullet\min(.42,.40)\bullet\min(.42,.48)$

$0.058212 \ge .32\bullet.40\bullet.42$

$0.058212 \ge 0.05376$

If A, B and C are antagonistic- intuitionistic fuzzy H-ideals in division BG-algebras X, the Cartesian product of $(A + B) \times_1 C$ is also an antagonistic- intuitionistic fuzzy H-ideals in division BG-algebras X.

Thoerem 3.7: If A, B and C are antagonistic- intuitionistic fuzzy H-ideals in division BG-algebras X, then the Cartesian product of $(A \cap B) \times_2 C$ is also an antagonistic- intuitionistic fuzzy H-ideals in division BG-algebras X.

Proof : Let A , B and C are antagonistic- intuitionistic fuzzy H-ideals in division BG-algebras X.

(i) $\sigma_A(1) \le \sigma_A(x)$, $\tau_A(1) \ge \tau_A(x)$, $\sigma_B(1) \le \sigma_B(x)$, $\tau_B(1) \ge \tau_B(x)$

$\sigma_C(1) \le \sigma_C(x)$, $\tau_C(1) \ge \tau_C(x)$

(ii) $\sigma_A(x \div z) \le \text{Max}\{\sigma_A(x\div(y \div z)), \sigma_A(y)\}$

$\sigma_B(x \div z) \le \text{Max}\{\sigma_B(x\div(y \div z)), \sigma_B(y)\}$

$\sigma_C(x \div z) \le \text{Max}\{\sigma_C(x\div(y \div z)), \sigma_C(y)\}$

(iii) $\tau_A(x \div z) \ge \text{mini}\{\tau_A(x\div(y \div z)), \tau_A(y)\}$

$\tau_B(x \div z) \ge \text{mini}\{\tau_B(x\div(y \div z)), \tau_B(y)\}$

$\tau_C(x \div z) \ge \text{mini}\{\tau_C(x\div(y \div z)), \tau_C(y)\}$

Now (i) mini $\{\sigma_A(1), \sigma_B(1)\} + \sigma_C(1)$ - mini $\{\sigma_A(1), \sigma_B(1)\} \bullet \sigma_C(1) \leq$
mini $\{\sigma_A(x), \sigma_B(x)\} + \sigma_C(x)$ – mini $\{\sigma_A(x), \sigma_B(x)\} \bullet \sigma_C(x)$ and
max $\{\tau_A(1), \tau_B(1)\} \bullet \tau_C(1) \geq$ max $\{\tau_A(x), \tau_B(x)\} \bullet \tau_C(x)$

$\sigma_{(A \cap B)\times_2 C}(1) \leq \sigma_{(A \cap B)\times_2 C}(x)$ and $\tau_{(A \cap B)\times_2 C}(1) \geq \tau_{(A \cap B)\times_2 C}(x)$

(ii) $\sigma_{(A \cap B)}(x \div z) + \sigma_C(x \div z) - \sigma_{(A \cap B)}(x \div z) + \sigma_C(x \div z) \leq \max\{\{\sigma_{(A \cap B)}(x \div (y \div z)), \sigma_{(A \cap B)}(y)\} + \max\{\sigma_C(x \div (y \div z)), \sigma_C(y)\} - \max\{\{\sigma_{(A \cap B)}(x \div (y \div z)), \sigma_{(A \cap B)}(y)\} \bullet \max\{\sigma_C(x \div (y \div z)), \sigma_C(y)\}$

mini $\{\sigma_A(x \div z), \sigma_B(x \div z)\} + \sigma_C(x \div z)$ - mini $\{\sigma_A(x \div z), \sigma_B(x \div z)\} \bullet \sigma_C(x \div z) \leq$

max { mini $[\sigma_A(x \div (y \div z)), \sigma_B(x \div (y \div z))]$, mini $[\sigma_A(y), \sigma_B(y)]\} + \max\{\sigma_C(x \div (y \div z)), \sigma_C(y)\} - \max\{$ mini $[\sigma_A(x \div (y \div z)), \sigma_B(x \div (y \div z))]$, mini $[\sigma_A(y), \sigma_B(y)]\} \bullet \max\{\sigma_C(x \div (y \div z)), \sigma_C(y)\}$

$\sigma_{(A \cap B)\times_2 C}(x \div z) \leq \max\{$ mini $[\sigma_A(x \div (y \div z)), \sigma_B(x \div (y \div z))], [\sigma_A(y), \sigma_B(y)]\} + \max\{\sigma_C(x \div (y \div z)), \sigma_C(y)\} - \max\{$ mini $[\sigma_A(x \div (y \div z)), \sigma_B(x \div (y \div z) \bullet \sigma_C(x \div (y \div z))], \text{mini } [\sigma_A(y), \sigma_B(y)] \bullet \sigma_C(y)\}$

$\sigma_{(A \cap B)\times_2 C}(x \div z) \leq \max\{\sigma_{(A \cap B)\times_2 C}(x \div (y \div z)), \sigma_{(A \cap B)\times_2 C}(y)\}$

(iii) $\tau_{(A \cap B)}(x \div z) \bullet \tau_C(x \div z) \geq \text{mini}\{\{\tau_{(A \cap B)}(x \div (y \div z)), \tau_{(A \cap B)}(y)\} \bullet \text{mini}\{\tau_C(x \div (y \div z)), \tau_C(y)\}$

max $\{\tau_A(x \div z), \tau_B(x \div z)\} \bullet \tau_C(x \div z) \geq \text{mini}\{\max\{\tau_A(x \div (y \div z)), \tau_B(x \div (y \div z))\}, \max\{\tau_A(y), \tau_B(y)\} \bullet \tau_C(x \div (y \div z)), \tau_C(y)\}$

$\tau_{(A \cap B)\times_2 C}(x \div z) \geq \text{mini}\{\max\{\{\tau_A(x \div (y \div z)), \tau_B(x \div (y \div z))\}, \{\tau_A(y), \tau_B(y)\}\} \bullet \{\tau_C(x \div (y \div z)), \tau_C(y)\}\}$

$\tau_{(A \cap B)\times_2 C}(x \div z) \geq \text{mini}\{\max\{\tau_A(x \div (y \div z)), \tau_B(x \div (y \div z))\} \bullet \tau_C(x \div (y \div z)),$
$\max\{\tau_A(y), \tau_B(y)\} \bullet \tau_C(y)\}$

$\tau_{(A \cap B)\times_2 C}(x \div z) \geq \text{mini}\{\tau_{(A \cap B)\times_2 C}(x \div (y \div z)), \tau_{(A \cap B)\times_2 C}(y)\}$

If A, B and C are antagonistic- intuitionistic fuzzy H-ideals in division BG-algebras X , the Cartesian product of $(A \cap B) \times_2 C$ is also an antagonistic- intuitionistic fuzzy H-ideals in division BG-algebras X.

Example: Let X={0,1,a,b} with the following Cayley table:

Table 1 : Cayley table for division BG-algebra.

÷	0	1	a	B
0	1	0	0	0
1	1	1	1	1
a	a	a	1	A
b	b	b	b	1

We define an antagonistic- intuitionistic fuzzy set A, then $A = < x, \sigma_A, \tau_A >$ by routine calculation A is antagonistic intuitionstic fuzzy H-ideal in division BG-algebras X.

X	0	1	a	b
σ_A	.64	.53	.66	.68
τ_A	.36	.46	.33	.32

We define an antagonistic- intuitionistic fuzzy set B, then $B = < x, \sigma_B, \tau_B >$ by routine calculation B is antagonistic intuitionstic fuzzy H-ideal in division BG-algebras X.

X	0	1	a	b
σ_B	.62	.51	.57	.58
τ_B	.30	.47	.42	.40

We define an antagonistic- intuitionistic fuzzy set C, then $C = < x, \sigma_C, \tau_C >$ by routine calculation C is antagonistic intuitionistic fuzzy H-ideal in division BG-algebras X.

X	0	1	a	b
σ_C	.66	..44	.55	.53
τ_C	.35	.50	.42	.48

If A ,B and C are antagonistic intuitionistic fuzzy H-ideal of division BG-algebras X. Obviously, the Cartesian product $(A \cap B) \times_2$. Since x = 0, y = 1, z = a.

Now $\sigma_A(1) \leq \sigma_A(0)$, $\tau_A(1) \geq \tau_A(0)$ and $\sigma_B(1) \leq \sigma_B(0)$, $\tau_B(1) \geq \tau_B(0)$

mini $\{\sigma_A(1), \sigma_B(1)\} + \sigma_C(1) - \text{mini}\{\sigma_A(1), \sigma_B(1)\} \bullet \sigma_C(1) \leq \text{mini}\{\sigma_A(0), \sigma_B(0)\} + \sigma_C(0) - \text{mini}\{\sigma_A(0), \sigma_B(0)\} \bullet \sigma_C(0)$
mini (.53,.51)+.44 – mini(.53,.51)•.44≤ mini(.64,.62) +.66 – mini(.64,.62)•.66
$0.7256 \leq 0.8708$
$\max\{\tau_A(1), \tau_B(1)\} \bullet \tau_C(1) \geq \max\{\tau_A(0), \tau_B(0)\} \bullet \tau_C(0)$
max(.46,.47)•.54 ≥ max{(.36,.30)•.33
.47•.54 ≥ .36•.33
0..2538 ≥ 0.108
(ii) $\sigma_{(A \cap B)\times_2 C}(x \div z) \leq \max\{\sigma_{(A \cap B)\times_2 C}(x \div (y \div z)), \sigma_{(A \cap B)\times_2 C}(y)\}$
mini $\{\sigma_A(0 \div a), \sigma_B(0 \div a)\} + \sigma_C(0 \div a) - \text{mini}\{\sigma_A(0 \div a), \sigma_B(0 \div a)\} \bullet \sigma_C(0 \div a) \leq$
$\max\{\text{mini}[\sigma_A(0 \div (1 \div a)), \sigma_B(0 \div (1 \div a))], \text{mini}[\sigma_A(1), \sigma_B(1)]\} + \max\{\sigma_C(0 \div (1 \div a)), \sigma_C(1)\} - \max\{\text{mini}[\sigma_A(0 \div (1 \div a)), \sigma_B(0 \div (1 \div a))], \text{mini}[\sigma_A(1), \sigma_B(1)]\} \bullet \max\{\sigma_C(0 \div (1 \div a)), \sigma_C(1)\}$
mini(.64,.62) + .66 – mini(.64,.62)•.66 ≤ max{mini(.64,.62),mini(.53,.51)+max(.64,.44)-max{mini(.64,.62)•.66,mini(.53,.51).44}
.62+.66-.62•.66 ≤ max(.62,51)+.66-max{(.62•.66),(.51•44)}
$0.8708 \leq 0.8708$
(iii) $\tau_{(A \cap B)}(x \div z) \bullet \tau_C(x \div z) \geq \text{mini}\{\{\tau_{(A \cap B)}(x \div (y \div z)), \tau_{(A \cap B)}(y)\} \bullet \text{mini}\{\tau_C(x \div (y \div z)), \tau_C(y)\}$
$\max\{\tau_A(0 \div a), \tau_B(0 \div a)\} \bullet \tau_C(0 \div a) \geq \text{mini}\{\max\{\tau_A(0 \div (1 \div a)), \tau_B(0 \div (1 \div a))\},$
$\max\{\tau_A(1), \tau_B(1)\} \bullet \{\tau_C(0 \div (1 \div a)), \tau_C(1)\}\}$
max(.36,.30) •.33 ≥ mini{max(.36,.30)•.33,max(.46,.47)•.54}
0.1188 ≥ mini(.36•.33,.47•.54)
0.1188 ≥ mini(0.1188,0.2538)
0.1188 ≥ 0.1188
If A, B and C are antagonistic- intuitionistic fuzzy H-ideals in division BG-algebras X, the Cartesian product of $(A \cap B) \times_2 C$ is also an antagonistic- intuitionistic fuzzy H-ideals in division BG-algebras X.

Thoerem 3.7: If A, B and C are antagonistic- intuitionistic fuzzy H-ideals in division BG-algebras X, then the Cartesian product of $(A \cup B) \times_3 C$ is also an antagonistic- intuitionistic fuzzy H-ideals in division BG-algebras X.

Proof : Let A , B and C are antagonistic- intuitionistic fuzzy H-ideals in division BG-algebras X.

(i) $\sigma_A(1) \leq \sigma_A(x)$, $\tau_A(1) \geq \tau_A(x)$, $\sigma_B(1) \leq \sigma_B(x)$, $\tau_B(1) \geq \tau_B(x)$
$\sigma_C(1) \leq \sigma_C(x)$, $\tau_C(1) \geq \tau_C(x)$

(ii) $\sigma_A(x \div z) \leq \text{Max}\{\sigma_A(x \div (y \div z)), \sigma_A(y)\}$

$\sigma_B(x \div z) \leq \max\{\sigma_B(x \div (y \div z)), \sigma_B(y)\}$
$\sigma_C(x \div z) \leq \max\{\sigma_C(x \div (y \div z)), \sigma_C(y)\}$
(iii) $\tau_A(x \div z) \geq \text{mini}\{\tau_A(x \div (y \div z)), \tau_A(y)\}$
$\tau_B(x \div z) \geq \text{mini}\{\tau_B(x \div (y \div z)), \tau_B(y)\}$
$\tau_C(x \div z) \geq \text{mini}\{\tau_C(x \div (y \div z)), \tau_C(y)\}$

Now (i) $\max\{\sigma_A(1), \sigma_B(1)\} \bullet \sigma_C(1) \leq \max\{\sigma_A(x), \sigma_B(x)\} \bullet \sigma_C(x)$,
$\text{mini}\{\tau_A(1), \tau_B(1)\} \bullet \tau_C(1) \geq \text{mini}\{\tau_A(x), \tau_B(x)\} \bullet \tau_C(x)$ $\sigma_{(A \cup B)\times_3 C}(1) \leq \sigma_{(A \cup B)\times_3 C}(x)$, $\tau_{(A \cup B)\times_3 C}(1) \geq \tau_{(A \cup B)\times_3 C}(x)$

(ii) $\sigma_{(A \cup B)}(x \div z) \bullet \sigma_C(x \div z) \leq \max\{\{\sigma_{(A \cup B)}(x \div (y \div z)), \sigma_{(A \cup B)}(y)\} \bullet \max\{\sigma_C(x \div (y \div z)), \sigma_C(y)\}$
$\max\{\sigma_A(x \div z), \sigma_B(x \div z)\} \bullet \sigma_C(x \div z) \leq \max\{\max[\sigma_A(x \div (y \div z)), \sigma_B(x \div (y \div z))], \max[\sigma_A(y), \sigma_B(y)]\} \bullet \max\{\sigma_C(x \div (y \div z)), \sigma_C(y)\}$
$\sigma_{(A \cup B)\times_3 C}(x \div z) \leq \max\{\max[\sigma_A(x \div (y \div z)), \sigma_B(x \div (y \div z))], [\sigma_A(y), \sigma_B(y)] \bullet [\sigma_C(x \div (y \div z)), \sigma_C(y)]\}$
$\sigma_{(A \cup B)\times_3 C}(x \div z) \leq \max\{\sigma_{(A \cup B)\times_3 C}(x \div (y \div z)), \sigma_{(A \cup B)\times_3 C}(y)\}$

(iii) $\tau_{(A\cup B)}(x \div z) + \tau_C(x \div z) - \tau_{(A\cup B)}(x \div z) \bullet \tau_C(x \div z) \geq \text{mini}\{\{\tau_{(A\cup B)}(x \div (y \div z)), \tau_{(A\cup B)}(y)\} + \text{mini}\{\tau_C(x \div (y \div z)), \tau_C(y)\} - \text{mini}\{\tau_{(A\cup B)}(x \div (y \div z)), \tau_{(A\cup B)}(y)\}$

$\bullet \text{mini}\{\tau_C(x \div (y \div z)), \tau_C(y)\}$

$\text{mini}\{\tau_A(x \div z), \tau_B(x \div z)\} + \tau_C(x \div z) - \text{mini}\{\tau_A(x \div z), \tau_B(x \div z)\} \bullet \tau_C(x \div z) \geq \text{mini}$

$\{\text{mini}[\tau_A(x \div (y \div z)), \tau_B(x \div (y \div z))], \text{mini}[\tau_A(y), \tau_B(y)]\} + \text{mini}\{\tau_C(x \div (y \div z)), \tau_C(y)\} - \text{mini}\{\text{mini}[\tau_A(x \div (y \div z)), \tau_B(x \div (y \div z))], \text{mini}[\tau_A(y), \tau_B(y)]\} \bullet \text{mini}\{\tau_C(x \div (y \div z)), \tau_C(y)\}$

$\tau_{(A\cup B)\times_3 C}(x \div z) \geq \text{mini}\{\text{mini}[\tau_A(x \div (y \div z)), \tau_B(x \div (y \div z))], [\tau_A(y), \tau_B(y)]\} + \text{mini}\{\tau_C(x \div (y \div z)), \tau_C(y)\} - \text{mini}\{\text{mini}[\tau_A(x \div (y \div z)), \tau_B(x \div (y \div z)] \bullet \tau_C(x \div (y \div z))\}, \text{mini}[\tau_A(y), \tau_B(y)] \bullet \tau_C(y)\}$

$\tau_{(A\cup B)\times_3 C}(x \div z) \geq \text{mini}\{\{\text{mini}\{[\tau_A(x \div (y \div z)), \tau_B(x \div (y \div z))], [\tau_A(y), \tau_B(y)]\} + \tau_C(x \div (y \div z)), \tau_C(y)\} - \text{mini}\{\text{mini}\{[\tau_A(x \div (y \div z)), \tau_B(x \div (y \div z)], [\tau_A(y), \tau_B(y)]\} \bullet [\tau_C(x \div (y \div z)), \tau_C(y)]\}$

$\tau_{(A\cup B)\times_3 C}(x \div z) \geq \text{mini}\{\tau_{(A\cup B)\times_3 C}(x \div (y \div z)), \tau_{(A\cup B)\times_3 C}(y)\}$

If A, B and C are antagonistic- intuitionistic fuzzy H-ideals in division BG-algebras X , the Cartesian product of $(A \cup B) \times_3 C$ is also an antagonistic- intuitionistic fuzzy H-ideals in division BG-algebras X.

Example: Let X={0,1,a,b} with the following Cayley table:

Table 1 : Cayley table for division BG-algebra.

÷	0	1	a	B
0	1	0	0	0
1	1	1	1	1
a	a	a	1	A
b	b	b	b	1

We define an antagonistic- intuitionistic fuzzy set A, then $A = < x, \sigma_A, \tau_A >$ by routine calculation A is antagonistic intuitionstic fuzzy H-ideal in division BG-algebras X.

X	0	1	a	b
σ_A	.64	.53	.66	.68
τ_A	.36	.46	.33	.32

We define an antagonistic- intuitionistic fuzzy set B, then $B = < x, \sigma_B, \tau_B >$ by routine calculation B is antagonistic intuitionstic fuzzy H-ideal in division BG-algebras X.

X	0	1	a	b
σ_B	.62	.51	.57	.58
τ_B	.30	.47	.42	.40

We define an antagonistic- intuitionistic fuzzy set C, then $C = < x, \sigma_C, \tau_C >$ by routine calculation C is antagonistic intuitionistic fuzzy H-ideal in division BG-algebras X.

X	0	1	a	b
σ_C	.66	..44	.55	.53
τ_C	.35	.50	.42	.48

If A ,B and C are antagonistic intuitionistic fuzzy H-ideal of division BG-algebras X. Obviously, the Cartesian product $(A \cap B) \times_2$. Since x = a, y = b, z = 0.

Now $\sigma_A(1) \leq \sigma_A(a)$, $\tau_A(1) \geq \tau_A(a)$ and $\sigma_B(1) \leq \sigma_B(a)$, $\tau_B(1) \geq \tau_B(a)$

$\max\{\sigma_A(1), \sigma_B(1)\} \bullet \sigma_C(1) \leq \max\{\sigma_A(a), \sigma_B(a)\} \bullet \sigma_C(a)$

$\max(.53,.51) \bullet .44 \leq \max(.66,.57) \bullet .55$

$.53 \bullet .44 \leq .66 \bullet .55$

$0.2332 \leq 0.363$ and

$\text{mini}\{\tau_A(1), \tau_B(1)\} + \tau_C(1) - \text{mini}\{\tau_A(1), \tau_B(1)\} \bullet \tau_C(1) \leq \text{mini}\{\tau_A(a), \tau_B(a)\} + \tau_C(a) - \text{mini}\{\tau_A(a), \tau_B(a)\} \bullet \tau_C(a)$

$\text{mini}(.46,.47) + .54 - \text{mini}(.46,.47) \bullet .54 \geq \text{mini}\{(.33,42) + .42 - \text{mini}(.33,.42) \bullet .42$

$.46 \bullet .54 - .46 \bullet .54 \geq .33 + .42 - .33 \bullet .42$

$0..7516 \geq 0.6184$

(ii) $\sigma_{(A\cup B)\times_3 C}(x \div z) \leq \max\{\sigma_{(A\cup B)\times_3 C}(x \div (y \div z)), \sigma_{(A\cup B)\times_3 C}(y)\}$

$\max\{\sigma_A(a \div 0), \sigma_B(a \div 0)\} \bullet \sigma_C(a \div 0) \le \max\{\max[\sigma_A(a \div (b \div 0)), \sigma_B(a \div (b \div 0))], \max[\sigma_A(b), \sigma_B(b)]\} \bullet \max\{\sigma_C(a \div (b \div 0)), \sigma_C(b)\}$

$\max\{\sigma_A(a), \sigma_B(a)\} \bullet \sigma_C(a) \le \max\{\max[\sigma_A(a), \sigma_B(a)], \max[\sigma_A(b), \sigma_B(b)]\} \bullet \max\{\sigma_C(a), \sigma_C(b)\}$

$\max(.66,.57) \bullet .55 \le \max\{\max(.66,.57) \bullet .55, \max i(.68,.58) \bullet .51$

$.66 \bullet .55 \le \max\{.66 \bullet .51, .68 \bullet .51\}$

$0.363 \le \max\{0.363, 0.3468\}$

$0.363 \le 0.363$

(iii) $\tau_{(A \cup B) \times_3 C}(x \div z) \ge \min i\{\tau_{(A \cup B) \times_3 C}(x \div (y \div z)), \tau_{(A \cup B) \times_3 C}(y)\}$

$\min\{\tau_A(a \div 0), \tau_B(a \div 0)\} + \tau_C(a \div 0) - \min\{\tau_A(a \div 0), \tau_B(a \div 0)\} \bullet \tau_C(a \div 0) \ge \min$
$\{\min[\tau_A(a \div (b \div 0)), \tau_B(a \div (b \div 0))], \min[\tau_A(b), \tau_B(b)]\} + \min\{\tau_C(a \div (b \div 0)), \tau_C(b)\} - \min\{\min[\tau_A(a \div (b \div 0)), \tau_B(a \div (b \div 0))], \min[\tau_A(b), \tau_B(b)]\} \bullet \min\{\tau_C(a \div (b \div 0)), \tau_C(b)\}$

$\min\{\tau_A(a), \tau_B(a)\} + \tau_C(a) - \min\{\tau_A(a), \tau_B(a)\} \bullet \tau_C(a) \ge \min$
$\{\min[\tau_A(a), \tau_B(a)], \min[\tau_A(b), \tau_B(b)]\} + \min\{\tau_C(a), \tau_C(b)\} - \min\{\min[\tau_A(a), \tau_B(a)], \min[\tau_A(b), \tau_B(b)]\} \bullet \min\{\tau_C(a), \tau_C(b)\}$

$\min(.33,.42) + .42 - \min(.33,.42) \bullet .42 \ge \min\{\min\{(.33,.32),(.42,.40)\} + \min(.42,.48) - \min\{(.33,.32),(.42,.40)\}\} \bullet \min(.42,.48)\}$

$.33 + .42 - .33 \bullet .42 \ge \min\{\{\min(.33,.42), \min(.32,.40)\} + \min(.42,.48) - \min\{i(.33,.42),(.32,.40)\} \bullet \min(.42,.48)$

$0.75 - 0.1386 \ge \min(.33,.32) + .42 - \min(.32,.42) \bullet .42$

$0.6114 \ge 0.74 - 0.1344$

$0.6114 \ge 0.6054$

If A, B and C are antagonistic- intuitionistic fuzzy H-ideals in division BG-algebras X, the Cartesian product of $(A \cup B) \times_3 C$ is also an antagonistic- intuitionistic fuzzy H-ideals in division BG-algebras X.

Thoerem 3.9: If A, B and C are antagonistic- intuitionistic fuzzy H-ideals in division BG-algebras X, then the Cartesian product of $(A \bullet B) \times_4 C$ is also an antagonistic- intuitionistic fuzzy H-ideals in division BG-algebras X.

Proof : Let A , B and C are antagonistic- intuitionistic fuzzy H-ideals in division BG-algebras X.

(i) $\sigma_A(1) \le \sigma_A(x)$, $\tau_A(1) \ge \tau_A(x)$, $\sigma_B(1) \le \sigma_B(x)$, $\tau_B(1) \ge \tau_B(x)$
$\sigma_C(1) \le \sigma_C(x)$, $\tau_C(1) \ge \tau_C(x)$

(ii) $\sigma_A(x \div z) \le \text{Max}\{\sigma_A(x \div (y \div z)), \sigma_A(y)\}$
$\sigma_B(x \div z) \le \max\{\sigma_B(x \div (y \div z)), \sigma_B(y)\}$
$\sigma_C(x \div z) \le \max\{\sigma_C(x \div (y \div z)), \sigma_C(y)\}$

(iii) $\tau_A(x \div z) \ge \min i\{\tau_A(x \div (y \div z)), \tau_A(y)\}$
$\tau_B(x \div z) \ge \min i\{\tau_B(x \div (y \div z)), \tau_B(y)\}$
$\tau_C(x \div z) \ge \min i\{\tau_C(x \div (y \div z)), \tau_C(y)\}$

Now (i) $\min\{[\sigma_A(1) \bullet \sigma_B(1)], \sigma_C(1)\} \le \min\{[\sigma_A(x) \bullet \sigma_B(x)], \sigma_C(x)\}$,
$\max\{[\tau_A(1) + \tau_B(1) - \tau_A(1) \bullet \tau_B(1)], \tau_C(1)\} \ge \max\{[\tau_A(x) + \tau_B(x) - \tau_A(x) \bullet \tau_B(x)], \tau_C(x)\}$

$\sigma_{(A \bullet B) \times_4 C}(1) \le \sigma_{(A \bullet B) \times_4 C}(x)$, $\tau_{(A \bullet B) \times_4 C}(1) \ge \tau_{(A \bullet B) \times_4 C}(x)$

(ii) $\min\{\sigma_{(A \bullet B)}(x \div z), \sigma_C(x \div z)\} \le \min\{\max\{\sigma_{(A \bullet B)}(x \div (y \div z)), \sigma_{(A \bullet B)}(y)\}, \max\{\sigma_C(x \div (y \div z)), \sigma_C(y)\}\}$
$\min\{[\sigma_A(x \div z) \bullet \sigma_B(x \div z)], \sigma_C(x \div z)\} \le \min\{\max\{[\sigma_A(x \div (y \div z)) \bullet \sigma_B(x \div (y \div z))], [\sigma_A(y) \bullet \sigma_B(y)]\}, \max[\sigma_C(x \div (y \div z)), \sigma_C(y)]\}$
$\sigma_{(A \bullet B) \times_4 C}(x \div z) \le \min\{\max\{[\sigma_A(x \div (y \div z)) \bullet \sigma_B(x \div (y \div z))], [\sigma_A(y) \bullet \sigma_B(y)], [\sigma_C(x \div (y \div z)), \sigma_C(y)]\}\}$
If $A \subset B$ and $B \subset A$
$\sigma_{(A \bullet B) \times_4 C}(x \div z) \le \max\{\min\{[\sigma_A(x \div (y \div z)) \bullet \sigma_B(x \div (y \div z))], [\sigma_A(y) \bullet \sigma_B(y)], [\sigma_C(x \div (y \div z)), \sigma_C(y)]\}\}$

$\sigma_{(A\bullet B)\times_4 C}(x \div z) \leq \max\{\sigma_{(A\bullet B)\times_4 C}(x \div (y \div z)), \sigma_{(A\bullet B)\times_4 C}(y)\}$

(iii) $\max\{\tau_{(A\bullet B)}(x \div z), \tau_C(x \div z)\} \geq \max\{\min\{\tau_{(A\bullet B)}(x \div (y \div z)), \{\tau_{(A\bullet B)}(y)\}, \min\{\tau_C(x \div (y \div z)), \tau_C(y)\}\}$

$\max\{[\tau_A(x \div z) + \tau_B(x \div z) - \tau_A(x \div z) \bullet \tau_B(x \div z)], \tau_C(x \div z)\} \geq \max\{\min\{[\tau_A(x \div (y \div z)) + \tau_B(x \div (y \div z)) - \tau_A(x \div (y \div z)) \bullet \tau_B(x \div (y \div z))], [\tau_A(y) + \tau_B(y) - \tau_A(y) \bullet \tau_B(y)], \min[\tau_C(x \div (y \div z)), \tau_C(y)]\}\}$

$\tau_{(A\bullet B)\times_4 C}(x \div z) \geq \max\{\min\{[\tau_A(x \div (y \div z)) + \tau_B(x \div (y \div z) - \tau_A(x \div (y \div z)) \bullet \tau_B(x \div (y \div z))], [\tau_A(y) + \tau_B(y) - \tau_A(y) \bullet \tau_B(y)], [\tau_C(x \div (y \div z)), \tau_C(y)]\}\}$

If $A \subset B$ and $B \subset A$

$\tau_{(A\bullet B)\times_4 C}(x \div z) \geq \min\{\max\{\{[\tau_A(x \div (y \div z)) + \tau_B(x \div (y \div z) - \tau_A(x \div (y \div z)) \bullet \tau_B(x \div (y \div z))], [\tau_A(y) + \tau_B(y) - \tau_A(y) \bullet \tau_B(y)], [\tau_C(x \div (y \div z)), \tau_C(y)]\}\}$

$\tau_{(A\bullet B)\times_4 C}(x \div z) \geq \min\{\max\{\{[\tau_A(x \div (y \div z)) + \tau_B(x \div (y \div z) - \tau_A(x \div (y \div z)) \bullet \tau_B(x \div (y \div z))], \tau_C(x \div (y \div z))\}, \max\{[\tau_A(y) + \tau_B(y) - \tau_A(y) \bullet \tau_B(y)], \tau_C(y)]\}\}$

$\tau_{(A\bullet B)\times_4 C}(x \div z) \leq \min\{\tau_{(A\bullet B)\times_4 C}(x \div (y \div z)), \{\tau_{(A\bullet B)\times_4 C}(y)\}$

If A, B and C are antagonistic- intuitionistic fuzzy H-ideals in division BG-algebras X , the Cartesian product of $(A\bullet B) \times_4 C$ is also an antagonistic- intuitionistic fuzzy H-ideals in division BG-algebras X.

Example: Let X={0,1,a,b} with the following Cayley table:

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X	0	1	a	b
σ_A	.64	.53	.66	.68
τ_A	.36	.46	.33	.32

We define an antagonistic- intuitionistic fuzzy set B, then $B = < x, \sigma_B, \tau_B >$ by routine calculation B is antagonistic intuitionstic fuzzy H-ideal in division BG-algebras X.

X	0	1	A	b
σ_B	.62	.51	.57	.58
τ_B	.30	.47	.42	.40

We define an antagonistic- intuitionistic fuzzy set C, then $C = < x, \sigma_C, \tau_C >$ by routine calculation C is antagonistic intuitionistic fuzzy H-ideal in division BG-algebras X.

X	0	1	A	b
σ_C	.66	..44	.55	.53
τ_C	.35	.50	.42	.48

If A ,B and C are antagonistic intuitionistic fuzzy H-ideal of division BG-algebras X. Obviously, the Cartesian product $(A \bullet B) \times_2 C$. Since x = b, y = 1, z = a.

Now $\sigma_A(1) \leq \sigma_A(b)$, $\tau_A(1) \geq \tau_A(b)$ and $\sigma_B(1) \leq \sigma_B(b)$, $\tau_B(1) \geq \tau_B(b)$

(i) $\sigma_{(A\bullet B)\times_4 C}(1) \leq \sigma_{(A\bullet B)\times_4 C}(b)$ and $\tau_{(A\bullet B)\times_4 C}(1) \geq \tau_{(A\bullet B)\times_4 C}(b)$

$\min\{[\sigma_A(1) \bullet \sigma_B(1)], \sigma_C(1)\} \leq \min\{[\sigma_A(b) \bullet \sigma_B(b)], \sigma_C(b)\}$,

$\min(.53\bullet.51,.44) \leq \min(.68\bullet.58,.51)$

$\min(0.2703,.44) \leq \min(0.3944,.51)$

$0.2703 \leq 0.3944$

$\max\{[\tau_A(1) + \tau_B(1) - \tau_A(1) \bullet \tau_B(1)], \tau_C(1)\} \geq \max\{[\tau_A(b) + \tau_B(b) - \tau_A(b) \bullet \tau_B(b)], \tau_C(b)\}$

$\max(.46+.47 - .46\bullet.47,.54) \geq \max(.32+.40 - .32\bullet.40,.48)$

$\max(.7318,.54) \geq \max(.592,.48)$

$0.7137 \leq 0.592$

(ii) $\sigma_{(A\bullet B)\times_4 C}(b \div a) \leq \max\{\sigma_{(A\bullet B)\times_4 C}(b \div (1 \div a)), \sigma_{(A\bullet B)\times_4 C}(1)\}$

$\min\{\sigma_{(A\bullet B)}(b \div a), \sigma_C(b \div a)\} \leq \min\{\max\{\sigma_{(A\bullet B)}(b \div (1 \div a)), \sigma_{(A\bullet B)}(1)\}, \max\{\sigma_C(b \div (1 \div a)), \sigma_C(1)\}\}$

$\min\{[\sigma_A(b \div a) \bullet \sigma_B(b \div a)], \sigma_C(b \div a)\} \leq \min\{\max\{[\sigma_A(b \div (1 \div a)) \bullet \sigma_B(b \div (1 \div a))], [\sigma_A(1) \bullet \sigma_B(1)]\}, \max[\sigma_C(b \div (1 \div a)), \sigma_C(1)]\}$

$\min\{[\sigma_A(b) \bullet \sigma_B(b)], \sigma_C(b)\} \leq \min\{\max\{[\sigma_A(b) \bullet \sigma_B(b)], [\sigma_A(1) \bullet \sigma_B(1)]\}, \max[\sigma_C(b), \sigma_C(1)]\}$

$\min(.68\bullet.40,.50) \leq \max\{\min(.68\bullet.40,.51), \min(.53\bullet.51,.44)\}$

$\min(.272,.51) \leq \max\{\min(.272,.51), \min(..2703,.44)\}$

$0.272 \leq \max(.272,.2703)$

$0.272 \leq 0.272$

(iii) $\tau_{(A\bullet B)\times_4 C}(b \div a) \leq \min\{\tau_{(A\bullet B)\times_4 C}(b \div (1 \div a)), \{\tau_{(A\bullet B)\times_4 C}(1)\}$

$\max\{\tau_{(A\bullet B)}(b \div a), \tau_C(b \div a\} \geq \min\{\max\{\tau_{(A\bullet B)}(b \div (1 \div a)), \{\tau_{(A\bullet B)}(1)\}, \min\{\tau_C(b \div (1 \div a)), \tau_C(1)\}\}$

$\max\{[\tau_A(b \div a) + \tau_B(b \div a) - \tau_A(b \div a) \bullet \tau_B(b \div a)], \tau_C(b \div a)\} \geq \min\{\max\{[\tau_A(b \div (1 \div a)) + \tau_B(b \div (1 \div a)) - \tau_A(b \div (1 \div a)) \bullet \tau_B(b \div (1 \div a))], [\tau_A(1) + \tau_B(1) - \tau_A(1) \bullet \tau_B(1)], \min[\tau_C(b \div (1 \div a)), \tau_C(1)]\}$

$\max\{[\tau_A(b) + \tau_B(b) - \tau_A(b) \bullet \tau_B(b)], \tau_C(b)\} \geq \min\{\max\{\tau_A(b)) + \tau_B(b) - \tau_A(b) \bullet \tau_B(b), \tau_C(b)\}, \max\{\tau_A(1) + \tau_B(1) - \tau_A(1) \bullet \tau_B(1), \tau_C(1)\}\}$

$\max(.32+.40-.32\bullet.40,.48) \geq \min\{\max\{(.32+.40-.32\bullet.40,.48), (.46+.47-.46\bullet.47,.54)\}\}$

$\max(.5664,.48) \geq \min(.5664,.7138)$

$0.5664 \geq 0.5664$

If A, B and C are antagonistic- intuitionistic fuzzy H-ideals in division BG-algebras X, the Cartesian product of $(A\bullet B) \times_4 C$ is also an antagonistic- intuitionistic fuzzy H-ideals in division BG-algebras X

Thoerem 3.10: If A, B and C are antagonistic- intuitionistic fuzzy H-ideals in division BG-algebras X, then the Cartesian product of $(A@B) \times_5 C$ is also an antagonistic- intuitionistic fuzzy H-ideals in division BG-algebras X.

Proof : Let A , B and C are antagonistic- intuitionistic fuzzy H-ideals in division BG-algebras X.

(i) $\sigma_A(1) \leq \sigma_A(x)$, $\tau_A(1) \geq \tau_A(x)$, $\sigma_B(1) \leq \sigma_B(x)$, $\tau_B(1) \geq \tau_B(x)$
$\sigma_C(1) \leq \sigma_C(x)$, $\tau_C(1) \geq \tau_C(x)$

(ii) $\sigma_A(x \div z) \leq \text{Max}\{\sigma_A(x \div (y \div z)), \sigma_A(y)\}$
$\sigma_B(x \div z) \leq \max\{\sigma_B(x \div (y \div z)), \sigma_B(y)\}$
$\sigma_C(x \div z) \leq \max\{\sigma_C(x \div (y \div z)), \sigma_C(y)\}$

(iii) $\tau_A(x \div z) \geq \min\{\tau_A(x \div (y \div z)), \tau_A(y)\}$
$\tau_B(x \div z) \geq \min\{\tau_B(x \div (y \div z)), \tau_B(y)\}$
$\tau_C(x \div z) \geq \min\{\tau_C(x \div (y \div z)), \tau_C(y)\}$

Now (i) $\max\{\frac{[\sigma_A(1) + \sigma_B(1)]}{2}, \sigma_C(1)\} \leq \max\{\frac{[\sigma_A(x) + \sigma_B(x)]}{2}, \sigma_C(x)\}$,
$\text{Mini}\{\frac{[\tau_A(1) + \tau_B(1)]}{2}, \tau_C(1)\} \geq \min\{\frac{[\tau_A(x) + \tau_B(x)]}{2}, \tau_C(x)\}$
$\sigma_{(A@B)\times_5 C}(1) \leq \sigma_{(A@B)\times_5 C}(x)$, $\tau_{(A@B)\times_5 C}(1) \geq \tau_{(A@B)\times_5 C}(x)$

(ii) $\max\{\frac{[\sigma_A(x \div z) + \sigma_B(x \div z)]}{2}, \sigma_C(x \div z)\} \leq \max\{\max\{\frac{[\sigma_A(x \div (y \div z)) + \sigma_B(x \div (y \div z))]}{2}, \sigma_C(x \div (y \div z))\}, \max\{\frac{[\sigma_A(y) + \sigma_B(y)]}{2}, \sigma_C(y)\}\}$

$\max\{\frac{[\sigma_A(x \div z) + \sigma_B(x \div z)]}{2}, \sigma_C(x \div z)\} \leq \max\{\max\{\frac{[\sigma_A(x \div (y \div z)) + \sigma_B x \div (y \div z)}{2}, \frac{[\sigma_A(y) + \sigma_B(y)]}{2}\}, \max\{\sigma_C(x \div (y \div z)), \sigma_C(y)\}\}$

$\sigma_{(A@B)\times_5 C}(x \div z) \leq \max\{\max\{\frac{[\sigma_A(x \div (y \div z)) + \sigma_B(x \div (y \div z))]}{2}, \frac{[\sigma_A(y) + \sigma_B(y)]}{2}\}, \sigma_C(x \div (y \div z)), \sigma_C(y)\}\}$

$\sigma_{(A@B)\times_5 C}(x \div z) \leq \max\{\sigma_{(A@B)\times_5 C}(x \div (y \div z)), \sigma_{(A@B)\times_5 C}(y)\}$

(iii) $\text{mini}\{\frac{[\tau_A(x \div z) + \tau_B(x \div z)]}{2}, \tau_C(x \div z)\} \geq \text{mini}\{\text{mini}\{\frac{[\tau_A(x \div (y \div z)) + \tau_B(x \div (y \div z))]}{2}, \tau_C(x \div (y \div z))\}, \text{mini}\{\frac{[\tau_A(y) + \tau_B(y)]}{2}, \tau_C(y)\}\}$

$\text{mini}\{\frac{[\tau_A(x \div z) + \tau_B(x \div z)]}{2}, \tau_C(x \div z)\} \geq \text{mini}\{\text{mini}\{\frac{[\tau_A(x \div (y \div z)) + \tau_B(x \div (y \div z))]}{2}, \frac{[\tau_A(y) + \tau_B(y)]}{2}\}, \text{mini}\{\tau_C(x \div (y \div z)), \tau_C(y)\}\}$

$\tau_{(A@B)\times_5 C}(x \div z) \geq \text{mini}\{\text{mini}\{\frac{[\tau_A(x \div (y \div z)) + \tau_B(x \div (y \div z))]}{2}, \frac{[\tau_A(y) + \tau_B(y)]}{2}\}, \{\tau_C(x \div (y \div z)), \tau_C(y)\}\}$

$\tau_{(A@B)\times_5 C}(x \div z) \geq \text{mini}\{\tau_{(A@B)\times_5 C}(x \div (y \div z)), \sigma_{(A@B)\times_5 C}(y)\}$

If A, B and C are antagonistic- intuitionistic fuzzy H-ideals in division BG-algebras X , the Cartesian product of (A@B) $\times_5$ C is also an antagonistic- intuitionistic fuzzy H-ideals in division BG-algebras X.

Example: Let X={0,1,a,b} with the following Cayley table:

Table 1 : Cayley table for division BG-algebra.

÷	0	1	a	B
0	1	0	0	0
1	1	1	1	1
a	a	a	1	A
b	b	b	b	1

We define an antagonistic- intuitionistic fuzzy set A, then A = < x, σ_A, τ_A > by routine calculation A is antagonistic intuitionstic fuzzy H-ideal in division BG-algebras X.

X	0	1	A	b
σ_A	.64	.53	.66	.68
τ_A	.36	.46	.33	.32

We define an antagonistic- intuitionistic fuzzy set B, then B = < x, σ_B, τ_B > by routine calculation B is antagonistic intuitionstic fuzzy H-ideal in division BG-algebras X.

X	0	1	A	b
σ_B	.62	.51	.57	.58
τ_B	.30	.47	.42	.40

We define an antagonistic- intuitionistic fuzzy set C, then C = < x, σ_C, τ_C > by routine calculation C is antagonistic intuitionistic fuzzy H-ideal in division BG-algebras X.

X	0	1	A	b
σ_C	.66	..44	.55	.53
τ_C	.35	.50	.42	.48

If A ,B and C are antagonistic intuitionistic fuzzy H-ideal of division BG-algebras X. Obviously, the Cartesian product (A • B) $\times_2$C . Since x = 1, y = a, z = b.

Now $\sigma_A(1) \leq \sigma_A(1)$, $\tau_A(1) \geq \tau_A(1)$ and $\sigma_B(1) \leq \sigma_B(1)$, $\tau_B(1) \geq \tau_B(1)$

(i) $\sigma_{(A@B)\times_5 C}(1) \leq \sigma_{(A@B)\times_5 C}(1)$ and $\tau_{(A@B)\times_5 C}(1) \geq \tau_{(A@B)\times_5 C}(1)$

$\max\{\frac{[\sigma_A(1) + \sigma_B(1)]}{2}, \sigma_C(1)\} \leq \max\{\frac{[\sigma_A(1) + \sigma_B(1)]}{2}, \sigma_C(1)\}$

$\max(.52,.44) \leq \max(.52,.44)$

$.52 \leq .52$ and

$\text{mini}\{\frac{[\tau_A(1) + \tau_B(1)]}{2}, \tau_C(1)\} \geq \text{mini}\{\frac{[\tau_A(1) + \tau_B(1)]}{2}, \tau_C(1)\}$

$\text{mini}(.465,.54) \geq \text{mini}(.465,.54)$

$.465 \geq .465$

(ii) $\sigma_{(A@B)\times_5 C}(1 \div b) \leq \max\{\sigma_{(A@B)\times_5 C}(1 \div (a \div b)), \sigma_{(A@B)\times_5 C}(a)\}$

$\max\{\frac{[\sigma_A(1 \div b) + \sigma_B(1 \div b)]}{2}, \sigma_C(1 \div b)\} \leq \max\{\max\{\frac{[\sigma_A(1 \div (a \div b)) + \sigma_B(1 \div (a \div b))]}{2}, \frac{[\sigma_A(a) + \sigma_B(a)]}{2}\}, \max\{\sigma_C(1 \div (a \div b)), \sigma_C(a)\}\}$

$\max\{\frac{[\sigma_A(1)+\sigma_B(1)]}{2}, \sigma_C(1)\} \le \max\{\max\{\frac{[\sigma_A(1)+\sigma_B(1)}{2}, \frac{[\sigma_A(a)+\sigma_B(a)]}{2}\}, \max\{\sigma_C(1), \sigma_C(a)\}\}$

$\max(.52,.44) \le \max\{\max(.52,.44),(.615,.55)\}$

$.52 \le \max(.52,.615)$

$0.52 \le 0.615$

(iii) $\tau_{(A@B)\times_5 C}(1\div b) \ge \min\{\tau_{(A@B)\times_5 C}(1\div(a\div b)), \sigma_{(A@B)\times_5 C}(a)\}$

$\min\{\frac{[\tau_A(1\div b)+\tau_B(1\div b)]}{2}, \tau_C(1\div b)\} \ge \min\{\min\{\frac{[\tau_A(1\div(a\div b))+\tau_B(1\div(a\div b))]}{2}, \frac{[\tau_A(a)+\tau_B(a)]}{2}\}, \min\{\tau_C(1\div(a\div b)), \tau_C(a)\}\}$

$\min\{\frac{[\tau_A(1)+\tau_B(1)]}{2}, \tau_C(1)\} \ge \min\{\min\{\frac{[\tau_A(1)+\tau_B(1)]}{2}, \frac{[\tau_A(a)+\tau_B(a)]}{2}\}, \min\{\tau_C(1), \tau_C(a)\}\}$

$\min(.465,.54) \ge \min\{\min(.465,.54),\min(.375,.42)\}$

$0.465 \ge \min(.465,.375)$

$0.465 \ge 0.375$

If A, B and C are antagonistic- intuitionistic fuzzy H-ideals in division BG-algebras X , the Cartesian product of $(A@B)\times_5 C$ is also an antagonistic- intuitionistic fuzzy H-ideals in division BG-algebras X.

4.Conclusions:

In the present paper, we have introduced the concept of Cartesian product over antagonistic intuitionistic fuzzy *H*-ideals in division *BG*-algebras and investigated some of its essential properties. We think this work would enhance the scope for further study in this field of intuitionistic fuzzy sets.

It is our hope that this work is going to impact the upcoming research works in this field of *BG*-algebras with a new horizon of interest and innovation.

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Effects of Brownian Motion and Thermophoresis Phenomenon On MHD 3-D Dual Solutions Eyring–Powell Nanofluid Flow

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Abstract: *Here, author analyzed 3-D dual solutions of MHD Eyring –Powell nanofluid flow with viscous dissipation and heat generation/ absorption. Similarity transformation is used to convert the governing partial differential equations of the flow into nonlinear coupled ordinary differential. Runge-Kutta-Fehlberg-scheme via shooting technique is considered to give in the numerical solutions for the model. The effect of non-dimensional Eyring- Powell nanofluid on velocity, temperature and concentration outlines are discussed and offered through graphs. It is found that with growing effect of material fluid parameter, Brownian motion, and Thermophoresis parameter, dual solution of fluid, temperature and concentration profile increases.*

Keywords*:* Eyring-Powell Nanofluid; Heat generation or absorption; MHD;Viscous dissipation

Nomenclature

B Heat source or Sink parameter

C species concentration

C_{fx}, C_{fy} skin friction coefficients in the x and y directions

C_∞ species concentration far away from the wall

C_p specific heat at constant pressure

D_B Brownian diffusion coefficient

D_T Thermophoretic diffusion coefficient

Ec Eckert number

Ec_x Local Eckert number based on variable x

Ec_y Local Eckert number based on variable y

Le Lewis number

M Magnetic parameter

Nb Brownian motion parameter

Nt Thermophoresis parameter

Nu_x Local Nusselt number

Pr Prandtl number

q_w heat flux per unit area of the plate

q_m mass flux per unit area of the plate

Re_L Reynolds number

Sh_x Local Sherwood number

T Temperature in the boundary layer

T_∞ Temperature of the fluid far away from the wall

U, V Shrinking velocities, respectively

W Suction velocity

x,y,z coordinate directions

Greek Symbols

Γ dimensionless material fluid parameters

η transformed variable

ρ density of the fluid ν kinematic viscosity

μ dynamic viscosity of the fluid

I. Introduction and Literature Review

Being non-newtonian fluids, Eyring-powell nanofluids attracts the focus of many researchers to research towards this direction. Due to its complex fluid models, it has several advantages among other fluids models in day-to-day life which includes simplicity, ease of computation, physical robustness and additionally it correctly reduces to Newtonian behavior for low and high shear stress. Examples of such models are human blood, ketchup, toothpaste etc. (Gaffar et al., 2016) presented mathematical Eyring-Powell fluid model in a non-Darcy, isotropic, homogeneous porous medium with hall currents and ion slip currents to analyze the nonlinear, non-isothermal, MHD free convective boundary layer. Other recent studies of Eyring-Powell fluid which make a major breakdown in research world includes (Rehman et al. ,2016, Siddiqui et al. , 2014, Hayat et al., 2015, Agbaje et al.,2016, Hayat et al., 2014, Krishna et al., 2016).

Over the past few decades, Nanofluids which are colloidal suspension of nanoparticles in base fluids have been reported to possess higher thermal conductivity, e.g, it could be useful as coolants in automobiles and electronic industries. Due to several industrial applications of nanofluids, many researchers like Timofeeva et al. , 2011, Yu and Xie, 2012, Dharmalingam et al., 2014, Sreelakshmy et al. , 2014, De et al., 2017, Mondal et al. , 2017) etc found their utmost interest to work in this area.

The study of Eyring-Powell Nanofluid has not been given much attention so far. So, author aims to investigate MHD 3-D dual solutions of Eyring –Powell Nanofluid flow with viscous dissipation and heat generation/ absorption effects. A suitable similarity transformation has been employed to convert governing equation into non-linear coupled ordinary differential equation and then solved numerically with fifth-order-Runge-Kutta-Fehlberg- scheme using shooting technique. The present paper was grouped into seven sections including nomenclature, introduction, mathematical formulation, numerical method, result and discussion, conclusion and reference. In nomenclature section notations used in the paper was given and literature survey was included in introduction part. Problem formulation of Eyring-Powell Nanofluid model was discussed in mathematical formulation part and procedure of Runge-Kutta-Fehlberg scheme was explained in numerical method. The effects of physical parameters were explained graphically in results and discussion section and the whole conclusions drawn from present investigated problem were given in conclusion part. Finally all the references cited in this work were incorporated in reference section.

II. Mathematical Formulation

We reflect on 3-D MHD viscous incompressible electrically conducting Eyring-Powell Nanofluid due to stretching/ shrinking sheet coincide with the plane z=0 and the flow is restricted to z>0. The x and y axes are chosen along the length and breadth of the sheet and z axes is perpendicular to it. A constant magnetic field with strength B_0 is applied in the z- direction which is portrayed in physical model and co-ordinate system shown in Fig.1. The magnetic Reynolds number is assumed to be minute, so that the induced magnetic field is ignored and suction is applied normal to sheet to hold the vorticity. Other fluid properties are assumed constant throughout the motion. The Cauchy stress tensor in Powell- Eyring fluid is given by:

$$\tau_{ij} = \mu \frac{\partial u_i}{\partial x_j} + \frac{1}{\beta} sinh^{-1}\left(\frac{1}{\gamma}\frac{\partial u_i}{\partial x_j}\right),$$

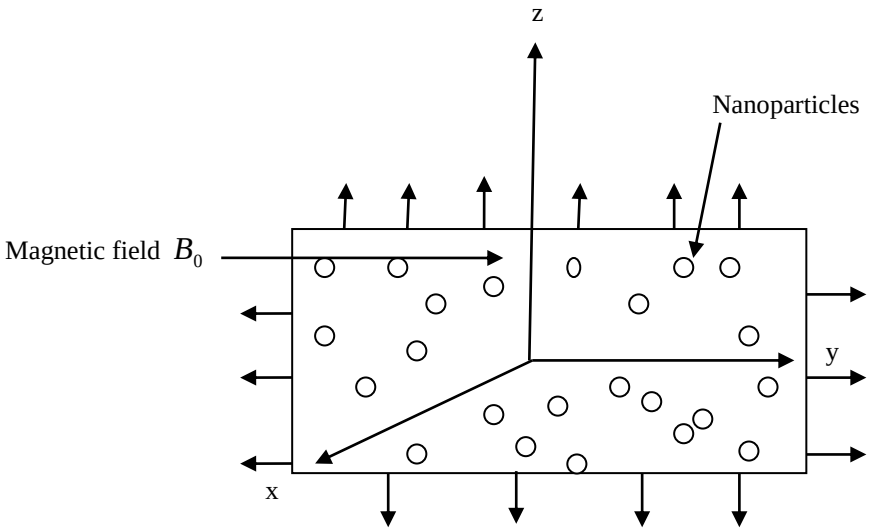


Fig.1 Physical Model Co-ordinate System of Eyring-Powell Nanofluid

Where μ is the viscosity coefficient, β and γ are the material fluid parameters of the Eyring-Powell fluid model. Under the usual boundary-layer approximations, the basic governing boundary layer equations with viscous dissipation and heat source/sink are given as (see Rajotia and Jat [14]):

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0,$$

(1)

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\left(\nu+\frac{1}{\rho\beta\gamma}\right)\frac{\partial^2 u}{\partial z^2}-\frac{1}{2\rho\beta\gamma^3}\left(\frac{\partial u}{\partial z}\right)^2-\frac{\sigma}{\rho}B_0^2 u, \tag{2}$$

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\left(\nu+\frac{1}{\rho\beta\gamma}\right)\frac{\partial^2 v}{\partial z^2}-\frac{1}{2\rho\beta\gamma^3}\left(\frac{\partial v}{\partial z}\right)^2-\frac{\sigma}{\rho}B_0^2 v$$

(3)

$$u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial z}+\left(\nu+\frac{1}{\rho\beta\gamma}\right)\frac{\partial^2 w}{\partial z^2}-\frac{1}{2\rho\beta\gamma^3}\left(\frac{\partial w}{\partial z}\right)^2$$

(4)

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}+w\frac{\partial T}{\partial z}=\frac{k}{\rho c_p}\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right)+\frac{\mu}{\rho c}\left[\begin{array}{l}2\left\{\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2+\left(\frac{\partial w}{\partial z}\right)^2\right\}+\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2+\\ \left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)^2\end{array}\right]$$

$$+\frac{Q}{\rho c_p}(T-T_\infty)+\tau\left[D_B\frac{\partial C}{\partial z}\frac{\partial T}{\partial z}+\frac{D_T}{T_\infty}\left(\frac{\partial T}{\partial Z}\right)^2\right]$$

(5)

$$u\frac{\partial C}{\partial x}+v\frac{\partial C}{\partial y}+w\frac{\partial C}{\partial z}=D_B\frac{\partial^2 C}{\partial z^2}+\frac{D_T}{T_\infty}\frac{\partial^2 T}{\partial z^2},$$

(6)

All the parameters of Eqs. (1)- (6) are defined in the Nomenclature section.

The appropriate boundary conditions are:

$$z=0:\quad u=-U=-ax,\quad v=-V=-ay,\quad w=-W\quad T=T_w,\quad C=C_w,$$

$$z\to\infty:\qquad u\to 0,\qquad v\to 0,\qquad T\to T_\infty,\qquad C\to C_\infty.$$

(7)

where $a>0$ is the shrinking constant, U and V are the shrinking velocities, $W>0$ is the suction velocity, T_w is the sheet temperature, C_w is the sheet concentration, and T_∞ and C_∞ is the free stream temperature and concentration, respectively.

The surface velocity is denoted by $u_s(x,t)=\frac{bx}{(1-at)}$, whereas the surface temperature $T_s(x,t)=T_\infty+T_{ref}\frac{bx^2}{2v}(1-at)^{-3/2}$ and the surface concentration $C_s(x,t)=C_\infty+C_{ref}\frac{bx^2}{2v}(1-at)^{-3/2}$. Here b (stretching rate) and a are positive constants having dimension time $^{-1}$. Also T_{ref}, C_{ref} are constant reference temperature and concentration respectively.

In order to obtain similarity solution of the problem we introduce the following non-dimensional variables (see Rajotia and Jat, 2014):

$$\eta=\sqrt{\frac{a}{\nu}}z,\quad u=axf'(\eta),\quad v=ayf'(\eta),\quad w=-2\sqrt{a\nu}\,f(\eta),$$

$$\theta(\eta)=\frac{T-T_\infty}{T_w-T_\infty},\qquad \phi(\eta)=\frac{C-C_\infty}{C_w-C_\infty}.$$

(8)

Eq. (1) is identically satisfied by the similarity transformation while Eq. (4) becomes:

$$\frac{p}{\rho}=\nu\frac{\partial w}{\partial z}-\frac{w^2}{2}+\text{constant},$$

(9)

Eqs. (2), (3),(5) and (6) reduce to the Eqs. (10)- (12),

$$(1+\Gamma)f'''+2ff''-f'^2-\Gamma\beta f''^2 f'''-Mf'=0,$$

(10)

$$\theta''+\Pr\left[2f\theta'+B\theta+12Ecf''+(Ec_x+Ec_y)f'^2+Nb\theta'\phi'+Nt\theta'^2\right]=0,$$

(11)

$$\phi''+\frac{Nt}{Nb}\theta''+Lef\phi'=0,$$

(12)

The corresponding boundary conditions (7) reduce to the following non-dimensional form:

$$\eta = 0: \quad f = S, \quad f' = -1, \quad \theta = 1, \quad \phi = 1,$$

$$\eta = \infty: \quad f' \to 0, \quad \theta \to 0, \quad \phi \to 0.$$

(13)

Where $S = \frac{W}{2\sqrt{a\nu}}$ is the suction parameter, $\Gamma = \frac{1}{\mu\beta\gamma}$ and $\beta = \frac{\rho u^3}{\mu x y^2}$ are the dimensionless material fluid parameters, $M = \frac{\sigma B_0^2}{\rho a}$ is the magnetic parameter, $Nb = \frac{\tau D_B (C_w - C_\infty)}{\nu}$ is the Brownian motion parameter, $Nt = \frac{\tau D_T (T_w - T_\infty)}{\nu T_\infty}$ is the thermophoresis parameter, $Le = \frac{\nu}{D_B}$ is the Lewis number, $\Pr = \frac{\mu c_p}{k}$ is the Prandtl number, $B = \frac{Q}{a\rho c_p}$ is the Heat Source $(B > 0)$ or Sink $(B < 0)$ parameter, $Ec = \frac{\mu a}{\rho c_p (T_w - T_\infty)}$ is the Eckert number,

$Ec_x = \frac{a^2 x^2}{c_p (T_w - T_\infty)}$ and $Ec_y = \frac{a^2 y^2}{c_p (T_w - T_\infty)}$ are the Local Eckert numbers based on the variables x and y, respectively.

The quantities of physical interest, namely, the local skin-friction coefficients C_f on the surface along the x and y directions, which are denoted by C_{fx} and C_{fy}, respectively, and the local Nusselt number Nu and the local Sherwood number Sh_x are given by:

$$C_{fx} = \frac{\tau_{wx}}{\rho U^2/2} = \frac{\mu\left(\frac{\partial u}{\partial z}\right)_{z=0}}{\rho U^2/2} = \frac{2}{\sqrt{\mathrm{Re}_x}} f''(0),$$

$$C_{fy} = \frac{\tau_{wy}}{\rho V^2/2} = \frac{\mu\left(\frac{\partial v}{\partial z}\right)_{z=0}}{\rho V^2/2} = \frac{2}{\sqrt{\mathrm{Re}_y}} f''(0),$$

$$Nu = -\frac{x\left(\frac{\partial T}{\partial z}\right)_{z=0}}{(T_w - T_\infty)} = -\sqrt{\mathrm{Re}_x}\,\theta'(0),$$

$$Sh_x = -\frac{x\left(\frac{\partial C}{\partial z}\right)_{z=0}}{(C_w - C_\infty)} = -\sqrt{\mathrm{Re}_x}\,\phi'(0).$$

Where τ_{wx} and τ_{wy} are the wall shear stresses along the x and y directions, respectively and $\mathrm{Re}_x = \frac{Ux}{\nu}$ and $\mathrm{Re}_y = \frac{Vy}{\nu}$ are the local Reynolds number.

III. Discussion of the Results

Fifth-order Runge-Kutta Scheme are considered to solve (10)- (12) with the boundary conditions (13) numerically using shooting scheme for the entire values of physical parameters like $\Gamma, \beta, M, B, Ec, Ec_x, Ec_y, Nb, NtLe, S, \mathrm{Pr}$. Here we consider dual solutions for unsteady Powell-Eyring Nanofluid flow over an stretching/shrinking sheet. The results are presented graphically in Figures 2-4 and conclusions are drawn to analyze the effects of Eyring –Powell parameters and Nanofluid parameters on velocity, temperature and concentration distribution that have significant effects. To measure validity of present code, author makes comparison of $-\theta'(0)$ (see Table 1) with Ishak et al. [15], Chen [16], and Pal and Mondal [17].

Table 1: Comparison of $-\theta'(0)$ with Ishak et al. [15], Chen [16], and Pal and Mondal [17] for various values of Pr and $\Gamma = \beta = M = B = Ec = Ec_x = Ec_y = Nt = Le = S = 0, Nb = 0.1$

Pr	Ishak et al. [15]	Chen [16]	Pal and Mondal [17]	Present Results
1.0	1.3333	1.33334	1.333333	1.181805
3.0	2.5097	2.50997	2.509715	2.469093
10.0	4.7969	4.79689	4.796871	4.713146

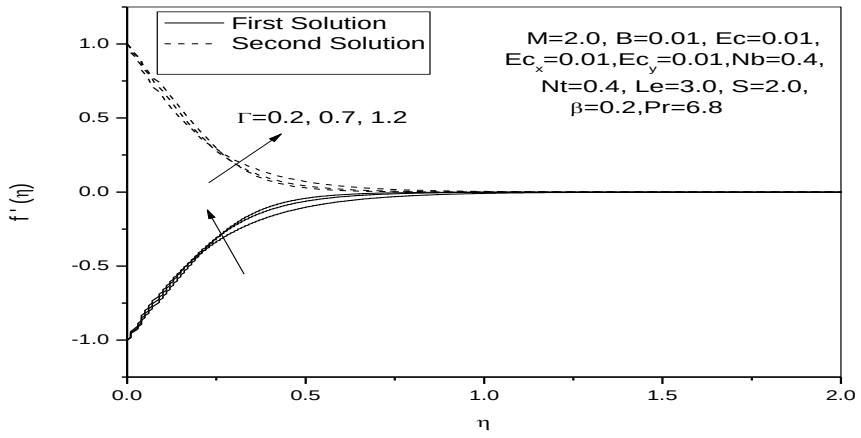


Fig 2. Dual velocity outline for different values of Γ

It can be noticed from Fig.2 that dual velocity outline increases for both first and second solutions with increasing effects on material parameters Γ. The physical reason is that due to rising values of Γ elevates the fluid velocity and thus velocity boundary layer enhanced.

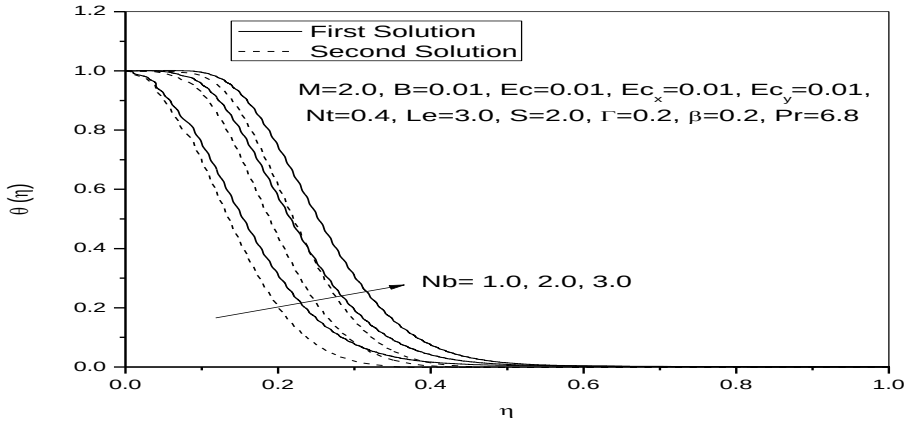


Fig.3. Dual temperature outline for different values of Nb

Fig. 3 depicts the nature of dual temperature for various value of Nb . Brownian motion is the random particles movement. Due to advanced molecular activity at superior values of Nb , transport of heat through thermal diffusion occur and fluid motion increase. Also with raise in temperature in the boundary layer, the kinetic energy of the particles enhance and thus increase the Brownian motion parameter as the particles start moving with a greater speed.

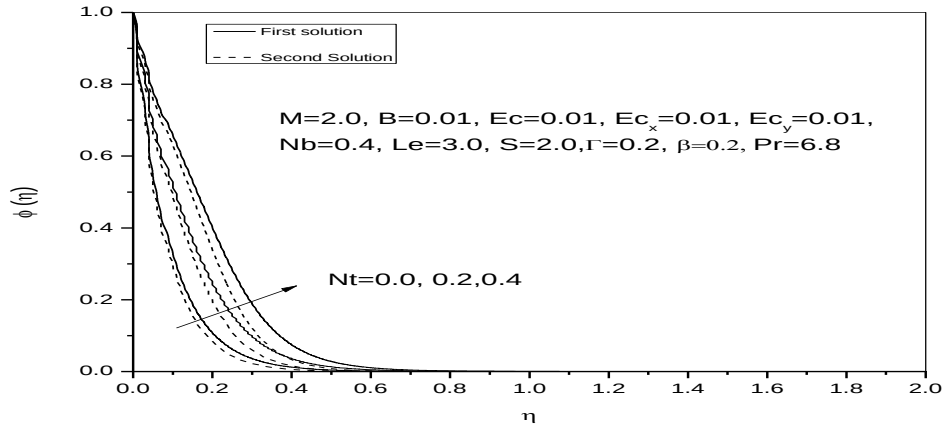


Fig.4. Dual temperature outline for different values of Nt

It can be observed from Fig. 4 that with increase in Nt leads to enhance the concentration outlines for both solutions. The reasons for this happening are that due to increase in Nt results in increasing temperature difference between the stretching surface and the ambient fluid.

IV. Conclusions

Numerical investigation on 3-D MHD dual solutions of Eyring Powell Nanofluid flow due to viscous dissipation and heat generation/absorption had been discussed. The following conclusion can be drawn from the graphical representation of Eyring Powell Nanofluid parameters:

i. With enhancement in Γ , dual velocity outline of both the solution increased.

ii. Due to rising effects of Nb , dual temperature outline enhanced.

iii. Increasing Nt , leads to rise in dual concentration outlines for both first and second solutions.

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$G_{\alpha,\beta}$ over antagonistic intuitionistic fuzzy sub implicative ideals and antagonistic intuitionistic fuzzy sub commutative ideals of subtraction G-algebra.

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Abstract: *The notions of $G_{\alpha,\beta}$ over antagonistic intuitionistic fuzzy sub implicative ideals, antagonistic intuitionistic fuzzy positive implicative ideals and antagonistic intuitionistic fuzzy sub commutative ideals of subtraction G-algebras are introduced. The characterization properties of antagonistic intuitionistic fuzzy sub implicative ideals, antagonistic intuitionistic fuzzy positive implicative ideals and antagonistic intuitionistic fuzzy sub commutative idealsare obtained. We study the effect of modal operators in antagonistic intuitionistic fuzzy ideals of subtraction G-algebras and the effect of modal operators on antagonistic intuitionistic fuzzy ideals of subtraction G-algebras under some interesting properties.*

Keywords: *Antagonistic intuitionistic-fuzzy subtraction G-algebra, Antagonistic intuitionistic-fuzzy sub implicative ideals, Antagonistic intuitionistic-fuzzy sub commutative ideals, Antagonistic intuitionistic fuzzy positive implicative ideals. Modal operator,some properties*.

I. Introduction

In 1966, Imai and Iseki [12] introduced two classes of abstract algebras, viz, BCK/BCI algebras. Bandaru and Rafi [8] introduced a new notion called G-algebra and Senapati et ai. [19] applied the fuzzy notions of G-algebras.The fuzzy G-algebras have been extensively investigated by several researches.After the introduction of fuzzy sets by Zadeh [20], there have been a number of generalizations of this fundamental concept. The notion of intuitionistic fuzzy sets introduced by Atanassov [1] is one among them.The concept of intuitionistic fuzzy subsets was introduced by Atanassov[1]. Which is generalization of the notion of fuzzy sets[17].The intuitionistic fuzzy modal operatorwas introduced by Atanassov[1] which are analogous to the modal logic operators of necessity and possibility. For BCK-algebras, Y. B. Jun et al. [13,14] introduced the notions of fuzzy positive implicative ideals and fuzzy commutative ideals, Y.L.Liu and J. Meng et al [15] sub implicative ideals and sub commutative ideals. Y.L.Liu, S.Y.Liu and J. Meng et al [16] fuzzy sub implicative ideals and fuzzy sub commutative ideals of BCI algebra.C.Ragavan and A.Solairaj, [17] some new results on intuitionistic fuzzy H-ideals in BCI algebra, Malaya Journal of mathematic 3(4) (2015),563-579. S.Senthilkumar, Eswari prem and C.Ragavan [18] intuitionistic fuzzy translation of anti-intuitionistic fuzzy T-idealsof subtraction BCK/BCI algebras, Malaya journal of matematik 6(3) (2018),701-710. Further the extension of all the operators is the operator $G_{\alpha,\beta}$ called (α,β) modal operator. The effect of all the modal operator on Antagonistic intuitionistic -fuzzy sub implicative ideals, Antagonistic intuitionistic -fuzzy sub commutative ideals, Antagonistic intuitionistic fuzzy positive implicative ideals. A lot of operators were defined and studied in [4,6,7].

Here in this paper, we study the effect of modal operators $G_{\alpha,\beta}$ in particular (α,β) modal operators on Antagonistic intuitionistic -fuzzy subtraction G-algebras and some properties are investigated.

II. 2. Preliminaries

Definition [8] A Subtraction G-algebra we mean a nonempty set X with a binary operation - and a constant 0 satisfying the following conditions:

(F_1) x- x = 0,

(F_2) x – (x - y) = y, for all x, y ϵ X

Definition [19] (*G*-subalgebra) A non-empty subset S of a subtraction G-algebra X is called a subtraction G-subalgebra of X if x-y ∈S [01].

Definition [20]: A fuzzy set *f* of a universe *X* is a function from *X* to the unit closed interval [0, 1], that is *f*: *X* → [0, 1].

Definition[1]: An intuitionistic fuzzy set A in a finite universe of discourse $X = \{x_1, x_2, x_3, x_4, \cdots x_n\}$ is given by $A = \{<x, \boldsymbol{\Psi_A}(x), \boldsymbol{\Omega_A}(x) >: x \in X\}$, Where $\boldsymbol{\Psi_A}: X \to [0, 1]$ and $\boldsymbol{\Omega_A}: X \to [0, 1]$ such that $0 \leq \boldsymbol{\Psi_A}(x) + \boldsymbol{\Omega_A}(x) \leq 1$. The number $\boldsymbol{\Psi_A}(x)$ and $\boldsymbol{\Omega_A}(x)$ denote the degree of membership and non-membership of $x \in X$ to A, respectively. For each IFS A in X, if $\boldsymbol{\pi_A}(x) = 1 - \boldsymbol{\Psi_A}(x) - \boldsymbol{\Omega_A}(x)$, for all $x \in X$

Definition: A nonempty subset I of X is called an ideal of X if (I_1): $0 \in I$, and (I_2): $x - y \in I$ and $y \in I$ imply $x \in I$.

Definition: A fuzzy subset Ψ_A of X is said to be a fuzzy ideal of X if it satisfies

(F_3) $\Psi_A(0) \geq \Psi_A(x)$ for all $x \in X$,

(F_4) $\Psi_A(x) \geq \min \{\Psi_A(x - y), \Psi_A(y)\}$ for all $x, y \in X$.

Definition: An intuitionistic fuzzy set $A = \{x, \Psi_A(x), \Omega_A(x): x \in X\}$ in X is called an intuitionistic fuzzy ideal of X if it satisfies

(F_5) $\Psi_A(0) \geq \Psi_A(x)$, $\Omega_A(0) \leq \Omega_A(x)$,

(F_6) $\Psi_A(x) \geq \min \{\Psi_A(x -y), \Psi_A(y)\}$ and

(F_7) $\Omega_A(x) \leq \max\{\Omega_A(x -y), \Omega_A(y)\}$ for all $x, y \in X$.

Definition: An intuitionistic fuzzy set $A = \{x, \Psi_A(x), \Omega_A(x): x \in X\}$ in X is called an antagonistic –intuitionistic fuzzy ideal of X if it satisfies

(F_8) $\Psi_A(0) \leq \Psi_A(x)$, $\Omega_A(0) \geq \Omega_A(x)$,

(F_9) $\Psi_A(x) \leq \max \{\Psi_A(x - y), \Psi_A(y)\}$ and

(F_{10}) $\Omega_A(x) \geq \min \{\Omega_A(x - y), \Omega_A(y)\}$ for all $x, y \in X$.

Definition [9]: A nonempty subset I of X is called a positive implicative ideal (i.e., weakly positive implicative ideal) of X if it satisfies (I_1) $0 \in I$ and (I_3): $((x-z)-z) -(y-z) \in I$ and $y \in I$ imply $x-z \in I$ [11].

Definition [15]: A nonempty subset I of X is called a sub-implicative ideal of X if it satisfies

(I_1) $0 \in I$ and

(I_3): $((x - (x - y)) - (y - x)) - z \in I$ and $z \in I$ imply $y-(y-x) \in I$ [9].

Definition [15]: A nonempty subset I of X is called a sub-commutative ideal of X if it satisfies (I_1) $0 \in I$ and

(I_4): $(y - (y - (x - (x - y)))) - z \in I$ and $z \in I$ imply $x - (x - y) \in I$ [9].

Definition: An antagonistic-fuzzy Subset $\boldsymbol{\Psi_A}$ of X is called an antagonistic -fuzzy sub-implicative ideal of X if it satisfies

(F_{11}) $\boldsymbol{\Psi_A}(0) \leq \boldsymbol{\Psi_A}(x)$ for all $x \in X$, and

(F_{12}) $\boldsymbol{\Psi_A}(y - (y - x)) \leq \max \{\boldsymbol{\Psi_A} (((x - (x - y)) - (y - x)) - z), \boldsymbol{\Psi_A}(z)\}$ for all $x, y, z \in X$.

Definition: An antagonistic-fuzzy subset Ψ_A of X is called an antagonistic -fuzzy sub-commutative ideal (briefly, AFSC-ideals) of X if it satisfies (F_1) and

(F_{13}) $\Psi_A(x - (x - y)) \leq \max \{\Psi_A((y - (y - (x - (x - y)))) - z), \Psi_A(z)\}$ for all $x, y, z \in X$.

Definition: An antagonistic-fuzzy subset μ_A of X is called an antagonistic -fuzzy positive implicative ideal of X if it satisfies

(F_{14}) $\Psi_A(0) \leq \Psi_A(x)$ and

(F_{15}) $\Psi_A(x – z) \leq \max \{\Psi_A((x -z)- z)- (y - z)), \Psi_A(y)\}$ for all $x, y, z \in X$.

Definition: An antagonistic -fuzzy Subset $(\boldsymbol{\Psi_A}, \boldsymbol{\Omega_A})$ of X is called an antagonistic -intuitionistic fuzzy sub-implicative ideal of X if it satisfies

(F_{16}) $\boldsymbol{\Psi_A}(0) \leq \boldsymbol{\Psi_A}(x), \boldsymbol{\Omega_A}(\boldsymbol{0}) \geq \boldsymbol{\Omega_A}(\boldsymbol{x})$ for all $x \in X$, and

(F_{17}) $\boldsymbol{\Psi_A}(y - (y - x)) \leq \max \{\boldsymbol{\Psi_A} (((x - (x - y)) - (y - x)) - z), \boldsymbol{\Psi_A}(z)\}$ for all $x, y, z \in X$.

(F_{18}) $\Omega_A(y - (y - x)) \geq \min \{\Omega_A (((x - (x - y)) - (y - x)) - z), \Omega_A(z)\}$ for all $x, y, z \in X$.

Definition: An anti-fuzzy subset $(\Psi_A, \Omega_A)$of X is called an antagonistic - intuitionistic fuzzy sub-commutative ideal (briefly, AFSC-ideals) of X if it satisfies

(F_{19}) $\boldsymbol{\Psi_A}(0) \leq \boldsymbol{\Psi_A}(x), \boldsymbol{\Omega_A}(\boldsymbol{0}) \geq \boldsymbol{\Omega_A}(\boldsymbol{x})$ for all $x \in X$, and

(F_{20}) $\Psi_A(x - (x - y)) \leq \max \{\Psi_A((y - (y - (x - (x - y)))) - z), \Psi_A(z)\}$ for all $x, y, z \in X$

(F_{21}) $\Omega_A(x - (x - y)) \geq \min \{\Omega_A ((y - (y - (x - (x - y)))) - z), \Omega_A(z))\}$ for all $x, y, z \in X$.

Definition: An antagonistic-fuzzy subset (Ψ_A, Ω_A) of X is called an antagonistic - intuitionistic fuzzy positive implicative ideal of X if it satisfies

(F_{22})$\Psi_A(0) \leq \Psi_A(x), \Omega_A(0) \geq \Omega_A(x)$ for all x ∈ X, and

(F_{23}) $\Psi_A(x - z) \leq \max\{\Psi_A((x - z) - z) - (y - z)), \Psi_A(y)\}$ for all x, y, z ∈ X.

(F_{24}) $\Omega_A(x - z) \geq \min\{\Omega_A((x - z) - z) - (y - z)), \Omega_A(y)\}$ for all x, y, z ∈ X.

3.$G_{\alpha,\beta}$Over Antagonistic-Intuitionistic Fuzzy Sub Implicative Ideal of subtraction G-algebras

Definition [6]: Given an intuitionistic fuzzy set A, we define the operator

$$G_{\alpha,\beta}(A) = \{(x, \alpha.\Psi_A(x), \beta.\Omega_A(x))/x \in E\}.$$

Theorem: If $G_{\alpha,\beta}(A)$ and $G_{\alpha,\beta}(B)$ are antagonistic-intuitionistic fuzzy sub-implicative ideal of subtraction G-algebra then $G_{\alpha,\beta}(A \cap B)$ is also antagonistic-intuitionistic fuzzy sub-implicative ideal of subtraction G-algebra.

Proof: Given $G_{\alpha,\beta}(A)$ and $G_{\alpha,\beta}(B)$ are antagonistic-intuitionistic fuzzy sub-implicative ideal of X.

(1) $\boldsymbol{\Psi_A}(0) \leq \boldsymbol{\Psi_A}(x), \boldsymbol{\Omega_A}(\mathbf{0}) \geq \boldsymbol{\Omega_A}(x)$ for all x ∈ X, and

(2) $\boldsymbol{\Psi_A}(y - (y - x)) \leq \max\{\boldsymbol{\Psi_A}(((x - (x - y)) - (y - x)) - z), \boldsymbol{\Psi_A}(z)\}$ for all x, y, z ∈X.

(3) $\Omega_A(y - (y - x)) \geq \min\{\Omega_A(((x - (x - y)) - (y - x)) - z), \Omega_A(z)\}$ for all x, y, z ∈X.

(4) $\Psi_B(0) \leq \Psi_B(x), \Omega_B(0) \geq \Omega_B(x)$ for all x ∈ X, and

(5) $\boldsymbol{\Psi_B}(y - (y - x)) \leq \max\{\boldsymbol{\Psi_B}(((x - (x - y)) - (y - x)) - z), \boldsymbol{\Psi_B}(z)\}$ for all x, y, z ∈X.

(6) $\Omega_B(y - (y - x)) \geq \min\{\Omega_B(((x - (x - y)) - (y - x)) - z), \Omega_B(z)\}$ for all x, y, z ∈X.

Case 1:

$$\min\{\Psi_A(0), \Psi_B(0)\} \leq \min\{\Psi_A(x), \Psi_B(x)\}$$
$$G_{\alpha,\beta}\{\min\{\Psi_A(0), \Psi_B(0)\}\} \leq G_{\alpha,\beta}\{\min\{\Psi_A(x), \Psi_B(x)\}\}$$
$$\min\{\alpha\Psi_A(0), \alpha\Psi_B(0)\} \leq \min\{\alpha\Psi_A(x), \alpha\Psi_B(x)\}$$
$$G_{\alpha,\beta}(A \cap B)(0) \leq G_{\alpha,\beta}(A \cap B)(x)$$
$$\max\{\Omega_A(0), \Omega_B(0)\} \geq \max\{\Omega_A(x), \Omega_B(x)\}$$
$$G_{\alpha,\beta}\max\{\Omega_A(0), \Omega_B(0)\} \geq G_{\alpha,\beta}\max\{\Omega_A(x), \Omega_B(x)\}$$
$$\max\{\beta\Omega_A(0), \beta\Omega_B(0)\} \geq \max\{\beta\Omega_A(x), \beta\,\Omega_B(x)\}$$
$$G_{\alpha,\beta}(A \cap B)(0) \geq G_{\alpha,\beta}(A \cap B)(x)$$

Case 2:

Min $\{\Psi_A(y-(y-x)), \Psi_B(y-(y-x)) \leq \min\{\max\{\Psi_A(((x-(x-y))-(y-x)-z)), \Psi_A(z)\}, \max\{\Psi_B(((x-(x-y))-(y-x)-z)), \Psi_B(z)\}\}$

If A$\subset B$

$G_{\alpha,\beta}$Min $\{\Psi_A(y-(y-x)), \Psi_B(y-(y-x)) \leq G_{\alpha,\beta}\max\{\min\{\Psi_A(((x-(x-y))-(y-x)-z)), \Psi_A(z)\}, \min\{\Psi_B(((x-(x-y))-(y-x)-z)), \Psi_B(z)\}\}$

Min $\{\alpha\Psi_A(y-(y-x)), \alpha\Psi_B(y-(y-x)) \leq G_{\alpha,\beta}\max\{\min\{\alpha\Psi_A(((x-(x-y))-(y-x)-z)), \alpha\Psi_A(z)\}, \min\{\alpha\Psi_B(((x-(x-y))-(y-x)-z)), \alpha\Psi_B(z)\}\}$

Min $\{\alpha\Psi_A(y-(y-x)), \alpha\Psi_B(y-(y-x)) \leq G_{\alpha,\beta}\max\{\min\{\alpha\Psi_A(((x-(x-y))-(y-x)-z)), \alpha\Psi_B(((x-(x-y))-(y-x)-z))\}, \min\{\alpha\Psi_A(z), \alpha\Psi_B(z)\}\}$

$G_{\alpha,\beta}(A \cap B)(y-(y-x)) \leq \max\{G_{\alpha,\beta}(A \cap B)(((x-(x-y))-(y-x)-z)), G_{\alpha,\beta}(A \cap B)(z)\}$

Case 3:

$\max\{\Omega_A(y-(y-x)), \Omega_B(y-(y-x))\} \geq \min\{\max\{\Omega_A(((x-(x-y))-(y-x)-z)), \Omega_A(z)\}, \min\{\Omega_B(((x-(x-y))-(y-x)-z)), \Omega_B(z)\}\}$

$G_{\alpha,\beta}\max\{\Omega_A(y-(y-x)), \Omega_B(y-(y-x))\} \geq G_{\alpha,\beta}\min\{\max\{\Omega_A(((x-(x-y))-(y-x)-z)), \Omega_A(z)\}, \max\{\Omega_B(((x-(x-y))-(y-x)-z)), \Omega_B(z)\}\}$

$\max\{\beta\Omega_A(y-(y-x)), \beta\Omega_B(y-(y-x))\} \geq \min\{\max\{\beta\Omega_A(((x-(x-y))-(y-x)-z)), \beta\Omega_A(z)\}, \{\beta\Omega_B(((x-(x-y))-(y-x)-z)), \beta\Omega_B(z)\}\}$

$\max\{\beta\Omega_A(y-(y-x)), \beta\Omega_B(y-(y-x))\} \geq \min\{\max\{\beta\Omega_A(((x-(x-y))-(y-x)-z)), \beta\Omega_B(((x-(x-y))-(y-x)-z))\}, \{\beta\Omega_A(z), \beta\Omega_B(z)\}\}$

$G_{\alpha,\beta}(A \cap B)(y-(y-x)) \geq \min\{G_{\alpha,\beta}(A \cap B)(((x-(x-y))-(y-x)-z)), G_{\alpha,\beta}(A \cap B)(z)\}$

Therefore A∩ B is an antagonistic-intuitionistic fuzzy sub-implicative ideal of X.

Example: Let X = {0, a, b, c} be a subtraction G-algebra with the following Cayley table.

-	0	A	B	c
0	0	A	B	c
a	0	0	B	c
b	0	A	0	c
c	0	A	B	0

We define an anti- fuzzy set A then A = < X, Ψ_A, Ω_A> by routine calculation A is an antagonistic-intuitionistic fuzzy sub-implicative ideal of X.

X	0	A	B	C
Ψ_A	.40	.42	.62	.75
Ω_A	.50	.45	.29	.21

We define an anti- fuzzy set B then B = < X, Ψ_B, Ω_B> by routine calculation B is an antagonistic-intuitionistic fuzzy sub-implicative ideal of X.

X	0	A	b	C
Ψ_B	.51	.53	.61	.72
Ω_B	.47	.42	.31	.26

If A and B are antagonistic-intuitionistic fuzzy sub-implicative ideal of X. obviously the A∩ B, Since x= b, y = c, z = a,α=.41,β=.46.

$$\min\{\alpha\Psi_A(0), \alpha\Psi_B(0)\} \leq \min\{\alpha\Psi_A(x), \alpha\Psi_B(x)\}$$
$$\min\{.41(.40), .41(.51)\} \leq \min\{.41(.62), .41(.61)\}$$
$$\min\{.164, .2091\} \leq \min\{.2542, .2501\}$$
$$.164 \leq .2501$$
$$\max\{\beta\Omega_A(0), \beta\Omega_B(0)\} \geq \max\{\beta\Omega_A(x), \beta\,\Omega_B(x)\}$$
$$\max\{.46(,50), .46(.47)\} \geq \max\{.46(.29), .46(.31)\}$$
$$\max\{.23, .2162\} \geq \max\{.1334, .1426\}$$
$$.23 \geq .1426$$

Min{$\alpha\Psi_A$(y-(y-x)),$\alpha\Psi_B$(y-(y-x)) ≤ $G_{\alpha,\beta}$max{min{$\alpha\Psi_A$(((x-(x-y))-(y-x)-z)),$\alpha\Psi_B$(((x-(x-y))-(y-x)-z)) ,min{$\alpha\Psi_A$(z),$\alpha\Psi_B$(z)}}

$$\min\{.41\Psi_A(b), .41\Psi_B(b)\} \leq \max\{\min\{.41\Psi_A(a), .41\Psi_B(a)\}, \min\{.41\Psi_A(a), .41\Psi_B(a)\}\}$$
$$\min\{.41(.62), .41(.61)\} \leq \max\{\min\{.41(.42), .41(.53)\}, \min\{.41(.42), .41(.53)\}\}$$
$$\min\{.2542, .2501\} \leq \max\{\min\{.1722, .2173\}, \min\{.1722, .2173\}\}$$
$$.2501 \not\leq .1722$$

Max{$\beta\Omega_A$(y-(y-x)),$\beta\Omega_B$(y-(y-x))} ≥min{max{$\beta\Omega_A$(((x-(x-y))-(y-x)-z)),$\beta\Omega_B$(((x-(x-y))-(y-x)-z))}, min{ $\beta\Omega_A$(z),$\beta\Omega_B$(z)}}

Max{.46Ω_A(b),.46Ω_B(b)} ≥min{max{.46Ω_A(a),.46Ω_B(a)}, min{.46Ω_A(a),.46Ω_B(a)}}

$$\max\{.46(.29), .46(.31)\} \geq \min\{\max\{.46(.45), .46(.42\}, \min\{.46(.45), .46\,(.42\}\}$$
$$\max\{.1334, .1426\} \geq \min\{\max\{.207, .1932\}, \max\{.207, .1932\}\}$$
$$.1426 \not\geq .207$$

Therefore, the given two intersection of antagonistic-intuitionistic fuzzy sub-implicative ideal of X is not an antagonistic-intuitionistic fuzzy sub-implicative ideal of X.

Theorem: If $G_{\alpha,\beta}$(A) and $G_{\alpha,\beta}$(B) are antagonistic-intuitionistic fuzzy sub-implicative ideal of subtraction G-algebra then $G_{\alpha,\beta}$(A∪B) is also antagonistic-intuitionistic fuzzy sub-implicative ideal of subtraction G-algebra.

Proof: Given $G_{\alpha,\beta}$(A) and $G_{\alpha,\beta}$(B) are antagonistic-intuitionistic fuzzy sub-implicative ideal of X.

(1) $\boldsymbol{\Psi_A}(0) \leq \boldsymbol{\Psi_A}(x), \boldsymbol{\Omega_A}(\mathbf{0}) \geq \boldsymbol{\Omega_A}(x)$ for all x ∈ X, and

(2) $\boldsymbol{\Psi_A}(y - (y - x)) \leq \max\{\boldsymbol{\Psi_A}(((x - (x - y)) - (y - x)) - z), \boldsymbol{\Psi_A}(z)\}$ for all x, y, z ∈X.

(3) $\Omega_A(y - (y - x)) \geq \min\{\Omega_A(((x - (x - y)) - (y - x)) - z), \Omega_A(z)\}$ for all x, y, z ∈X.

(4) $\Psi_B(0) \leq \Psi_B(x), \Omega_B(0) \geq \Omega_B(x)$ for all x ∈ X, and

(5) $\boldsymbol{\Psi_B}(y - (y - x)) \leq \max\{\boldsymbol{\Psi_B}(((x - (x - y)) - (y - x)) - z), \boldsymbol{\Psi_B}(z)\}$ for all x, y, z ∈X.

(6) $\Omega_B(y - (y - x)) \geq \min\{\Omega_B(((x - (x - y)) - (y - x)) - z), \Omega_B(z)\}$ for all x, y, z ∈X.

Case 1:

$$\max\{\Psi_A(0), \Psi_B(0)\} \leq \max\{\Psi_A(x), \Psi_B(x)\}$$
$$G_{\alpha,\beta}\{\max\{\Psi_A(0), \Psi_B(0)\}\} \leq G_{\alpha,\beta}\{\max\{\Psi_A(x), \Psi_B(x)\}\}$$
$$\max\{\alpha\Psi_A(0), \alpha\Psi_B(0)\} \leq \max\{\alpha\Psi_A(x), \alpha\Psi_B(x)\}$$
$$G_{\alpha,\beta}(A \cup B)(0) \leq G_{\alpha,\beta}(A \cup B)(x)$$
$$\min\{\Omega_A(0), \Omega_B(0)\} \geq \min\{\Omega_A(x), \Omega_B(x)\}$$
$$G_{\alpha,\beta}\min\{\Omega_A(0), \Omega_B(0)\} \geq G_{\alpha,\beta}\min\{\Omega_A(x), \Omega_B(x)\}$$
$$\min\{\beta\Omega_A(0), \beta\Omega_B(0)\} \geq \min\{\beta\Omega_A(x), \beta\,\Omega_B(x)\}$$
$$G_{\alpha,\beta}(A \cup B)(0) \geq G_{\alpha,\beta}(A \cup B)(x)$$

Case 2:

Max{Ψ_A(y-(y-x)),Ψ_B(y-(y-x) ≤max{max{Ψ_A(((x-(x-y))-(y-x)-z)),Ψ_A(z)},max{Ψ_B(((x-(x-y))-(y-x)-z)),Ψ_B(z)}}

$G_{\alpha,\beta}$Max{Ψ_A(y-(y-x)),Ψ_B(y-(y-x)) ≤ $G_{\alpha,\beta}$max{max{Ψ_A(((x-(x-y))-(y-x)-z)),Ψ_A(z)},max{Ψ_B(((x-(x-y))-(y-x)-z)),Ψ_B(z)}}

Max{$\alpha\Psi_A$(y-(y-x)),$\alpha\Psi_B$(y-(y-x)) ≤ $G_{\alpha,\beta}$max{max{$\alpha\Psi_A$(((x-(x-y))-(y-x)-z)),$\alpha\Psi_A$(z)},max{$\alpha\Psi_B$(((x-(x-y))-(y-x)-z)),$\alpha\Psi_B$(z)}}

Max{$\alpha\Psi_A$(y-(y-x)),$\alpha\Psi_B$(y-(y-x)) ≤ $G_{\alpha,\beta}$max{max{$\alpha\Psi_A$(((x-(x-y))-(y-x)-z)),$\alpha\Psi_B$(((x-(x-y))-(y-x)-z)),max{$\alpha\Psi_A$(z),$\alpha\Psi_B$(z)}}

$G_{\alpha,\beta}$(A∪B) (y-(y-x)) ≤ max{ $G_{\alpha,\beta}$(A∪B) (((x-(x-y))-(y-x)-z)),$G_{\alpha,\beta}$(A∪B) (z)}

Case 3:

Min{Ω_A(y-(y-x)),Ω_B(y-(y-x))} ≥min{min{Ω_A(((x-(x-y))-(y-x)-z)),Ω_A(z)},min{Ω_B(((x-(x-y))-(y-x)-z)),Ω_B(z)}}

$G_{\alpha,\beta}$ min{ Ω_A (y-(y-x)), Ω_B (y-(y-x))} ≥ $G_{\alpha,\beta}$ min{min{ Ω_A (((x-(x-y))-(y-x)-z)), Ω_A (z)},min{ Ω_B (((x-(x-y))-(y-x)-z)),Ω_B(z)}}

Min{ $\beta\Omega_A$ (y-(y-x)), $\beta\Omega_B$ (y-(y-x))} ≥ min{min{ $\beta\Omega_A$ (((x-(x-y))-(y-x)-z)), $\beta\Omega_A$ (z)},min{ $\beta\Omega_B$ (((x-(x-y))-(y-x)-z)),$\beta\Omega_B$(z)}}

Min{ $\beta\Omega_A$ (y-(y-x)), $\beta\Omega_B$ (y-(y-x))} ≥ min{min{ $\beta\Omega_A$ (((x-(x-y))-(y-x)-z)), $\beta\Omega_B$ (((x-(x-y))-(y-x)-z))}, min{ $\beta\Omega_A$(z),$\beta\Omega_B$(z)}}

$G_{\alpha,\beta}$(A∪B) (y-(y-x)) ≥ min{ $G_{\alpha,\beta}$(A∪B) (((x-(x-y))-(y-x)-z)),$G_{\alpha,\beta}$(A∪B) (z)}

Therefore A∪B is an antagonistic-intuitionistic fuzzy sub-implicative ideal of X.

Example: Let X = {0, a, b, c} be a subtraction G-algebra with the following Cayley table.

-	0	a	B	C
0	0	a	B	C
a	0	0	B	C
b	0	a	0	C
c	0	a	B	0

We define an anti- fuzzy set A then A = < X, Ψ_A, Ω_A> by routine calculation A is an antagonistic-intuitionistic fuzzy sub-implicative ideal of X.

X	0	a	B	C
Ψ_A	.40	.42	.62	.75
Ω_A	.50	.45	.29	.21

We define an anti- fuzzy set B then B = < X, Ψ_B, Ω_B> by routine calculation B is an antagonistic-intuitionistic fuzzy sub-implicative ideal of X.

X	0	a	B	C
Ψ_B	.51	.53	.61	.72
Ω_B	.47	.42	.31	.26

If A and B are antagonistic-intuitionistic fuzzy sub-implicative ideal of X. obviously the A∪ B, Since x= b, y = c, z = a,α=.41,β=.46.

$$\max\{\alpha\Psi_A(0), \alpha\Psi_B(0)\} \leq \max\{\alpha\Psi_A(x), \alpha\Psi_B(x)\}$$
$$\max\{.41(.40), .41(.51)\} \leq \max\{.41(.62), .41(.61)\}$$
$$\max\{.164, .2091\} \leq \max\{.2542, .2501\}$$
$$.2091 \leq .2542$$
$$\min\{\beta\Omega_A(0), \beta\Omega_B(0)\} \geq \min\{\beta\Omega_A(x), \beta\,\Omega_B(x)\}$$
$$\min\{.46(,50), .46(.47)\} \geq \min\{.46(.29), .46(.31)\}$$
$$\min\{.23, .2162\} \geq \min\{.1334, .1426\}$$
$$.2162 \geq .1334$$

Max{ $\alpha\Psi_A$ (y-(y-x)), $\alpha\Psi_B$ (y-(y-x)) ≤ $G_{\alpha,\beta}$ max{max{ $\alpha\Psi_A$ (((x-(x-y))-(y-x)-z)), $\alpha\Psi_B$ (((x-(x-y))-(y-x)-z)) ,max{$\alpha\Psi_A$(z),$\alpha\Psi_B$(z)}}

Max{$.41\Psi_A$(b),$.41\Psi_B$(b)≤ $G_{\alpha,\beta}$max{max{$.41\Psi_A$(a),$.41\Psi_B$(a), max{$.41\Psi_A$(a),$.41\Psi_B$(a)}}

$$\max\{.41(.62), .41(.61)\} \leq \max\{\max\{.41(.42), .41(.53)\}, \max\{.41(.42), .41(.53)\}\}$$
$$\max\{.2542, .2501\} \leq \max\{\max\{.1722, .2173\}, \min\{.1722, .2173\}\}$$
$$.2542 \not\leq .2173$$

Min{$\beta\Omega_A$(y-(y-x)),$\beta\Omega_B$(y-(y-x))} ≥min{min{$\beta\Omega_A$(((x-(x-y))-(y-x)-z)),$\beta\Omega_B$(((x-(x-y))-(y-x)-z))}, min{ $\beta\Omega_A$(z),$\beta\Omega_B$(z)}}

$$\min\{.46\Omega_A(b), .46\Omega_B(b)\} \geq \min\{\min\{.46\Omega_A(a), .46\Omega_B(a)\}, \min\{.46\Omega_A(\boldsymbol{a}), .46\,\Omega_B(\boldsymbol{a})\}\}$$
$$\min\{.46(.29), .46(.31)\} \geq \min\{\min\{.46(.45), .46(.42\}, \min\{.46(.45), .46\,(.42\}\}$$
$$\min\{.1334, .1426\} \geq \min\{\min\{.207, .207\}, \min\{.207, .207\}\}$$
$$.1334 \not\geq .207$$

Therefore, the given two union of antagonistic-intuitionistic fuzzy sub-implicative ideal of X is not an antagonistic-intuitionistic fuzzy sub-implicative ideal of X.

4.$G_{\alpha,\beta}$ Over Antagonistic-Intuitionistic Fuzzy Sub Commutative Ideal of subtraction G-algebras

Theorem: If $G_{\alpha,\beta}$(A) and $G_{\alpha,\beta}$(B) are antagonistic-intuitionistic fuzzy sub-commutative ideal of subtraction G-algebra then $G_{\alpha,\beta}$(A+B) is also antagonistic-intuitionistic fuzzy sub-commutative ideal of subtraction G-algebra.

Proof: Given $G_{\alpha,\beta}$(A) and $G_{\alpha,\beta}$(B) are antagonistic-intuitionistic fuzzy sub-commutative ideal of X.

(1) $\boldsymbol{\Psi_A}(0) \leq \boldsymbol{\Psi_A}(x), \boldsymbol{\Omega_A}(\mathbf{0}) \geq \boldsymbol{\Omega_A}(x)$ for all x ∈ X, and

(2) Ψ_A (x - (x - y)) ≤ max {Ψ_A ((y - (y - (x - (x - y)))) - z),Ψ_A(z)} for all x, y, z ∈ X.

(3)Ω_A (x - (x - y)) ≥min {Ω_A ((y - (y - (x - (x - y)))) - z),Ω_A(z))} for all x, y, z ∈ X.

(4) $\boldsymbol{\Psi_B}(0) \leq \boldsymbol{\Psi_B}(x), \boldsymbol{\Omega_B}(\mathbf{0}) \geq \boldsymbol{\Omega_B}(x)$ for all x ∈ X, and

(5) Ψ_B (x - (x - y)) ≤ max {Ψ_B ((y - (y - (x - (x - y)))) - z),Ψ_B(z)} for all x, y, z ∈ X.

(6)Ω_B (x - (x - y)) ≥min {Ω_B ((y - (y - (x - (x - y)))) - z),Ω_B(z))} for all x, y, z ∈ X.

Case1:

$$\Psi_A(0) + \Psi_B(0) - \Psi_A(0).\Psi_B(0) \leq \Psi_A(x) + \Psi_B(x) - \Psi_A(x).\Psi_B(x)$$
$$G_{\alpha,\beta}\{\Psi_A(0) + \Psi_B(0) - \Psi_A(0).\Psi_B(0)\} \leq G_{\alpha,\beta}\{\Psi_A(x) + \Psi_B(x) - \Psi_A(x).\Psi_B(x)\}$$
$$\{\alpha\Psi_A(0) + \alpha\Psi_B(0) - \alpha\Psi_A(0).\alpha\Psi_B(0)\} \leq \{\alpha\Psi_A(x) + \alpha\Psi_B(x) - \alpha\Psi_A(x).\alpha\Psi_B(x)\}$$
$$\{\alpha\Psi_A(0) + \alpha\Psi_B(0) - \alpha^2\Psi_A(0).\Psi_B(0)\} \leq \{\alpha\Psi_A(x) + \alpha\Psi_B(x) - \alpha^2\Psi_A(x).\Psi_B(x)\}$$
$$G_{\alpha,\beta}(A+B)(0) \leq G_{\alpha,\beta}(A+B)(x)\}$$
$$\Omega_A(0).\Omega_B(0) \geq \Omega_A(x).\Omega_B(x)$$
$$G_{\alpha,\beta}\{\Omega_A(0).\Omega_B(0)\} \geq G_{\alpha,\beta}\{\Omega_A(x).\Omega_B(x)\}$$
$$\{\beta\Omega_A(0).\beta\Omega_B(0)\} \geq \{\beta\Omega_A(x).\beta\Omega_B(x)\}$$
$$\{\beta^2\Omega_A(0).\Omega_B(0)\} \geq \{\beta^2\Omega_A(x).\Omega_B(x)\}$$
$$G_{\alpha,\beta}(A+B)(0) \geq G_{\alpha,\beta}(A+B)(x)\}$$

Case2:

$$\{\Psi_A(x-(x-y)) + \Psi_B(x-(x-y)) - \Psi_A(x-(x-y)).\Psi_B(x-(x-y))\} \leq \max\{[\Psi_A((y-(y-(x-(x-y)))-z), \Psi_A(z) + \max \Psi_B((y-(y-(x-(x-y)))-z), \Psi_B(z)] - \max\{\Psi_A((y-(y-(x-(x-y)))-z), \Psi_A(z)\}.\ max\{\Psi_B((y-(y-(x-(x-y)))-z), \Psi_B(z)\}\}$$
$$G_{\alpha,\beta}\{\Psi_A(x-(x-y)) + \Psi_B(x-(x-y)) - \Psi_A(x-(x-y)).\Psi_B(x-(x-y))\} \leq G_{\alpha,\beta}\max\{\{\Psi_A((y-(y-(x-(x-y)))-z) + \Psi_B((y-(y-(x-(x-y)))) - \Psi_A((y-(y-(x-(x-y)))-z).\Psi_B((y-(y-(x-(x-y)))), \Psi_A(z)+\Psi_B(z)-\Psi_A(z).\Psi_B(z)\}\}$$
$$\{\alpha\Psi_A(x-(x-y)) + \alpha\Psi_B(x-(x-y)) - \alpha\Psi_A(x-(x-y)).\alpha\Psi_B(x-(x-y))\} \leq \max\{\{\alpha\Psi_A((y-(y-(x-(x-y)))-z) + \alpha\Psi_B((y-(y-(x-(x-y)))-z) - \alpha\Psi_A((y-(y-(x-(x-y)))-z).\alpha\Psi_B((y-(y-(x-(x-y)))-z), \alpha\Psi_A(z) + \alpha\Psi_B(z) - \alpha\Psi_A(z).\alpha\Psi_B(z)\}\}$$
$$\{\alpha\Psi_A(x-(x-y)) + \alpha\Psi_B(x-(x-y)) - \alpha^2\Psi_A(x-(x-y)).\Psi_B(x-(x-y))\} \leq \max\{\{\alpha\Psi_A((y-(y-(x-(x-y)))-z) + \alpha\Psi_B((y-(y-(x-(x-y)))-z) - \alpha^2\Psi_A((y-(y-(x-(x-y)))-z).\Psi_B((y-(y-(x-(x-y)))-z)\}, \{\alpha\Psi_A(z) + \alpha\Psi_B(z) - \alpha^2\Psi_A(z).\Psi_B(z)\}\}$$
$$G_{\alpha,\beta}(A+B)(x-(x-y)) \leq \max\{G_{\alpha,\beta}(A+B)((y-(y-(x-(x-y)))-z),\ G_{\alpha,\beta}(A+B)(z)\}$$

Case3:

$$\Omega_A(x-(x-y)).\Omega_B(x-(x-y)) \geq \min\{\Omega_A((y-(y-(x-(x-y))))-z), \Omega_A(z)\}.\ \min\{\Omega_B((y-(y-(x-(x-y))))-z), \Omega_B(z)\}$$
$$\Omega_A(x-(x-y)).\Omega_B(x-(x-y)) \geq \min\{\Omega_A((y-(y-(x-(x-y))))-z).\Omega_B((y-(y-(x-(x-y))))-z\},\ \min\{\Omega_A(z).\Omega_B(z)\}$$
$$G_{\alpha,\beta}\{\Omega_A(x-(x-y)).\Omega_B(x-(x-y))\} \geq G_{\alpha,\beta}\{\min\{\Omega_A((y-(y-(x-(x-y))))-z).\Omega_B((y-(y-(x-(x-y))))-z,\},\ \min\{\Omega_A(z).\Omega_B(z)\}\}$$
$$\{\beta\Omega_A(x-(x-y)).\beta\Omega_B(x-(x-y))\} \geq \min\{\beta\Omega_A((y-(y-(x-(x-y))))-z).\ \beta\Omega_B((y-(y-(x-(x-y))))-z\},\ \min\{\alpha\Omega_A(z)).\alpha\Omega_B(z)\}$$
$$\{\beta^2\Omega_A(x-(x-y)).\Omega_B(x-(x-y))\} \geq \min\{\{\beta^2\Omega_A((y-(y-(x-(x-y))))-z).\Omega_B((y-(y-(x-(x-y))))-z,\}, \{\beta^2\Omega_A(z)).\alpha\Omega_B(z\}\}$$
$$G_{\alpha,\beta}(A+B)\ (x-(x-y)) \geq \min\{G_{\alpha,\beta}(A+B)\ ((y-(y-(x-(x-y)))), G_{\alpha,\beta}(A+B)(z)\}$$

Therefore, the two numbers of A+ B is an antagonistic- intuitionistic fuzzy sub-commutative ideal in subtraction G-algebra of X

Example: Let X = {0, a, b, c} be a subtraction G-algebra with the following Cayley table.

-	0	a	B	C
0	0	a	B	C
A	0	0	B	C
B	0	a	0	C
C	0	a	B	0

We define an anti- fuzzy set A then A = < X, Ψ_A, Ω_A> by routine calculation A is an antagonistic-intuitionistic fuzzy sub-implicative ideal of X.

X	0	a	B	C
Ψ_A	.40	.42	.62	.75
Ω_A	.50	.45	.29	.21

We define an anti- fuzzy set B then B = < X, Ψ_B, Ω_B> by routine calculation B is an antagonistic-intuitionistic fuzzy sub-implicative ideal of X.

X	0	a	B	C
Ψ_B	.51	.53	.61	.72
Ω_B	.47	.42	.31	.26

If A and B are antagonistic-intuitionistic fuzzy sub-implicative ideal of X. obviously the A+ B, Since x= b, y = c, z = a,α=.41,β=.46.

Case1:

$$\{\alpha\Psi_A(0) + \alpha\Psi_B(0) - \alpha^2\Psi_A(0).\Psi_B(0)\} \leq \{\alpha\Psi_A(x) + \alpha\Psi_B(x) - \alpha^2\Psi_A(x).\Psi_B(x)\}$$
$$\{.41(.40) + .41(.51) - .41^2(.40)(.51)\} \leq \{.41(.62) + .41(.61) - .41^2(.62)(.61)\}$$
$$.3388 \leq .4407$$
$$\{\beta^2\Omega_A(0).\Omega_B(0)\} \geq \{\beta^2\Omega_A(x).\Omega_B(x)\}$$
$$\{.46^2(.50)(.47)\} \geq \{.46^2(.29)(.31)\}$$
$$.0497 \geq .0192$$

Case2:

$$\{\alpha\Psi_A(x-(x-y)) + \alpha\Psi_B(x-(x-y)) - \alpha^2\Psi_A(x-(x-y)).\Psi_B(x-(x-y))\} \leq \max\{\{\alpha\Psi_A((y-(y-(x-(x-y)))-z) + \alpha\Psi_B((y-(y-(x-(x-y)))-z) - \alpha^2\Psi_A((y-(y-(x-(x-y)))-z).\Psi_B((y-(y-(x-(x-y)))-z)\},\{\alpha\Psi_A(z) + \alpha\Psi_B(z) - \alpha^2\Psi_A(z).\Psi_B(z)\}\}$$
$$\{.41\Psi_A(c) + .41\Psi_B(c) - .41^2\Psi_A(c).\Psi_B(c)\} \leq \max\{\{.41\Psi_A(a) + .41\Psi_B((a) - .41^2\Psi_A(a).\Psi_B(a)\},\{.41\Psi_A(a) + .41\Psi_B(a) - .41^2\Psi_A(a).\Psi_B(a)\}\}$$
$$.41(.75) + .41(.73) - .41^2(.75)(.73)\} \leq \max\{\{.41(.42) + .41(.53) - .41^2(.42).(53)\}, \{.41(.42) + .41(.53) - .41^2(.42)(.53)\}\}$$
$$.5148 \leq \max\{.3521,.3521\}$$
$$.5148 \not\leq .3521$$

Case3:

$$\{\beta^2\Omega_A(x-(x-y)).\Omega_B(x-(x-y))\} \geq \min\{\{\beta^2\Omega_A((y-(y-(x-(x-y))))-z).\Omega_B((y-(y-(x-(x-y))))-z,\},\{\alpha^2\Omega_A(z)).\alpha\Omega_B(z\}\}$$
$$\{.46^2\Omega_A(c).\Omega_B(c)\} \geq \min\{\{.46^2\Omega_A(a).\Omega_B(a)\}, \{.46^2\Omega_A(a)).\Omega_B(a)\}\}$$
$$\{.46^2(.21)(.26)\} \geq \min\{\{.46^2(.45)(.42)\}, \{.46^2(.45)(.42)\}$$
$$.0116 \geq \min\{.0399,.0399\}$$
$$.0116 \geq .0399$$

Therefore, the given two addition of antagonistic-intuitionistic fuzzy sub-implicative ideal of X is an antagonistic-intuitionistic fuzzy sub-implicative ideal of X.

Theorem: If $G_{\alpha,\beta}$(A) and $G_{\alpha,\beta}$(B) are antagonistic-intuitionistic fuzzy sub-commutative ideal of subtraction G-algebra then $G_{\alpha,\beta}$(A•B) is also antagonistic-intuitionistic fuzzy sub-commutative ideal of subtraction G-algebra.

Proof: Given $G_{\alpha,\beta}$(A) and $G_{\alpha,\beta}$(B) are antagonistic-intuitionistic fuzzy sub-commutative ideal of X.

(1) $\boldsymbol{\Psi_A}(0) \leq \boldsymbol{\Psi_A}(x), \boldsymbol{\Omega_A}(\mathbf{0}) \geq \boldsymbol{\Omega_A}(\boldsymbol{x})$ for all x ∈ X, and

(2) $\Psi_A(x-(x-y)) \leq \max\{\Psi_A((y-(y-(x-(x-y))))-z), \Psi_A(z)\}$ for all x, y, z ∈ X.

(3) $\Omega_A(x-(x-y)) \geq \min\{\Omega_A((y-(y-(x-(x-y))))-z), \Omega_A(z))\}$ for all x, y, z ∈ X.

(4) $\boldsymbol{\Psi_B}(0) \leq \boldsymbol{\Psi_B}(x), \boldsymbol{\Omega_B}(\mathbf{0}) \geq \boldsymbol{\Omega_B}(\boldsymbol{x})$ for all x ∈ X, and

(5) $\Psi_B(x-(x-y)) \leq \max\{\Psi_B((y-(y-(x-(x-y))))-z), \Psi_B(z)\}$ for all x, y, z ∈ X.

(6) $\Omega_B(x-(x-y)) \geq \min\{\Omega_B((y-(y-(x-(x-y))))-z), \Omega_B(z))\}$ for all x, y, z ∈ X.

Case1:

$$\{\Psi_A(0).\Psi_B(0)\} \leq \{\Psi_A(x).\Psi_B(x)\}$$
$$G_{\alpha,\beta}\{\Psi_A(0).\Psi_B(0)\} \leq G_{\alpha,\beta}\{\Psi_A(x).\Psi_B(x)\}$$
$$\{\alpha\Psi_A(0).\alpha\Psi_B(0)\} \leq \{\alpha\Psi_A(x).\alpha\Psi_B(x)\}$$
$$\{\alpha^2\Psi_A(0).\Psi_B(0)\} \leq \{\alpha^2\Psi_A(x).\Psi_B(x)\}$$
$$G_{\alpha,\beta}(A.B)(0) \leq G_{\alpha,\beta}(A.B)(x)$$
$$\{\Omega_A(0)+\Omega_B(0) - \Omega_A(0).\Omega_B(0)\} \geq \{\Omega_A(x)+\Omega_B(x)-\Omega_A(x).\Omega_B(x)\}$$
$$G_{\alpha,\beta}\{\Omega_A(0)+\Omega_B(0) - \Omega_A(0).\Omega_B(0)\} \geq G_{\alpha,\beta}\{\Omega_A(x)+\Omega_B(x)-\Omega_A(x).\Omega_B(x)\}$$
$$\{\beta\Omega_A(0)+\beta\Omega_B(0) - \beta\Omega_A(0).\beta\Omega_B(0)\} \geq \{\beta\Omega_A(x)+\beta\,\Omega_B(x)-\beta\Omega_A(x).\beta\Omega_B(x)\}$$
$$\{\beta\Omega_A(0)+\beta\Omega_B(0) - \beta^2\Omega_A(0).\Omega_B(0)\} \geq \{\beta\Omega_A(x)+\beta\,\Omega_B(x)-\beta^2\Omega_A(x).\Omega_B(x)\}$$
$$G_{\alpha,\beta}(A.B)(0) \geq G_{\alpha,\beta}(A.B)(x)$$

Case2:

$\{\Psi_A (x - (x - y)).\Psi_B (x - (x - y))\} \leq \{\max \{\Psi_A ((y - (y - (x - (x - y)))) - z),\Psi_A(z)\}.\max \{\Psi_B ((y - (y - (x - (x - y)))) - z),\Psi_B(z)\}\}$

$\{\Psi_A (x - (x - y)).\Psi_B (x - (x - y))\} \leq \max \{\Psi_A ((y - (y - (x - (x - y)))) - z).\Psi_B ((y - (y - (x - (x - y)))) - z)\}, \max\{\Psi_A(z).\Psi_B(z)\}\}$

$\{\Psi_A (x - (x - y)).\Psi_B (x - (x - y))\} \leq \max \{\{\Psi_A ((y - (y - (x - (x - y)))) - z).\Psi_B ((y - (y - (x - (x - y)))) - z)\}, \{\Psi_A(z).\Psi_B(z)\}\}$

$G_{\alpha,\beta}\{\Psi_A (x - (x - y)).\Psi_B (x - (x - y))\} \leq G_{\alpha,\beta} \max \{\{\Psi_A ((y - (y - (x - (x - y)))) - z).\Psi_B ((y - (y - (x - (x - y)))) - z)\}, \{\Psi_A(z).\Psi_B(z)\}\}$

$\{\alpha\Psi_A (x - (x - y)).\ \alpha\Psi_B (x - (x - y))\} \leq \max \{\{\alpha\Psi_A ((y - (y - (x - (x - y)))) - z).\alpha\ \Psi_B ((y - (y - (x - (x - y)))) - z)\}, \{\alpha\Psi_A(z).\ \alpha\Psi_B(z)\}\}$

$\{\alpha^2\Psi_A (x - (x - y)).\ \Psi_B (x - (x - y))\} \leq \max \{\{\alpha^2\Psi_A ((y - (y - (x - (x - y)))) - z).\Psi_B ((y - (y - (x - (x - y)))) - z)\}, \{\alpha^2\Psi_A(z).\Psi_B(z)\}\}$

$G_{\alpha,\beta}(A.B) (x - (x - y)\} \leq \max\{G_{\alpha,\beta}(A.B) ((y - (y - (x - (x - y)))) - z),G_{\alpha,\beta}(A.B)(z)\}$

Case3:

$\Omega_A (x - (x - y))+\Omega_B (x - (x - y))-\Omega_A (x - (x - y)).\Omega_B (x - (x - y)) \geq \min \{\Omega_A ((y - (y - (x - (x - y)))) - z),\Omega_A(z))\}+\min \{\Omega_B ((y - (y - (x - (x - y)))) - z),\Omega_B(z))\}-\min \{\Omega_A ((y - (y - (x - (x - y)))) - z),\Omega_A(z))\}.\min \{\Omega_B ((y - (y - (x - (x - y)))) - z),\Omega_B(z))\}$

$\Omega_A (x - (x - y))+\Omega_B (x - (x - y))-\Omega_A (x - (x - y)).\Omega_B (x - (x - y)) \geq \{\min \{\Omega_A ((y - (y - (x - (x - y)))) - z)+\Omega_B ((y - (y - (x - (x - y)))) - z)-\Omega_A ((y - (y - (x - (x - y)))) - z).\Omega_B ((y - (y - (x - (x - y)))) - z),\Omega_B(z))\},\min\{\Omega_A(z))+\Omega_B(z))-\Omega_A(z)).\Omega_B(z))\}\}$

$\Omega_A (x - (x - y))+\Omega_B (x - (x - y))-\Omega_A (x - (x - y)).\Omega_B (x - (x - y)) \geq \min \{\{\Omega_A ((y - (y - (x - (x - y)))) - z)+\Omega_B ((y - (y - (x - (x - y)))) - z)-\Omega_A ((y - (y - (x - (x - y)))) - z).\Omega_B ((y - (y - (x - (x - y)))) - z),\Omega_B(z))\},\{\Omega_A(z))+\Omega_B(z))-\Omega_A(z)).\Omega_B(z))\}\}$

$G_{\alpha,\beta}\{\Omega_A (x - (x - y))+\Omega_B (x - (x - y))-\Omega_A (x - (x - y)).\Omega_B (x - (x - y))\} \geq \min G_{\alpha,\beta} \{\{\Omega_A ((y - (y - (x - (x - y)))) - z)+\Omega_B ((y - (y - (x - (x - y)))) - z)-\Omega_A ((y - (y - (x - (x - y)))) - z).\Omega_B ((y - (y - (x - (x - y)))) - z)\},\{\Omega_A(z))+\Omega_B(z))-\Omega_A(z)).\Omega_B(z))\}\}$

$\{\beta\Omega_A (x - (x - y))+\beta\Omega_B (x - (x - y))-\beta\Omega_A (x - (x - y)).\ \beta\Omega_B (x - (x - y)) \geq \min \{\{\beta\Omega_A ((y - (y - (x - (x - y)))) - z)+\beta\Omega_B ((y - (y - (x - (x - y)))) - z)-\beta\Omega_A ((y - (y - (x - (x - y)))) - z).\beta\Omega_B ((y - (y - (x - (x - y)))) - z)\},\{\beta\Omega_A(z))+\beta\Omega_B(z))-\beta\Omega_A(z)).\beta\Omega_B(z))\}\}$

$\{\beta\Omega_A (x - (x - y))+\beta\Omega_B (x - (x - y))-\beta^2\Omega_A (x - (x - y)).\Omega_B (x - (x - y)) \geq \min \{\{\beta\Omega_A ((y - (y - (x - (x - y)))) - z)+\beta\Omega_B ((y - (y - (x - (x - y)))) - z)-\beta^2\Omega_A ((y - (y - (x - (x - y)))) - z).\Omega_B ((y - (y - (x - (x - y)))) - z)\},\{\beta\Omega_A(z))+\beta\Omega_B(z))-\beta^2\Omega_A(z)).\Omega_B(z))\}\}$

$G_{\alpha,\beta}(A.B) (x - (x - y)\} \geq \min\{G_{\alpha,\beta}(A.B) ((y - (y - (x - (x - y)))) - z),G_{\alpha,\beta}(A.B)(z)\}$

Therefore, the two numbers of A. B is an antagonistic- intuitionistic fuzzy sub-commutative ideal in subtraction G-algebra of X.

Example: Let X = {0, a, b, c} be a subtraction G-algebra with the following Cayley table.

-	0	A	B	C
0	0	A	B	C
A	0	0	B	C
B	0	A	0	C
C	0	A	B	0

We define an anti- fuzzy set A then A = < X, Ψ_A, Ω_A> by routine calculation A is an antagonistic-intuitionistic fuzzy sub-implicative ideal of X.

X	0	a	b	c
Ψ_A	.40	.42	.62	.75
Ω_A	.50	.45	.29	.21

We define an anti- fuzzy set B then B = < X, Ψ_B, Ω_B> by routine calculation B is an antagonistic-intuitionistic fuzzy sub-implicative ideal of X.

X	0	a	b	c
Ψ_B	.51	.53	.61	.72
Ω_B	.47	.42	.31	.26

If A and B are antagonistic-intuitionistic fuzzy sub-implicative ideal of X. obviously the A. B, Since x= b, y = c, z = a,α=.41,β=.46.

Case1:

$\{\alpha^2\Psi_A(0).\Psi_B(0)\} \leq \{\alpha^2\Psi_A(x).\Psi_B(x)\}$

$\{.41^2(.40)(.51)\} \leq .41^2(.62)(.61)\}$

$$.0343 \leq .0636$$

$$\{\beta\Omega_A(0)+\beta\Omega_B(0) - \beta^2\Omega_A(0).\Omega_B(0)\} \geq \{\beta\Omega_A(x)+\beta\,\Omega_B(x)-\beta^2\Omega_A(x).\Omega_B(x)\}$$

$$\{.46(.50) + .46(.47) - .46^2(.46)(.47)\} \geq \{.46(.29) + .46\,(.31) - .46^2(.29)\,(.31)\}$$

$$.4605 \geq .2569$$

Case2:

$$\{\alpha^2\Psi_A\,(x - (x - y)).\,\Psi_B\,(x - (x - y))\} \leq \max\{\{\alpha^2\Psi_A\,((y - (y - (x - (x - y)))) - z).\Psi_B\,((y - (y - (x - (x - y)))) - z)\}, \{\alpha^2\Psi_A(z).\Psi_B(z)\}\}$$

$$\{.41^2\Psi_A\,(b - (b - c)).\,\Psi_B\,(b - (b - c))\} \leq \max\{\{.41^2\Psi_A\,((c - (c - (b - (b - c)))) - a).\Psi_B\,((c - (c - (b - (b - c)))) - a)\},\ \{.41^2\Psi_A(a).\Psi_B(a)\}\}$$

$$\{.41^2\Psi_A\,(c).\,\Psi_B\,(c)\} \leq \max\{\{.41^2\Psi_A\,(a).\Psi_B\,(a)\},\ \{.41^2\Psi_A(a).\Psi_B(a)\}\}$$

$$\{.41^2(.75).\,(.73\} \leq \max\{\{.41^2(.62)(.61)\},\{.41^2(.62).61)\}$$

$$.0920 \not\leq .0636$$

Case3:

$$\{.46\Omega_A\,(b - (b - c)) + .46\Omega_B\,(b - (b - c)) - .46^2\Omega_A\,(b - (b - c)).\Omega_B\,(b - (b - c)) \geq \min\{\{.46\Omega_A\,((c - (c - (b - (b - c)))) - a) + .46\Omega_B\,((c - (c - (b - (b - c)))) - a) - .46^2\Omega_A\,((c - (c - (b - (b - c)))) - a).\Omega_B\,((c - (c - (b - (b - c)))) - a)\},\{.46\Omega_A(a)) + .46\Omega_B(a)) - .46^2\Omega_A(a)).\Omega_B(a))\}\}$$

$$\{.46\Omega_A\,(c) + .46\Omega_B\,(c) - .46^2\Omega_A\,(c).\Omega_B\,(c) \geq \min\{\{.46\Omega_A\,(a) + .46\Omega_B\,(a) - .46^2\Omega_A\,(a).\Omega_B\,(a)\},\{.46\Omega_A(a)) + .46\Omega_{*B}(a)) - .46^2\Omega_A(a)).\Omega_B(a))\}\}$$

$$.46(.21) + .46(.26) - .46^2(.21).26) \geq \min\{\{.46(.45) + .46(.42) - .46^2(.45).\,(.42)\},\ \{.46(.45) + .46(.42) - .46(.45.).\,(.41)\}\}$$

$$.2047 \not\geq .3603$$

Therefore, the given two products of antagonistic-intuitionistic fuzzy sub-implicative ideal of X is not an antagonistic-intuitionistic fuzzy sub-implicative ideal in subtraction G algebra of X.

5.$G_{\alpha,\beta}$ Over Antagonistic-Intuitionistic Fuzzy positive Implicative Ideal of subtraction G-algebras

Theorem: If $G_{\alpha,\beta}$(A) and $G_{\alpha,\beta}$(B) are antagonistic-intuitionistic fuzzy positive implicative ideal of subtraction G-algebra then $G_{\alpha,\beta}$(A@B) is also antagonistic-intuitionistic fuzzy positive implicative ideal of subtraction G-algebra.

Proof: Given $G_{\alpha,\beta}$(A) and $G_{\alpha,\beta}$(B) are antagonistic-intuitionistic fuzzy positive ideal of X.

(1) $\Psi_A\,(0) \leq \Psi_A\,(x)$, $\Omega_A(0) \geq \Omega_A(x)$ for all x ∈ X, and

(2) $\Psi_A\,(x - z) \leq \max\{\Psi_A\,((x - z) - z) - (y - z)), \Psi_A(y)\}$ for all x, y, z ∈ X.

(3) $\Omega_A(x - z) \geq \min\{\Omega_A\,((x - z) - z) - (y - z)), \Omega_A(y)\}$ for all x, y, z ∈ X.

(4) $\Psi_B\,(0) \leq \Psi_B\,(x)$, $\Omega_B(0) \geq \Omega_B(x)$ for all x ∈ X, and

(5) $\Psi_B\,(x - z) \leq \max\{\Psi_B\,((x - z) - z) - (y - z)), \Psi_B(y)\}$ for all x, y, z ∈ X.

(6) $\Omega_B(x - z) \geq \min\{\Omega_B\,((x - z) - z) - (y - z)), \Omega_B(y)\}$ for all x, y, z ∈ X.

Case1:

$$\{\Psi_A(0) + \Psi_B(0)\}/2 \leq \{\Psi_A(x) + \Psi_B(x)\}/2$$

$$G_{\alpha,\beta}\{\Psi_A(0) + \Psi_B(0)\}/2 \leq G_{\alpha,\beta}\{\Psi_A(x) + \Psi_B(x)\}/2$$

$$\{\alpha\Psi_A(0) + \alpha\Psi_B(0)\}/2 \leq \{\alpha\Psi_A(x) + \alpha\Psi_B(x)\}/2$$

$$\alpha\,\{\Psi_A(0) + \Psi_B(0)\}/2 \leq \alpha\,\{\Psi_A(x) + \Psi_B(x)\}/2$$

$$G_{\alpha,\beta}(A@B)(0) \leq G_{\alpha,\beta}(A@B)(x)$$

$$\{\Omega_A(0) + \Omega_B(0)\}/2 \geq \{\Omega_A(x) + \Omega_B(x)\}/2$$

$$G_{\alpha,\beta}\{\Omega_A(0) + \Omega_B(0)\}/2 \geq G_{\alpha,\beta}\{\Omega_A(x) + \Omega_B(x)\}/2$$

$$\{\beta\Omega_A(0) + \beta\Omega_B(0)\}/2 \geq \{\beta\Omega_A(x) + \beta\Omega_B(x)\}/2$$

$$\beta\,\{\Omega_A(0) + \Omega_B(0)\}/2 \geq \beta\,\{\Omega_A(x) + \Omega_B(x)\}/2$$

$$G_{\alpha,\beta}(A@B)\,(0) \geq G_{\alpha,\beta}(A@B)\,(x)$$

Case2:

$$\{\Psi_A(x-z) + \Psi_B(x-z)\}/2 \leq \{\max\{\Psi_A((x-z)-z)-(y-z), \Psi_A(y)\} + \max\{\Psi_B((x-z)-z)-(y-z), \Psi_B(y)\}\}/2$$

$$\{\Psi_A(x-z) + \Psi_B(x-z)\}/2 \leq \{\max\{\Psi_A((x-z)-z) - (y-z) + \Psi_B((x-z)-z)-(y-z)\},\ \max\{\Psi_A(y) + \Psi_B(y)\}\}/2$$

$$\{\Psi_A(x-z) + \Psi_B(x-z)\}/2 \leq \max\{\Psi_A((x-z)-z)-(y-z) + \Psi_B((x-z)-z)-(y-z)\}/2, \max\{\Psi_A(y) + \Psi_B(y)\}/2$$

$$G_{\alpha,\beta}\{\Psi_A(x-z) + \Psi_B(x-z)\}/2 \leq G_{\alpha,\beta}\{\max\{\Psi_A((x-z)-z)-(y-z) + \Psi_B((x-z)-z)-(y-z)\}/2, \max\{\Psi_A(y) + \Psi_B(y)\}/2\}$$

$$\{\alpha\Psi_A(x-z) + \alpha\Psi_B(x-z)\}/2 \leq \max\{\alpha\Psi_A((x-z)-z)-(y-z) + \alpha\Psi_B((x-z)-z)-(y-z)\}/2, \{\alpha\Psi_A(y) + \alpha\Psi_B(y)\}/2$$

$$\alpha\{\Psi_A(x-z) + \Psi_B(x-z)\}/2 \leq \max\{\alpha\{\Psi_A((x-z)-z)-(y-z) + \Psi_B((x-z)-z)-(y-z)\}/2, \alpha\{\Psi_A(y) + \Psi_B(y)\}/2\}$$

$$G_{\alpha,\beta}(A@B)(x\text{-}z) \leq \max\{G_{\alpha,\beta}(A@B)\Psi_A((x\text{-}z)\text{-}z)\text{-}(y\text{-}z), G_{\alpha,\beta}(A@B)(y)\}$$

Case3:

$$\{\Omega_A(x\text{-}z)+\Omega_B(x\text{-}z)\}/2 \geq \{\min\{\Omega_A((x\text{-}z)\text{-}z)\text{-}(y\text{-}z), \Omega_A(y)\} + \min\{\Omega_B((x\text{-}z)\text{-}z)\text{-}(y\text{-}z), \Omega_B(y)\}\}/2$$

$$\{\Omega_A(x\text{-}z)+\Omega_B(x\text{-}z)\}/2 \geq \{\min\{\Omega_A((x\text{-}z)\text{-}z)\text{-}(y\text{-}z)+\Omega_B((x\text{-}z)\text{-}z)\text{-}(y\text{-}z)\}, \min\{\Omega_A(y)+\Omega_B(y)\}\}/2$$

$$\{\Omega_A(x\text{-}z)+\Omega_B(x\text{-}z)\}/2 \geq \min\{\Omega_A((x\text{-}z)\text{-}z)\text{-}(y\text{-}z)+\Omega_B((x\text{-}z)\text{-}z)\text{-}(y\text{-}z)\}/2, \min\{\Omega_A(y)+\Omega_B(y)\}/2$$

$$G_{\alpha,\beta}\{\Omega_A(x\text{-}z)+\Omega_B(x\text{-}z)\}/2 \geq G_{\alpha,\beta}\{\min\{\Omega_A((x\text{-}z)\text{-}z)\text{-}(y\text{-}z)+\Omega_B((x\text{-}z)\text{-}z)\text{-}(y\text{-}z)\}/2, \max\{\Omega_A(y)+\Omega_B(y)\}/2\}$$

$$\{\beta\alpha\Omega_A(x\text{-}z)+\beta\Omega_B(x\text{-}z)\}/2 \geq \min\{\beta\Omega_A((x\text{-}z)\text{-}z)\text{-}(y\text{-}z)+\beta\Omega_B((x\text{-}z)\text{-}z)\text{-}(y\text{-}*z)\}/2, \{\beta\Psi_A(y)+\beta\Omega_B(y)\}/2$$

$$\beta\{\Omega_A(x\text{-}z)+\Omega_B(x\text{-}z)\}/2 \geq \min\{\beta\{\Omega_A((x\text{-}z)\text{-}z)\text{-}(y\text{-}z)+\Omega_B((x\text{-}z)\text{-}z)\text{-}(y\text{-}z)\}/2, \{\Omega_A(y)+\Omega_B(y)\}/2\}$$

$$G_{\alpha,\beta}(A@B)(x\text{-}z) \geq \min\{G_{\alpha,\beta}(A@B)\Psi_A((x\text{-}z)\text{-}z)\text{-}(y\text{-}z), G_{\alpha,\beta}(A@B)(y)\}$$

Therefore, the two numbers of A@B are an antagonistic- intuitionistic fuzzy positive implicative ideal in subtraction G-algebra of X.

Example: Let X = {0, a, b, c} be a subtraction G-algebra with the following Cayley table.

-	0	A	B	C
0	0	A	B	C
a	0	0	B	C
b	0	A	0	C
c	0	A	B	0

We define an anti- fuzzy set A then A = < X, Ψ_A, Ω_A> by routine calculation A is an antagonistic-intuitionistic fuzzy sub-implicative ideal of X.

X	0	A	B	c
Ψ_A	.40	.42	.62	.75
Ω_A	.50	.45	.29	.21

We define an anti- fuzzy set B then B = < X, Ψ_B, Ω_B> by routine calculation B is an antagonistic-intuitionistic fuzzy sub-implicative ideal of X.

X	0	A	B	c
Ψ_B	.51	.53	.61	.72
Ω_B	.47	.42	.31	.26

If A and B are antagonistic-intuitionistic fuzzy sub-implicative ideal of X. obviously the A@B, Since x= b, y = c, z = a,α=.41,β=.46.

Case1:

$$\alpha\ \{\Psi_A(0)+\Psi_B(0)\}/2 \leq\ \alpha\ \{\Psi_A(x)+\Psi_B(x)\}/2$$

$$.41\{.40+.51\}/2 \leq \{.41\ \{.62+.61\}\}/2$$

$$.1865 \leq .2521$$

$$\beta\ \{\Omega_A(0)+\Omega_B(0)\}/2 \geq\ \beta\ \{\Omega_A(x)+\Omega_B(x)\}/2$$

$$.46\ \{.50+.47\}/2 \geq\ .46\ \{.29+.31\}/2$$

$$.2231 \geq .138$$

Case2:

$$\alpha\{\Psi_A(x\text{-}z)+\Psi_B(x\text{-}z)\}/2 \leq \max\{\alpha\{\Psi_A((x\text{-}z)\text{-}z)\text{-}(y\text{-}z)+\Psi_B((x\text{-}z)\text{-}z)\ \text{-}(y\text{-}z)\}/2, \alpha\{\Psi_A(y)+\Psi_B(y)\}/2\}$$

$$.41\{\Psi_A(a)+\Psi_B(a)\}/2 \leq \max\{.41\{\Psi_A(a)+\Psi_B(a)\}/2, \alpha\{\Psi_A(c)+\Psi_B(c)\}/2\}$$

$$.41\{.42+.53\}/2 \leq \max\ \{.41(.42+.53)\}/2, .41(.29+.61)/2\}$$

$$.1947 \leq \max\ \{.1947, .1845\}$$

$$.1947 \leq .1947$$

Case3:

$$\beta\{\Omega_A(x\text{-}z)+\Omega_B(x\text{-}z)\}/2 \geq \min\{\beta\{\Omega_A((x\text{-}z)\text{-}z)\text{-}(y\text{-}z)+\Omega_B((x\text{-}z)\text{-}z)\text{-}(y\text{-}z)\}/2, \{\Omega_A(y)+\Omega_B(y)\}/2\}$$

$$.46\{\Omega_A(x\text{-}z)+\Omega_B(x\text{-}z)\}/2 \geq \min\{.46\{\Omega_A((x\text{-}z)\text{-}z)\text{-}(y\text{-}z)+\Omega_B((x\text{-}z)\text{-}z)\text{-}(y\text{-}z)\}/2, \{\Omega_A(y)+\Omega_B(y)\}/2\}$$

$$.46\{\Omega_A(a)+\Omega_B(a)\}/2 \geq \min\{.46\{\Omega_A(a)+\Omega_B(a)\}/2, \{\Omega_A(c)+\Omega_B(c)\}/2\}$$

$$.46(.45+.42)/2 \geq \min\{.46\{.45+.42\}/2, .46\{.21+.26\}/2\}$$

$$.2001 \geq \min\{.2001, .1001\}$$

$$.2001 \geq .1001$$

Therefore, the given A@B of antagonistic-intuitionistic fuzzy sub-implicative ideal of X is an antagonistic-intuitionistic fuzzy sub-implicative ideal is subtraction G algebra of X.

Theorem: If $G_{\alpha,\beta}$(A) and $G_{\alpha,\beta}$(B) are antagonistic-intuitionistic fuzzy positive implicative ideal of subtraction G-algebra then $G_{\alpha,\beta}$(A$B) is also antagonistic-intuitionistic fuzzy positive implicative ideal of subtraction G-algebra.

Proof: Given $G_{\alpha,\beta}$(A) and $G_{\alpha,\beta}$(B) are antagonistic-intuitionistic fuzzy positive implicative ideal of X.

$$(1)\ \Psi_A(0) \le \Psi_A(x),\ \Omega_A(0) \ge \Omega_A(x) \text{ for all } x \in X, \text{ and}$$
$$(2)\ \Psi_A(x-z) \le \max\{\Psi_A((x-z)-z)-(y-z)), \Psi_A(y)\} \text{ for all } x, y, z \in X.$$
$$(3)\ \Omega_A(x-z) \ge \min\{\Omega_A((x-z)-z)-(y-z)), \Omega_A(y)\} \text{ for all } x, y, z \in X.$$
$$(4)\ \Psi_B(0) \le \Psi_B(x),\ \Omega_B(0) \ge \Omega_B(x) \text{ for all } x \in X, \text{ and}$$
$$(5)\ \Psi_B(x-z) \le \max\{\Psi_B((x-z)-z)-(y-z)), \Psi_B(y)\} \text{ for all } x, y, z \in X.$$
$$(6)\ \Omega_B(x-z) \ge \min\{\Omega_B((x-z)-z)-(y-z)), \Omega_B(y)\} \text{ for all } x, y, z \in X.$$

Case1:

$$\sqrt{\Psi_A(0).\Psi_B(0)} \le \sqrt{\Psi_A(x).\Psi_B(x)}$$
$$G_{\alpha,\beta}\sqrt{\Psi_A(0).\Psi_B(0)} \le G_{\alpha,\beta}\sqrt{\Psi_A(x).\Psi_B(x)}$$
$$\sqrt{\alpha\Psi_A(0).\alpha\Psi_B(0)} \le \sqrt{\alpha\Psi_A(x).\alpha\Psi_B(x)}$$
$$\alpha\sqrt{\Psi_A(0).\Psi_B(0)} \le \alpha\sqrt{\Psi_A(x).\Psi_B(x)}$$
$$G_{\alpha,\beta}(A\$B)(0) \le G_{\alpha,\beta}(A\$B)(x)$$
$$\sqrt{\Omega_A(0).\Omega_B(0)} \ge \sqrt{\Omega_A(x).\Omega_B(x)}$$
$$G_{\alpha,\beta}\sqrt{\Omega_A(0).\Omega_B(0)} \ge G_{\alpha,\beta}\sqrt{\Omega_A(x).\Omega_B(x)}$$
$$\sqrt{\beta\Omega_A(0).\beta\Omega_B(0)} \ge \sqrt{\beta\Omega_A(x).\beta\Omega_B(x)}$$
$$\beta\sqrt{\Omega_A(0).\Omega_B(0)} \ge \beta\sqrt{\Omega_A(x).\Omega_B(x)}$$
$$G_{\alpha,\beta}(A\$B)(0) \ge G_{\alpha,\beta}(A\$B)(x)$$

Case2:

$$\sqrt{\Psi_A(x-z).\Psi_B(x-z)} \le$$
$$\sqrt{\max\left\{\Psi_A(((x-z)-z)-(y-z)), \Psi_A(y)\right\}.\max\{\Psi_B(((x-z)-z)-(y-z)), \Psi_B(y)\}}$$
$$\sqrt{\Psi_A(x-z).\Psi_B(x-z)} \le$$
$$\sqrt{\max\{\Psi_A(((x-z)-z)-(y-z)).\Psi_B(((x-z)-z)-(y-z))\}, \max\{\Psi_A(y).\Psi_B(y)\}}$$
$$\sqrt{\Psi_A(x-z).\Psi_B(x-z)} \le$$
$$max\sqrt{\{\Psi_A(((x-z)-z)-(y-z)).\Psi_B(((x-z)-z)-(y-z))\}, \{\Psi_A(y).\Psi_B(y)\}}$$
$$G_{\alpha,\beta}\sqrt{\Psi_A(x-z).\Psi_B(x-z)} \le$$
$$maxG_{\alpha,\beta}\sqrt{\left\{\Psi_A(((x-z)-z)-(y-z)).\Psi_B(((x-z)-z)-(y-z))\right\}, \{\Psi_A(y).\Psi_B(y)\}}$$
$$\sqrt{\boldsymbol{\alpha}\Psi_A(x-z).\alpha\Psi_B(x-z)} \le$$
$$max\sqrt{\{\alpha\Psi_A(((x-z)-z)-(y-z)).\alpha\Psi_B(((x-z)-z)-(y-z))\}, \{\alpha\Psi_A(y).\alpha\Psi_B(y)\}}$$
$$\boldsymbol{\alpha}\sqrt{\Psi_A(x-z).\Psi_B(x-z)} \le$$
$$max\boldsymbol{\alpha}\sqrt{\{\Psi_A(((x-z)-z)-(y-z)).\Psi_B(((x-z)-z)-(y-z))\}, \{\Psi_A(y).\Psi_B(y)\}}$$
$$G_{\alpha,\beta}(A\$B)(x-z) \le \max\{G_{\alpha,\beta}(A\$B)(((x-z)-z)-(y-z), G_{\alpha,\beta}(A\$B)(y)\}$$

Case3:

$$\sqrt{\Omega_A(x-z).\Omega_B(x-z)}$$

$$\geq \sqrt{\max\left\{\Omega_A\left(((x-z)-z)-(y-z)\right),\Omega_A(y)\right\}.\max\left\{\Psi_B\left(((x-z)-z)-(y-z)\right),\Omega_B(y)\right\}}$$

$$\sqrt{\Omega_A(x-z).\Omega_B(x-z)}$$

$$\geq \sqrt{\min\left\{\Omega_A(((x-z)-z)-(y-z)),\Omega_B(((x-z)-z)-(y-z))\right\}.\min\{\Omega_A(y),\Omega_B(y)\}}$$

$$\sqrt{\Omega_A(x-z).\Omega_B(x-z)}$$

$$\geq min\sqrt{\{\Omega_A(((x-z)-z)-(y-z)),\Omega_B(((x-z)-z)-(y-z))\}.\{\Omega_A(y),\Omega_B(y)\}}$$

$$G_{\alpha,\beta}\sqrt{\Omega_A(x-z).\Omega_B(x-z)}$$

$$\geq minG_{\alpha,\beta}\sqrt{\left\{\Omega_A\left(((x-z)-z)-(y-z)\right),\Omega_B(((x-z)-z)-(y-z))\right\}.\{\Omega_A(y),\Omega_B(y)\}}$$

$$\sqrt{\boldsymbol{\beta}\Omega_A(x-z).\beta\Omega_B(x-z)}\leq$$

$$max\sqrt{\{\beta\Omega_A(((x-z)-z)-(y-z)).\beta\Omega_B(((x-z)-z)-(y-z))\},\{\beta\Omega_A(y),\beta\Omega_B(y)\}}$$

$$\boldsymbol{\beta}\sqrt{\Omega_A(x-z).\Omega_B(x-z)}\leq$$

$$max\boldsymbol{\beta}\sqrt{\{\Omega_A(((x-z)-z)-(y-z)).\Omega_B(((x-z)-z)-(y-z))\},\{\Omega_A(y).\Omega_B(y)\}}$$

$$G_{\alpha,\beta}(A\$B)(x-z)\leq\max\{G_{\alpha,\beta}(A\$B)\left(((x-z)-z\right)-(y-z),G_{\alpha,\beta}(A\$B)(y)\}$$

Therefore, the two numbers of $A\$$ B is an antagonistic- intuitionistic fuzzy positive implicative ideal in subtraction G-algebra of X.

Example: Let X = {0, a, b, c} be a subtraction G-algebra with the following Cayley table.

-	0	A	B	C
0	0	A	B	C
A	0	0	B	C
B	0	A	0	C
C	0	A	B	0

We define an anti- fuzzy set A then A = < X, Ψ_A, Ω_A> by routine calculation A is an antagonistic-intuitionistic fuzzy sub-implicative ideal of X.

X	0	A	B	C
Ψ_A	.40	.42	.62	.75
Ω_A	.50	.45	.29	.21

We define an anti- fuzzy set B then B = < X, Ψ_B, Ω_B> by routine calculation B is an antagonistic-intuitionistic fuzzy sub-implicative ideal of X.

X	0	a	B	c
Ψ_B	.51	.53	.61	.72
Ω_B	.47	.42	.31	.26

If A and B are antagonistic-intuitionistic fuzzy sub-implicative ideal of X. obviously the A\$B, Since x= b, y = c, z = a,α=.41,β=.46.

$$\alpha\sqrt{\Psi_A(0).\Psi_B(0)}\leq\alpha\sqrt{\Psi_A(x).\Psi_B(x)}$$

$$.41\sqrt{(.40).(.51)}\leq.41\sqrt{(.62).(.31)}$$

$$.1851\not\leq.1797$$

$$\boldsymbol{\alpha}\sqrt{\Psi_A(x-z).\Psi_B(x-z)}\leq$$

$$max\boldsymbol{\alpha}\sqrt{\{\Psi_A(((x-z)-z)-(y-z)).\Psi_B(((x-z)-z)-(y-z))\},\{\Psi_A(y).\Psi_B(y)\}}$$

$$.41\sqrt{\Psi_A(\mathrm{b}-\mathrm{a}).\Psi_B(\mathrm{b}-\mathrm{a})}\leq$$

$$max\alpha\sqrt{\{\Psi_A(((\mathrm{b}-\mathrm{a})-\mathrm{a})-(\mathrm{c}-\mathrm{a})).\Psi_B(((\mathrm{b}-\mathrm{a})-\mathrm{a})-(\mathrm{c}-\mathrm{a}))\},\{\Psi_A(\mathrm{c}).\Psi_B(\mathrm{c})\}}$$

$$.41\sqrt{\Psi_A(\mathrm{a}).\Psi_B(\mathrm{a})}\leq max.41\sqrt{\{\Psi_A(\mathrm{a}).\Psi_B(\mathrm{a}))\},\{\Psi_A(\mathrm{c}).\Psi_B(\mathrm{c})\}}$$

$$.41\sqrt{(.42).(.53)}\leq max.41\sqrt{(.42).(.53))\},\{(.75).(.73)\}}$$

$$.1934\leq max(.1934,.3033)$$

$$.1934\leq .3034$$

$$\alpha\sqrt{\Psi_A(\mathrm{x}-\mathrm{z}).\Psi_B(\mathrm{x}-\mathrm{z})}\leq$$

$$max\alpha\sqrt{\{\Psi_A(((\mathrm{x}-\mathrm{z})-\mathrm{z})-(\mathrm{y}-\mathrm{z})).\Psi_B(((\mathrm{x}-\mathrm{z})-\mathrm{z})-(\mathrm{y}-\mathrm{z}))\},\{\Psi_A(\mathrm{y}).\Psi_B(\mathrm{y})\}}$$

$$.41\sqrt{\Psi_A(\mathrm{z}).\Psi_B(\mathrm{z})}\leq max.41\sqrt{\{\Psi_A(\mathrm{z}).\Psi_B(\mathrm{z})\},\{\Psi_A(\mathrm{y}).\Psi_B(\mathrm{y})\}}$$

$$.41\sqrt{(.4).(.51)}\leq max.41\sqrt{(.40).(.51)\},\{(.75).(.73)\}}$$

$$.1851\leq \max(.1852$$

$$\boldsymbol{\beta}\sqrt{\Omega_A(\mathrm{x}-\mathrm{z}).\Omega_B(\mathrm{x}-\mathrm{z})}\leq$$

$$max\beta\sqrt{\{\Omega_A(((\mathrm{x}-\mathrm{z})-\mathrm{z})-(\mathrm{y}-\mathrm{z})).\Omega_B(((\mathrm{x}-\mathrm{z})-\mathrm{z})-(\mathrm{y}-\mathrm{z}))\},\{\Omega_A(\mathrm{y}).\Omega_B(\mathrm{y})\}}$$

$$.46\sqrt{\Omega_A(\mathrm{z}).\Omega_B(\mathrm{z})}\leq max.46\sqrt{\{\Omega_A(\mathrm{z})).\Omega_B(\mathrm{z})\},\{\Omega_A(\mathrm{y}).\Omega_B(\mathrm{y})\}}$$

$$.46\sqrt{(.50).(.47)}\leq max.46\sqrt{\{(.50).(.47)\},(.21)(.26)\}}$$

$$.2229\leq \max(.2229,.1075)$$

$$.2229\leq .2229$$

Therefore, the given A$B of antagonistic-intuitionistic fuzzy sub-implicative ideal of X is an antagonistic-intuitionistic fuzzy sub-implicative ideal in subtraction G algebra of X.

Theorem: If $G_{\alpha,\beta}(\mathrm{A})$ and $G_{\alpha,\beta}(\mathrm{B})$ are antagonistic-intuitionistic fuzzy positive implicative ideal of subtraction G-algebra then $G_{\alpha,\beta}(\mathrm{A}*\mathrm{B})$ is also antagonistic-intuitionistic fuzzy positive implicative ideal of subtraction G-algebra.

(1) $\Psi_A(0)\leq \Psi_A(\mathrm{x})$, $\Omega_A(0)\geq \Omega_A(x)$ for all $\mathrm{x}\in \mathrm{X}$, and

(2) $\Psi_A(\mathrm{x}-\mathrm{z})\leq \max\{\Psi_A((\mathrm{x}-\mathrm{z})-\mathrm{z})-(\mathrm{y}-\mathrm{z})),\Psi_A(\mathrm{y})\}$ for all $\mathrm{x},\mathrm{y},\mathrm{z}\in \mathrm{X}$.

(3) $\Omega_A(\mathrm{x}-\mathrm{z})\geq \min\{\Omega_A((\mathrm{x}-\mathrm{z})-\mathrm{z})-(\mathrm{y}-\mathrm{z})),\Omega_A(\mathrm{y})\}$ for all $\mathrm{x},\mathrm{y},\mathrm{z}\in \mathrm{X}$.

(4) $\Psi_B(0)\leq \Psi_B(\mathrm{x})$, $\Omega_B(0)\geq \Omega_B(x)$ for all $\mathrm{x}\in \mathrm{X}$, and

(5) $\Psi_B(\mathrm{x}-\mathrm{z})\leq \max\{\Psi_B((\mathrm{x}-\mathrm{z})-\mathrm{z})-(\mathrm{y}-\mathrm{z})),\Psi_B(\mathrm{y})\}$ for all $\mathrm{x},\mathrm{y},\mathrm{z}\in \mathrm{X}$.

(6) $\Omega_B(\mathrm{x}-\mathrm{z})\geq \min\{\Omega_B((\mathrm{x}-\mathrm{z})-\mathrm{z})-(\mathrm{y}-\mathrm{z})),\Omega_B(\mathrm{y})\}$ for all $\mathrm{x},\mathrm{y},\mathrm{z}\in \mathrm{X}$.

Case1:

$$\frac{\Psi_A(0)+\Psi_B(0)}{2\{\Psi_A(0).\Psi_B(0)+1\}}\leq\frac{\Psi_A(\mathrm{x})+)\,\Psi_B(\mathrm{x})}{2\{\Psi_A(\mathrm{x}).\Psi_B(\mathrm{x})+1\}}$$

$$G_{\alpha,\beta}\frac{\Psi_A(0)+\Psi_B(0)}{2\{\Psi_A(0).\Psi_B(0)+1\}}\leq G_{\alpha,\beta}\frac{\Psi_A(\mathrm{x})+)\,\Psi_B(\mathrm{x})}{2\{\Psi_A(\mathrm{x}).\Psi_B(\mathrm{x})+1\}}$$

$$\frac{\alpha\Psi_A(0)+\alpha\Psi_B(0)}{2\{\alpha\Psi_A(0).\alpha\Psi_B(0)+1\}}\leq\frac{\alpha\,\Psi_A(\mathrm{x})+\alpha\Psi_B(\mathrm{x})}{2\{\alpha\Psi_A(\mathrm{x}).\alpha\Psi_B(\mathrm{x})+1\}}$$

$$\frac{\alpha\{\Psi_A(0)+\Psi_B(0)\}}{2\{\alpha^2\Psi_A(0).\Psi_B(0)+1\}}\leq\frac{\alpha\,\{\Psi_A(\mathrm{x})+\Psi_B(\mathrm{x})\}}{2\{\alpha^2\Psi_A(\mathrm{x}).\Psi_B(\mathrm{x})+1\}}$$

$$G_{\alpha,\beta}(A*B)(0)\leq G_{\alpha,\beta}(A*B)(x)$$

$$\frac{\Omega_A(0)+\Omega_B(0)}{2\{\Omega_A(0).\Omega_B(0)+1\}}\geq\frac{\Omega_A(\mathrm{x})+)\,\Omega_B(\mathrm{x})}{2\{\Omega_A(\mathrm{x}).\Omega_B(\mathrm{x})+1\}}$$

$$G_{\alpha,\beta}\frac{\Omega_A(0)+\Omega_B(0)}{2\{\Omega_A(0).\Omega_B(0)+1\}}\geq G_{\alpha,\beta}\frac{\Omega_A(\mathrm{x})+)\,\Omega_B(\mathrm{x})}{2\{\Psi_A(\mathrm{x}).\Psi_B(\mathrm{x})+1\}}$$

$$\frac{\beta\Omega_A(0)+\beta\Omega_B(0)}{2\{\beta\Omega_A(0).\beta\Omega_B(0)+1\}}\geq\frac{\beta\,\Omega_A(\mathrm{x})+\beta\Omega_B(\mathrm{x})}{2\{\beta\Omega_A(\mathrm{x}).\beta\Omega_B(\mathrm{x})+1\}}$$

$$\frac{\beta\{\Omega_A(0)+\Omega_B(0)\}}{2\{\beta^2\Omega_A(0).\Omega_B(0)+1\}}\geq\frac{\beta\,\{\Omega_A(\mathrm{x})+\Omega_B(\mathrm{x})\}}{2\{\beta^2\Omega_A(\mathrm{x}).\Omega_B(\mathrm{x})+1\}}$$

$$G_{\alpha,\beta}(A*B)(0) \geq G_{\alpha,\beta}(A*B)(x)$$

Case2:

$$\frac{\Psi_A(x-z)+\Psi_B(x-z)}{2\{\Psi_A(x-z).\Psi_B(x-z)+1\}} \leq \frac{\max\{\Psi_A((x-z)-z)-(y-z)),\Psi_A(y)\}+\max\{\Psi_B((x-z)-z)-(y-z)),\Psi_B(y)\}}{2\left\{\max\left\{\Psi_A\left((x-z)-z\right)-(y-z)\right),\Psi_A(y)\right\}.\max\left\{\Psi_B\left((x-z)-z\right)-(y-z)\right),\Psi_B(y)+1\right\}}$$

$$\frac{\Psi_A(x-z)+\Psi_B(x-z)}{2\{\Psi_A(x-z).\Psi_B(x-z)+1\}} \leq \frac{\max\left\{\Psi_A\left((x-z)-z\right)-(y-z)\right)+\{\Psi_B((x-z)-z)-(y-z))\},\max\{\Psi_A(y)+\Psi_B(y)\}}{2max\left\{\Psi_A\left((x-z)-z\right)-(y-z)\right).\Psi_B((x-z)-z)-(y-z))+1\},2\max\{\Psi_A(y).\Psi_B(y)+1\}}$$

$$G_{\alpha,\beta}\frac{\Psi_A(x-z)+\Psi_B(x-z)}{2\{\Psi_A(x-z).\Psi_B(x-z)+1\}} \leq G_{\alpha,\beta}\{\frac{\max\left\{\Psi_A\left((x-z)-z\right)-(y-z)\right)+\{\Psi_B((x-z)-z)-(y-z))\}}{2max\left\{\Psi_A\left((x-z)-z\right)-(y-z)\right).\Psi_B((x-z)-z)-(y-z))+1\}},\frac{\max\{\Psi_A(y)+\Psi_B(y)\}}{2\max\{\Psi_A(y).\Psi_B(y)+1\}}\}$$

$$\frac{\alpha\Psi_A(x-z)+\alpha\Psi_B(x-z)}{2\{\alpha\Psi_A(x-z).\alpha\Psi_B(x-z)+1\}} \leq \{\frac{\max\left\{\alpha\Psi_A\left((x-z)-z\right)-(y-z)\right)+\alpha\Psi_B((x-z)-z)-(y-z))\}}{2max\left\{\alpha\Psi_A\left((x-z)-z\right)-(y-z)\right).\alpha\Psi_B((x-z)-z)-(y-z))+1\}},\frac{\max\{\alpha\Psi_A(y)+\alpha\Psi_B(y)\}}{2\max\{\alpha\Psi_A(y).\alpha\Psi_B(y)+1\}}\}$$

$$\frac{\alpha\{\Psi_A(x-z)+(x-z)\}}{2\{\alpha^2\Psi_A(x-z).\Psi_B(x-z)+1\}} \leq \{\frac{\max\alpha\left\{\Psi_A\left((x-z)-z\right)-(y-z)\right)+\Psi_B((x-z)-z)-(y-z))\}}{2max\left\{\alpha^2\Psi_A\left((x-z)-z\right)-(y-z)\right).\Psi_B((x-z)-z)-(y-z))+1\}},\frac{\max\alpha\{\Psi_A(y)+\Psi_B(y)\}}{2\max\{\alpha^2\Psi_A(y).\Psi_B(y)+1\}}\}$$

$$\frac{\alpha\{\Psi_A(x-z)+(x-z)\}}{2\{\alpha^2\Psi_A(x-z).\Psi_B(x-z)+1\}} \leq \max\{\frac{\alpha\left\{\Psi_A\left((x-z)-z\right)-(y-z)\right)+\Psi_B((x-z)-z)-(y-z))\}}{2\left\{\alpha^2\Psi_A\left((x-z)-z\right)-(y-z)\right).\Psi_B((x-z)-z)-(y-z))+1\}},\frac{\alpha\{\Psi_A(y)+\Psi_B(y)\}}{2\{\alpha^2\Psi_A(y).\Psi_B(y)+1\}}\}$$

$$G_{\alpha,\beta}(A*B)(x-z) \leq \max\{G_{\alpha,\beta}(A*B)\left((x-z)-z\right)-(y-z),G_{\alpha,\beta}(A*B)(y)\}$$

Case3:

$$\frac{\Omega_A(x-z)+\Omega_B(x-z)}{2\{\Omega_A(x-z).\Omega_B(x-z)+1\}} \geq \frac{\min\{\Omega_A((x-z)-z)-(y-z)),\Omega_A(y)\}+\min\{\Omega_B((x-z)-z)-(y-z)),\Omega_B(y)\}}{2\left\{\min\left\{\Omega_A\left((x-z)-z\right)-(y-z)\right),\Omega_A(y)\right\}.\min\left\{\Omega_B\left((x-z)-z\right)-(y-z)\right),\Omega_B(y)+1\right\}}$$

$$\frac{\Omega_A(x-z)+\Omega_B(x-z)}{2\{\Omega_A(x-z).\Omega_B(x-z)+1\}} \geq \frac{\min\left\{\Omega_A\left((x-z)-z\right)-(y-z)\right)+\Omega_B((x-z)-z)-(y-z))\},\min\{\Omega_A(y)+\Omega_B(y)\}}{2min\left\{\Omega_A\left((x-z)-z\right)-(y-z)\right).\Omega_B((x-z)-z)-(y-z))+1\},2\min\{\Omega_A(y).\Omega_B(y)+1\}}$$

$$G_{\alpha,\beta}\frac{\Omega_A(x-z)+\Omega_B(x-z)}{2\{\Omega_A(x-z).\Omega_B(x-z)+1\}} \geq G_{\alpha,\beta}\{\frac{\min\left\{\Omega_A\left((x-z)-z\right)-(y-z)\right)+\Omega_B((x-z)-z)-(y-z))\}}{2min\left\{\Omega_A\left((x-z)-z\right)-(y-z)\right).\Omega_B((x-z)-z)-(y-z))+1\}},\frac{\min\{\Omega_A(y)+\Omega_B(y)\}}{2\min\{\Omega_A(y).\Omega_B(y)+1\}}\}$$

$$\frac{\beta\Omega_A(x-z)+\beta\Omega_B(x-z)}{2\{\beta\Omega_A(x-z).\beta\Omega_B(x-z)+1\}} \geq \{\frac{\min\left\{\beta\Omega_A\left((x-z)-z\right)-(y-z)\right)+\beta\Omega_B((x-z)-z)-(y-z))\}}{2min\left\{\beta\Omega_A\left((x-z)-z\right)-(y-z)\right).\beta\Omega_B((x-z)-z)-(y-z))+1\}},\frac{\min\{\beta\Omega_A(y)+\beta\Omega_B(y)\}}{2\min\{\beta\Omega_A(y).\beta\Omega_B(y)+1\}}\}$$

$$\frac{\beta\{\Omega_A(x-z)+(x-z)\}}{2\{\beta^2\Omega_A(x-z).\Omega_B(x-z)+1\}} \geq \{\frac{\min\beta\left\{\Omega_A\left((x-z)-z\right)-(y-z)\right)+\Omega_B((x-z)-z)-(y-z))\}}{2min\left\{\beta^2\Omega_A\left((x-z)-z\right)-(y-z)\right).\Omega_B((x-z)-z)-(y-z))+1\}},\frac{\min\beta\{\Omega_A(y)+\Omega_B(y)\}}{2\min\{\beta^2\Omega_A(y).\Omega_B(y)+1\}}\}$$

$$\frac{\beta\{\Omega_A(x-z)+(x-z)\}}{2\{\beta^2\Omega_A(x-z).\Omega_B(x-z)+1\}} \geq \min\{\frac{\beta\left\{\Omega_A\left((x-z)-z\right)-(y-z)\right)+\Omega_B((x-z)-z)-(y-z))\}}{2\left\{\beta^2\Omega_A\left((x-z)-z\right)-(y-z)\right).\Omega_B((x-z)-z)-(y-z))+1\}},\frac{\beta\{\Omega_A(y)+\Omega_B(y)\}}{2\{\beta^2\Omega_A(y).\Omega_B(y)+1\}}\}$$

$$G_{\alpha,\beta}(A*B)(x-z) \geq \min\{G_{\alpha,\beta}(A*B)\left((x-z)-z\right)-(y-z),G_{\alpha,\beta}(A*B)(y)\}$$

Therefore, the two numbers of $A*B$ is an antagonistic- intuitionistic fuzzy positive implicative ideal in subtraction G-algebra of X.

Example: Let X = {0, a, b, c} be a subtraction G-algebra with the following Cayley table.

-	0	A	B	C
0	0	A	B	C
A	0	0	B	C
B	0	A	0	C
C	0	A	B	0

We define an anti- fuzzy set A then A = < X, Ψ_A, Ω_A> by routine calculation A is an antagonistic-intuitionistic fuzzy sub-implicative ideal of X.

X	0	A	B	C
Ψ_A	.40	.42	.62	.75
Ω_A	.50	.45	.29	.21

We define an anti- fuzzy set B then B = < X, Ψ_B, Ω_B> by routine calculation B is an antagonistic-intuitionistic fuzzy sub-implicative ideal of X.

X	0	A	B	C
Ψ_B	.51	.53	.61	.72
Ω_B	.47	.42	.31	.26

If A and B are antagonistic-intuitionistic fuzzy sub-implicative ideal of X. obviously the A∗ B, Since x= b, y = c, z = a,α=.41,β=.46.

Case1:

$$\frac{\alpha\{\Psi_A(0) + \Psi_B(0)\}}{2\{\alpha^2\Psi_A(0).\Psi_B(0)+1\}} \leq \frac{\alpha\{\Psi_A(x) + \Psi_B(x)\}}{2\{\alpha^2\Psi_A(x).\Psi_B(x)+1\}}$$

$$\frac{.41\{(.40) + (.51)\}}{2\{.41^2(.40).(.51)+1\}} \leq \frac{.41\{(.62) + (.61)\}}{2\{.41^2(.62).(.61)+1\}}$$

$$\frac{.3731}{2.0685} \leq \frac{.5043}{2.1271}$$

.1804≤.2371

$$\frac{\beta\{\Omega_A(0) + \Omega_B(0)\}}{2\{\beta^2\Omega_A(0).\Omega_B(0)+1\}} \geq \frac{\beta\{\Omega_A(x) + \Omega_B(x)\}}{2\{\beta^2\Omega_A(x).\Omega_B(x)+1\}}$$

$$\frac{.46\{(.50) + (.47)\}}{2\{.46^2(.50)(.47)+1\}} \geq \frac{.46\{(.29) + (.31)\}}{2\{.46^2(.29)(.31)+1\}}$$

$$\frac{.4462}{2.0995} \geq \frac{.276}{2.0380}$$

.2125≥ .1354

Case2:

$$\frac{\alpha\{\Psi_A(x-z)+(x-z)\}}{2\{\alpha^2\Psi_A(x-z).\Psi_B(x-z)+1\}} \leq \max\left\{\frac{\alpha\{\Psi_A((x-z)-z)-(y-z))+\Psi_B((x-z)-z)-(y-z))\}}{2\{\alpha^2\Psi_A((x-z)-z)-(y-z)).\Psi_B((x-z)-z)-(y-z))+1\}}, \frac{\alpha\{\Psi_A(y)+\Psi_B(y)\}}{2\{\alpha^2\Psi_A(y).\Psi_B(y)+1\}}\right\}$$

$$\frac{.41\{(.42)+(.53)\}}{2\{.41^2(.42)(.53)+1\}} \leq \max\left\{\frac{.41\{(.42)+(.53)\}}{2\{.41^2(.42)(.53)+1\}}, \frac{.41\{(.75)+(.73)\}}{2\{.41^2(.75)(.73)+1\}}\right\}$$

$$\frac{.3895}{2.0745} \leq \max\left\{\frac{.3895}{2.0745}, \frac{.6068}{2.1841}\right\}$$

.1877 ≤max(.1878,.2778)

.1877≤.2778

Case3:

$$\frac{\beta\{\Omega_A(x-z)+(x-z)\}}{2\{\beta^2\Omega_A(x-z).\Omega_B(x-z)+1\}} \geq \min\left\{\frac{\beta\{\Omega_A((x-z)-z)-(y-z))+\Omega_B((x-z)-z)-(y-z))\}}{2\{\beta^2\Omega_A((x-z)-z)-(y-z)).\Omega_B((x-z)-z)-(y-z))+1\}}, \frac{\beta\{\Omega_A(y)+\Omega_B(y)\}}{2\{\beta^2\Omega_A(y).\Omega_B(y)+1\}}\right\}$$

$$\frac{.46\{\Omega_A(z)+\Omega_B(z)\}}{2\{\beta^2\Omega_A(z).\Omega_B(z)+1\}} \geq \min\left\{\frac{.46\{\Omega_A(z)+\Omega_B(z)\}}{2\{.46^2\Omega_A(z).\Omega_B(z)+1\}}, \frac{.46\{\Omega_A(y)+\Omega_B(y)\}}{2\{.46^2\Omega_A(y).\Omega_B(y)+1\}}\right\}$$

$$\frac{.46\{(..45)+(.42)\}}{2\{.46^2(.45)(.42)+1\}} \geq \min\left\{\frac{.46\{(.45)+(.42)\}}{2\{.46^{2(.45).42)}+1\}}, \frac{.46\{(.21)+(.26)\}}{2\{.46^2(.21)(.26)+1\}}\right\}$$

$$\frac{.4002}{2.0799} \geq \min\left\{\frac{.4002}{2.0799}, \frac{.2162}{2.0231}\right\}$$

.1924≥min(.1924,.1068)

.1924≥.1068

Therefore, the given A*B of antagonistic-intuitionistic fuzzy sub-implicative ideal of X is an antagonistic-intuitionistic fuzzy sub-implicative ideal in subtraction G algebra of X.

7.Conclusion

G-algebras is an important class of logical algebras. Many logical algebras can be represented in G-algebras. For example, Boolean algebras are equivalent to the bounded implicative g-algebras [8]. In this paper we proposed the concepts of antagonistic intuitionistic fuzzy sub implicative ideals, antagonistic intuitionistic fuzzy positive implicative ideals and antagonistic intuitionistic fuzzy sub commutative ideals in BCI-algebras, established the relations between some modal operators used in antagonistic intuitionistic fuzzy sub implicative ideals, antagonistic intuitionistic fuzzy positive implicative ideals and antagonistic intuitionistic fuzzy sub commutative ideals.

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Altering distance function on some fixed point theorems in fuzzy metric spaces

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***Abstract**: In this paper, we prove some fixed point theorems in strong fuzzy metric spaces through altering distance function and illustrate an example.*

***Keywords**: t-norm, fuzzy metric, strong fuzzy metric space, Cauchy's sequence, altering distance function*,

I. Introduction

The theory of fuzzy set has been invented by Zadeh in 1965[12], extensively by many authors applied this theory in different fields such as Control theory, Engineering Sciences, Neural networks, etc. In 1975[7], the notion of fuzzy metric space has been initiated by Kramosil and Michalek. Later on, George and Veeramani [2] gives the modified the concept of fuzzy metric spaces due to Kramosil and Michalek and analysed a Hausdorff topology of fuzzy metric spaces. Recently, Gregori et al. [5] gave many interesting examples of fuzzy metric in the sense of George and Veeramani and have also applied these fuzzy metrics to color image processing. In 1988, Grabiec [4] proved an analog of the popular Banach contraction theorem and Edelstein fixed point theorems in fuzzy metric spaces. Grabiec prove the above theory in the Cauchy sequence of fuzzy metric spaces. M.S.Khan et al. propounded a new notion of Banach fixed point theorem in metric spaces by presents a control function is called an altering distance function in 1984[6]. Recently, Shen et al.[10] introduced the notion of control function in fuzzy metric space $(X, M, *)$ is given by

$\varphi\big(M(Tu,Tv,t)\big) \leq \alpha(t).\varphi\big(M(u,v,t)\big),\ \forall u,v \in X, t>0$,

Obtained fixed point result for self-mapping$T$.

In this paper, we prove existence and uniqueness of fixed point theorems by using altering distance function in strong fuzzy metric spaces and illustrate an example.

II. Preliminaries

In this section we discuss the basic definitions and properties.

Definition 2.1

A fuzzy set $\tilde{A}$ is defined by $\tilde{A} = \{(x, \mu_A(x)) : x \in A, \mu_A(x) \in [0,1]\}$. In the pair $(x, \mu_A(x))$, the first element x belongs to the classical set A, the second element $\mu_A(x)$ belongs to the interval $[0,1]$ is called the membership function.

Definition 2.2

A binary operation $*: [0,1] \times [0,1] \to [0,1]$ is a continuous t-norm if it satisfies the following conditions:

(1) $*$ is associative and commutative;

(2) $*$ is continuous;

(3) $a * 1 = a$ for all $a \in [0,1]$;

(4) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, for all $a, b, c, d \in [0,1]$.

Examples 2.3

i. Lukasievicz t-norm: $a * b = max\{a + b - 1, 0\}$

ii. Product t-norm: $a * b = a.b$

iii. Minimum t-norm: $a * b = min(a, b)$

Definition 2.4

A fuzzy metric space is an ordered triple $(X, M, *)$ such that X is a nonempty set, $*$ is a continuous t-norm and M is a fuzzy set on $X \times X \times (0, \infty) \to [0, 1]$ satisfies the following conditions: for each $x, y, z \in X$ and $s, t > 0$

(KM1) $M(x, y, 0) = 0$;
(KM2) $M(x, y, t) = 1$ if and only if $x = y, t > 0$;
(KM3) $M(x, y, t) = M(y, x, t)$;
(KM4) $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)$;
(KM5) $M(x, y, .)$: $[0, \infty) \to [0, 1]$ is left-continuous.
Then M is called a fuzzy metric on X.

Definition 2.5

A fuzzy metric space is an ordered triple such that X is a nonempty set, $*$ is a continuous t-norm and M is a fuzzy set on $X \times X \times (0, \infty) \to [0, 1]$ satisfies the following conditions: for each $x, y, z \in X$ and $s, t > 0$

(GV1) $M(x, y, t) > 0$, $\forall\, t > 0$;
(GV2) $M(x, y, t) = 1$ if and only if $x = y, t > 0$;
(GV3) $M(x, y, t) = M(y, x, t)$;
(GV4) $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)$;
(GV5) $M(x, y, .)$: $(0, \infty) \to [0, 1]$ is continuous.
Then M is called a fuzzy metric on X.

Definition 2.6

Let $(X, M, *)$ be a fuzzy metric space. M is said to be strong if it satisfies the following axiom instead of definition (2.5) in (GV4):

(GV4′) $M(x, z, t) \geq M(x, y, t) * M(y, z, t), x, y, z \in X, t > 0$,
then $(X, M, *)$ is called a strong fuzzy metric space.

Definition 2.7

Let $(X, M, *)$ be a fuzzy metric space, for $t > 0$ an open ball $B(x, r, t)$ with a centre $x \in X$ and a radius $0 < r < 1$ is defined by

$$B(x, r, t) = \{y \in X : \mathrm{M(x, y, t)} > 1 - r\}.$$

A subset $A \subset X$ is called open if for each $x \in A$, there exist $t > 0$ and $0 < r < 1$ such that $B(x, r, t) \subset A$. Let τ denote the family of all open subsets of X. Then τ is topology on X, called the topology induced by the fuzzy metric M.

Definition 2.8

Let $(X, M, *)$ be a fuzzy metric space

(i) A sequence $\{x_n\}$ in X is said to be convergent to a point x in $(X, M, *)$ if $\lim_{n \to \infty} \mathrm{M(x, y, t)} = 1$ for all $t > 0$.
(ii) A sequence $\{x_n\}$ in X is called a Cauchy sequence in $(X, M, *)$ if for each $0 < \in < 1$ and $t > 0$, there exists $n_0 \in N$ such that $M(x_n, x_m, t) > 1 - \in$ for each $n, m \geq n_0$
(iii) A fuzzy metric space in which every Cauchy sequence is convergent itself is said to be complete.
(iv) A fuzzy metric space in which every sequence has a convergent subsequence is said to be compact.

Lemma 2.9

Let $(X, M, *)$ be a fuzzy metric space. For all u, $v \in X$, $M(u, v, .)$ is non-decreasing function.

Proof

If $M(u, v, t) > M(u, v, s)$ for some $0 < t < s$.

Then $M(u, v, t) * M(v, v, s - t) \leq M(u, v, s) < M(u, v, t)$,
Thus $M(u, v, t) < M(u, v, t) < M(u, v, t)$, (since $M(v, v, s - t) = 1$)
which is a contradiction.

Definition 2.10

A function $\varphi: [0,1] \to [0,1]$ is a called control function (an altering distance function) if it satisfies the following properties:

(AD1) φ is strictly decreasing and continuous;

(AD2) $\varphi(\lambda) \geq 0, \forall\, \lambda \neq 1$ if $\varphi(\lambda) = 0$ if and only if $\lambda = 1$.

It is obvious, $\lim_{\lambda \to 1^-} \varphi(\lambda) = \varphi(1) = 0$.

III. Main Results

Theorem 3.1

Let $(X, M, *)$ be a complete strong FMS with continuous t-norm $*$ and let $T: X \to X$. Suppose if there exists an altering distance function φ from [0,1] to [0,1] and $\alpha: (0, \infty) \to (0,1)$, for any $t > 0$ such that

$$\varphi\big(M(Tu, Tv, t)\big) \leq \alpha(t).\varphi(L(u, v, t)), \tag{3.1.1}$$

where $L(u, v, t) = M(u, Tu, t) * M(v, Tv, t) * M(u, v, t)$.

For each $u, v \in X$ and $u \neq v$. Then T has an unique fixed point in X.

Proof

Let u be any arbitrary point in X and define a sequence $\{u_n\} \in X$ such that $u_{n+1} = Tu_n$ and

$\forall\, n \in \mathbb{N} \cup \{0\}, t > 0$.

Assume that $u_{n+1} = Tu_n = u_n$ for some $n \in \mathbb{N} \cup \{0\}$, then u_n is a fixed point of T.

Suppose $u_{n+1} \neq u_n$ and $0 < M(u_n, u_{n+1}, t) < 1$ for each n and $t > 0$

We can obtain $\varphi\big(M(u_n, u_{n+1}, t)\big) = \varphi(M(Tu_{n-1}, Tu_n, t))$

$$\leq \alpha(t)\,.\,\varphi(L(u_{n-1}, u_n\,, t)) \tag{3.1.2}$$

Where $L(u_{n-1}, u_n\,, t) = M(u_{n-1}, Tu_{n-1}, t) * M(u_n, Tu_n, t) * M(u_{n-1}, u_n, t)$

$$= M(u_{n-1}, u_n, t) * M(u_n, u_{n+1}, t) * M(u_{n-1}, u_n, t)$$

$L(u_{n-1}, u_n\,, t) = M(u_{n-1}, u_n, t) * M(u_n, u_{n+1}, t)$ (since $a * b = min\,(a, b)$)

If $L(u_{n-1}, u_n\,, t) = M(u_n, u_{n+1}, t)$

From (3.1.2) becomes

$$\varphi\big(M(u_n, u_{n+1}, t)\big) \leq \alpha(t)\,.\,\varphi\big(M(u_n, u_{n+1}, t)\big) < \varphi\big(M(u_n, u_{n+1}, t)\big)$$

This shows contradiction.

Therefore $L(u_{n-1}, u_n\,, t) = M(u_{n-1}, u_n, t)$

Thus we have

$$\varphi\big(M(u_n, u_{n+1}, t)\big) \leq \alpha(t)\,.\,\varphi\big(M(u_{n-1}, u_n, t)\big) < \varphi\big(M(u_{n-1}, u_n, t)\big) \tag{3.1.3}$$

Since φ is strictly decreasing function.

This shows that the sequence $\{M(u_n, u_{n+1}, t)\}$ is non-decreasing for every $t > 0$.

Hence $\lim_{n \to \infty} M(u_n, u_{n+1}, t) = M(u, u, t)$ and suppose that $0 < M(u, u, t) < 1$

Then $M(u_n, u_{n+1}, t) \leq M(u, u, t)$ implies that

$$\varphi\big(M(u_n, u_{n+1}, t)\big) \leq \alpha(t)\,.\,\varphi\big(M(u_{n-1}, u_n, t)\big) < \varphi\big(M(u_{n-1}, u_n, t)\big)$$

Suppose that $n \to \infty$ and φ is continuous we have

$$\varphi\big(M(u, u, t)\big) \leq \alpha(t)\,.\,\varphi\big(M(u, u, t)\big) < \varphi\big(M(u, u, t)\big)$$

Which is a contradiction.

Therefore $M(u, u, t) = 1$. That is the sequence $\{M(u_n, u_{n+1}, t)\}$ converges to 1 for any $t > 0$

Now we prove that the sequence $\{u_n\}$ is a Cauchy's sequence.

Assume that $\{u_n\}$ is not a Cauchy's sequence for any $0 < \epsilon < 1$, $t > 0$ then there exists sequence

$\{u_{n_k}\}$ and $\{u_{m_k}\}$ where $n_k, m_k \geq n$ and $n_k, m_k \in \mathbb{N}$ $(n_k > m_k)$

Such that $M(u_{n_k}, u_{m_k}, t) \leq 1 - \epsilon$ (3.1.4)

Let n_k be least integer exceeding m_k satisfying the above property

That is, $M(u_{n_k-1}, u_{m_k-1}, t) > 1 - \epsilon$ and $M(u_{n_k-1}, u_{m_k}, t) > 1 - \epsilon, n_k, m_k \in \mathbb{N}$ and $t > 0$ (3.1.5)

For each $n \in \mathbb{N} \cup \{0\}$

Suppose that $(1 - \epsilon) \geq M(u_{n_k}, u_{m_k}, t)$

$$(1 - \epsilon) \geq M(u_{n_k}, u_{n_k-1}, t) * M(u_{n_k-1}, u_{m_k}, t) \quad \text{(using GV4}') \quad (3.1.6)$$

substituting (3.1.5)

$$(1 - \epsilon) \geq M(u_{n_k}, u_{n_k-1}, t) * (1 - \epsilon) \quad (3.1.7)$$

Suppose that $k \to \infty$ then $M(u_{n_k}, u_{m_k}, t)$ converges to $(1 - \epsilon)$, for every $t > 0$.

Put $u = u_{n_k-1}$ and $v = u_{m_k-1}$ in (3.1.1) then we obtain

$$\varphi(M(Tu_{n_k-1}, Tu_{m_k-1}, t)) \leq \alpha(t) \,.\, \varphi(L(u_{n_k-1}, u_{m_k-1}, t)), \quad (3.1.8)$$

where $L(u_{n_k-1}, u_{m_k-1}, t) = M(u_{n_k-1}, Tu_{n_k-1}, t) * M(u_{m_k-1}, Tu_{m_k-1}, t) * M(u_{n_k-1}, u_{m_k-1}, t)$

$$L(u_{n_k-1}, u_{m_k-1}, t) = M(u_{n_k-1}, u_{n_k}, t) * M(u_{m_k-1}, u_{m_k}, t) * M(u_{n_k-1}, u_{m_k-1}, t).$$

Using (3.1.5) we obtain

$$L(u_{n_k-1}, u_{m_k-1}, t) > M(u_{n_k-1}, u_{n_k}, t) * M(u_{m_k-1}, u_{m_k}, t) * (1 - \epsilon) \quad (3.1.9)$$

From (3.1.8) we have

$$\varphi(M(u_{n_k}, u_{m_k}, t)) \leq \alpha(t) \,.\, \varphi(M(u_{n_k-1}, u_{n_k}, t) * M(u_{m_k-1}, u_{m_k}, t) * (1 - \epsilon))$$

$$\varphi(1 - \epsilon) \leq \alpha(t) \,.\, \varphi(M(u_{n_k-1}, u_{n_k}, t) * M(u_{m_k-1}, u_{m_k}, t) * (1 - \epsilon)). \quad (3.1.10)$$

Taking $k \to \infty$ in the above inequality

$$\varphi(1 - \epsilon) \leq \alpha(t) \,.\, \varphi(1 - \epsilon) < \varphi(1 - \epsilon)$$

which is a contradiction. Hence $\{u_n\}$ is a Cauchy's sequence.

Since X is a complete fuzzy metric space and there exists an element $z \in X$ such that $\lim_{n\to\infty} u_n = z$, that is $M(u_n, z, t) = 1$ as $n \to \infty$.

Put $u = u_{n-1}$ and $v = z$ in (1) we get,

$$\varphi(M(Tu_{n-1}, Tz, t)) \leq \alpha(t).\, \varphi(L(u_{n-1}, z, t)) \quad (3.1.11)$$

where $L(u_{n-1}, z, t) = M(u_{n-1}, Tu_{n-1}, t) * M(z, Tz, t) * M(u_{n-1}, z, t)$

$$L(u_{n-1}, z, t) = M(u_{n-1}, u_n, t) * M(z, Tz, t) * M(u_{n-1}, z, t)$$

$$\varphi(M(Tu_{n-1}, Tz, t)) \leq \alpha(t).\, \varphi(M(u_{n-1}, u_n, t) * M(z, Tz, t) * M(u_{n-1}, z, t)) \quad (3.1.12)$$

taking $k \to \infty$ in (3.1.12)

$$\varphi(M(z, Tz, t)) \leq \alpha(t).\, \varphi(M(z, z, t) * M(z, z, t) * M(z, z, t))$$

$$\varphi(M(z, Tz, t)) \leq \alpha(t).\, \varphi(M(z, z, t)) = \alpha(t).\, \varphi(1) = 0$$

Since we obtain $\varphi(M(z, Tz, t)) = 0$ by the condition (AD2)

Which is shows that $M(z, Tz, t) = 1$ this implies $Tz = z$.

Therefore z is a fixed point of T.

To prove Uniqueness,

Suppose that w is an another fixed point of T, that is $Tw = w$ where $w \neq z$

Now, show that $w = z$,

From (3.1.1), we have

$$\varphi\big(M(w,z,t)\big) = \varphi\big(M(Tw,Tz,t)\big) \leq \alpha(t).\varphi(L(w,z,t)) \quad (3.1.13)$$

where $L(w,z,t) = M(w,Tw,t) * M(z,Tz,t) * M(w,z,t)$

$$L(w,z,t) = M(w,w,t) * M(z,z,t) * M(w,z,t)$$

$$L(w,z,t) = 1 * 1 * M(w,z,t) = M(w,z,t)$$

From (3.1.13) becomes

$$\varphi\big(M(w,z,t)\big) = \varphi\big(M(Tw,Tz,t)\big) \leq \alpha(t).\varphi\big(M(w,z,t)\big) < \varphi\big(M(w,z,t)\big)$$

which is a contradiction.

Here φ is non decreasing sequence. Hence $z = w$ is the fixed point of T.

Theorem 3.2

Let $(X, M, *)$ be a complete strong FMS with continuous t-norm $*$ and let $T: X \to X$. Suppose if there exists an altering distance function φ and let $\alpha: (0,\infty) \to (0,1)$, for any $t > 0$ such that

$$\varphi\big(M(Tu,Tv,t)\big) \leq \alpha(t).\varphi(L(u,v,t)), \quad (3.2.1)$$

where $L(u,v,t) = M(u,Tu,t) * M(v,Tv,t) * M(u,Tv,t) * M(Tu,v,t) * M(u,v,t)$.

For every $u, v \in X$ and $u \neq v$. Then T has an unique fixed point in X.

Proof

Let u be any arbitrary point in X and define a sequence $\{u_n\} \in X$ such that $u_{n+1} = Tu_n$ and

$\forall\, n \in \mathbb{N} \cup \{0\}, t > 0$.

Assume that $u_{n+1} = Tu_n = u_n$ for some $n \in \mathbb{N} \cup \{0\}$, then u_n is a fixed point of T.

Suppose $u_{n+1} \neq u_n$ and $0 < M(u_n, u_{n+1}, t) < 1$ for each n and $t > 0$

We can obtain $\varphi\big(M(u_n,u_{n+1},t)\big) = \varphi(M(Tu_{n-1},Tu_n,t))$

$$\leq \alpha(t)\,.\varphi(L(u_{n-1},u_n\,,t)) \quad (3.2.2)$$

$$L(u_{n-1},u_n\,,t) = M(u_{n-1},Tu_{n-1},t) * M(u_n,Tu_n,t) * M(u_{n-1},Tu_n,t) * M(Tu_{n-1},u_n,t) * M(u_{n-1},u_n,t)$$

$$= M(u_{n-1},u_n,t) * M(u_n,u_{n+1},t) * M(u_{n-1},u_{n+1},t) * M(u_n,u_n,t) * M(u_{n-1},u_n,t)$$

$L(u_{n-1},u_n\,,t) = M(u_{n-1},u_n,t) * M(u_n,u_{n+1},t) * M(u_{n-1},u_{n+1},t)$ (since $a * b = min\,(a,b)$)

Here $M(u_{n-1},u_{n+1},t) = M(u_{n-1},u_n,t) * M(u_n,u_{n+1},t)$ (Using GV4')

Then we have $L(u_{n-1},u_n\,,t) = M(u_{n-1},u_n,t) * M(u_n,u_{n+1},t)$

If $L(u_{n-1},u_n\,,t) = M(u_n,u_{n+1},t)$

From (3.2.2) becomes

$$\varphi\big(M(u_n,u_{n+1},t)\big) \leq \alpha(t)\,.\varphi\big(M(u_n,u_{n+1},t)\big) < \varphi\big(M(u_n,u_{n+1},t)\big)$$

This shows contradiction.

Therefore $L(u_{n-1},u_n\,,t) = M(u_{n-1},u_n,t)$

Thus we have

$$\varphi\big(M(u_n,u_{n+1},t)\big) \leq \alpha(t)\,.\varphi\big(M(u_{n-1},u_n,t)\big) < \varphi\big(M(u_{n-1},u_n,t)\big) \quad (3.2.3)$$

Since φ is strictly decreasing function.

This shows that the sequence $\{M(u_n,u_{n+1},t)\}$ is non-decreasing for every $t > 0$.

Hence, $\lim_{n\to\infty} M(u_n,u_{n+1},t) = M(u,u,t)$ and suppose that $0 < M(u,u,t) < 1$

Then $M(u_n, u_{n+1}, t) \leq M(u, u, t)$ implies that

$$\varphi\big(M(u_n, u_{n+1}, t)\big) \leq \alpha(t)\,.\,\varphi\big(M(u_{n-1}, u_n, t)\big) < \varphi\big(M(u_{n-1}, u_n, t)\big)$$

Suppose that $n \to \infty$ and φ is continuous we have

$$\varphi\big(M(u, u, t)\big) \leq \alpha(t)\,.\,\varphi\big(M(u, u, t)\big) < \varphi\big(M(u, u, t)\big)$$

Which is a contradiction.

Therefore $M(u, u, t) = 1$. That is the sequence $\{M(u_n, u_{n+1}, t)\}$ converges to 1 for any $t > 0$

Now we prove that the sequence $\{u_n\}$ is a Cauchy's sequence.

Assume that $\{u_n\}$ is not a Cauchy's sequence for any $0 < \epsilon < 1$, $t > 0$ then there exists sequence $\{u_{n_k}\}$ and $\{u_{m_k}\}$ where $n_k, m_k \geq n$ and $n_k, m_k \in \mathbb{N}$ $(n_k > m_k)$

Such that $M\big(u_{n_k}, u_{m_k}, t\big) \leq 1 - \epsilon$ (3.2.4)

Let n_k be least integer exceeding m_k satisfying the above property

That is, $M\big(u_{n_k-1}, u_{m_k-1}, t\big) > 1 - \epsilon$ and $M\big(u_{n_k-1}, u_{m_k}, t\big) > 1 - \epsilon, n_k, m_k \in \mathbb{N}$ and $t > 0$ (3.2.5)

For each $n \in \mathbb{N} \cup \{0\}$

Suppose that $(1 - \epsilon) \geq M\big(u_{n_k}, u_{m_k}, t\big)$

$$(1 - \epsilon) \geq M\big(u_{n_k}, u_{n_k-1}, t\big) * M\big(u_{n_k-1}, u_{m_k}, t\big) \qquad \text{(using GV4}') \qquad (3.2.6)$$

substituting (3.2.5) in the above inequality,

$$(1 - \epsilon) \geq M\big(u_{n_k}, u_{n_k-1}, t\big) * (1 - \epsilon) \qquad (3.2.7)$$

Suppose that $k \to \infty$ then $M\big(u_{n_k}, u_{m_k}, t\big)$ converges to $(1 - \epsilon)$, for every $t > 0$.

Put $u = u_{n_k-1}$ and $v = u_{m_k-1}$ in (1) then we obtain

$$\varphi\big(M(Tu_{n_k-1}, Tu_{m_k-1}, t)\big) \leq \alpha(t)\,.\,\varphi(L(u_{n_k-1}, u_{m_k-1}, t)), \qquad (3.2.8)$$

$$L(u_{n_k-1}, u_{m_k-1}, t) = M\big(u_{n_k-1}, Tu_{n_k-1}, t\big) * M\big(u_{m_k-1}, Tu_{m_k-1}, t\big) * M\big(u_{n_k-1}, Tu_{m_k-1}, t\big) * M\big(Tu_{n_k-1}, u_{m_k-1}, t\big) * M\big(u_{n_k-1}, u_{m_k-1}, t\big)$$

$$L(u_{n_k-1}, u_{m_k-1}, t) = M\big(u_{n_k-1}, u_{n_k}, t\big) * M\big(u_{m_k-1}, u_{m_k}, t\big) * M\big(u_{n_k-1}, u_{m_k}, t\big) * M\big(u_{n_k}, u_{m_k-1}, t\big) * M\big(u_{n_k-1}, u_{m_k-1}, t\big).$$

Using (3.2.5) we obtain

$$L(u_{n_k-1}, u_{m_k-1}, t) > M\big(u_{n_k-1}, u_{n_k}, t\big) * M\big(u_{m_k-1}, u_{m_k}, t\big) * (1 - \epsilon) * M\big(u_{n_k}, u_{m_k-1}, t\big) * (1 - \epsilon) \quad (3.2.9)$$

From (3.2.8) we have

$$\varphi\big(M(u_{n_k}, u_{m_k}, t)\big) \leq \alpha(t)\,.\,\varphi\Big(M\big(u_{n_k-1}, u_{n_k}, t\big) * M\big(u_{m_k-1}, u_{m_k}, t\big) * (1 - \epsilon) * M\big(u_{n_k}, u_{m_k-1}, t\big)\Big)$$

$$\varphi(1 - \epsilon) \leq \alpha(t)\,.\,\varphi\Big(M\big(u_{n_k-1}, u_{n_k}, t\big) * M\big(u_{m_k-1}, u_{m_k}, t\big) * M\big(u_{n_k}, u_{m_k-1}, t\big) * (1 - \epsilon)\Big) \qquad (3.2.10)$$

Taking $k \to \infty$ in the above inequality

$$\varphi(1 - \epsilon) \leq \alpha(t)\,.\,\varphi(1 - \epsilon) < \varphi(1 - \epsilon)$$

which is a contradiction. Hence $\{u_n\}$ is a Cauchy's sequence.

Since X is a complete fuzzy metric space and there exists an element $z \in X$ such that $\lim_{n\to\infty} u_n = z$,

that is, $M(u_n, z, t) = 1$ as $n \to \infty$.

Put $u = u_{n-1}$ and $v = z$ in (1) we get

$$\varphi\big(M(Tu_{n-1}, Tz, t)\big) \leq \alpha(t).\,\varphi(L(u_{n-1}, z, t)) \qquad (3.2.11)$$

where $L(u_{n-1},z,t)=M(u_{n-1},Tu_{n-1},t)*M(z,Tz,t)*M(Tu_{n-1},z,t)*M(u_{n-1},Tz,t)*M(u_{n-1},z,t)$

$$L(u_{n-1},z,t)=M(u_{n-1},u_n,t)*M(z,Tz,t)*M(u_n,z,t)*M(u_{n-1},Tz,t)*M(u_{n-1},z,t)$$

$$\varphi\big(M(Tu_{n-1},Tz,t)\big)\le\alpha(t).\varphi\big(M(u_{n-1},u_n,t)*M(z,Tz,t)*M(u_n,z,t)*M(u_{n-1},Tz,t)*M(u_{n-1},z,t)\big) \quad (3.2.12)$$

taking $k\to\infty$ in (3.2.12)

$$\varphi\big(M(z,Tz,t)\big)\le\alpha(t).\varphi\big(M(z,z,t)*M(z,z,t)*M(z,z,t)*M(z,z,t)*M(z,z,t)\big)$$

$$\varphi\big(M(z,Tz,t)\big)\le\alpha(t).\varphi\big(M(z,z,t)\big)=\alpha(t).\varphi(1)=0$$

Since we obtain $\varphi\big(M(z,Tz,t)\big)=0$ by the condition (AD2)

Which is shows that $M(z,Tz,t)=1$ this implies $Tz=z$.

Therefore z is a fixed point of T.

To prove Uniqueness,

Suppose that w is another fixed point of T, that is $Tw=w$ where $w\neq z$

Now show that $w=z$,

From (3.2.1) we have

$$\varphi\big(M(w,z,t)\big)=\varphi\big(M(Tw,Tz,t)\big)\le\alpha(t).\varphi(L(w,z,t)) \quad (3.2.13)$$

where $L(w,z,t)=M(w,Tw,t)*M(z,Tz,t)*M(Tw,z,t)*M(w,Tz,t)*M(w,z,t)$

$$L(w,z,t)=M(w,w,t)*M(z,z,t)*M(w,z,t)*M(w,z,t)*M(w,z,t)$$

$$L(w,z,t)=1*1*M(w,z,t)=M(w,z,t)$$

From (3.2.13) becomes

$$\varphi\big(M(w,z,t)\big)=\varphi\big(M(Tw,Tz,t)\big)\le\alpha(t).\varphi\big(M(w,z,t)\big)<\varphi\big(M(w,z,t)\big)$$

which is a contradiction.

Here φ is non decreasing sequence. Hence $z=w$ is the fixed point of T.

Example 3.3

Let $X=\{P,Q,R,S,U\}$ is a subset of $\mathbb{R}^2$ defined by where $P(0,0),Q=(0,1),R=(1,2),S=(0,1)$ and $T=(1,3)$. $\varphi\big(M(u,v,t)\big)=1-\big(M(u,v,t)\big)^{\frac{1}{2}}$ for all $M(u,v,t)\in[0,1]$ and $M(u,v,t)=\frac{t}{t+d(u,v)}$ for all $t>0$.

Here $(X,M,*)$ is a complete FMS with t-norm $a*b=\min(a,b)$ and $T:X\to X$ is given by

$T(P)=T(Q)=T(R)=T(S)=P, T(U)=Q$ and α is the function from $(0,\infty)$ to $(0,1)$ is defined by

$$\alpha(t)=\begin{cases}1-e^{-\frac{t}{4}}, & t\in(0,2]\\ \dfrac{t}{t+1}, & t\in(2,\infty)\end{cases}$$

The function satisfies the formula (1) so by the theorem T has a unique fixed point on X.

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Analysis of an M/M/1 Queueing System with Working Vacation and Impatient Customers

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Abstract*: In this paper, we consider an infinite capacity M/M/1 queuing model under working vacation policy where the arrivals balk with a probability and may renege due to impatience during the working vacation. We derived the probability generating functions of the steady state probabilities and obtained explicit expressions of the queue sizes when the server is in a regular busy period and in a working vacation period. Various performance measures such as the expected system size, the rate of customer loss due to balking and reneging, and the expected waiting time of a customer served are deduced. Stochastic decomposition structures of the queue length and waiting time are derived. The effects of the parameters on system performance measures has been numerically illustrated.*

Keywords*: M/M/1 queue, Impatient customers, Working vacation, Balking.*

I. Introduction

Queuing models with vacation have been thoroughly examined in Ke et al. (2010), Doshi (1986), Tian and Zhang (2006), etc. In these vacation models sever is supposed to cease the service completely during a vacation period. However, there are many situations where the server does not remain completely inactive during a vacation. But provides service to the queue at a lower rate. This type of vacation policy is called working vacation (WV) and was first introduced by Servi and Finn (2002) in an M/M/1 queuing model.

This study analysis the model for two WV policies, the multiple working vacation (MWV) and single working vacation (SWV). In MWV policy, the server continue to take vacation till he finds the system non-empty at a vacation completion instant. SWV policy is different from the multiple working vacation policy in a way that when the server return from their working vacation and if there are no customers in the system, the server stays idle until the first arrival occurs rather than taking another vacation. Tian et al. (2008) considered an M/M/1/WV model with single working vacation obtained the stochastic decomposition structures of the expected queue length and expected waiting time. Liu et al. (2007) derived the stochastic decompositions of Servi and Finn (2002) model. Kim et al. (2003), Wu and Takagi (2006) extended the study of Servi and Finn (2002) to an M/G/1 queue with working vacation. Baba (2005) generalized this work to a GI/M/1 queue with working vacation. Banik et al. (2007) analyzed the finite buffer exhaustive service GI/M/1/N queue with MWV and calculated a series of numerical results. Banik (2006) studied the GI/M/1/N and GI/M/1/ ∞ queuing models for single working vacation. Laxmi and Yesuf (2011) analysed a batch service GI/M/1 queue with exponential WV. Recently, Shakir Majid and P. Manoharan (2017, 2018) derived the steady state solution of M/M/c queue with WV by utilizing PGF method.

Many situations occur in day today life where the customers tend to get disappointed by waiting in a lengthy queue. As a result, the customers either decide to depart from the system without joining the queue (i.e., balk) or leave after joining the queue without getting served due to impatience (i.e., renege). These phenomena are observed in situations involving impatient telephone switchboard customers, hospital emergency rooms handling critical patients, inventory systems that store perishable goods, etc. Altman and Yechiali (2006, 2008) studied queuing models with impatient customers where the cause of impatience becomes the server's vacation and unavailability of server upon arrival. The M/M/1, M/G/1 and M/M/c queues were analysed in Altman and Yechiali (2006), whereas M/M/ ∞ queue was studied in Altman and Yechiali (2008). Yechiali (2007) investigated the queueing model with system disasters where the customers turned to be impatient only when the system is down. Yue et al. (2012) investigated the impatient customers in a Markovian single server model with MWV. Recently, Selvaraju and Goswami (2013) studied an M/M/1 queue with single and multiple working vacation and impatient customers.

The paper is presented as follows. In Section II, we present the description of the model. In Section III, we construct the model as a quasi-birth–death process and form the steady state equations. We obtained the explicit

expressions for the various performance measures. Section IV presented the stochastic decomposition results of the stationary queue length and waiting time. Finally, some numerical illustrations are given in Section V.

II. Model description

We consider an infinite capacity M/M/1 queuing model with working vacation, where the server provides service to the customers at a reduced rate rather than stopping the service completely during his vacation period. The customers arrive according to a Poisson process with parameter λ. The arriving customers either decides to join the queue with probability b or balk with probability $\bar{b}$=1-b. However, when the system is empty, an arriving customer joins the system with probability 1. The server serves the customers at an exponential rate μ_b during a non-vacation period and service discipline is first come first served (FCFS). The server begins a working vacation as soon as the system becomes unoccupied. The arriving customers during working vacation are served at a rate lower than the regular service rate. The service times during working vacation and vacation times are also assumed to be exponentially distributed with rates μ_v and θ, respectively. We present a combined technique in order to investigate both single and multiple working vacation models and defined an indicator function such that

$$\delta=\begin{cases}0, & \textit{System in MVW policy}\\ 1, & \textit{System in SVW policy}\end{cases}$$

A customer waiting in a queue becomes impatient if he finds the server in working vacation period i.e. if he finds the server serving at rate μ_v, he activates an exponentially distributed impatient timer T with parameter ξ. The customer exits the queue and never returns if its service has not been completed before the timer T expires. Thus to conclude that only those customers whose arrival occurs during a WV of the server, are impatient. This type of impatient policy is different from that of the impatient policy studied in Altman and Yechiali (2006) in which all arriving customers become impatient during the vacation period, since a pure vacation policy is considered. The interarrival times, service times, vacation duration times and impatient time are all taken to be mutually independent.

Stationary Probability

At time t, let L(t) represent the total number of customers in the system and J(t) represents the state of the server with

$$J(t)=\begin{cases}0, & \textit{if the server is in working vacation period}\\ 1, & \textit{if the server is in normal busy period}\end{cases}$$

Then the pair $\{L(t), J(t); t \geq 0\}$ defines a two dimensional continuous time Markov chain with state space $E=\{(n, j); n \geq 0, j=0; n \geq 1-\delta, j=1\}$. Let $P_{ij} = \lim_{t\to\infty} P\{L(t)=n, J(t)=j\}$ $n \geq 0, j=0$; $n \geq 1-\delta, j=1$ represent the stationary probabilities of the Markov process $\{L(t), J(t), t \geq 0\}$, then the steady state equations are,

$$(\lambda+\delta\theta)P_{00} = \mu_v P_{10} + \mu_b P_{11} \quad (1)$$

$$(\lambda b+\theta+\mu_v)P_{10} = \lambda P_{00} + (\mu_v+\xi)P_{20} \quad (2)$$

$$(\lambda b+\theta+\mu_v+(n-1)\xi)P_{n,0} = \lambda b P_{n-1,0} + (\mu_v+n\xi)P_{n+1,0}, n \geq 0 \quad (3)$$

$$\lambda P_{01} = \delta\theta P_{00} \quad (4)$$

$$(\lambda b+\mu_b)P_{11} = \lambda P_{01} + \mu_b P_{21} + \theta P_{10} \quad (5)$$

$$(\lambda b+\mu_b)P_{n,1} = \lambda P_{n-1,1} + \mu_b P_{n+1,0} + \theta P_{n,0}, n \geq 2 \quad (6)$$

Using the probability generating functions, the stationary probabilities are derived by solving the system of equations (1) to (6).

Define

$$P_0(z) = \sum_{n=0}^{\infty} z^n P_{n,0}$$

$$P_1(z) = \delta P_{01} + \sum_{n=1}^{\infty} P_{n,1}$$

with $P_0(1) + P_1(1) = 1$ and $P_0^{'}(z) = \frac{d}{dz} P_0(z) = \sum_{n=1}^{\infty} n z^{n-1} P_{n,0}$

Multiplying equation (1), (2) and (3) by 1, z and z^n respectively and summing over n, we get

$$\xi z(1-z)P_0^{'}(z) - [(1-z)(\lambda bz - \mu_v + \xi) + z\theta]P_0(z) + A_1 z - A_2(1-z) + A_3 z^2 = 0 \tag{7}$$

where

$$A_1 = (1-\delta)\theta P_{00} - \lambda b P_{00} + \mu_b P_{11}, \quad A_2 = (\mu_v - \xi)P_{00}, \quad A_3 = \lambda \bar{b} P_{00}$$

Similarly, from equations (4), (5) and (6), we get

$$(\lambda bz - \mu_b)(1-z)P_1(z) = z\theta P_0(z) - (\mu_b P_{11} + \theta P_{00})z + \frac{\delta^2 \theta}{\lambda}(\lambda bz - \mu_b)(1-z)P_{00} + \delta\theta P_{00} z^2 \tag{8}$$

III. Solution of Differential Equation

Equation (7) can be expressed as

$$P_0^{'}(z) + \left[\frac{-(\lambda bz - \mu_v + \xi)}{\xi z} - \frac{\theta}{\xi(1-z)}\right] P_0 z = \frac{-A_1}{\xi(1-z)} + \frac{A_2}{\xi z} - \frac{z}{\xi(1-z)} A_3 \tag{9}$$

for $\xi = 0, z \neq 0, z \neq 1$

In order to solve the above differential equation we multiply it both sides by I.F= $e^{\frac{-\lambda nz}{\xi}} z^{\frac{\mu_v}{\xi}-1}(1-z)^{\frac{\theta}{\xi}}$ and integrating from 0 to z, we get

$$P_0(z) = \frac{-A_1 T_1(z) + A_2 T_2(z) - A_3 T_3(z)}{\xi z^{\frac{\mu_v}{\xi}-1}(1-z)^{\frac{\theta}{\xi}}} \tag{10}$$

where

$$T_1(z) = \int_0^z e^{\frac{\lambda n(z-x)}{\xi}} z^{\frac{\mu_v}{\xi}-1}(1-x)^{\frac{\theta}{\xi}-1} dx$$

$$T_2(z) = \int_0^z e^{\frac{\lambda n(z-x)}{\xi}} z^{\frac{\mu_v}{\xi}-1}(1-x)^{\frac{\theta}{\xi}} dx$$

$$T_3(z) = \int_0^z e^{\frac{\lambda n(z-x)}{\xi}} z^{\frac{\mu_v}{\xi}}(1-x)^{\frac{\theta}{\xi}-1} dx$$

Remark 1. Taking $\xi = 0, b = 1$ and $\delta = 0$ in equation (7), we get

$$P_0(z) = \frac{\mu_v(1-z)P_{00} - z(\theta P_{00} + \mu_b P_{11})}{\lambda z^2 - z(\lambda + \mu_v + \theta) + \mu_v}$$

which match with Servi and Finn (2002).

Taking limit as $z \to 1$ in equation (10), we have

$$\lim_{z\to 1} P_0(z) = \frac{1}{\xi}\left[-A_1 T_1(1) + A_2 T_2(z) - A_2 T_3(1)\right] \lim_{z\to 1}(1-z)^{\frac{-\theta}{\xi}}$$

Since $P_0(1)=\sum_{n=0}^{\infty}P_{n,0}>0$ and $\lim_{z\to 1}(1-z)^{\frac{-\theta}{\xi}}\to\infty$, we must have

$$-A_1T_1(1)+A_2T_2(1)-A_3T_3(1)=0$$

Putting A_1, A_2 and A_3, we get

$$P_{11}=\left\{\left(\frac{\mu_v-\xi}{\mu_b}\right)\frac{T_2(1)}{T_1(1)}-\frac{\lambda\bar{b}}{\mu_b}\frac{T_3(1)}{T_1(1)}-\left[\frac{(1-\delta)\theta-\lambda\bar{b}}{\mu_b}\right]\right\}P_{00} \tag{11}$$

Applying equation (11) in equation (10), we get,

$$P_0(z)=\frac{P_{00}}{\xi}\left\{(\mu_v-\xi)\left[T_2(z)-\frac{T_2(1)T_1(z)}{T_1(1)}\right]P_{00}-\frac{\lambda\bar{b}T_3(1)T_1(z)}{T_1(1)}-\lambda\bar{b}T_3(1\right\}(1-z)^{\frac{-\theta}{\xi}}z^{1-\frac{\mu_v}{\xi}} \tag{12}$$

Using equation (11) in equation (8), we get

$$P_1(z)=\left\{z\theta P_0(z)+\left[\begin{array}{l}\lambda\bar{b}z\left(1-\frac{T_3(z)}{T_1(1)}\right)-\frac{(\mu_v-\xi)T_2(1)z}{T_1(1)}+z\theta\delta(1+z)\\+\frac{\delta^2\theta}{\lambda}(\lambda bz-\mu_b)(1-z)\end{array}\right]P_{00}\right\}(1-z)^{-1}(\lambda bz-\mu_b)^{-1} \tag{13}$$

Theorem 3.1. *If $\rho<1$ and $0<\xi<\mu_v$, then the probability $P_{0,0}$ is given by*

$$P_{00}=\frac{(\theta+\xi)(\mu_b-\lambda b)}{\theta+(\mu_v+\xi)+\theta\delta(\theta+\xi)\left[1+\frac{\delta}{\lambda}(\mu_b-\lambda b)\right]+\lambda\bar{b}\theta+\left[\left(\frac{\mu_v-\xi}{\theta}\right)\frac{T_2(1)}{T_1(1)}-\frac{\lambda\bar{b}}{\theta}\frac{T_3(1)}{T_1(1)}\frac{\lambda\bar{b}}{\theta}\right]\times\left[\theta(\lambda b-\mu_v+\xi)+(\theta+\xi)(\mu_b-\lambda b)\right]} \tag{14}$$

Proof. Applying L'Hospital's rule in equation (10), we get

$$\lim_{z\to 1}P_0(z)=\frac{1}{\xi}\lim_{z\to 1}\frac{[-(\lambda\bar{b}P_{00}-(1-\delta)\theta P_{00}-\mu_bP_{11})z+(\mu_v-\xi)P_{00}(1-z)-\lambda\bar{b}P_{00}z^2]}{\left(\frac{\mu_v}{\xi}-1\right)(1-z)-\frac{\theta}{\xi}z}$$

which implies

$$\theta P_0(1)=(1-\delta)\theta P_{00}+\mu_bP_{11} \tag{15}$$

From (8), we get

$$\lim_{z\to 1}P_1(z)=\lim_{z\to 1}\frac{\left\{\theta P_0(z)+\theta zP_0^{'}(z)-(\theta P_{00}+\mu_bP_{11})+\delta^2\theta\lambda[(1-z)\lambda b-(\lambda bz-\mu_b)]P_{00}+2\theta\delta P_{00}z\right\}}{\lambda z(1-z)-(\lambda bz-\mu_b)}$$

Applying (15), we get

$$P_1(1)=\frac{\theta P_0^{'}(1)+\frac{\delta^2\theta}{\lambda}(\mu_b-\lambda b)P_{00}+\delta\theta P_{00}}{\mu_b-\lambda b}$$

which gives

$$P_0^{'}(1)=\left(\frac{\mu_b-\lambda b}{\theta}\right)P_1(1)-\frac{\delta^2}{\lambda}(\mu_b-\lambda b)P_{00}-\delta P_{00} \tag{16}$$

Applying L'Hospital's rule in (7), we get

$$\lim_{z\to 1} P_0^{'}(z) = \lim_{z\to 1} \frac{\left\{\begin{array}{l}[\lambda b(1-2z)+\mu_v-\xi+\theta]P_0(z)+[(1-z)(\lambda bz-\mu_v+\xi)+z\theta]P_0^{'}(z)\\ -[(1-\delta)\theta P_{00}-\lambda\bar{b}P_{00}+\mu_b P_{11}]-(\mu_v-\xi)P_{00}-2\lambda\bar{b}P_{00}z\end{array}\right\}}{\xi(1-2z)}$$

Applying (15), we obtain,

$$P_0^{'}(1) = \frac{(\lambda b-\mu_v+\xi)P_0(1)+(\mu_v-\xi)P_{00}+\lambda\bar{b}P_{00}}{\theta+\xi} \tag{17}$$

Equations (16) and (17) gives

$$\left(\frac{\mu_b-\lambda b}{\theta}\right)P_1(1)-\frac{\delta^2}{\lambda}(\mu_b-\lambda b)P_{00}-\delta P_{00} = \frac{(\lambda b-\mu_v+\xi)P_0(1)+(\mu_v-\xi)P_{00}+\lambda\bar{b}P_{00}}{\theta+\xi}$$

Which implies

$$\left[\theta(\lambda b-\mu_v+\xi)+(\mu_v-\lambda b)(\theta+\xi)\right]\left[(1-\delta)P_{00}+\frac{\mu_b}{\theta}P_{11}\right]$$

$$+\frac{\delta^2}{\lambda}\theta(\mu_b-\lambda b)(\theta+\xi)P_{00}+\delta\theta(\theta+\xi)P_{00}+\theta(\mu_v-\xi)P_{00}+\lambda\bar{b}\theta P_{00} = (\mu_b-\lambda b)(\theta+\xi)$$

Applying (11), we get

$$P_{00} = \frac{(\theta+\xi)(\mu_b-\lambda b)}{\theta+(\mu_v+\xi)+\theta\delta(\theta+\xi)\left[1+\frac{\delta}{\lambda}(\mu_b-\lambda b)\right]+\lambda\bar{b}\theta+\left[\left(\frac{\mu_v-\xi}{\theta}\right)\frac{T_2(1)}{T_1(1)}-\frac{\lambda\bar{b}}{\theta}\frac{T_3(1)}{T_1(1)}\frac{\lambda\bar{b}}{\theta}\right]\times\left[\theta(\lambda b-\mu_v+\xi)+(\theta+\xi)(\mu_b-\lambda b)\right]}$$

Performance measures

From equation (15), the probability that the system is in working vacation is

$$P_0(1) = \left[\left(\frac{\mu_v-\xi}{\theta}\right)\frac{T_2(1)}{T_1(1)}-\frac{\lambda\bar{b}}{\theta}\frac{T_3(1)}{T_1(1)}+\frac{\lambda\bar{b}}{\theta}\right]P_{00} \tag{18}$$

And the probability that the system is in normal busy period is

$$P_1(1) = 1-P_0(1) = 1-\left[\left(\frac{\mu_v-\xi}{\theta}\right)\frac{T_2(1)}{T_1(1)}-\frac{\lambda\bar{b}}{\theta}\frac{T_3(1)}{T_1(1)}-\frac{\lambda\bar{b}}{\theta}\right]P_{00} \tag{19}$$

The expected number of customers in the system during a working vacation period is

$$E(L_0) = P_0^{'}(1)$$

$$= \left(\frac{\mu_v-\lambda b}{\theta}\right)\left\{\left[1-\left(\frac{\mu_v-\xi}{\theta}\right)\frac{T_2(1)}{T_1(1)}-\frac{\lambda\bar{b}}{\theta}\frac{T_3(1)}{T_1(1)}+\frac{\lambda\bar{b}}{\theta}\right]P_{00}\right\}-\frac{\delta^2}{\lambda}(\mu_b-\lambda b)P_{00}-\delta P_{00} \tag{20}$$

Differentiate equation (18) and using L'Hospital's rule, we obtain

$$E(L_1) = P_1^{'}(1)$$

$$= [-\theta P_0^{''}(1)-2\theta\delta(1+b)P_{00}](\mu_b-\lambda b)-\frac{2\theta(\mu_b-\lambda b)P_0^{'}(1)+\mu_b\lambda(\mu_b P_{11}-2\theta\delta P_{00})}{2(\lambda b-\mu_b)^2} \tag{21}$$

Differentiate equation (7) twice at z=1, we obtain

$$P_0^{''}(1) = \frac{2(\lambda b-\mu_v+\theta)P_0^{'}(1)+2\lambda bP_0(1)+2\lambda\bar{b}P_{00})}{\theta+2\xi} \tag{22}$$

Substituting equation (22) into (21), we get

$$E(L_1)=\frac{\{\theta(1+\lambda b-\mu_v-\theta)E(L_0)+\lambda\theta bP_0(1)+\lambda\theta\bar{b}P_{00}+\theta\delta(1+b)P_{00}\}\lambda b\mu_b+2\lambda b(\mu_b P_{11}-2\theta\delta P_{00})}{(\lambda b-\mu_b)^2} \quad (23)$$

where $P_{11}, P_0(1)$ and $E(L_0)$ are given by equation (11), (18) and (20) respectively
Hence the expected system length is obtain by

$$E(L)=E(L_0)+E(L_1)$$

As the instantaneous balking rate is $\lambda\bar{b}$, therefore the expected balking rate (B.R) is obtained by

$$B.R=\sum_{n=1}^{\infty}\lambda\bar{b}P_{n,0}+\sum_{n=1}^{\infty}\lambda\bar{b}P_{n,1}=\lambda\bar{b}-\bar{b}(\lambda+\delta^2\theta)P_{00}$$

Since the instantaneous reneging rate during working vacation is n ξ , therefore the expected reneging rate (R.R) is obtained by

$$R.R=\sum_{n=1}^{\infty}n\xi\pi_{n,0}=\xi E(L_0)$$

Hence, the expected rate of customer loss (L.R) is given by
L.R=B.R+R.R

$$=\left[\xi\left(\frac{\mu_b-\lambda b}{\theta}\right)\left\{1-\left[\left(\frac{\mu_v-\xi}{\theta}\right)\frac{T_2(1)}{T_1(1)}-\frac{\lambda\bar{b}}{\theta}\frac{T_3(1)}{T_1(1)}+\frac{\lambda\bar{b}}{\theta}\right]-\frac{\delta^2}{\lambda}(\mu_b-\lambda b)-\delta\right\}-\bar{b}(\lambda+\delta^2\theta)\right]P_{00}+\lambda\bar{b}$$

IV. Stochastic Decomposition

Theorem 4.1. If $\rho<1$, the steady state queue length can be decomposed into the sum of independent random variables: $L=L_0+L_d$, where L_0 is the steady state queue length of the M/M/1 queue with balking and L_d is the additional queue length due to the effect of working vacation with its probability generating function

$$L_d(z)=\frac{1}{1-\rho}\left[\begin{array}{l}\left(1-\rho z-\frac{z\theta}{\mu_b(1-z)}\right)P_0(z)+\frac{\delta^2\theta P_{00}}{\lambda}(1-\rho z)\\ -\frac{\delta^2\theta P_{00}}{\mu_b}\frac{z^2}{(1-z)}+\left(\frac{\theta P_0(1)+\delta\theta P_{00}}{\mu_b}\frac{z}{1-z}\right)\end{array}\right] \quad (24)$$

Proof. Consider

$$\begin{aligned}L(z)&=P_0(z)+P_1(z)\\&=\left[1+\frac{z\theta}{(\lambda bz-\mu_b)(1-z)}\right]P_0(z)+\frac{\delta\theta P_{00}^2-(\mu_b P_{11}+\theta P_{00})z}{(\lambda bz-\mu_b)(1-z)}+\frac{\delta^2\theta}{\lambda}P_{00}\\&=\left(\frac{\mu_b-\lambda b}{\mu_b-\lambda bz}\right)\left[\begin{array}{l}\left(\frac{\mu_b-\lambda bz}{\mu_b-\lambda b}-\frac{z\theta}{(\mu_b-\lambda b)(1-z)}\right)P_0(z)+\frac{\delta^2\theta P_{00}}{\lambda}\left(\frac{\mu_b-\lambda bz}{\mu_b-\lambda b}\right)\\ -\frac{\delta^2\theta P_{00}}{\mu_b}\frac{z^2}{(1-z)}-\frac{\theta P_0(1)+\delta\theta P_{00}}{\lambda b-\mu_b}\end{array}\right]\\&=\left(\frac{1-\rho}{1-\rho z}\right)L_d(z)\end{aligned} \quad (25)$$

$L_d(z)$ can be expanded as series,

$$L_d(z)=\frac{1}{1-\rho}\left[\sum_{n=0}^{\infty}P_{n,0}z^n-\rho\sum_{n=0}^{\infty}P_{n,0}z^{n+1}+\frac{\theta}{\mu_b}\sum_{n=1}^{\infty}\sum_{j=0}^{\infty}P_{n-j,0}z^n+\frac{\delta\theta P_{00}}{\mu_b}(1-\delta b)z+\frac{\delta^2\theta P_{00}}{\lambda}\right]=\sum_{n=0}^{\infty}t_n z^n$$

where $t_0 = \left(1+\frac{\delta^2\theta}{\lambda}\right)\frac{P_{00}}{1-\rho}$, $\quad t_1 = \frac{1}{1-\rho}\left[P_{10} - \rho P_{00} + \frac{\theta}{\mu_b}\sum_{j=0}^{\infty} P_{j+1,0}\frac{\delta\theta P_{00}}{\mu_b}(1-\delta b)\right]$

$$t_n = \frac{1}{1-\rho}\left[P_{n,0} - \rho P_{n-1,0} + \frac{\theta}{\mu_b}\sum_{j=0}^{\infty} P_{n+j,0}\right],\ n \geq 2$$

Now, we prove that $\sum_{n=0}^{\infty} t_n = 1$ for $t_n \in [0,1]$.

$$\sum_{n=0}^{\infty} t_n = \frac{1}{1-\rho}\left[\sum_{n=0}^{\infty} P_{n,0} - \rho\sum_{n=0}^{\infty} P_{n-1,0} + \frac{\theta}{\mu_b}\sum_{n=1}^{\infty}\sum_{j=0}^{\infty} P_{n+j,0} + \frac{\delta\theta P_{00}}{\mu_b}(1-\delta b)z + \frac{\delta^2\theta P_{00}}{\lambda}\right]$$

$$= \frac{1}{1-\rho}\left[(1-\rho)\sum_{n=0}^{\infty} P_{n,0} + \frac{\theta}{\mu_b}\sum_{n=1}^{\infty} nP_{n,0} + \frac{\delta\theta P_{00}}{\mu_b}(1-\delta b)z + \frac{\delta^2\theta P_{00}}{\lambda}\right]$$

Applying (16), we get

$$\sum_{n=0}^{\infty} t_n = P_0(1) + \frac{1}{1-\rho}\left\{\frac{\theta}{\mu_b}\left[\left(\frac{\mu_v - \lambda b}{\theta}\right)(1-P_0(1)) - \frac{\delta^2}{\lambda}(\mu_v - \lambda b)P_{00} - \delta P_{00}\right] + \frac{\delta\theta P_{00}}{\mu_b}(1-\delta b) + \frac{\sigma^2\theta}{\lambda}P_{00}\right\}$$

$$= P_0(1) + \frac{1}{1-\rho}\left[(1-\rho)(1-P_0(1)) - \frac{\delta^2\theta}{\lambda\mu_b}(\mu_b - \lambda b)P_{00} - \frac{\delta\theta P_{00}}{\mu_b} + \frac{\delta\theta P_{00}}{\mu_b}(1-\delta b) + \frac{\sigma^2\theta}{\lambda}P_{00}\right]$$

$$= P_0(1) + (1-P_0(1)) - \frac{\sigma^2\theta}{\lambda}P_{00} - \frac{\delta\theta P_{00}}{1-\rho} + \frac{\delta\theta P_{00}}{\mu_b}\frac{(1-\delta b)}{1-\rho} + \frac{\delta^2\theta P_{00}}{\lambda(1-\rho)} = 1$$

Hence $L_d(z)$ is a probability generating function.

Theorem 4.2. If $\rho < 1$, the steady state waiting time can be decomposed into a sum of two independent random variables $W = W_0 + W_d$, where W_0 is the steady state waiting time of the M/M/1 queue with balking and W_d is the additional delay due to the effect of working vacation with its Laplace Stieltjes transform (LST)

$$W_d^*(s) = \frac{1}{(\mu_b - \lambda)s}\begin{bmatrix}((\mu_b - \lambda b + s)s - \theta(\lambda - s))P_0\left(1-\frac{s}{\lambda}\right) - \frac{\delta\theta P_{00}}{\lambda}(\lambda - s)^2 \\ +(\theta P_0(1) + \delta\theta P_{00})(\lambda - s) + \frac{\delta^2\theta P_{00}}{\lambda}(\mu_b - \lambda b + sb)\end{bmatrix}$$

Proof. The relationship between the probability generating function of L and the LST of waiting time is given by

$$L(z) = W^*(\lambda(1-z))$$

Assume that $s = \lambda(1-z)$, So $z = 1 - \frac{s}{\lambda}$ and $1 - z = \frac{s}{\lambda}$. Applying the relations in equation (24) we obtain the desired result.

Taking $\lim_{s\to 0} W_d^*(s)$, we get

$$\lim_{s\to 0} W_d^*(s) = \frac{1}{(\mu_b - \lambda b)}\left[(\mu - \lambda b + \theta)P_0(1) + \theta P_0^{'}(1) - \theta P_0(1) + \delta\theta\left(1 + \delta\frac{\mu - \lambda b}{\lambda}\right)P_{00}\right]$$

Applying equation (16), we get $\lim_{s\to 0} W_d^*(s) = 1$. This proves that $W_d^*(s)$ is a random variable of the additional waiting time.

V. Numerical Results

The main findings in this study are itemized as

- Figure 1 illustrates the impact of arrival rate λ on E[L] in MWV model for different values of b. The mean queue length expectedly increases with the increase of λ for any b. Furthermore, it is clear that system length increases with b. This is due to the fact that the probability that the customer joins the queue increases as b increases.
- Figure 2 describes the variation of λ with L.R. in MWV model for different values of ξ. L.R. increases with the increase of λ because higher the number customers in the system, greater are the chances that they would balk and vacant the system. Also it is clear that the number the reneging customers increases with the increase in ξ resulting in the increase of L.R.
- Figure 3 presents the effect of impatient rate ξ on E[L] and L.R in MWV model. It demonstrates that E[L] and L.R. decreases and increases respectively with the increase in ξ. The value of ξ at which E[L] and L.R. are minimum and maximum respectively is given by the point of intersection of the two curves. Hence, the managers of the queuing system can optimize E[L] or L.R. by taking the necessary steps with the knowledge of the reneging rate.
- Figure 4 presents the comparison between the system sizes of both the models. With the increase in the impatient rate ξ the difference between the systems sizes increases. Hence, the MWV and SWV models give the same system sizes if the ξ is small but MWV model will give better performance for the higher impatient rates. Therefore, the impatience rate performs a significant role in the performance of the system.

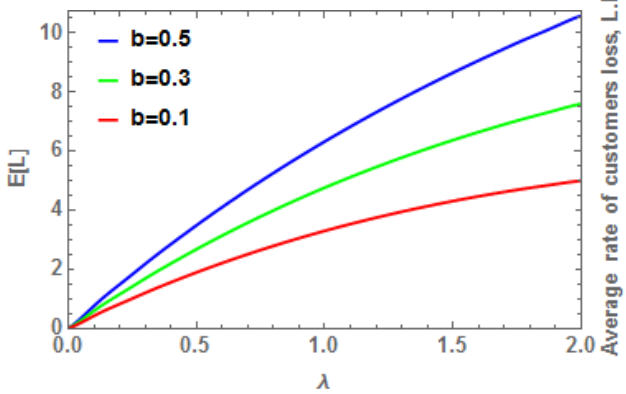


Fig 1. Mean queue length vs λ with μ=6 , η=4.5 , ξ=1.5 , θ=2 .

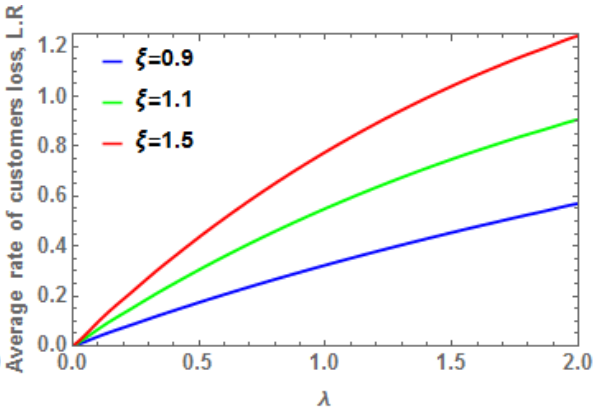


Fig 2. L.R vs λ with μ=6 , η=4.5 , b=0.8 , θ=2

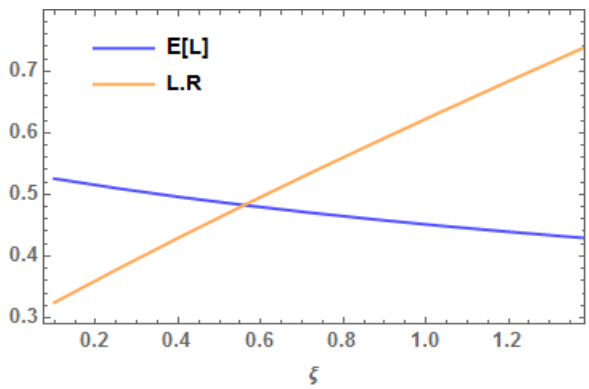


Fig 3. Effect of ξ on E[L] and L.R with μ=2.5 , η=1.5 , λ=1.5 , θ=0.5 .

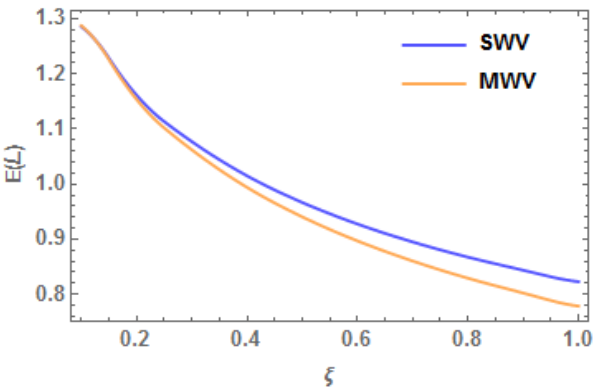


Fig 4. Comparison between models with μ=2.5 , η=1.5 , λ=1.5 , θ=0.18 .

VI. References

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On Achromatic Number of Double Fan Graphs

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***Abstract**: Let G be a nontrivial connected graph, an achromatic coloring of a simple nontrivial graph is the proper vertex coloring such that each pair of color classes is adjacent by at least one edge. The achromatic number of G denoted by Ψ(G), is the largest possible number of colors in an achromatic coloring of G. In this paper, we obtain the achromatic number of central, middle, total, line and jump graph of double fan graph.*

***Keywords**: chromatic number, achromatic number, achromatic coloring, central graph, middle graph, total graph, line graph and jump graph.*

I. Introduction

In this paper, we are concerned with finite, undirected, loop less graph without multiple edges. Let G = (V(G), E(G)) be a graph. For two vertices p and q of G, if there is an edge e joining them, we say p and q are adjacent. Here, both p and q are end vertices of e, and p and e or q and e are said to be incident. A coloring of a graph G is a partitioning of the vertex set V into color classes. A proper coloring or chromatic vertex coloring of a graph G with vertex set V and edge set E is a map f : V(G)→S such that f(p)≠f(q) whenever pq ∈ E(G). The smallest number of colors used in such a coloring is called the chromatic number of G, denoted by χ(G). Similarly, a proper achromatic coloring of a graph G assigns colors to each vertex of G such that for each color class C_i, N[C_i] contains representatives of every color class. The maximum number of color classes in a proper achromatic partition of G is the achromatic number of G, and is denoted by Ψ(G) [1], [2]. Hedetniemi [3] conjectured that pseudoachromatic number and achromatic number are equal for all trees. But Keith. J. Edwards disproves by giving an infinite family of trees for which pseudoachromatic number strictly exceeds the achromatic number. Computing the achromatic number of a general graph was proved to be NP complete by Yannakakis and Gavril [9] in 1980. In 1976, Pavol Hell and Donald J.Miller [5] who found the achromatic number of Paths and Cycles.

The Central graph [7], [8] C(G) of graph G is attained from G by subdividing every edge of G precisely once and then joining every point of vertices of the original graph which were previously non - adjacent.

The Middle graph of G which is represented by M(G). M(G) has vertex set V(G) ∪ E(G). Two vertices p,q of M(G) are neighboring in M(G) in case one of the following conditions holds.

(i) p,q are within E(G) and p,q are neighboring in G
(ii) p is within V(G), q is within E(G), and p,q are neighboring in G.

Let G be a graph with vertex set V(G) and edge set E(G). Total graphs are generality of Line graphs. The Total graph of a graph G, represented by T(G) is exemplified as follows. The vertex set of T(G) is V(G) ∪ E(G). Two vertex p,q in the vertex set of T(G) are neighboring in T(G) in case one of the following holds.

(i) p,q are within V(G) and p is neighboring to q in G.
(ii) p,q are within E(G) and p,q are neighboring in G.
(iii) p is within V(G), q is within E(G) and p,q are incident in G.

The line graph L(G) of G is the graph whose vertex set is E(G) in which two vertices are adjacent if and only if they are adjacent in G [5], [8].

For a nonempty graph G, the jump graph of G is noted as J(G), whose vertices are edges of G and the where two vertices of this jump graph J(G) are adjacent if and only if they are nonadjacent in G [4]. Also the jump graph J(G) is the complement of line graph L(G) of G.

A fan graph $F_{m,n}$ is defined as the join of two graphs $\overline{K}_m$+P_n, where $\overline{K}_m$ is the empty graph on m nodes and P_n is the path graph on n nodes. Let (X,Y) be the bipartition of $F_{m,n}$ with |X| = m and |Y| = n. In the case of m = 1 shows that the regular fan graph $F_{1,n}$ whereas $F_{2,n}$ shows that the double fan graph in the case of m = 2.

2. Achromatic Coloring of Central Graph of F (2, n)

Algorithm 2.1

Input: The number n of F (2, n).

Output: Assigning achromatic coloring for the vertices in C(F(2, n)).

begin
for i = 1 to n+2
{
$v_1 = \{v_i\}$
C (v_i) = i
}
for i = 1 to n
{
$v_2 = \{x_i\}$
C (x_i) = i+2
}
for i = 1 to n-1
{
$v_3 = \{y_i\}$
C (y_i) = i
}
for i = 1 to n
{
$v_4 = \{z_i\}$
C (z_i) = n+2-i
}

Theorem 2.1

For any positive integer n $\geq$ 3, $\Psi[C(F(2,n))] = 2+n$.

Proof

Let v_1 , v_2 … v_{n+2} be the vertices for F(2,n). Let the vertex set of F(2,n) is defined as $V[F(2,n)] = \{v_i: 1 \leq i \leq n+2\}$. Consider C(F(2,n)). By the definition of central graph, subdividing each edges (v_1v_{i+1}) into x_i for $1 \leq i \leq n$, (v_iv_{i+1}) into y_i for $1 \leq i \leq n-1$, and $(v_{n+2}v_{i+1})$ into z_i for $1 \leq i \leq n$. Thus the vertex set of C(F(2,n)) is defined as follows:

$V[C(F(2,n))] = \{v_i: 1 \leq i \leq n+2\} \cup \{x_i : 1 \leq i \leq n\} \cup \{y_i: 1 \leq i \leq n-1\} \cup \{z_i: 1 \leq i \leq n\}$.

In C(F(2,n)) the vertices v_i for $2 \leq i \leq n+1$ induces a clique of order n+1. Thus we have $\Psi [(C(F(2,n))] \leq n+2$.

Consider the color class $C = \{c_1,c_2,\ldots,c_n,c_{n+1},c_{n+2}\}$. Now assign a proper vertex coloring to the vertices of C(F(2,n) by using the Algorithm 2.1 to make the coloring as achromatic. By considering the algorithm, this coloring process will accommodate all the pairs of the class. Thus we have $\Psi[(C(F(2,n))] \geq n+2$.

Hence $\Psi[C(F(2,n))] = n+2$.

3. Achromatic Coloring of Middle Graph of F (2, n)

Algorithm 3.1

Input: The number n of F (2, n).

Output: Assigning achromatic coloring for the vertices in M(F(2,n)).

begin
for i = 1
{
$v_1 = \{v_i\}$
M (v_i) = i
}
for i = 2 to n+1
{
M (v_i) = i+1
}
for i = n+2

{
M (v_i) = i+2
}
for i =1 to n
{
v_2={x_i}
M (x_i) = i+4
}
for i = 1 to 4
{
v_3 = {y_i}
M (y_i) = i
}
for i = 5 to n-1
{
M (y_i) = i-4
}
for i = 1 to n-3
{
v_3 = {z_i}
M (z_i) = i+1
}
for i = n-2 to n
{
M (z_i) = i+3
}

Theorem 3.1
For any positive integer n $\geq$ 6, Ψ[M(F(2,n))] = n+4.
Proof

Let v_1 ,v_2 ,…,v_n,v_{n+1},v_{n+2} be the vertices for F (2, n). Let the vertex set of F (2, n) is defined as V[F(2,n)] = {v_i: 1 ≤ i ≤n+2}. By the definition of middle graph, subdividing each edges (v_iv_{i+1}) into x_i for 1 ≤ i ≤ n, (v_iv_{i+1}) into y_i for 1 ≤ i ≤ n-1 and ($v_{n+2}v_i$) into z_i for 1≤ i ≤ n. Thus the vertex set of V[M(F(2,n))] can be written as,

V[M(F(2,n))] = {v_i: 1 ≤ i ≤n+2} ∪ {x_i: 1 ≤ i ≤n} ∪ {y_i: 1 ≤ i ≤ n-1} ∪ {z_i: 1 ≤ i ≤n}.

In M(F(2,n)) for the vertices x_i (1 ≤ i ≤ n) induces a clique of order n+1 and the vertices z_i(1 ≤ i ≤ n) induces a clique of order n+1.

Thus we have Ψ[F(2, n)] ≤ n+4.

Now consider the color class C = {c_1,c_2,…,c_n,…c_{n+4}}. By assigning a proper coloring to the vertices of M[F(2,n)] by using the Algorithm 3.1, which will satisfies the condition of achromatic coloring and an easy check shows that Ψ[(M(F(2, n))] ≥ n+4 .

Hence we have Ψ[M(F(2, n))] = n+4.

4. Achromatic Coloring of Total Graph of F (2, n)

Theorem 4.1
For any positive integer n ≥ 6, Ψ[T(F(2,n))] = n+4.
Proof
The proof is same as Theorem 3.1

5. Achromatic Coloring of Line Graph of F (2, n)

Algorithm 5.1
Input: The number n of F (2, n).
Output: Assigning achromatic coloring for the vertices in L(F(2,n)).
begin
for i = 1 to n+2
{
v_1={v_i}
L (v_i) = i

}
for i = 1 to n-4
{
$v_2 = \{u_i\}$
L (u_i) = n-i
}
for i = 1 to 3
{
$v_3 = \{w_i\}$
L (w_i) = n+i-1
}
for i = 1 to n-2
{
$v_3 = \{x_i\}$
L (x_i) = i
}

Theorem 5.1
For any positive integer n $\geq$ 5, Ψ[L(F(2, n))] = n+2.
Proof:
The vertex set for F (2, n) is
V [F(2,n)] = $\{v_i: 1 \leq i \leq n+2\} \cup \{u_i: 1 \leq i \leq n-4\} \cup \{w_i : 1 \leq i \leq n-2\}$. By the definition of line graph, each edges of F (2, n) taken to be as vertex in L(F(2,n)). By the definition of line graph, the edge set in F (2, n) corresponds to the vertex set of L(F(2,n)) respectively. i.e., V [L(F(2,n))] = $\{v_i : 1 \leq i \leq n+2\} \cup \{u_i : i = n-4\} \cup \{w_i : 1 \leq i \leq 3\} \cup \{x_i : 1 \leq i \leq n-2\}$. Consider the color class C = $\{c_1, c_2, \ldots, c_{n+2}\}$ for L(F(2,n)). By using the Algorithm 5.1, we have assigned n+2 color. In case we assign the color more than n+2, we have to show that Ψ[L(F(2,n))] $\geq$ n+2. So assign the color c_{n+3} to the vertex w_3, then the coloring accommodate all the pairs except (c_1,c_{n+2}), (c_2,c_{n+2}), (c_n,c_{n+2}) and (c_{n+2}, c_{n+3}). By avoiding such pairs we need to reduce one color from the maximum number of colors for producing the achromatic number. Hence by choosing the maximum number of color not exceeding n+2, will result the procedure of achromatic coloring. Thus Ψ[L(F(2,n))] $\leq$ n+2.
Hence we have Ψ[L(F(2,n))] = n+2.

6. Achromatic Coloring of Jump Graph of F (2, n)
Algorithm 6.1
Input: The number n of F (2, n).
Output: Assigning achromatic coloring for the vertices in J(F(2,n)).
begin
for i = 1
{
$v_1 = \{v_i\}$
J (v_i) = i
}
for i = 2 to n-1
{
J (v_i) = n+2-i
}
for i =1 to n+1
{
$v_2 = \{u_i\}$
J (u_i) = i
}
for i =1 to n-1
{
$v_3 = \{w_i\}$
J (w_i) = i+1
}

Theorem 6.1
For any positive integer n ≥ 3, we have Ψ[J(F(2,n))] = n+1.
Proof

The vertex set for F (2, n) is,

V[F(2,n)] = $\{v_i: 1 \leq i \leq n-1\} \cup \{u_i: 1 \leq i \leq n+1\} \cup \{w_i: 1 \leq i \leq n-1\}$.

By the definition of jump graph, the complement of the edge set in F(2,n) corresponds to the vertex set of L(F(2,n)) respectively.

i.e., V [L(F(2,n))] = $\{v_i : 1 \leq i \leq n-1\} \cup \{u_i : 1 \leq i \leq n+1\} \cup \{w_i : 1 \leq i \leq n-1\}$.

Consider the color class C = $\{c_1, c_2, \ldots, c_{n+1}\}$ for J(F(2,n)). By using the Algorithm 6.1, we have assigned n+1 color. In case , we assign the color more than n+1, we need to show that $\Psi[J(F(2,n))] \geq n+1$. But for the case of maximization, we fail to receive one pair (c_i, c_{n+1})($1 \leq i \leq n$). This contradicts the definition of achromatic coloring. So we need to reduce one color from the maximum number of colors not exceeding n+1 colors and this will result the condition of achromatic coloring. Thus $\Psi[J(F(2,n))] \leq n+1$. Hence we have $\Psi[J(F(2,n))] = n+1$.

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E – Super Vertex Magic Labeling of Some Families of Graph

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Abstract: *Let G be a finite simple graph with p vertices and q edges. A vertex magic total labeling is a bijection from* $V(G) \cup E(G)$ *to the consecutive integers* $1,2,3,\dots,p+q$ *with the property that every* $u \in V(G)$, $f(u) + \sum_{v \in N(u)} f(uv) = k$ *for some constant k. Such a labeling is E-Super if* $f[E(G)] = 1,2,3,\dots,q$. *A graph G is called E-Super vertex magic if it admits a E-Super vertex magic labeling. In this paper, we establish the E-Super vertex magic graph of some classes of* $H_{2r+1,4n}$ *graph,* $H_{4,2n+5}$ *graph,* $P_1 \otimes PG_n$ *graph,* $P_2 \otimes H_s$ *graph,* $P_2 \otimes P_3$ *graph, parity graph, parachute graph, Herschel graphs, and corona graphs.*

Keywords: *Graph Labeling, Magic Labeling, E-Super Vertex Magic, Herschel Graph, Parity Graph.*

1. Introduction

Let G be a finite, undirected graph with no loops and multiple edges. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. A magic graph is a graph whose edges are labeled by positive integers, so that the sum over the edges incident with any vertex is the same independent of the choice of vertex. A graph is vertex magic if its vertices can be labeled so that the sum on any edges is the same. It is total magic if its edges and vertices can be labeled so that the vertex plus the sum of labels on edges incident with that vertex is a constant.

Sedlacek (1963) introduced the concept of magic labeling. Suppose that G is a graph with q edges we say that G is magic if the edges of G can be labeled by the numbers $1,2,3,\dots,q$ so that the sum of labels of all the edges incident with any vertex is same. Macdougall et.al.(2002) introduced the notion of vertex magic total labeling. If G is a finite simple undirected graph with p vertices and q edges, then a vertex magic total labeling is a bijection from $V(G) \cup E(G)$ to the integers $1,2,3,\dots,p+q$ with the property that for every u in $V(G)$.

A **Graph Labeling** is the assignment of values or labels, to the edges or vertices, or both of the graph labels by an integer values. A graph is said to be **Magic Graph** whose vertices and edges are labeled by positive integers, so that the sum over the edges incident with the vertex has the same. A bipartite undirected graph with 11 vertices and 18 edges is called as **Herschel Graph**, which is denoted as H_s. A **Parity Graph** is a graph in which every two induced paths between the same two vertices have the same equality either both paths have odd length, or both have even length. A graph G with p vertex and q edges is called **E-super vertex magic** if there exist a bijection $f: E \to \{1,2,3,\dots,q\}$ and $f: V \to \{p+1, p+2, \dots, p+q\}$ and for this labeling there is some constant k such that for each vertex p.

Theorem 1.1[2] If G has a E-super vertex magic total labeling, then $k = q + \frac{(p+1)}{2} + \frac{q(q+1)}{p}$.

Theorem 1.2[2] If G has a super vertex magic total labeling, then $k = \frac{(p+q)(p+q+1)}{p} - \frac{(p+1)}{2}$.

Lemma 1.1[4] If a non-trivial graph G is an V-super vertex magic total, then the magic constant k is given by $k = 2q + \frac{(p+1)}{2} - \frac{q(q+1)}{p}$.

Lemma 2.2[3] If G is a vertex magic graph with v vertices and e edges, then $\frac{(p+q)(p+q+1)}{2p} + \frac{E_{sum}}{p} = k$.

II. Main Results

Theorem: 2.1

The graph $H_{2r+1,4n}$ has a E-super vertex magic total labeling if $r \geq 1$ and $n \geq 2$.

Proof:

Case:1

Let $H_{2r+1,4n}$ be the graph with each vertex of degree is $2r+1$, if $r+1$ and then $4n$ vertices and $6n$ edges. Suppose $H_{2r+1,4n}$ has a E-super vertex magic total labeling, then by theorem 2.1

$$k = q + \frac{(p+1)}{2} + \frac{q(q+1)}{p}$$

In graph, $p = 4n$ and $q = 6n$

$$k = 6n + \frac{(4n+1)}{2} + \frac{6n(6n+1)}{4n}$$

Since $\frac{4n+1}{2}$ and $\frac{3(6n+1)}{2}$ is not an integer. But the summation value is integer.

$$k = 17n + 2$$

This k is an integer for any value of $n \geq 2$.

$$k = integer$$

Case:2

Let $H_{2r+1,4n}$ be the graph with each vertex of degree is $2r+1$, if $r = 2$ and then $4n$ vertices and $10n$ edges.

Suppose $H_{2r+1,4n}$ has a E-super vertex magic total labeling, then by theorem 2.1

$$k = q + \frac{(p+1)}{2} + \frac{q(q+1)}{p}$$

In graph, $p = 4n$ and $q = 10n$

$$k = 10n + \frac{(4n+1)}{2} + \frac{10n(10n+1)}{4n}$$

Since $\frac{4n+1}{2}$ and $\frac{5(10n+1)}{2}$ is not an integer. But the summation value is integer.

$$k = 37n + 3$$

This k is an integer for any value of $n \geq 2$.

$$k = integer.$$

Case:3

Let $H_{2r+1,4n}$ be the graph with each vertex of degree is $2r+1$, if $r = 3$ and then $4n$ vertices and $6n$ edges.

Suppose $H_{2r+1,4n}$ has a E-super vertex magic total labeling, then by theorem 2.1

$$k = q + \frac{(p+1)}{2} + \frac{q(q+1)}{p}$$

In graph, $p = 4n$ and $q = 14n$

$$k = 14n + \frac{(4n+1)}{2} + \frac{14n(14n+1)}{4n}$$

Since $\frac{4n+1}{2}$ and $\frac{7(14n+1)}{2}$ is not an integer. But the summation value is integer.

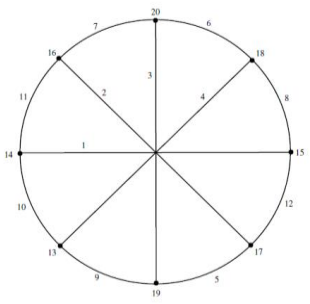
$$k = 65n + 4$$

This k is an integer for any value of $n \geq 2$.

$$k = Integer.$$

Hence $H_{2r+1,4n}$ has a E-super vertex magic total labeling

The Graph $H_{3,8}$ is a E-Super Vertex Magic with $K = 36$



Theorem: 2.2

The graph $H_{4,2n+5}$ has a E-super vertex magic total labeling if $n \geq 1$.

Proof:

Let $H_{4,2n+5}$ be a regular graph with each vertex of degree is 4. The vertices are $2n+5$ and edges are $4n+10$.

Number of edges$= 2 *$ Number of vertices.

Suppose $H_{4,2n+5}$ has a E-super vertex magic total labeling, then by theorem 2.1

$$k = q + \frac{(p+1)}{2} + \frac{q(q+1)}{p}$$

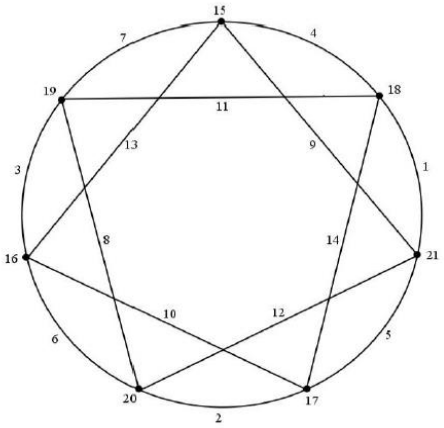
In graph, $p = 2n+5$ and $q = 2 * p = 4n+10$

$$k = 4n + 10 + \frac{(2n+5+1)}{2} + \frac{(4n+10)(4n+11)}{2n+5}$$

$$k = 13n + 35$$

This is an integer if $n \geq 1$. Hence $H_{4,2n+5}$ has a E-super vertex magic total labeling.

The Graph $H_{4,7}$ is a E-Super Vertex Magic with $K = 48$



Theorem: 2.3

The corona graph $P_1 \otimes PG_n$ of path graph with one vertex and parachute graph has a E-super vertex magic total labeling if $n = 3$.

Proof:

Let PG_n be a parachute graph with $2n$ vertices and $3n-1$ edges.

The corona $P_1 \otimes PG_n$ has a E-super vertex magic total labeling, then by theorem 2.1

$$k = q + \frac{(p+1)}{2} + \frac{q(q+1)}{p}$$

Hence, $p = 2n+1$ and q is

The edges of graph $P_1 \otimes PG_n$= Edges of parachute G + Edges of corona P_1.

$$q = (3n-1) + (2n)$$
$$= 5n - 1$$

Hence

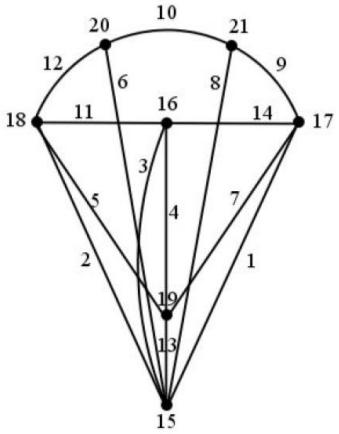
$$k = 5n - 1 + \frac{(2n+1+1)}{2} + \frac{(5n-1)(5n)}{2n+1}$$

$$k = 6n + \frac{(5n-1)(5n)}{2n+1}$$

$k = Integer$ if $n = 3$

Hence $P_1 \otimes PG_3$ the graph is E-super vertex magic total labeling.

If $n = 3$. The corona graph $P_1 \otimes PG_n$ has E-super vertex magic graph with magic constant $k = 48$.



Theorem: 2.4

The corona graph $P_2 \otimes H_S$ has a E-super vertex magic total labeling.

Proof:

Let P_2 be a path graph with 2 vertices and 1 edge and let H_S be Herschel graph with 11 vertices and 18 edges.

The corona graph,

The vertices of graph $P_2 \otimes H_S =$ Vertices of path $+2 *$ (vertices of herschel)

$$P_2 \otimes H_S = 2 + 2 * (11)$$
$$= 24$$

And

The edges of graph $P_2 \otimes H_S =$ Edges of path$+2 *$ (Edges of Herschel $+$ vertices of herschel)

$$P_2 \otimes H_S = 1 + 2 * (18 + 11)$$
$$= 59$$

The corona graph $P_2 \otimes H_S$ be a 24 vertices and 59 edges.

Suppose that $P_2 \otimes H_S$ has E-super vertex magic total labeling, then by theorem 2.1

$$k = q + \frac{(p+1)}{2} + \frac{q(q+1)}{p}$$

Here $p = 24$ and $q = 59$

$$k = 59 + \frac{(24+1)}{2} + \frac{59(59+1)}{24}$$
$$k = 59 + \frac{(25)}{2} + \frac{3540}{24}$$

Since $\frac{25}{2}$ and $\frac{3540}{24}$ is not an integer. But the summation value is integer.

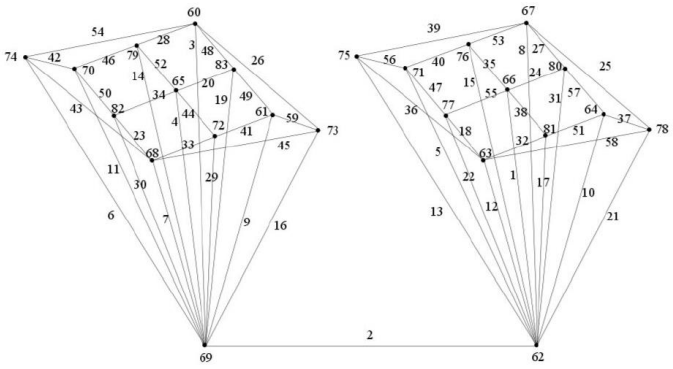
Then

$$k = 219$$
$$k = Integer.$$

Hence $P_2 \otimes H_S$ the graph is E-super vertex magic total labeling.

The corona graph $P_2 \otimes H_S$ has a E-super vertex magic with magic constant k=219.

The corona graph $P_2 \otimes H_S$ has a E-super vertex magic with magic constant $k = 219$.



Theorem: 2.5

The corona graph $P_2 \otimes P_3$ has a E-super vertex magic total labeling.

Proof:

Let P_n be a path graph with n vertices and n-1 edges
Let P_2 be a path graph with 2 vertices and 1 edge and
Let P_3 be a path graph with 3 vertices and 2 edges
The corona graph,
The vertices of graph $P_2 \otimes P_3$ = Vertices of path $P_2 + 2 *$ (vertices of path P_3)

$$P_2 \otimes P_3 = 2 + 2 * (3)$$
$$= 8$$

And
The edges of graph $P_2 \otimes P_3$ =Edges of path $P_2 + 2 *$ (Edges of $P_3 +$ vertices of P_3)

$$P_2 \otimes P_3 = 1 + 2 * (2 + 3)$$
$$= 11$$

The corona graph $P_2 \otimes P_3$ be a 8 vertices and 11 edges.
Suppose that $P_2 \otimes P_3$ has E-super vertex magic total labeling, then by theorem 2.1

$$k = q + \frac{(p+1)}{2} + \frac{q(q+1)}{p}$$

Here $p = 8$ and $q = 11$

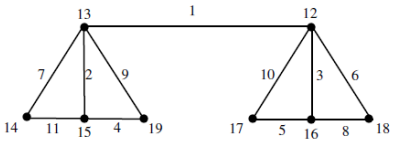
$$k = 11 + \frac{(8+1)}{2} + \frac{11(11+1)}{8}$$

Since $\frac{9}{2}$ and $\frac{33}{2}$ is not an integer. But the summation value is integer.
Then

$$k = 32$$
$$k = Integer$$

Hence $P_2 \otimes P_3$ the graph is E-super vertex magic total labeling.
The corona graph $P_2 \otimes P_3$ has a E=super vertex magic with magic constant k=219.

The corona graph $P_2 \otimes P_3$ has a E-super vertex magic with magic constant $k = 219$.



Theorem:2.6

The parity graph P_a has a E-super vertex magic total labeling.

Proof:

Let P_a be a parity graph with 8 vertices and 12 edges. Suppose P_a has a E-super vertex magic total labeling, then by theorem 2.1

$$k = q + \frac{(p+1)}{2} + \frac{q(q+1)}{p}$$

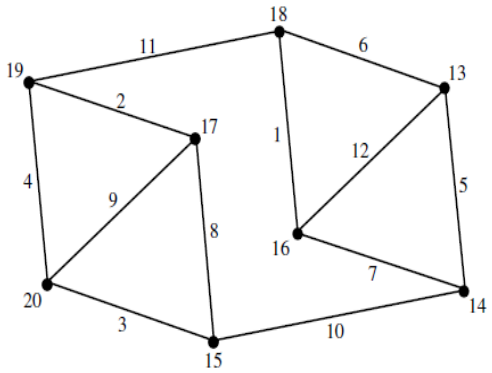
Here $p = 8$ and $q = 12$

$$k = 12 + \frac{(8+1)}{2} + \frac{12(12+1)}{8}$$

Since $\frac{9}{2}$ and $\frac{39}{2}$ is not an integer. But the summation value is integer.
Then

$$k = 36$$
$$k = Integer$$

Hence P_a the graph is E-super vertex magic total labeling.
The parity graph $P_a$has a E-super vertex magic with magic constant $k = 36$.



Theorem:2.7

If $H_{m,n}(m = 3, n = 4(j + 1)$ if j=1,2....) is a E-super vertex magic graph, then Labeling of outer edges are $\frac{p}{2} + 1, \frac{p}{2} + 2, \ldots q$ and inner edges are 1,2,...,$\frac{p}{2}$.

Proof:

If G is E-super vertex magic labeling the vertex labeling are $q + 1, q + 2, \ldots, p + q$. The edges labeling of graph G is $1,2,3, \ldots, q$. The edge labeling of $H_{m,n}$ has two partitions into inner and outer edges. Let $H_{m,4n}$ be a regular graph with m degree and $4n$ edges. For $n = 2$. Suppose G is E-super vertex magic graph, then the labeling of vertices are $q + 1, q + 2, \ldots, p + q$. Therefore, the only possibility for labeling of edges are $1,2,3, \ldots, q$. Each vertex has at least one outer and one inner edge. The outer edges are(number of vertex) then the minimum edge value are $1,2,3, \ldots, p$. The remaining edges are inner edges then the maximum edge values are $p + 1, p + 2, \ldots, q$. Hence $H_{m,n}$ is E-super vertex labeling.

If $H_{m,n}(m = 3, n = 4(j + 1)$ if j=1,2....) is a E-super vertex magic graph, Suppose we consider the graph j=1, then the graph is $H_{3,8}$. The Graph $H_{3,8}$ is a E-Super Vertex Magic with $k = 36$.

Theorem:2.8

If $H_{m,4n}(m = 2i + 1$ if $i = 2, \ldots. n \geq 2)$ is a E-super vertex magic graph, then labeling of outer edges are $1,2, \ldots, p$ and inner edges are $p + 1, p + 2, \ldots . q$.

Proof:

If G is E-super vertex magic labeling the vertex labeling are $q + 1, q + 2, \ldots, p + q$. The edges labeling of graph G is $1,2,3, \ldots, q$. The edges labeling of $H_{m,4n}$ has two partitions into inner and outer edges. Let $H_{m,4n}$ be a regular graph with m degree and $4n$ edges. For $n = 2$, suppose G is E-super vertex magic graph, then the labeling of vertices are $q + 1, q + 2, \ldots, p + q$. Therefore, the only possibility for labeling of edges are $1,2,3, \ldots, q$. Each vertex has at least one outer and one outer and one inner edges. The outer edges are(number of vertex) then the minimum edge values are $1,2,3, \ldots, p$. The remaining edges are inner edges then the maximum edge values are $p + 1, p + 2, \ldots, q$. Hence $H_{m,n}$ is E-super vertex labeling.

$H_{m,4n}(m = 2i + 1$ if $i = 2, \ldots. n \geq 2)$. Suppose we consider the graph $i = 2$ if $m = 5$, and $n = 2$. Then the graph is $H_{5,8}$.

III. Conclusion

This paper addresses labeling graphs in such a way that they are E-Super vertex magic graph. Exploring E-Super vertex magic for magic numbers is an interesting problem for all graphs. We are able to obtain a E-super magic constant for the graphs namely Herschel graph, Parity graph, Corona graph and etc.,

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Analysis of Prey - Predator Model with Holling Type II Functional Response in Discrete time

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***Abstract**: In this paper, the dynamical behavior of a Discrete time Prey - Predator model is investigated. Stability of the fixed points investigated for the system. Numerical simulations are used to verify the correctness of the analytical results. The system undergoes period doubling bifurcation at certain parametric values for positive interior fixed point and chaos exist in the system. Finally the system is found to be sensitive for the initial conditions.*

***Keywords**: Fractional Order, Predator - Prey system, Stability, Bifurcation.*

I. Introduction

Mathematical modeling of interactions between species has drawn the attention of researchers. The investigation of the population dynamics with Functional Response has attracted the researchers in recent decades. In type II functional response, the rate of prey consumption by a predator rises as prey density increases, but eventually levels off at a asymptote at which the rate of consumption remains constant regardless of increases in prey density.

II.Discrete time Prey - Predator System

Let us consider the prey-predator model describing the interactions among two species by the following system of difference equations:

$$\begin{aligned} x(t+1) &= rx(t)\left[1-\frac{x(t)}{k}\right]-\frac{ax(t)y(t)}{1+amx(t)} \\ y(t+1) &= \frac{ax(t)y(t)}{1+amx(t)}-\beta y(t) \end{aligned} \tag{1}$$

where $r,a,k,\beta,m>0$ are positive constants. Prey with no predators grows logistically to its environmental capacity K, with an intrinsic birth rate constant r, β is the death rate of predator, a is attacking rate and m is the handling time. While x and y represent the densities of the prey and predator populations respectively.

The fixed points of system (1) are:

- The trivial state $E_0=(0,0)$
- The axial state $E_1=\left(\frac{k(r-1)}{r},0\right)$
- The study state of coexistence

$$E_2=\left(-\frac{(\beta-1)}{a(\beta m-m+1)},\frac{akm(\beta r-\beta-r+1)+ak(r-1)+r(\beta-1)}{a^2(\beta m-m+1)^2k}\right).$$

III. Stability Analysis

Next, we study the local stability of the fixed points. The Jacobian matrix of (1) evaluated at any fixed point (x^*,y^*) is

$$J(x^*,y^*)=\begin{bmatrix} r\left(1-\frac{2x^*}{k}\right)-\frac{ay^*}{(1+amx^*)^2} & -\frac{ax^*}{1+amx^*} \\ \frac{ay^*}{(1+amx^*)^2} & \frac{ax^*}{1+amx^*}-\beta \end{bmatrix}.$$

At E_0, $J(E_0)=\begin{bmatrix} r & 0 \\ 0 & -\beta \end{bmatrix}$. The corresponding eigenvalues are $\lambda_1 = r$ and $\lambda_2 = -\beta$.

At E_1, $J(E_1)=\begin{bmatrix} r\left(1-\dfrac{2k\left(\dfrac{r-1}{r}\right)}{k}\right) & -\dfrac{ak(r-1)}{\left(r+amk(r-1)\right)} \\ 0 & \dfrac{ak(r-1)}{\left(r+amk(r-1)\right)}-\beta \end{bmatrix}$. Eigenvalues for $J(E_1)$ are $\lambda_1 = 2-r$ and $\lambda_2 = \dfrac{ak(r-1)}{r+amk(r-1)}-\beta$.

Hence *i) The trivial fixed point E_0 is locally asymptotically stable if $r<1$ and $\beta<1$, If $r>1$ and $\beta>1$ then E_0 is unstable.*

ii) The semi-trivial fixed point E_1 is locally asymptotically stable when $r<3$ and $\beta<1+\dfrac{ak(1-r)}{r+amk(r-1)}$.

E_1 is Unstable when $r>3$ and $\beta>1+\dfrac{ak(1-r)}{r+amk(r-1)}$.

Now, we focus on the positive fixed point (x^*, y^*). The Jacobian matrix at E_2 is

$$J(E_2)=\begin{bmatrix} \delta & \beta-1 \\ \gamma & 1-2\beta \end{bmatrix}.$$

where $\delta=\left(r\left[1+\dfrac{\dfrac{2(\beta-1)}{a(\beta m-m+1)}}{k}\right]-\dfrac{ay^*}{\dfrac{1}{(\beta m-m+1)^2}}\right), \gamma=\dfrac{ak(\beta r-\beta-r+1)+ak(r-1)+r(\beta-1)}{ak}$

Thus the matrix $J(E_2)$ yields the characteristic equation

$$P(\lambda)=\lambda^2-(trE_2)\lambda+detE_2=\lambda^2-(1-2\beta+\delta)\lambda+\left[(\delta)(1-2\beta)+(\beta-1)(\gamma)\right]=0 \quad (2)$$

It follows from the well-known *Jury conditions*, that the modulus of all roots of equation (2) is less than one if and only if the conditions

$$P(1)>0 \;\; P(-1)>0 \;\; and \;\; \left|detJ_{E_2}\right|<1$$

hold.

First, we obtain that $P(1)>0$ holds if and only if $\gamma>\dfrac{\delta-2\beta-\left[(\delta)(1-2\beta)\right]}{\beta-1}$. On the other hand $P(-1)>0$ holds if and only if $\gamma>\dfrac{2\beta-\delta-2-\left[(\delta)(1-2\beta)\right]}{\beta-1}$. Finally, $detE_2<1$ holds if and only if $\gamma<\dfrac{1-\left[(\delta)(1-2\beta)\right]}{\beta-1}$.

Theorem 1. *The positive fixed point (x^*, y^*) of the prey - predator system (1) is locally asymptotically stable if $\gamma>\dfrac{\delta-2\beta-\left[(\delta)(1-2\beta)\right]}{\beta-1}$ and $\gamma<\dfrac{1-\left[(\delta)(1-2\beta)\right]}{\beta-1}$ holds.*

The next result follows from Theorem (1) immediately.

Corollary 2. *The positive fixed point (x^*, y^*) of model (1) is unstable if $\gamma<\dfrac{\delta-2\beta-\left[(\delta)(1-2\beta)\right]}{\beta-1}$* ***(or)*** *$\gamma>\dfrac{1-\left[(\delta)(1-2\beta)\right]}{\beta-1}$ holds.*

IV. Numerical Simulations

In this section, some dynamic aspect of the system is examined numerically. We plot the bifurcation diagram for the system (1) in particular ranges. The parameters are assigned for the different values $r\in(2.0,4.0)$,

$a = 0.6, k = 0.9,\ d = 0.5$ and $\beta = 0.7$. The bifurcation diagram together with time plots of the prey population for $r = 3.33, 3.51, 3.58, 3.78$ and $r = 3.91$ respectively is shown below. In time plots we see that 2–period orbit, 4-period orbits and 8-period orbits occur for different values of r. At last, the 2^n period orbits disappear and the dynamical behaviors are non-periodic and chaotic with increasing of r.

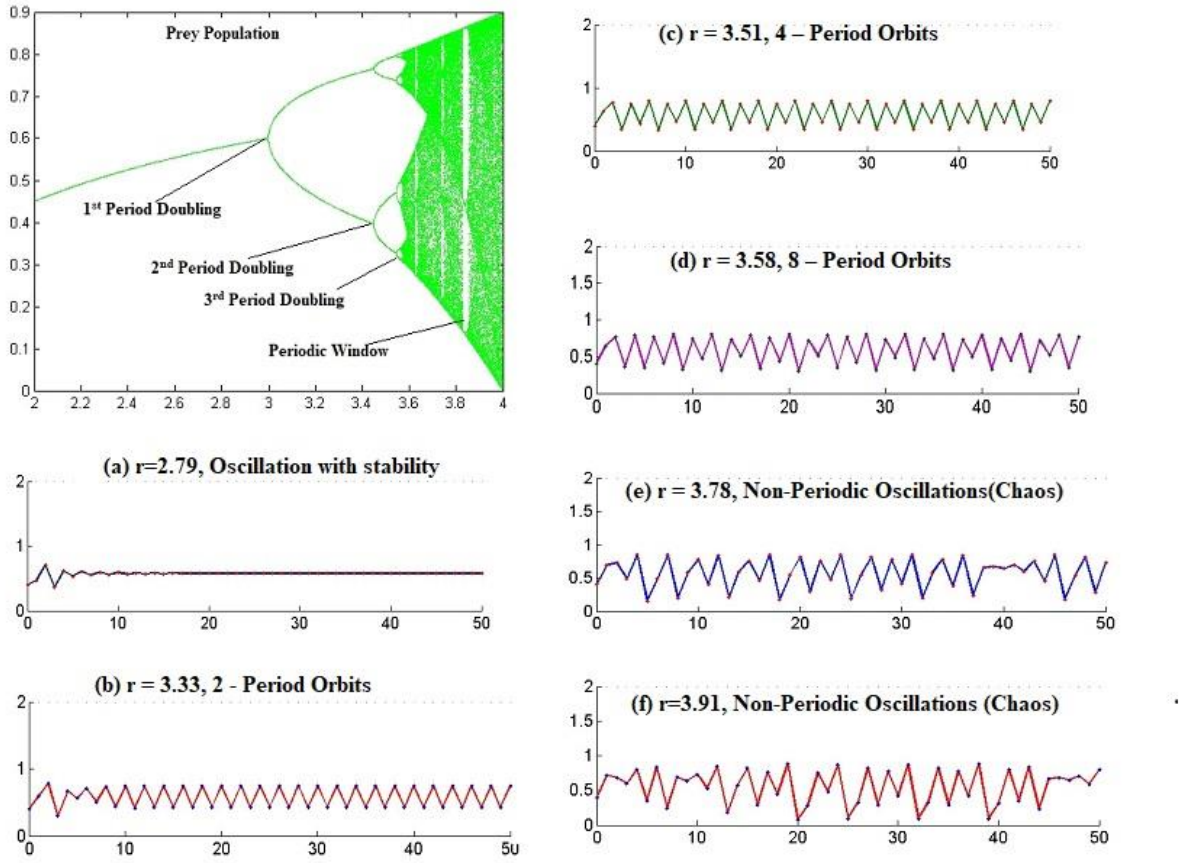


Fig 1. Bifurcation diagram and time plots for the Prey population

Now, we see the changes occuring in the dynamical behavior of prey population as there is sensitive dependence to initial conditions.

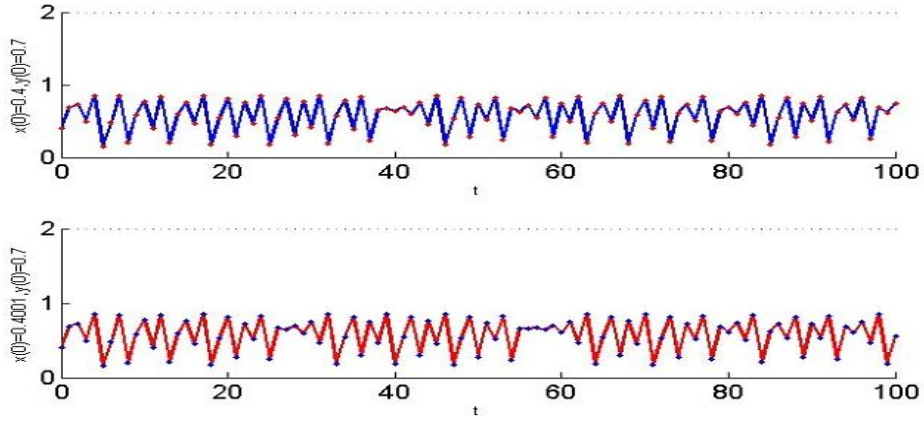


Fig 2: Sensitive dependence on initial conditions.

The sensitivity to initial conditions is a characteristic of chaos. We take two sets of parameter values $r = 3.78, a = 0.6, m = 0.5,\ k = 0.9$ and $\beta = 0.3$ in the axial fixed point F_1 with the sensitive dependence on initial conditions $(x_0 = 0.4, y_0 = 0.7)$ and $(x_0 = 0.4001, y_0 = 0.7)$.

Figure 3 (a). is the bifurcation of the system (1). Local magnifications of periodic windows of Fig 3(a) are shown in 3(b) to 3(d). Hence from the period doubling bifurcation we see that the dynamic behavior of the system changes from stability to chaos then after certain period again it undergoes the process of stability to chaos.

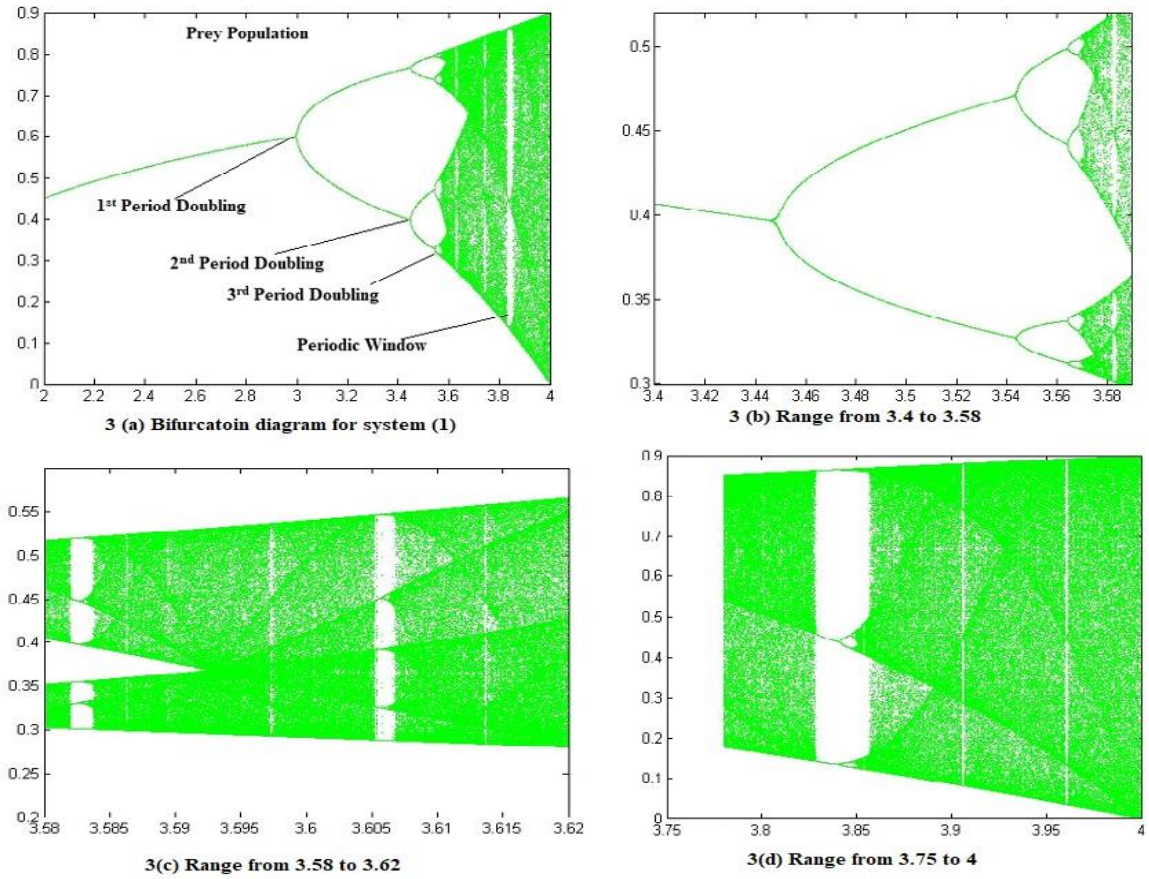


Fig 3. Bifurcation diagram and local magnification of Prey Population

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Fuzzy derivations KU-Bi-ideals on KU algebras

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***Abstract**: In this paper, we introduce a concept of bi-ideals, KU-bi-ideals,fuzzy KU-bi-ideals,left (right) derivations in KU-bi-ideals, fuzzy left(right) derivations in KU-bi-ideals and fuzzy derivations in KU-bi-ideals on KU algebras. We give the concepts of the image and pre-image of fuzzy left(right) derivations in KU-bi-ideals under homomorphism of KU algebras and investigated some of its properties. Further,we have proved that every image and pre-image of fuzzy left(right) derivations in KU-bi-ideals under homomorphism of KU algebras are fuzzy left(right) derivations KU-bi-ideals.*

***Keywords**: Bi-ideal, KU-bi-ideal, fuzzy KU-bi-ideal,left derivations of KU-bi-ideal,right derivations of KU -bi-ideal, derivations of KU-bi-ideal,fuzzy left derivations of KU-bi-ideal,fuzzy right derivations of KU-bi-ideal,fuzzy derivations of KU-bi-ideal.*

I. Introduction

The concept of a fuzzy derivative was first introduced by S.S.L.Chang and Zadeh.In 1965,Zadeh defined the fuzzy set and then many authors investigated this concept and applied it on different branches of mathematics.After that 1966,a new structure in mathematics was obtained by Imai and Iseki called BCK-algebra.Then many studies appeared on BCK\BCI-algebra and its properties.The concept of derivation which applied firstly on rings and near rings presented by so many authors. Subsequently,Jun and Xin proposed derivation on BCI-algebra and many results were obtained.Recently,Aboujabal and Alshehri discussed the derivation on BCK-algebra.Moreover,in 2007 they defined a left derivation on BCK-algebras and investigated a regular left derivation. In this chapter,we introduce the notion of fuzzy left(right) derivations of KU-bi-ideals in KU algebras.The homomorphic image(pre-image) of fuzzy left(right) derivations of KU-bi-ideals in KU algebras under homomorphism of a KU-algebras are discussed.Many related results and theorems have been derived.

II.Preliminaries

In this section,we recall some basic definitions and results that needed for my work.

Definition 2.1: Let X be a set with a binary operation * and a constant 0.(X,*,0) is called KU-algebra if the following axioms hold:$\forall$ x,y,z$\in$X:

(KU_1) (x*y)*[(y*z)*(x*z)]=0

(KU_2) (x*0)=0

(KU_3) 0*x=x

(KU_4) if x*y=0=y*x implies x=y

Define a binary relation $\leq$ by :$x \leq y \leftrightarrow y*x=0$,we can prove that (X,$\leq$) is a partially ordered set.

In this chapter,X will denote a KU-algebra.

Corollary 2.2: In KU-algebra the following identities are true for all x,y,z$\in$X:

(i) z*z=0

(ii) z*(x*z)=0

(iii) If $x \leq y$ implies $y*z \leq x*z$

(iv) z*(y*x)=y*(z*x)

(v) y*[(y*x)*x]=0

Definition 2.3: A subset S of KU-algebra X is called sub algebra of X if x*y$\in$S,whenever x,y$\in$S

Definition 2.4: A non empty subset A of KU-algebra X is called ideal of X if it is satisfied the following conditions:

(i) 0$\in$A

(ii) y*z$\in$A,y$\in$A implies z$\in$A $\forall$y,z$\in$X.

Definition 2.5: A non-empty subset A of a KU-algebra X is called an KU ideal of X if it satisfies the following conditions:

(i) $0 \in A$

(ii) $x*(y*z) \in A, y \in A$ implies $x*z \in A, \forall x,y,z \in X$.

Definition 2.6: Let X be a KU-algebra, a fuzzy set μ in X is called a fuzzy sub-algebra if it satisfies:

(i) $\mu(0) \geq \mu(x)$

(ii) $\mu(x) \geq \min\{\mu(x*y), \mu(y)\}$ $\forall x,y \in X$.

Definition 2.7: Let X be a KU-algebra, a fuzzy set μ in X is called a fuzzy KU-ideal if it satisfies:

(i) $\mu(0) \geq \mu(x)$

(ii) $\mu(x*z) \geq \min\{(x*(y*z)), \mu(y)\}$.

Definition 2.8: For elements x and y of KU-algebra (X,*,0), we denote $x \wedge y = (x*y)*y$.

Definition 2.9: Let X be a KU-algebra. A self map $d:X \rightarrow X$ is a left-right derivation of X if it satisfies the identity $d(x*y)=(d(x)*y) \wedge (x*d(y))$ $\forall x,y \in X$. If d satisfies the identity $d(x*y)= (x*d(y)) \wedge (d(x)*y)$ $\forall x,y \in X$. d is called right-left derivation of X. Moreover, if d is both (l,r) and (r,l) – derivation then d is called a derivation of X.

Definition 2.10: A derivation of KU-algebra is said to be regular if d(0)=0.

Lemma 2.11: A derivation d of KU-algebra X is regular.

Example 2.12: Let X={0,1,2,3,4} be a set in which the operation * is defined as follows:

*	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	2	4
2	0	0	0	1	4
3	0	0	0	0	4
4	0	1	1	1	0

we can prove that (X,*,0) is a KU-algebra. Define a map $d:X \rightarrow X$ by

$$d(x)=\begin{cases}0 & if\ x = 0,1,2,3\\ 4 & if\ x = 4\end{cases}$$

Then it is easy to show that d is both a (l,r) and (r,l)-derivation of X.

Proposition 2.13: Let X be a KU-algebra with partial order ≤, and let d be a derivation of X.Then the following hold $\forall x,y \in X$:

(i) $d(x) \leq x$

(ii) $d(x*y) \leq d(x)*y$

(iii) $d(x*y) \leq x*d(y)$

(iv) $d(x*d(x)) = 0$

(v) $d^{-1}(0) = \{x \in X \backslash d(x)=0\}$ is a sub algebra of X.

Definition 2.14: Let X be a KU-algebra and d be a derivation of X. Denote $Fix_d(x)=\{x \in X/d(x)=x\}$.

Proposition 2.15: Let X be a KU-algebra and d be a derivation of X. Then $Fix_d(x)$ is a sub algebra of X.

III. Fuzzy KU-Bi-ideals of KU-algebras

In this section, we introduce the new concepts of bi-ideal,KU-bi-ideal, fuzzy KU-bi-ideals of KU algebras**.**

Definition 3.1: A non-empty subset A of a KU-algebra X is called bi-ideal of X if it satisfies the following conditions:

(i) $0 \in A$

(ii) $(y*u)*(z*v) \in A, (y*u) \in A$ implies $(z*v) \in A$ $\forall y,z,u,v \in X$.

Definition 3.2: A non-empty subset A of a KU-algebra X is called KU-Bi-ideal of X if it satisfies the following conditions:

(i) $0 \in A$

(ii) $x*((y*u)*(z*v)) \in A, (y*u) \in A$ implies $x*(z*v) \in A$ $\forall x,y,z,u,v \in X$.

Definition 3.3: Let X be a KU- algebra, a fuzzy set μ in X is called fuzzy sub algebra if it satisfies the following conditions:

(i) $\mu(0) \geq \mu(x)$

(ii) $\mu(x) \geq \{\mu(x*(y*u)), \mu(y*u)\}$ $\forall x,y,u \in X$.

Deifnition 3.4: Let X be a KU-algebra, a fuzzy set μ in X is called a fuzzy KU-Bi-ideal of X if it satisfies the following conditions:

(i) $\mu(0) \geq \mu(x)$

(ii) $\mu(x*(z*v)) \geq \min\{\mu(x*((y*u)*(z*v)) , \mu(z*v)\} \quad \forall x,y,z,u,v \in X.$

IV. Fuzzy derivations KU-Bi-ideals of KU-algebras

In this section ,we will discuss and investigate a new notion called fuzzy-left derivations KU-bi-ideals of KU-algebras, fuzzy-right derivations KU-bi-ideals of KU-algebras and study several basic properties which are related to fuzzy left derivations KU-bi-ideals, fuzzy right derivations KU-bi ideals of KU algebras.

Definition 4.1: Let X be a KU-algebra and d:X→X be self map.A non empty subset A of a KU-algebra X is called the left derivations KU-Bi-ideal of X if it satisfies the following conditions:

(i) $0 \in A$

(ii) $d(x)*((y*u)*(z*v)) \in A$, $d(y*u) \in A$, implies $d(x*(z*v)) \in A \quad \forall x,y,z,u,v \in X.$

Definition 4.2: Let X be a KU-algebra and d:X→X be self map.A non empty subset A of a KU-algebra X is called the right derivations KU-Bi-ideal of X if it satisfies the following conditions:

(i) $0 \in A$

(ii) $(x)*d((y*u)*(z*v)) \in A$, $d(y*u) \in A$, implies $d(x*(z*v)) \in A \quad \forall\ x,y,z,u,v \in X.$

Definition 4.3: Let X be a KU-algebra and d:X→X be self map.A non empty subset A of a KU-algebra X is called the derivations KU-Bi-ideal of X if it satisfies the following conditions:

(i) $0 \in A$

(ii) $d(x*((y*u)*(z*v))) \in A$, $d(y*u) \in A$, implies $d(x*(z*v)) \in A \quad \forall x,y,z,u,v \in X.$

Definition 4.4: Let X be a KU-algebra and d:X→X be self map.A fuzzy set μ:X→[0,1] in X is called a fuzzy left derivations KU-Bi-ideal of X if it satisfies the following conditions:

(i) $\mu(d(0)) \geq \mu(d(x))$

(ii) $\mu(d(x*(z*v))) \geq \min\{\mu(d(x)*((y*u)*(z*v))), \mu(d(z*v))\} \quad \forall x,y,z,u,v \in X.$

Definition 4.5: Let X be a KU-algebra and d:X→X be self map.A fuzzy set μ:X→[0,1] in X is called a fuzzy right derivations KU-Bi-ideal of X if it satisfies the following conditions:

(i) $\mu(d(0)) \geq \mu(d(x))$

(ii) $\mu(d(x*(z*v))) \geq \min\{\mu((x)*d((y*u)*(z*v))), \mu(d(z*v))\} \quad \forall x,y,z,u,v \in X.$

Definition 4.6: Let X be a KU-algebra and d:X→X be self map.A fuzzy set μ:X→[0,1] in X is called a fuzzy derivations KU-Bi-ideal of X if it satisfies the following conditions:

(i) $\mu(d(0)) \geq \mu(d(x))$

(ii) $\mu(d(x*(z*v))) \geq \min\{\mu(d((x)*((y*u)*(z*v)))), \mu(d(z*v))\} \quad \forall x,y,z,u,v \in X.$

Example 4.7: Let X={0,a,b,c,d} be a KU-algebra with previous cayley table.(Example 2.12). Then (X,*,0) is a KU-algebra. Define a fuzzy set μ:X→[0,1] by μ(0)=0.2, μ(a)=0.3, μ(b)=0.4, μ(c)=0.4, μ(d)=0.5.Then it is easy to verify that μ is a fuzzy KU-Bi-ideal of X.

Example 4.8: Define a self map d:X→X by

$$d(x)=\begin{cases}0 & if\ x = 0,1,2,3\\ 4 & if\ x = 4\end{cases}$$

Define a fuzzy set μ:d(X)→[0,1] by μ(0)=0.1, μ(1)=0.2, μ(2)=0.3, μ(3)=0.4, μ(4)=0.5.Then it is easily checked that μ is a fuzzy left(right)-derivations KU-Bi-ideal of KU-algebra X.

Proposition 4.9: Let μ be a fuzzy left derivations KU-Bi-ideal of KU-algebra X, if the inequality, $x*(y*u) \leq d(z*v)$ holds in X, then $\mu(d(y*u)) \geq \min\{\mu(d(x)), \mu(z*v)\}$.

Proof. Let X be a KU-algebra and the inequality $x*(y*u) \leq d(z*v)$ holds in X,then $d(z*v)*(x*(y*u))=0$,$(z*v)*(x*(y*u))=0$ from(Prop:2,13 and defn:4.4) …… (A)

We have $\mu(d(z*v)*(y*u)) \geq \min\{\mu(d(z*v)*(x*(y*u))), \mu(d(x))\}=\min\{\mu(0), \mu(d(x))\}=\mu(d(x))$ ……(i)

Put z=0 in (i), we have $\mu(d(0*(y*u))=\mu(d(y*u)) \geq \min\{\mu(x*(y*u)), \mu(d(x))\}$ ……(ii)

But $\mu(x*(y*u)) \geq \min\{\mu(x*((z*v)*(y*u)),\mu(z*v))\}=\min\{\mu((z*v)*(x*(y*u)),\mu(z*v))\}$ (by coro:2.2(v))

$=\min\{\mu(0), \mu(z*v)\}$(by (A))

$=\mu(z*v)$……...(iii)

From (ii),(iii) we get $\mu(d(y*u)) \geq \min\{\mu(z*v),\mu(d(x))\}$.

Proposition 4.10: If μ is a fuzzy left derivations KU-Bi-ideal of KU-algebra X and if $x \leq d(z*v)$ then $\mu(d(x)) \geq \mu(d(z*v))$

Proof. Assume that μ is a fuzzy left derivations KU-Bi-ideal of KU-algebra X.

Let $x \le d(z*v)$,then $d(z*v)*x=0$, $(z*v)*x=0$ since $d(z*v) \le z*v$(from prop:2.13 (i)) and this together with $0*x=x$ and $\mu(0) \ge \mu(z*v)$, we get $\mu(d(0*x))=\mu(d(x)) \ge \min\{\mu(d(0)*((z*v)*x)), \mu(d(z*v))\}$

$= \min\{d(0)*0, \mu(d(z*v)\}$ (since $(z*v)*x=0$)

$= \min\{\mu(0), \mu(d(z*v))\}=\mu(d(z*v))$

Hence, $\mu(d(x)) \ge \mu(d(z*v))$.

Proposition 4.11: The intersection of any set of fuzzy left derivations KU-Bi-ideal of KU-algebra X is also fuzzy left derivations KU-Bi ideal.

Proof. Let $\{\mu_i\}$ be a family of fuzzy left derivations KU-Bi ideal of KU-algebra X, then for any $x,y,z,u,v \in X$, $(\cap\mu_i)(0) = \inf(\mu_i(0)) \ge \inf(\mu_i(d(x)))=(\cap\mu_i)(d(x)$ and

Now, $(\cap\mu_i)(d(x*(z*v)))=\inf(\mu_i(d(x*(z*v))) \ge \inf(\min\{ \mu_i(d(x)*((y*u)*(z*v)), \mu_i(d(z*v))\}$

$= \min\{\inf(\mu_i(d(x)*((y*u)*(z*v)), \mu_i(d(z*v)))\}$

$= \min\{(\cap\mu_i)(d(x)*((y*u)*(z*v))), (\cap\mu_i)(d(z*v))\}$.

Proposition 4.12: The intersection of any set of fuzzy right derivations KU-Bi-ideal of KU-algebra X is also fuzzy left derivations KU-Bi-ideal.

Proof. Let $\{\mu_i\}$ be a family of fuzzy right derivations KU-Bi-ideal of KU-algebra X, then for any $x,y,z,u,v \in X$, $(\cap\mu_i)(0) = \inf(\mu_i(0)) \ge \inf(\mu_i(d(x)))=(\cap\mu_i)(d(x)$ and

Now, $(\cap\mu_i)(d(x*(z*v)))=\inf(\mu_i(d(x*(z*v))) \ge \inf(\min\{ \mu_i((x)*d((y*u)*(z*v)), \mu_i(d(z*v))\}$

$= \min\{\inf(\mu_i((x)*d((y*u)*(z*v)), \mu_i(d(z*v)))\}$

$= \min\{(\cap\mu_i)((x)*d((y*u)*(z*v))), (\cap\mu_i)(d(z*v))\}$.

Theorem 4.13: Let μ be a fuzzy set in X then μ is a fuzzy left derivations KU-Bi-ideal of X if and only if it satisfies: For all $\alpha \in [0,1]$, $U(\mu,\alpha) \ne \varphi$ implies $U(\mu,\alpha)$ is a fuzzy left derivations KU-Bi-ideal of X……...(A), where $U(\mu,\alpha) = \{x \in X/\mu(d(x)) \ge \alpha\}$.

Proof. Assume that μ is a fuzzy left derivations KU-Bi-ideal of X, let $\alpha \in [0,1]$ be such that $U(\mu,\alpha) \ne \varphi$ and $x,y,z,u,v \in X$ such that $(z*v) \in U(\mu,\alpha)$,then $\mu(d(z*v)) \ge \alpha$…….(1) and so by

Defn.4.4,taking $z*v = 0$,then $\mu(d(0)) = \mu(d(x*0)) \ge \min\{\mu(d(x)*((y*u)*0)), \mu(d(0))\}$

$= \min\{\mu(d(x)*0), \mu(d(0))\}$ (since by KU2)

$= \min\{\mu(0), \mu(d(z*v))\}$

$= \mu(d(z*v)) \ge \alpha$ (since by (1)),

Therefore, $\mu(d(0)) \ge \alpha$ and hence $0 \in U(\mu,\alpha)$

Let $d(x)*((y*u)*(z*v)) \in U(\mu, \alpha)$ and $d(z*v) \in U(\mu, \alpha)$,

Then $\mu(d(d(x)*((y*u)*(z*v)))) \ge \alpha$ and $\mu(d(d(z*v))) \ge \alpha$……(2).

Consider,

$\mu(d(d(x*(z*v)))) \ge \min\{\mu(d(d(x)*((y*u)*(z*v)))), \mu(d(d(z*v)))\}$

$= \min\{\alpha, \alpha\}$(since by (2))

$= \alpha$

Therefore, $\mu(d(d(x*(z*v)))) \ge \alpha$ and hence $d(x*(z*v)) \in U(\mu,\alpha)$

Hence,$U(\mu,\alpha)$ is a fuzzy left derivations KU-Bi-ideal of X.

Conversly, suppose that $U(\mu,\alpha)$ is a fuzzy left derivations KU-Bi-ideal of X.

Let $x,y,z,u,v \in X$ be such that $\mu(d(x*(z*v))) < \min\{\mu(d(x)*((y*u)*(z*v))), \mu(d(z*v)))\}$,

Taking, $\beta_0 = 1/2\ \{\mu(d(x*(z*v))) + \min\{\mu(d(x)*((y*u)*(z*v))), \mu(d(z*v))\}$,

we have $\beta_0 \in [0,1]$ and $\mu(d(x*(z*v)))< \beta_0 < \min\{\mu(d(x)*((y*u)*(z*v))), \mu(d(z*v)))\}$,

$\mu(d(x*(z*v)))< \beta_0$ and $\min\{\mu(d(x)*((y*u)*(z*v))), \mu(d(z*v))))\} > \beta_0$, Which is a contradiction.

Therefore, μ is a fuzzy left derivations KU-Bi-ideal of X.

Theorem 4.15: Let μ be a fuzzy set in X then μ is a fuzzy right derivations KU-Bi-ideal of X if and only if it satisfies: For all $\alpha \in [0,1]$, $U(\mu,\alpha) \ne \varphi$ implies $U(\mu,\alpha)$ is a fuzzy right derivations KU-Bi ideal of X……...(A), where $U(\mu,\alpha) = \{x \in X/\mu(d(x)) \ge \alpha\}$

Proof. Assume that μ is a fuzzy right derivations KU-Bi ideal of X, let $\alpha \in [0,1]$ be such that $U(\mu,\alpha) \ne \varphi$ and $x,y,z,u,v \in X$ such that $(z*v) \in U(\mu,\alpha)$,then $\mu(d(z*v)) \ge \alpha$…….(1) and so by

Defn.4.4,taking $z*v = 0$,then $\mu(d(0)) = \mu(d(x*0)) \ge \min\{\mu(x*d((y*u)*0)), \mu(d(0))\}$

$= \min\{\mu(x*d(0)), \mu(d(0))\}$ (since by KU2)

$= \min\{\mu(0), \mu(d(z*v))\}$

$= \mu(d(z*v)) \ge \alpha$ (since by (1)),

Therefore, $\mu(d(0)) \ge \alpha$ and hence $0 \in U(\mu,\alpha)$

Let $x*d((y*u)*(z*v)) \in U(\mu, \alpha)$ and $d(z*v) \in U(\mu, \alpha)$,

Then $\mu(d(x*d((y*u)*(z*v)))) \ge \alpha$ and $\mu(d(d(z*v))) \ge \alpha$……(2).

Consider,

$\mu(d(d(x*(z*v)))) \geq \min\{\mu(d(x*d((y*u)*(z*v)))), \mu(d(d(z*v)))\}$
$= \min\{\alpha, \alpha\}$(since by (2))
$= \alpha$
Therefore, $\mu(d(d(x*(z*v)))) \geq \alpha$ and hence $d(x*(z*v)) \in U(\mu,\alpha)$
Hence,$U(\mu,\alpha)$ is a fuzzy left derivations KU-Bi-ideal of X.
Conversly, suppose that $U(\mu,\alpha)$ is a fuzzy left derivations KU-Bi-ideal of X.
Let $x,y,z,u,v \in X$ be such that $\mu(d(x*(z*v))) < \min\{\mu(x*d((y*u)*(z*v))), \mu(d(z*v)))\}$,
Taking, $\beta_0 = 1/2\ \{\mu(d(x*(z*v))) + \min\{\mu(x*d((y*u)*(z*v))), \mu(d(z*v))\}$,
we have $\beta_0 \in [0,1]$ and $\mu(d(x*(z*v))) < \beta_0 < \min\{\mu(x*d((y*u)*(z*v))), \mu(d(z*v)))\}$,
$\mu(d(x*(z*v))) < \beta_0$ and $\min\{\mu(x*d((y*u)*(z*v))), \mu(d(z*v)))\} > \beta_0$,
Which is a contradiction.
Therefore, μ is a fuzzy right derivations KU-Bi-ideal of X.

Proposition 4.16: If μ is a fuzzy left derivations KU-Bi-ideal of X, then $\mu(d(x)*(x*(y*u)) \geq \mu(d(z*v))$
Proof. Taking $z*v = x*(y*u)$ in the defn(4.4), we have
$\mu(d(x*(x*(y*u))) \geq \min\{\mu(d(x)*((y*u)*(x*(y*u)))), \mu(d(x*(y*u)))\}$
$= \min\{\mu(d(x)*(x*((y*u)*(y*u)))), \mu(d(x*(y*u)))\}$ (since by coro.2.2(v))
$= \min\{\mu(d(x)*(x*0)), \mu(d(x*(y*u)))\}$(since by coro 2.2(i))
$= \min\{\mu(d(x)*0), \mu(d(x*(y*u)))\}$
$= \min\{\mu(0), \mu(d(z*v))\}$
$= \mu(d(z*v)$
Hence, $\mu(d(x)*(x*(y*u)) \geq \mu(d(z*v))$
Proposition 4.17: If μ is a fuzzy right derivations KU-Bi-ideal of X, then $\mu((x)*d(x*(y*u)) \geq \mu(d(z*v))$
Proof. Taking $z*v = x*(y*u)$ in the defn(4.4), we have
$\mu(d(x*(x*(y*u))) \geq \min\{\mu((x)*((y*u)*d(x*(y*u)))), \mu(d(x*(y*u)))\}$
$= \min\{\mu((x)*d(x*((y*u)*(y*u)))), \mu(d(x*(y*u)))\}$ (since by coro.2.2(v))
$= \min\{\mu((x)*d(x*0)), \mu(d(x*(y*u)))\}$(since by coro 2.2(i))
$= \min\{\mu((x)*d(0)), \mu(d(x*(y*u)))\}$
$= \min\{\mu(0), \mu(d(z*v))\}$
$= \mu(d(z*v)$
Hence, $\mu((x)*d(x*(y*u)) \geq \mu(d(z*v))$.

V. Image(Pre-image) of fuzzy derivations KU-Bi-ideals under homomorphism

In this section, we introduce the concepts of the image and the pre-image of fuzzy left derivations KU-Bi-ideals and fuzzy right derivations KU-Bi ideals in KU-algebras under homomorphism.
Definition 5.1: Let f be a mapping from the set X to a set Y. If μ is a fuzzy subset of X, then the fuzzy subset β of Y defined by

$$f(\mu)(y)=\beta(y)=\begin{cases} \sup_{x \in f^{-1}(y)} \mu(x), & \text{if } f^{-1}(y) = \{x \in X,\ f(x) = y\} \neq \varphi \\ 0 & \text{otherwise} \end{cases}$$

is said to be the image of μ under f.
Similarly, if β is a fuzzy subset of Y, then the fuzzy subset $\mu=\beta \circ f$ in X(i.e the fuzzy subset defined by $\mu(x)=\beta(f(x))$ for all $x \in X$) is called the pre-image of β under f.
Theorem 5.2: An onto homomorphic pre-image of a fuzzy left derivations KU-Bi-ideal is also a fuzzy left derivations KU-Bi-ideal.
Proof. Let $f:X \to X'$ be an onto homomorphism of KU-algebras, β a fuzzy left derivations KU-Bi-ideal of X' and μ the pre-image of β under f, then $\beta(f(d(x))=\mu(d(x))$, for all $x \in X$. Let $x \in X$, then $\mu(d(0))=\beta(f(d(0)) \geq \beta(f(d(x))=\mu(d(x))$. Let $x,y,z,u,v \in X$, then
$\mu(d(x*(z*v))) = \beta(f\ (d(x*(z*v))))$
$\geq \min\{\beta(f(d(x)*(f(y*u)*f(z*v))), \beta(f(d(z*v)))\}$
$= \min\{\beta(f(d(x)*((y*u)*(z*v)), \beta(f(d(z*v)))\}$
$= \min\{\mu(d(x)*((y*u)*(z*v))), \mu(d(z*v))\}$
Hence, $\mu(d(x*(z*v)) \geq \min\{\ \mu(d(x)*((y*u)*(z*v))), \mu(d(z*v))\}$.
Theorem 5.3: An onto homomorphic pre-image of a fuzzy right derivations KU-Bi-ideal is also a fuzzy right derivations KU-Bi-ideal.
Proof. Let $f:X \to X'$ be an onto homomorphism of KU-algebras, β a fuzzy right derivations KU-Bi-ideal of X' and μ the pre-image of β under f, then $\beta(f(d(x))=\mu(d(x))$, for all $x \in X$. Let $x \in X$, then
$\mu(d(0))=\beta(f(d(0)) \geq \beta(f(d(x))=\mu(d(x))$. Let $x,y,z,u,v \in X$, then

$\mu(d(x*(z*v))) = \beta(f\ (d(x*(z*v))))$

$\geq \min\{\beta(f((x)*(f(d(y*u))*f(d(z*v))), \beta(f(d(z*v)))\}$

$= \min\{\beta(f((x)*d((y*u)*(z*v)), \beta(f(d(z*v)))\}$

$= \min\{\mu((x)*d((y*u)*(z*v))), \mu(d(z*v))\}$

Hence, $\mu(d(x*(z*v)) \geq \min\{\ \mu((x)*d((y*u)*(z*v))), \mu(d(z*v))\}$.

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Q-HOMOMORPHISM AND ANTI Q-HOMOMORPHISM OF LEVEL SUBHEMIRINGS OF Q-FUZZY SOFT SUBHEMIRINGS

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Abstract: *In this paper, we start the investigation of Q-fuzzy soft subhemirings by consolidating Q-homomorphism and anti Q-homomorphism hypothesis and level subhemirings of fuzzy set hypothesis. The thoughts of a Q-homomorphism fuzzy soft subhemirings and an optimistic fuzzy soft subhemiring are characterized, and a few related properties and auxiliary qualities are explored. At that point, the meanings of a level subhemirings of fuzzy soft function and a Q fuzzy homomorphism are exhibited, and the hypotheses on homomorphic image and homomorphic pre-picture are given. At long last, we present the meaning of a Q-homomorphism and anti Q-homomorphism of level subhemirings of Q-fuzzy soft subhemirings and look at some of its essential properties.*

Keywords: *Fuzzy sets, Q-homomorphism and anti-Q-homomorphism, hemiring, level subhemiring, Q-fuzzy soft subhemirings.*

I. Introduction

After the presentation of fuzzy sets by Zadeh.L.A [13], a few researchers investigated on the speculation of the idea of fuzzy set. Kumbhojkar H.V and Bapat M.S [2] characterized not really fuzzy ideals, Palaniappan.N and Arjunan. K [6] characterized the homomorphism, anti-homomorphism of a fuzzy and an anti- fuzzy ideals,Chandrasekhara Rao .K and V.Swaminathan [1] characterized the anti homomorphisms in near rings. We characterize the idea of Q-homomorphism and anti Q-homomorphism of level subhemirings of Q-fuzzy soft subhemirings.

II. Preliminaries

2.1 Definition:

A non-empty set R together with two binary operations denoted by + and. are called addition and multiplication which satisfy the following axioms are called a h**emiring**. (i) (R, +) is a semigroup and commutative with zero, (ii) (R, .) is a semigroup, (iii) (a+b).c = a.c + b.c and a. (b+c) = a.b + a.c, for all a, b, c ∈ R.

2.2 Definition:

Let R be a hemiring. A Q- fuzzy soft subset A of R is said to be a Q- fuzzy soft subhemiring(QFSSHR) of R if it satisfies the following conditions: (i) $\mu_{(F,A)}(x_{(F,A)} + y_{(F,A)},q) \geq \min\{ \mu_{(F,A)}(x_{(F,A)},q), \mu_{(F,A)}(y_{(F,A)},q) \}$, (ii) $\mu_{(F,A)}(x_{(F,A)}y_{(F,A)},q) \geq \min\{ \mu_{(F,A)}(x_{(F,A)},q), \mu_{(F,A)}(y_{(F,A)},q) \}$,for all $x_{(F,A)}$ and $y_{(F,A)}$ in R and q in Q.

2.3 Definition:

If (R, +, **.**) and (R^{l}, +, **.**) are any two hemirings and Q be a nonempty set. The function f : R x Q→ R^{l} x Q is called a **Q-homomorphism** if f(x+y,q) = f(x,q)+f(y,q) and f(xy,q)=f(x,q)f(y,q) , for all x and y in R and q in Q.

2.4 Definition:

If (R, +, **.**) and (R^{l}, +, **.**) are any two hemirings and Q be a nonempty set. Then the function f : R x Q → R^{l} x Q is called an **anti Q-homomorphism** if f(x+y,q) = f(y,q)+f(x,q) and f(xy,q)=f(y,q)f(x,q), for all x and y in R and q in Q.

2.5 Definition:

Let R and R^{l} be any two hemirings and Q be a nonempty set. Let f : R x Q → R^{l} x Q be any function and let (F,A) be a Q- fuzzy soft subhemiring in R, V be a Q- fuzzy soft subhemiring in f(R) = R^{l}, defined by $\mu_{(G,V)}(y_{(G,V)},q) = \mu_{(F,A)}(x_{(F,A)},q)$ for all $x_{(F,A)}$ in R , $y_{(G,V)}$ in R^{l} and q in Q. Then (F,A) is called a preimage of (G,V) under f and is denoted by $f^{-1}((G,V))$.

2.6 Definition:

Let (F,A) be a Q- fuzzy soft subhemiring of X. For α in [0, 1], the level subset of (F,A) is the set $\mu_{(F,A)\alpha} = \{ x_{(F,A)} \in X : \mu_{(F,A)}(x_{(F,A)},q) \geq \alpha \}$. This is called a **Q- fuzzy soft level subset** of A.

2.7 Definition:

Let (F,A) be a Q-fuzzy soft subhemiring of a hemiring (R, +, .). The level subhemiring $\mu_{(F,A)\alpha}$, for α in [0,1] such that $\alpha \leq \mu_{(F,A)}(0)$, $\alpha \leq \mu_{(F,A)}(1)$ is called **Q- fuzzy soft level subset** of A.

2.8 Definition:

Let (F,A) be a Q-fuzzy soft subhemiring of a hemiring (R, +, .). The level subhemiring $\mu_{(F,A)\alpha}$, for α in [0,1] such that $\alpha \leq \mu_{(F,A)}(0)$, $\alpha \leq \mu_{(F,A)}(1)$ is called **Q- fuzzy soft level soft subhemiring** of A.

III. SOME PROPERIES OF Q-HOMOMORPHISM AND ANTI Q-HOMOMORPHISM OF LEVEL SUBHEMIRINGS OF Q-FUZZY SOFT SUBHEMIRINGS

3.1 Theorem: Let (R, +, •) and (R^{l}, +, •) be any two hemirings. If $f : R \times Q \rightarrow R^{l} \times Q$ is a Q-homomorphism, then the Q-homomorphic image of a level subhemiring of a Q- fuzzy soft subhemiring of R is a level subhemiring of a Q-fuzzy soft subhemiring of R^{l}.

Proof:

Let (R, +, •) and (R^{l}, +, •) be any two hemirings and $f : R \times Q \rightarrow R^{l} \times Q$ be a Q-homomorphism. That is, $f(x+y ,q) = f(x ,q)+f(y ,q)$, for all x and y in R and $f(xy ,q) = f(x ,q)f(y ,q)$, for all x and y in R. Let $\mu_{(G,V)} = f(\mu_{(F,A)})$, where $\mu_{(F,A)}$is a Q-fuzzy soft subhemiring of R. Clearly $\mu_{(G,V)}$is a Q- fuzzy subhemiring of $R^{l} \times Q$. If $x_{(F,A)}$ and $y_{(F,A)}$ in R, then $f(x_{(G,V)},q)$ and $f(y_{(G,V)} ,q)$ in $R^{l} \times Q$. Let $\mu_{(F,A)\alpha}$ be a level subhemiring of A. Suppose $x_{(F,A)}$ and $y_{(F,A)}$ in $\mu_{(F,A)\alpha}$, then $x_{(F,A)}+y_{(F,A)}$ and $x_{(F,A)}y_{(F,A)}$ in $\mu_{(F,A)\alpha}$. That is, $\mu_{(F,A)} (x_{(F,A)} ,q) \geq \alpha$ and $\mu_{(F,A)} (y_{(F,A)} ,q) \geq \alpha$, $\mu_{(F,A)} (x_{(F,A)}+y_{(F,A)} ,q) \geq \alpha$, $\mu_{(F,A)} (x_{(F,A)}y_{(F,A)} ,q) \geq \alpha$. We have to prove that $f (\mu_{(F,A)\alpha})$ is a level subhemiring of $\mu_{(G,V)}$. Now, $\mu_{(G,V)} (f(x_{(G,V)}q)) \geq \mu_{(F,A)} (x_{(F,A)} ,q) \geq \alpha$, implies that $\mu_{(G,V)}(f(x_{(G,V)} ,q)) \geq \alpha$; $\mu_{(G,V)} (f(y_{(G,V)},q)) \geq \mu_{(F,A)} (y_{(F,A)} ,q) \geq \alpha$, implies that $\mu_{(G,V)} (f(y_{(G,V)},q)) \geq \alpha$, $\mu_{(G,V)} (f(x_{(G,V)},q)+f(y_{(G,V)},q)) = \mu_{(G,V)}(f(x_{(G,V)}+y_{(G,V)}, q))$, as f is a Q-homomorphism $\geq \mu_{(F,A)} (x_{(F,A)}+y_{(F,A)} , q) \geq \alpha$, which implies that $\mu_{(G,V)} (f(x_{(G,V)}+y_{(G,V)} , q)) \geq \alpha$, for all $f(x_{(G,V)}, q)$ and $f(y_{(G,V)} ,q)$ in $R^{l} \times Q$. $\mu_{(G,V)}(f(x_{(G,V)},q)f(y_{(G,V)} ,q)) = \mu_{(G,V)}(f(x_{(G,V)}y_{(G,V)} , q))$, as f is a Q-homomorphism $\geq \mu_{(F,A)} (x_{(G,V)}y_{(G,V)} ,q) \geq \alpha$, which implies that $\mu_{(G,V)} (f(x_{(G,V)},q) f(y_{(G,V)} ,q)) \geq \alpha$, for all $f(x_{(G,V)} ,q)$ and $f(y_{(G,V)} ,q) \neq 0^{l}$ in R^{l}. Therefore, $\mu_{(G,V)}(f(x_{(G,V)} ,q) + f(y_{(G,V)} ,q)) \geq \alpha$, for all $f(x_{(G,V)} ,q)$ and $f(y_{(G,V)} ,q)$ in R^{l} and $\mu_{(G,V)}(f(x_{(G,V)} ,q)(f(y_{(G,V)} ,q))) \geq \alpha$, for all $f(x_{(G,V)} ,q)$ and $f(y_{(G,V)},q) \neq 0^{l}$ in R^{l} and Hence $f ((\mu_{(F,A)})_{\alpha})$ is a level subhemiring of a Q- fuzzy soft subhemiring $\mu_{(G,V)}$ of a hemiring R^{l}.

3.2 Theorem: Let (R, +, •) and (R^{l}, +, •) be any two hemirings. If $f: R \times Q \rightarrow R^{l} \times Q$ is a Q-homomorphism, then the Q-homomorphic pre-image of a level subhemiring of a Q- fuzzy soft subhemiring of $R^{l} \times Q$ is a level subhemiring of a Q- fuzzy soft subhemiring of $R \times Q$.

Proof:

Let (R +, •) and (R^{l}, +, •) be any two hemirings and $f: R \times Q \rightarrow R^{l} \times Q$ be a Q-homomorphism. That is, $f(x+y ,q) = f(x ,q)+f(y ,q)$, for all x and y in R and $f(xy ,q) = f(x ,q)f(y ,q)$, for all x and y in R. Let $\mu_{(G,V)}= f(\mu_{(F,A)})$, where $\mu_{(G,V)}$is a Q- fuzzy soft subhemiring of R^{l} x Clearly $\mu_{(F,A)}$ is a Q- fuzzy soft subhemiring of R. Let $x_{(F,A)}$ and $y_{(F,A)}$ in R. Let $f((\mu_{(F,A)\alpha})$ be a level subhemiring of $\mu_{(G,V)}$. Suppose $f(x_{(F,A)}, q)$ and $f(_{(F,A)}y ,q)$ in $f(\mu_{(F,A)\alpha})$, That is, $\mu_{(G,V)} (f(x_{(G,V)} ,q)) \geq \alpha$; $\mu_{(G,V)} (f(y_{(G,V)} ,q)) \geq \alpha$; $\mu_{(G,V)} (f(x_{(G,V)} ,q)+f(y_{(G,V)} ,q)) \geq \alpha$, $\mu_{(G,V)} (f(x_{(G,V)} ,q)f(y_{(G,V)} ,q)) \geq \alpha$ We have to prove that $\mu_{(F,A)\alpha}$ is a level subhemiring of $\mu_{(F,A)}$. Now, $\mu_{(F,A)} (x_{(F,A)} ,q) = \mu_{(G,V)} (f(x_{(G,V)} ,q)) \geq \alpha$, implies that $\mu_{(F,A)} (x_{(F,A)} ,q) \geq \alpha$; $\mu_{(F,A)} (y_{(F,A)} ,q) = \mu_{(G,V)}(f(y_{(G,V)} ,q)) \geq \alpha$, implies that $\mu_{(F,A)} (y_{(F,A)} ,q) \geq \alpha$, we have $\mu_{(F,A)} (x_{(F,A)}+y_{(F,A)} , q) = \mu_{(G,V)} (f(x_{(G,V)}+y_{(G,V)} , q))$, $= \mu_{(G,V)} (f(x_{(G,V)} ,q)+f(y_{(G,V)} ,q))$, as f is a Q-homomorphism $\geq \alpha$, which implies that $\mu_{(F,A)} (x_{(F,A)}+y_{(F,A)} ,q) \geq \alpha$, for all $x_{(F,A)}$ and $y_{(F,A)}$ in R. And $\mu_{(F,A)} (x_{(F,A)}y_{(F,A)}, q) = \mu_{(G,V)}(f(x_{(G,V)}y_{(G,V)} ,q))$, $=\mu_{(G,V)} (f(x_{(G,V)} ,q)f(y_{(G,V)}q))$, as f is a Q-homomorphism $\geq \alpha$, which implies that $\mu_{(F,A)} (x_{(F,A)}y_{(F,A)} ,q) \geq \alpha$, for all $x_{(F,A)}$ and $y_{(F,A)} \neq 0$ in R. Hence $\mu_{(F,A)\alpha}$ is a level subhemiring of a Q- fuzzy soft subhemiring , $\mu_{(F,A)}$of R x Q.

3.3 Theorem: Any subhemiring H of a hemiring (R, +, .) can be realized as a level subhemiring of some Q-fuzzy soft subhemiring of R .

Proof:

Let (F,A) be the Q- fuzzy soft subset of a hemiring (R, +, .) defined by

$$\mu_{(F,A)} (x,q) = \begin{cases} \alpha & \text{if } x_{(F,A)} \in H,\ 0 < \alpha < 1 \\ 0 & \text{if } x_{(F,A)} \notin H \ , \end{cases}$$

where H is a subhemiring of R We claim that $\mu_{(F,A)}$is a Q- fuzzy soft subhemiring of R. Let $x_{(F,A)}$ and $y_{(F,A)}$ in R. If $x_{(F,A)}$ and $y_{(F,A)}$ in H, then $x_{(F,A)} + y_{(F,A)}$, $x_{(F,A)}\, y_{(F,A)}$ in H, since H is a subhemiring of R, $\mu_{(F,A)}(x_{(F,A)} + y_{(F,A)} ,q) = \alpha$, $\mu_{(F,A)}(x_{(F,A)}\, y_{(F,A)} ,q) = \alpha$, $\mu_{(F,A)}(x_{(F,A)},q) = \alpha$, $\mu_{(F,A)}(y_{(F,A)},q) = \alpha$. So, $\mu_{(F,A)}(x_{(F,A)} + y_{(F,A)}) \geq \min \{ \mu_{(F,A)}(x_{(F,A)},q), \mu_{(F,A)}(y_{(F,A)},q)\}$, for all $x_{(F,A)}$ and $y_{(F,A)}$ in R $\mu_{(F,A)}(x_{(F,A)}\, y_{(F,A)} ,q) \geq \min \{ \mu_{(F,A)}(x_{(F,A)},q), \mu_{(F,A)}(y_{(F,A)},q)\}$, for all $x_{(F,A)}$ and $y_{(F,A)} \neq e$ in R. If $x_{(F,A)}$ in H, $y_{(F,A)}$ not in H, then $x_{(F,A)} + y_{(F,A)}$, $x_{(F,A)}\, y_{(F,A)}$ not in H. Then, $\mu_{(F,A)}(x_{(F,A)} + y_{(F,A)} ,q) = 0$, $\mu_{(F,A)}(x_{(F,A)}\, y_{(F,A)} ,q) = 0$, $\mu_{(F,A)}(x_{(F,A)},q) = \alpha$, $\mu_{(F,A)}(y_{(F,A)},q) = 0$. Therefore, $\mu_{(F,A)}(x_{(F,A)} + y_{(F,A)} ,q) \geq \min\{ \mu_{(F,A)}(x_{(F,A)},q), \mu_{(F,A)}(y_{(F,A)},q) \}$, for all $x_{(F,A)}$ and $y_{(F,A)}$ in R $\mu_{(F,A)}(x_{(F,A)}\, y_{(F,A)} ,q) \geq \min \{ \mu_{(F,A)}(x_{(F,A)},q), \mu_{(F,A)}(y_{(F,A)} ,q) \}$, for all $x_{(F,A)}$ and $y_{(F,A)} \neq e$ in R. If $x_{(F,A)}$ and $y_{(F,A)}$ not in H, then $x_{(F,A)} + y_{(F,A)}$, $x_{(F,A)}\, y_{(F,A)}$ not belong to H. Clearly $\mu_{(F,A)}(x_{(F,A)} + y_{(F,A)} ,q) \geq \min \{ \mu_{(F,A)}(x_{(F,A)} ,q), \mu_{(F,A)}(y_{(F,A)} , q)\}$, for all $x_{(F,A)}$ and $y_{(F,A)}$ in R, $\mu_{(F,A)}(x_{(F,A)}\, y_{(F,A)} ,q) \geq \min \{ \mu_{(F,A)}(x_{(F,A)},q), \mu_{(F,A)}(y_{(F,A)} ,q)\}$, for all $x_{(F,A)}$ and $y_{(F,A)} \neq e$ in R. In any case, $\mu_{(F,A)}(x_{(F,A)} + y_{(F,A)},q) \geq \min \{ \mu_{(F,A)}(x_{(F,A)} ,q), \mu_{(F,A)}(y_{(F,A)} ,q) \}$, for all $x_{(F,A)}$ and $y_{(F,A)}$ in R and $\mu_{(F,A)}(x_{(F,A)}\, y_{(F,A)},q) \geq \min \{\mu_{(F,A)}(x_{(F,A)},q), \mu_{(F,A)}(y_{(F,A)},q) \}$, for all $x_{(F,A)}$ and $y_{(F,A)} \neq e$ in R and Thus in all the cases, $\mu_{(F,A)}$ is a Q- fuzzy soft subhemiring of R.

3.4 Theorem: Let (R, +, •) and (R^{l}, +, •) be any two hemirngs. If f: R x Q→R^{l} x Q is an anti Q-homomorphism, then the anti Q-homomorphic image of a level subhemiring of a Q-fuzzy soft subhemiring of R is a level subhemiring of a Q- fuzzy soft subhemiring of R^{l}.

Proof: It is trivial.

3.5 Theorem: Let (R, +, •) and (R^{l}, +, •) be any two fields. If f: R x Q → R^{l} x Q is an anti Q-homomorphism, then the anti Q-homomorphic pre-image of a level subhemiring of a Q-fuzzy soft subhemiring of R^{l} x Q is a level subfield of a Q- fuzzy soft subhemiring of R x Q.

Proof: It is trivial.

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Reliable passive controller design for a class of fuzzy time-delay systems against nonlinear actuator faults

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***Abstract**: This paper aims to design a passive controller to stabilize nonlinear systems with time-varying delay via Takagi-Sugeno (T-S) fuzzy model. The stabilization conditions are given in the form of linear matrix inequality (LMI) usingLyapunov- Krasovskii Functional (LKF). The necessary stability conditions are deduced and the stability analysisresults are summarized into a theorem. The effectiveness of the proposed controller designmethodology is finally demonstrated through numerical example on the truck trailer model. In addition, it shouldbe noted that the example taken is a realistic nonlinear fuzzy system, which proves that the reliable controller isone of the best in applications in real industrial processes.*

***Keywords**:Fuzzy systems; time-delay systems; reliable control; passive control; nonlinearactuator failures.*

I. Introduction

Today there exists a large number of literature study on stability analysis and design of Takagi-Sugeno (T-S) fuzzy systems. T-S fuzzy systems provides a natural, simple and effective design approach to complement linearand nonlinear control systems. In certain instances complete knowledge about the plant may not be available. Undersuch instances fuzzy logic control was found successful in applications of industrial processes. So over the pastdecades researchers have paid attention to the study of T-S fuzzy systems.T-S fuzzysystems helps to achieve resultsby smoothly blending a family of local linear models through fuzzy membership functions. Many results have beenreported in terms of all kinds of methods [3]. Thus, within the general framework of the (T-S) fuzzy models underlinear and nonlinear control systems, there has been a lot of research analysis based on LMI's [1]. [5] deals with anew synchronization strategy, namely a mixed fuzzy delayed feedback dissipativity for chaotic systems with externaldisturbance. [6] has made an attempt to review the applications of fuzzy logic based models in renewable energysystems namely solar, wind, bio-energy, micro-grid and hybrid applications. More recently S. He has studied in[7] the robust finite-time energy-to-peak filtering problem for a class of uncertain nonlinear markovian jumpingsystems (MJSs) with energy constraint external disturbances. The T-S fuzzy model was employed to represent thenonlinear MJSs. Based on all the above studies, it is found that fuzzy based models are extensively used as theyprovide realistic estimates. Here, in this paper we consider a nonlinear fuzzy system and consider the true signalof the control input to consist of linear and nonlinear faults.

In realistic cases, the output of the actuator/sensor cannot reflect the true information due to the externaldisturbances. Under those scenarios,the actuator fault with certain nonlinearitiesneed to be considered. Researchersmodeled the actuator fault as $u^F(t) = \varphi(t)$, where $u(t)$ is the true signal of the control input to be sent to the system,$u^F(t)$is the real signal that the system receives, $\varphi(t) = diag\{\varphi_1, \varphi_2, \dots, \varphi_m\}$is the fault factor. Different definitionsof the fault factor$u^F(t)$expresses different physical meaning. The researches have defined $\varphi_i, i = 1, 2, \dots, m$tosatisfy$\varphi_i \epsilon [0,1]$, and further considered that if $\varphi_i = 0$ it depicts a complete failure, and if $\varphi_i = 1$ it depictsthat the device works normally. In [2], $\varphi_{ii}$was considered as a Gaussian variable to characterize a randomization,whereas in [9], a discrete-time homogeneous Markov chain was used to represent the stochastic behaviour of actuatorfaults. The problem of finite-time reliable $L_2 - L_\infty / H_\infty$control for non-linear systems with actuator faults throughT-S fuzzy model approach was focussed in [8]. So in this paper we have considered a fuzzy system under mixedactuators cases.

In [11] the problem of admissibility analysis and mixed H_∞ and passive control synthesis for a class of singularsystems with Markovian jump and time delays has been addressed. The passivity and dissipativity problems ofsingular systems with time delays have been studied in [10]. On the other hand, the problem of H_∞filtering hasbeen studied for all sorts of systems with the development of a linear matrix inequality(LMI) approach. The main

objective of H_∞filtering is to design a suitable filter such that the bound of induced L_2norm of the operator to thefiltering error is less than a prescribed level. Passivity is an important issue in systems stability and control [12].Also, many number of practical systems need to be passive in order to reduce external noises and disturbanceseffectively. Kwon et al in [4] has derived results on the passivity problem for a class of uncertain neural networkswith both discrete and distributed time-varying delays. We propose a controller design, in which the disturbance ofthe system are reduced to a prescribed level using the passivity controller.

From all the above researches regarding the actuator failures it should be noted that the actuator fault neednot always be a single multiplicative fault. Therefore, for practical fault cases, we need a further study, whichgives rise to the motivation of the current study. Hence in this paper we have considered an actuator fault whichhas both a linear and a non-linear part. To the best of our knowledge, very little attention has been paid to theproblem of passivity analysis for T S fuzzy systems. Considering all these facts, in this paper we focus our study onanalysing the passive reliable controller for T-S fuzzy systems against mixed actuator cases. Sufficient conditions areinvestigated for the asymptotic stability of the state variables, satisfying a prescribed level of passivity performancemeasure, which is presented in terms of LMIs. These LMIs are further easily calculated using MATLAB toolbox.The stability is derived and further verified using truck-trailer model.

II. Problem Formulation

Consider a T−S fuzzy system with external disturbances and controlled input represented by the following fuzzyrules:

R_i :If$\xi_1(t)$ is about M_{1i}, $\xi_2(t)$ is about M_{2i}, ..., and $\xi_q(t)$ is about M_{qi}then

$$\dot{x}(\mathrm{t}) = A_i\,x(t) + A_{di}\,x\big(t-\tau(\mathrm{t})\big) + B_{1i}\,\omega(\mathrm{t}) + B_{2i}\,u^f(t),$$
$$z(t) = C_i x(t) + D_i u^f(t),$$
$$x(t) = \varphi(t), t \in [-\bar{\tau}, 0],$$

where $i = 1,2,\ldots.r,$ in which $M_{ji}$is the fuzzy set of $\xi_i(t)$, r is the number of If–Then fuzzy rules and $\xi_i(t), \in R^q$isthe decision variable vector, $x(t) \in R^n$is the system state vector, $u^f(t) \in R^m$is the control input with actuatorfault, $z(t) \in R^p$ is the output vector and $\omega(t)$ is the disturbance variable. $A_i, A_{di}, B_{1i}, B_{2i}, C_i, D_i$ are constant realmatrices that describe the nominal system with appropriate dimensions. $\tau(t)$ denotes time varying delay satisfying

$$0 \le \tau(t) \le \tau, \dot{\tau}(t) \le \mu. \qquad (1)$$

The global dynamic system is inferred as

$$\dot{x}(\mathrm{t}) = \textstyle\sum_{i=1}^{r} \mu_i\,(\xi(\mathrm{t}))(A_i\,x(t) + A_{di}\,x\big(t-\tau(\mathrm{t})\big) + B_{1i}\,\omega(\mathrm{t}) + B_{2i}\,u^f(t)),$$
$$z(t) = \textstyle\sum_{i=1}^{r} \mu_i\,(\xi(\mathrm{t}))(C_i\,x(t) + D_i\,u^f(t)),$$
$$x(t) = \varphi(t), t \in [-\bar{\tau}, 0],$$

where$\mu_i(\xi(\mathrm{t})) = \frac{\prod_{j=i}^{q} M_{ji}(\xi_j(\mathrm{t}))}{\sum_{i=1}^{r}\prod_{j=i}^{q} M_{ji}(\xi_j(\mathrm{t}))}$, in which $M_{ji}(\xi_j(\mathrm{t}))$ is the grade of membership of $\xi_j(\mathrm{t})$ in M_{ji}. Thenormalized activation function $M_i(\xi(\mathrm{t}))$ in relation with the i^{th}sub model is such that$\sum_{i=1}^{r} \mu_i\,(\xi(\mathrm{t})) = 1, 0 \le M_i\big(\xi(\mathrm{t})\big) \le 1, i = 1,2,\ldots,r$.By incorporating the fuzzy rule, the state feedback controller can be written as

$$u(t) = \sum_{j=1}^{r} \mu_j\,(\xi(t))\;K_j x(t).$$

Here we consider the actuator fault model as follows:

$$u^f(t) = \Xi_1 u(t) + g\big(u(t)\big), \qquad (2)$$

where$\Xi_1$denotes the control effectiveness factor. The actuator fault matrix satisfies the following condition$\Xi_1 = diag\{g_1, g_2, \ldots, g_n\}, g_l \in [\check{g}_l, \hat{g}_l], l = 1,2,\ldots,n, 0 \le \check{g}_l \le g_l \le \hat{g}_l \le 1$, where g_l is an unknown constant; $\check{g}_l$and $\hat{g}_l$represent the known lower and upper bound of g_l. Also, for simplicity, we define$\widehat{\Xi_1}$= diag{$\widehat{g_1}$, $\widehat{g_2}$,..., $\widehat{g_n}$},$\check{\Xi_1}$=diag{ $\check{g_1}$, $\check{g_2}$,..., $\check{g_n}$},$L = diag\{l_1, l_2 \ldots, ln\}, with\ -1 \le l_i \le 1, i = 1,2,,\ldots,n$. For $\Xi_1' = \frac{\widehat{\Xi_1}+\check{\Xi_1}}{2}$and $\Xi_0 = \frac{\widehat{\Xi_1}-\check{\Xi_1}}{2}$, wedefine

$$\Xi_1 = \Xi_1' + \Xi_0 L \qquad (3)$$

and the vector function $g(u(t)) = [g_1(u(t)), g_2(u(t)), \ldots, g_m(u(t))]^T$ which satisfies $|g_i(u(t))| \le \sqrt{\alpha_i}|u(t)|, i \in \{1,2,\ldots,m\}$. These class of faults do not vary linearly with the control input $u(t)$ due to the dead zoneof the actuator or some other nonlinear mechanical behaviour. It follows $g^T\big(u(t)\big)g\big(u(t)\big) \le u^T(t)\,\Xi_2 u(t)$, where $\Xi_2 = diag\{\alpha_1, \alpha_2, \ldots, \alpha_m\}$.This equation denotes that the actuator failure rate of the model varies withina bounded interval. By incorporating our mixed actuator controller, the T− S fuzzy closed loop system can bewritten as follows:

$$\dot{x}(\mathrm{t}) = A_\mu\,x(t) + A_{d\mu}\,x\big(t-\tau(\mathrm{t})\big) + B_{1\mu}\,\omega(\mathrm{t}) + B_{2\mu}\,\Xi_1 K_\mu\,x(t) + B_{2\mu}\,g(u(t)),$$
$$z(t) = C_\mu x(t) + D_\mu \Xi_1 K_\mu\,x(t) + D_\mu g(u(t)), \qquad (4)$$

where

$$A_\mu = \sum_{i=1}^{r} \mu_i(\xi(t)) \sum_{i=1}^{r} \mu_j(\xi(t)) A_i, A_{d\mu} = \sum_{i=1}^{r} \mu_i(\xi(t)) \sum_{i=1}^{r} \mu_j(\xi(t)) A_{di},$$

$$B_{1\mu} = \sum_{i=1}^{r} \mu_i(\xi(t)) \sum_{i=1}^{r} \mu_j(\xi(t)) B_{1i}, B_{2\mu} = \sum_{i=1}^{r} \mu_i(\xi(t)) \sum_{i=1}^{r} \mu_j(\xi(t)) B_{2i},$$

$$C_\mu = \sum_{i=1}^{r} \mu_i(\xi(t)) \sum_{i=1}^{r} \mu_j(\xi(t)) C_i,\ D_\mu = \sum_{i=1}^{r} \mu_i(\xi(t)) \sum_{i=1}^{r} \mu_j(\xi(t))\, D_i, K_\mu = \sum_{i=1}^{r} \mu_i(\xi(t)) \sum_{i=1}^{r} \mu_j(\xi(t)) K_j.$$

Lemma 2.1: System (4) is said to be very strictly passive (VSP) if there exists a constant $\delta > 0, \varepsilon > 0$ and ρ such that

$$2\int_0^{t*} z^T(t)\,\omega(t)dt \geqslant \rho + \delta \int_0^{t*} \omega^T(t)\,\omega(t)dt + \varepsilon \int_0^{t*} z^T(t) z(t) dt,$$

holds for all $t* \geqslant 0$.

Lemma above describes four classes of passivity notions. Obviously, if system (4) is VSP, then it is passive, input strictly passive (ISP) and output strictly passive (OSP). Thus, the VSP notion is more restrictive than the notionsof passivity, ISP and OSP. It is also obvious that, if we obtain a condition guaranteeing that system (4) is VSP,then, by using the same method, desired results on the passivity, ISP and OSP for the same system can be easilyobtained. Therefore, we only study the VSP problem in this paper.

III. Passivity Controller Design for T-S Fuzzy systems

For given positive scalars $\rho, \delta, \varepsilon, \Xi_2, \mu$ and the known actuator fault matrix Ξ_1, the closed-loop system (4) is VSPfor any time-varying delay satisfying (1), under the control input (2), if there exists symmetric positive definitematrices $Q > 0, L > 0$, and a real matrix $Y_j, j = 1, 2, \ldots, r$ such that the following LMI holds:

$$\Phi_{6\times6} < 0, \tag{5}$$

where $\Phi_{1,1} = 2A_\mu Q + 2B_{2\mu}\Xi_1 Y_j + L - 2Q, \Phi_{1,2} = A_{d\mu}L, \Phi_{1,3} = B_{1\mu} - XC_\mu^T - Y^T\Xi_1^T D_\mu^T,$
$\Phi_{1,4} = B_{2\mu}, \Phi_{1,5} = XC_\mu^T + Y^T\Xi_1^T D_\mu^T, \Phi_{1,6} = Y^T\Xi_2^T, \Phi_{2,2} = -(1-\mu)Q, \Phi_{3,3} = -\delta,$
$\Phi_{3,4} = -D_\mu, \Phi 4{,}4 = -\rho, \Phi_{4,5} = D_\mu^T, \Phi_{5,5} = -\varepsilon, \Phi_{6,6} = -\rho\Xi_2.$

Proof: In order to prove the required result, we define the following Lyapunov-Krasovskii functional (LKF) candidatefor the system

$$V(x(t)) = V_1(x(t)) + V_2(x(t))$$

$$where\ V_1(x(t)) = x^T(t)Px(t)\ and\ V_2(x(t)) = \int_{t-\tau(t)}^{t} x^T(s)Rx(s)ds.$$

Calculating the time derivatives $V(x(t))$along the trajectories of system we have

$$\dot{V}_1(x(t)) = 2x^T(t)P\dot{x}(t)$$

$$\text{and}\dot{V}_2(x(t)) = x^T(t)Rx(t) - (1-\mu)x^T(t-\tau(t))Rx(t-\tau(t)).$$

Let $J(t) = \varepsilon z^T(t)z(t) + \delta\omega^T(t)\omega(t) - 2z^T(t)\omega(t)$. Then under zero initial condition, it is easy to see that

$$J(t) \leq \varepsilon z^T(t)z(t) + \delta\omega^T(t)\omega(t) - 2z^T(t)\omega(t) + \dot{V}(x(t)) = \eta^T(t)\varphi_{5\times5}\,\eta(t),$$

where$\eta(t) = [x^T(t), x^T(t-\tau(t)), \omega^T(t), g^T(u(t)), z^T(t)]^T$and the elements of $\varphi_{5\times5}$are given by

$$\varphi_{1,1} = 2PA_\mu + 2PB_{2\mu}\Xi_1 K_\mu + R, \varphi_{1,2} = PA_{d\mu}, \varphi_{1,3} = PB_{1\mu} - C_\mu^T - K_\mu^T\Xi_1^T D_\mu^T, \varphi_{1,4} = PB_{2\mu}, \varphi_{1,5}$$
$$= C_\mu^T + K_\mu^T\Xi_1^T D_\mu^T, \varphi_{2,2} = -(1-\mu)R, \varphi_{3,3} = -\delta, \varphi_{4,5} = D_\mu^T, \varphi_{5,5} = -\varepsilon.$$

Applyingschur complement lemma and further by pre and post multiplying $by\ \{Q = P^{-1}, L = R^{-1}, I, I, I, I\}$, we obtainthe left hand side of (5).Hence if (5) holds, i.e., if$\varphi_{6\times6} < 0$ holds, then theorem is established. Therefore,byLemma 1, system (4) is VSP for any time-varying delay $\tau(t)$ satisfying (1).

IV. Numerical Example

In this section, we present an example to illustrate the effectiveness of the method proposed. Consider the following truck-trailer nonlinear system with a time−varying delay

$$\dot{x}_1(t) = \frac{-av\bar{t}}{Lt_0}x_1(t) - \frac{a_1 v\bar{t}}{Lt_0}x_1(t-\tau(t)) + \frac{v\bar{t}}{lt_0}u(t) + \omega(t)$$

$$\dot{x}_2(t) = \frac{av\bar{t}}{Lt_0}x_1(t) + \frac{a_1 v\bar{t}}{Lt_0}x_1(t-\tau(t))$$

$$\dot{x}_3(t) = \frac{v\bar{t}}{t_0}\sin[x_2(t) + \frac{av\bar{t}}{2L}x_1(t) + \frac{av\bar{t}}{2L}x_1(t-\tau(t))]$$

where$x_1(t)$ is the angle difference between truck and trailer; $x_2(t)$ is the angle of trailer; $x_3(t)$ represents the

vertical position of rear of trailer; $u(t)$ denotes the steering angle; $\omega(t)$ is the disturbance. In order to verify the obtained results, we borrow the model parameters from [13] as $l = 2.8, L = 5.5, v = -1.0, t = 2.0, t_0 = 0.5, a = 0.7, a_1 = 1 - a$ and $\tau(t)$ is a differentiable function. This system can be represented by the following T-S fuzzy model:

Plant Rule 1:If$\theta(t) = x_2(t) + \frac{av\bar{t}}{2L}x_1(t) + \frac{av\bar{t}}{2L}x_1(t - \tau(t))$is about 0 then

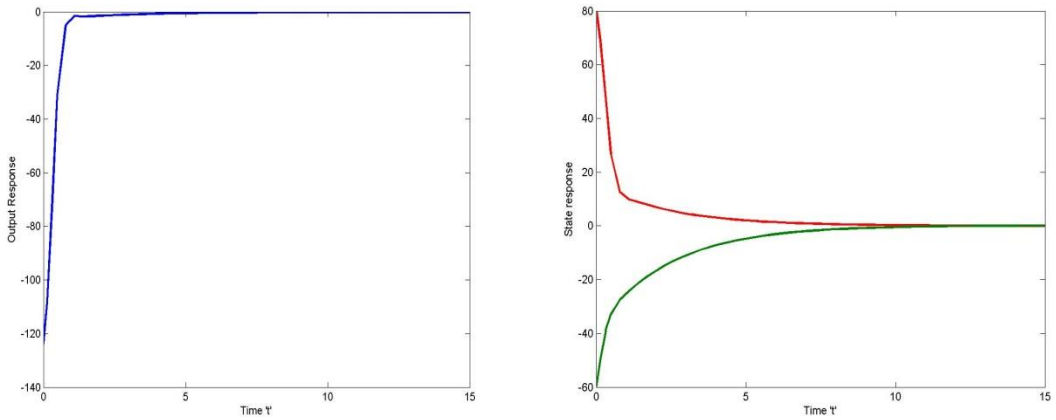
$$\dot{x}(t) = A_1 x(t) + A_{d1} x(t - \tau(t)) + B_{11}\,\omega(t) + B_{21}\,u(t).$$

Plant Rule 1:If$\theta(t) = x_2(t) + \frac{av\bar{t}}{2L}x_1(t) + \frac{av\bar{t}}{2L}x_1(t - \tau(t))$is about π or $-\pi$ then

$$\dot{x}(t) = A_2 x(t) + A_{d2} x(t - \tau(t)) + B_{12}\,\omega(t) + B_{22}\,u(t).$$

Here x(t) =$[x_1^T(t), x_2^T(t), x_3^T(t)]^T$, $A_1 = \begin{bmatrix} \frac{-av\bar{t}}{Lt_0} & 0 & 0 \\ \frac{av\bar{t}}{Lt_0} & 0 & 0 \\ \frac{-av^2\bar{t}^2}{2Lt_0} & \frac{v\bar{t}}{2Lt_0} & 0 \end{bmatrix}, A_2 = \begin{bmatrix} \frac{-av\bar{t}}{Lt_0} & 0 & 0 \\ \frac{av\bar{t}}{Lt_0} & 0 & 0 \\ \frac{-adv^2\bar{t}^2}{2Lt_0} & \frac{v\bar{t}}{2Lt_0} & 0 \end{bmatrix}$,

$A_{d1} = \begin{bmatrix} \frac{-bv\bar{t}}{Lt_0} & 0 & 0 \\ \frac{bv\bar{t}}{Lt_0} & 0 & 0 \\ \frac{-bv^2\bar{t}^2}{2Lt_0} & \frac{v\bar{t}}{2Lt_0} & 0 \end{bmatrix}, A_{d2} = \begin{bmatrix} \frac{-bv\bar{t}}{Lt_0} & 0 & 0 \\ \frac{bv\bar{t}}{Lt_0} & 0 & 0 \\ \frac{-bdv^2\bar{t}^2}{2Lt_0} & \frac{v\bar{t}}{2Lt_0} & 0 \end{bmatrix}$, $B_{21} = B_{22} = [\frac{v\bar{t}}{lt_0}\ 0\ 0]^T$, $B_{11} = B_{12} = [1\ 0\ 0]^T$with $a + b = 1$ and set $d = \frac{10\bar{t_0}}{\pi}$. The membership function is employed as $\mu_1(\xi(t)) = (1 - \frac{1}{1+\exp(\beta)}) \times (\frac{1}{1+\exp(\beta)})$ and$\mu_2(\xi(t)) = 1 - \mu_1(\xi(t))$, where $\beta = -3(\xi(t) - 0.5\pi)$. For the controller output, the matrices are chosenas$C_1 = C_2$=[0.01 0.1 0.01] and $D_1 = D_2 = \frac{v\bar{t}}{l}$. The disturbance input is assumed to be$\omega(t) = sin(0.1t)exp(-0.1t)$. In addition we choose $\mu = 0.1\ and\ \delta = 0.82$. By solving the LMI's in theorem the controller gains of the desired controller are obtained as$K_1 = [14.8093\ \ 12.8889\ \ 4.9171]\ and\ K_2 = [14.9548\ \ 12.9850\ \ 4.9742]$. The simulation results for the state and output responses of the system (4) against mixed actuator failures are presented below. This figure shows that our considered system is stable under the proposed controller.



V. Conclusion

In this paper, the VSP analysis and VSP controller design problem for T-S fuzzy systems with time-varying delays have been investigated. This controller consists of both the linear and the nonlinear parts, the system can be made stable at large. Moreover, the proposed results are validated by means of a practical example which shows the superiority of the designed controller.

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$\delta^* g\alpha$ -Continuous Functions in topological spaces

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Abstract: *In this paper, the authors introduce a new class of functions called $\delta^* g\alpha$ -continuous functions and $\delta^* g\alpha$ -irresolute functions in Topological spaces. Some of their properties and characterizations are investigated. Also, new class of mappings of $\delta^* g\alpha$ -homeomorphism and $\delta^* g\alpha$ c-homeomorphism using $\delta^* g\alpha$ - closed sets were introduced; their basic properties and relationship with other functions are also investigated.*

Keywords: *$\delta^* g\alpha$ -continuous functions; $\delta^* g\alpha$ -irresolute functions; $\delta^* g\alpha$ -homeomorphism; $\delta^* g\alpha$ c-homeomorphism.*

I. Introduction

Norman Levine [19] initiated the idea of continuous function in 1970. T.Noiri [30] introduced δ-continuous in 1980. The generalised continuous (briefly g-continuous) was studied by K.Balachandran [12] in 1991. Further many others contributed their research towards continuity. Julian Dontchev[8], R.Devi,et.al[23], R.sudha,et.al[25], J.Dontchev[9], J.Arokiarani,et.al[6] defined and analysed δg -continuous, αg -continuous, δg^* -continuous, $g\delta$ -continuous and gp -continuous respectively. Recently the authors have introduced $\delta^* g\alpha$ -closed sets and also investigated their basic properties and relationship with other existing sets. In this paper the authors define a new class of mappings called $\delta^* g\alpha$ -continuous functions and $\delta^* g\alpha$ - irresolute functions using $\delta^* g\alpha$ -closed set, and obtain their characterization and properties. Further a new class of generalization of homeomorphism called $\delta^* g\alpha$ -homeomorphism and $\delta^* g\alpha c$ -homeomorphism using $\delta^* g\alpha$ -closed sets were introduced, their basic properties and relationship with other functions are also investigated.

II.Preliminaries

Throughout this paper $(X,\tau),(Y,\sigma)$ and (Z,η) (or simply X, Yand Z) represent topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset A of X, $cl(A), int(A)$ and $X - A$ denote the closure of A, the interior of A and the complement of A respectively. Let us recall the following definitions, which are useful in the sequel.

*Definition*2.1:A subset A of (X,τ) is said to be

(i) Semi-open set [17] if $A \subseteq cl(int(A))$.

(ii) Pre-open set [1] if $A \subseteq int(cl(A))$.

(iii) α -open set [20] if $A \subseteq int(cl(int(A)))$.

(iv) Regular open set [14] if A=int(cl(A)).

(v) b-closed set [2] if $cl(int(A)) \cap int(cl(A)) \subseteq A$.

The complement of a semi-open (resp. pre-open, α -open, regular open, b-open set) set is called semi-closed (resp. semi-closed, α -closed, regular closed, b-closed set).

*Definition:2.2*The δ-interior [18] of a subset A of X is the union of all regular open set of X contained in A and is denoted by $\text{Int}_{\delta}(A)$. The subset A is called δ-open [18] if A = $\text{Int}_{\delta}(A)$, i.e. a set is δ-open if it is the union of regular open sets. The complement of a δ-open is called δ-closed. Alternatively, a set $A \subseteq (X,\tau)$ is called δ-closed [18] if $A = \text{cl}_{\delta}(A)$, where $\text{cl}_{\delta}(A) = \{x \in X : \text{int}(\text{cl}(U)) \neq \phi, U \in \delta \text{ and } x \in U\}$.

Definition 2.3: A subset A of (X,τ) is called

(i) a generalized closed (briefly g-closed) set[16] if $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and U is open set in (X,τ).

(ii) a generalized semi-closed (briefly gs-closed) set [26] if $\text{scl}(A) \subseteq U$ whenever $A \subseteq U$ and U is open set in (X,τ).

(iii) a α-generalized closed (briefly αg-closed) set [4] if $\alpha\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and U is open set in (X,τ).

(iv) a δ-generalized closed (briefly δg-closed) set [8] if $\text{cl}_{\delta}(A) \subseteq U$ whenever $A \subseteq U$ and U is open set in (X,τ).

(v) a generalized preclosed (briefly gp-closed) set [3] if $\text{pcl}(A) \subseteq U$ whenever $A \subseteq U$ and U is open set in (X,τ).

(vi) a generalized semi-preclosed (briefly gsp-closed) set [10] if $\text{spcl}(A) \subseteq U$ when-ever $A \subseteq U$ and U is open set in (X,τ).

(vii) a generalized δ- closed (briefly $g\delta$-closed) set [9] if $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and U is δ open set in (X,τ).

(viii) a δ generalized*- closed (briefly δg^*-closed) set [24] if $\text{cl}_{\delta}(A) \subseteq U$ whenever $A \subseteq U$ and U is δ open set in (X,τ).

(ix) a generalized δ- semi closed (briefly $g\delta s$-closed) set [11] if $\text{scl}(A) \subseteq U$ whenever $A \subseteq U$ and U is δ open set in (X,τ).

(x) a δ-generalized b-closed (briefly δgb-closed) set [28] if $\text{bcl}(A) \subseteq U$ whenever $A \subseteq U$ and U is δ open set in (X,τ).

(xi) a δ^* generalized α –closed(briefly $\delta^* g\alpha$-closed) set [13] if $\text{cl}_{\delta}(A) \subseteq U$ whenever $A \subseteq U$ and U is a $^*g\alpha$-open set in (X,τ).

The complement of a g-closed (resp. gs-closed, αg-closed, δg-closed, gp-closed, gsp-closed, $g\delta$-closed, $g\delta^*$-closed, $g\delta s$-closed and δgb-closed) set is called g-open (resp. gs-open, αg-open, δg-open, gp-open, gsp-open, $g\delta$-open, $g\delta^*$-open, $g\delta s$-open and δgb-open).

Definition: 2.4: A function $f : (X,\tau) \to (Y,\sigma)$ is called

(i) a generalized continuous (briefly g-continuous) [12] if $f^{-1}(V)$ is g-closed in (X,τ) for every closed in (Y,σ)

(ii) a generalized semi-continuous (briefly gs-continuous) [22] if $f^{-1}(V)$ is gs-closed in (X,τ) for every closed in (Y,σ)

(iii) a - generalized continuous (briefly αg-continuous) [23] if $f^{-1}(V)$ is αg-closed in (X,τ) for every closed in (Y,σ)

(iv) a -generalized continuous (briefly δg-continuous) [8] if $f^{-1}(V)$ is δg-closed in (X,τ) for every closed in (Y,σ)

(v) a generalized precontinuous (briefly gp-continuous) [6] if $f^{-1}(V)$ is gp-closed in (X,τ) for every closed in (Y,σ)

(vi) a generalized semi-precontinuous (briefly gsp-continuous) [10] if $f^{-1}(V)$ is gsp-closed in (X,τ) for every closed in (Y,σ)

(vii) a generalized- continuous (briefly $g\delta$ -continuous) [9] if $f^{-1}(V)$ is $g\delta$ -closed in (X,τ) for every closed in (Y,σ)

(viii) a generalized * continuous (briefly $g\delta^*$-continuous) [25] if $f^{-1}(V)$ is $g\delta^*$-closed in (X,τ) for every closed in (Y,σ)

(ix) a generalized- semi continuous (briefly $g\delta s$ -continuous) [27] if $f^{-1}(V)$ is $g\delta s$ -closed in (X,τ) for every closed in (Y,σ)

(x) a generalized precontinuous (briefly δgp -continuous) [7] if $f^{-1}(V)$ is δgp -closed in (X,τ) for every closed in (Y,σ)

(xi) a -generalized b-continuous (briefly δgb -continuous) [29] if $f^{-1}(V)$ is δgb -closed in (X,τ) for every closed in (Y,σ)

Definition 2.5: A space (X,τ) is called a ${}_{\alpha\delta}T^{**}_{\frac{3}{4}}g\alpha$ -space if every $\delta^* g\alpha$ -closed set is an δ -closed.

Definition 2.6 A function $f:(X,\tau)\to(Y,\sigma)$ is called

(i) g -Homeomorphism [5] if f is bijection, g -open and g -continuous.

(ii) gs -Homeomorphism [22] if f is bijection, gs -open and gs -continuous.

III. $\delta^* g\alpha$ -Continuous Function In Topological Spaces

The authors introduce the following definition.

Definition 3.1 A map $f:(X,\tau)\to(Y,\sigma)$ is said to be a delta star generalized alpha-continuous (briefly $\delta^* g\alpha$ -continuous)if $f^{-1}(V)$ is $\delta^* g\alpha$ - closed in (X,τ) for every closed in (Y,σ).

Theorem 3.2 Every $\delta^* g\alpha$ -continuous is gs -continuous. (resp αg -continuous, gsp -continuous, gp -continuous)

*Proof:*Let V be a closed set in (Y,σ). Since f is $\delta^* g\alpha$ -continuous map. $f^{-1}(V)$ is $\delta^* g\alpha$ -closed in (X,τ) . Since every $\delta^* g\alpha$ -closed set is gs -closed (resp αg -closed, gsp -closed, gp -closed), therefore $f^{-1}(V)$ is gs -closed (resp αg -closed, gsp -closed, gp -closed) in (X,τ) . Hence f is gs - continuous. (resp αg - continuous, gsp -continuous, gp -continuous)

Remark 3.3: The converse of the above theorem is not true as shown in the following example.

Example 3.4 Let $X=\{a,b,c\},\ \tau=\{\phi,X,\{a\},\{b\},\{a,b\}\};\sigma=\{\phi,X,\{a\},\{b,c\}\}$

Def $f:(X,\tau)\to(Y,\sigma)$ where f(a) = a; f(b) = b; f(c) = c

$gsC(X;\tau)=\{\phi,X,\{a\},\{b\},\{c\},\{b,c\},\{a,c\}\}$; $gspC(X;\tau)=\{\phi,X,\{a\},\{b\},\{c\}\ \{b,c\},\{a,c\}\}$;

$\delta^* g\alpha C(X;\tau)=\{\phi,X,\{c\},\{a,c\},\{b,c\}\}$

Here $f^{-1}[\{a\}]=\{a\}$ is not $\delta^* g\alpha$ -closed in (X,τ) therefore it is gs -continuous, gsp -continuous but not $\delta^* g\alpha$ -continuous.

*Example 3.5*Let $X=\{a,b,c\},\ \tau=\{\phi,X,\{a,b\}\};\sigma=\{\phi,X,\{a\},\{a,b\}\}$

Def $f:(X,\tau)\to(Y,\sigma)$ where f(a) = a; f(b) = c; f(c) = b

$gpC(X;\tau)=\{\phi,X,\{a\},\{b\},\{c\},\{b,c\},\{a,c\}\}$; $\delta^* g\alpha C(X;\tau)=\{\phi,X,\{c\},\{a,c\},\{b,c\}\}$

Here $f^{-1}[\{c\}]=\{b\}$ is not $\delta^* g\alpha$ -closed in (X,τ) therefore it is gp -continuousbut not $\delta^* g\alpha$ -continuous.

*Example 3.6*Let $X=\{a,b,c\},\ \tau=\{\phi,X,\{c\},\{a,c\}\};\sigma=\{\phi,X,\{a,c\}\}$

Def $f:(X,\tau)\to(Y,\sigma)$ where f(a) = a; f(b) = b; f(c) = c $\alpha gC(X;\tau)=\{\phi,X,\{a\},\{b\},\{a,b\},\{b,c\}\}$;

$\delta^* g\alpha C(X;\tau)=\{\phi,X,\{b\},\{a,b\},\{b,c\}\}$

Here $f^{-1}[\{a\}]=\{a\}$ is not $\delta^* g\alpha$ -closed in (X,τ) therefore it is αg -continuousbut not $\delta^* g\alpha$ -continuous.

Theorem 3.7 Every $\delta^* g\alpha$ -continuous is δg -continuous. (resp δgp -continuous, δg^* -continuous, $g\delta s$ -continuous, δgb -continuous)

*Proof:*Let V be a closed set in (Y, σ). Since f is $\delta^* g\alpha$ -continuous map, $f^{-1}(V)$ is $\delta^* g\alpha$ -closed in (X, τ). Since every $\delta^* g\alpha$ -closed set is δg -closed (resp δgp -closed, δg^* -closed, $g\delta s$ -closed, δgb -closed), therefore $f^{-1}(V)$ is δg -closed (resp δgp -closed, δg^* -closed, $g\delta s$ -closed, δgb -closed) in (X, τ). Hence f is δg -continuous. (resp δgp -continuous, δg^* -continuous, $g\delta s$ -continuous, δgb -continuous).

Remark 3.8: The converse of the above theorem is not true as shown in the following example.

*Example 3.9*Let $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}; \sigma = \{\phi, X, \{a\}\{b, c\}\}$

Def $f : (X, \tau) \to (Y, \sigma)$ where f(a) = a; f(b) = b; f(c) = c

$\delta gbC(X; \tau) = \{\phi, X, \{a\}, \{b\}, \{c\}, \{b, c\}, \{a, c\}\}$; $g\delta sC(X; \tau) = \{\phi, X, \{a\}, \{b\}, \{c\}, \{b, c\}, \{a, c\}\}$

$\delta^* g\alpha C(X; \tau) = \{\phi, X, \{c\}, \{a, c\}, \{b, c\}\}$

Here $f^{-1}[\{a\}] = \{a\}$ is not $\delta^* g\alpha$ -closed in (X, τ) therefore it is δgb -continuous, $g\delta s$ -continuousbut not $\delta^* g\alpha$ -continuous.

Example 3.10 Let $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}\}; \sigma = \{\phi, X, \{a\}, \{b\}, \{b, c\}\}$

Def $f : (X, \tau) \to (Y, \sigma)$ where f(a) = a; f(b) = b; f(c) = c

$\delta gC(X; \tau) = \{\phi, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}\}$;

$\delta gpC(X; \tau) = \{\phi, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}\}$;

$\delta g^* C(X; \tau) = \{\phi, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}\}$;

$\delta^* g\alpha C(X; \tau) = \{\phi, X, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}\}$

Here $f^{-1}[\{a\}] = \{a\}$ is not $\delta^* g\alpha$ -closed in (X, τ) therefore it is δg -continuous, δgp -continuous, δg^* -continuous but not $\delta^* g\alpha$ -continuous.

Theorem 3.11: The composition of two $\delta^* g\alpha$ -continuous map is also a $\delta^* g\alpha$ - continuous map.

Proof: Let $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \eta)$ be two $\delta^* g\alpha$ -continuous map. Let f be a closed set in (Z, η). Since g is a $\delta^* g\alpha$ -continuous map, $g^{-1}(f)$ is $\delta^* g\alpha$ -closed in (Y, σ). Since every $\delta^* g\alpha$ -closed is closed, $g^{-1}(f)$ is closed in (Y, σ).Since f is a $\delta^* g\alpha$ -continuous map, $f^{-1}(g^{-1}(f)) = gof(f)$ is $\delta^* g\alpha$ -closed in (X, τ), therefore gof is also $\delta^* g\alpha$ -continuous map.

Example 3.12: Let $X = \{a, b, c\} = Y, Z$, $\tau = \{\phi, X, \{a\}, \{c\}, \{a, c\}\}$; $\sigma = \{\phi, X, \{a\}, \{b\}\{a, b\}\}$;

$\eta = \{\phi, X, \{a\}\}$

Def. $f : (X, \tau) \to (Y, \sigma)$, $g : (Y, \sigma) \to (Z, \eta)$ where f(a) = a; f(b) = c; f(c) = b and

g(a) = a; g(b) = b; g(c) = c ; $\delta^* g\alpha C(X; \tau) = \{\phi, X, \{b\}, \{a, b\}, \{b, c\}\}$;

$\delta^* g\alpha C(Y; \sigma) = \{\phi, X, \{c\}, \{b, c\}, \{a, c\}\}$; $(gof)^{-1}[\{a, b\}] = f^{-1}\{b, c\} = \{b, c\}$

Here $\{b, c\}$ is $\delta^* g\alpha$ -closed in (Y, σ), also $\{a, b\}$ $\delta^* g\alpha$ -closed in (X, τ) Hence gof is $\delta^* g\alpha$ -continuous.

IV. $\delta^* g\alpha$ -Irresolute Function In Topological Spaces

The authors introduce the following definition.

*Definition 4.1*A map $f : (X, \tau) \to (Y, \sigma)$ is said to be a delta star generalizedalpha-irresolute (briefly $\delta^* g\alpha$ -irresolute) if $f^{-1}(V)$ is $\delta^* g\alpha$ -closed in (X, τ) for every $\delta^* g\alpha$ -closed in (Y, σ).

Theorem 4.2 Let $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \eta)$ be any twofunctions, then

(i) $gof : (X, \tau) \to (Z, \eta)$ is $\delta^* g\alpha$ -continuous if g is continuous and f is $\delta^* g\alpha$ -continuous.

(ii) $gof : (X, \tau) \to (Z, \eta)$ is $\delta^* g\alpha$ -irresolute if both g and f is $\delta^* g\alpha$ -irresolute.

(iii) $\text{gof} : (\text{X}, \tau) \to (\text{Z}, \eta)$ is $\delta^* g\alpha$ -continuous if g is $\delta^* g\alpha$ -continuous and f is $\delta^* g\alpha$ irresolute.

Proof:

(i) Let v be a closed set in (Z, η). Since g is continuous, $g^{-1}(V)$ is closed in (Y, σ). Since f is $\delta^* g\alpha$ -continuous, $\text{f}^{-1}(\text{g}^{-1}(\text{v})) = (\text{gof})^{-1}(\text{v})$ is g -closed in (X, τ) ,thereforegof is $\delta^* g\alpha$ -continuous.

(ii) Let v be a $\delta^* g\alpha$ closed set in (Z, η). Since g is $\delta^* g\alpha$ irresolute, $g^{-1}(V)$ is $\delta^* g\alpha$ closed in (Y, σ). Since f is $\delta^* g\alpha$ -irresolute, $\text{f}^{-1}(\text{g}^{-1}(\text{v})) = (\text{gof})^{-1}(\text{v})$ is $\delta^* g\alpha$ -closed in (X, τ) , Therefore gof is $\delta^* g\alpha$ -irresolute.

(iii) Let v be a closed set in (Z, η). Since g is $\delta^* g\alpha$ - continuous, $g^{-1}(V)$ is $\delta^* g\alpha$ - closed in (Y, σ). Since f is $\delta^* g\alpha$ - irresolute, $\text{f}^{-1}(\text{g}^{-1}(\text{v})) = (\text{gof})^{-1}(\text{v})$ is $\delta^* g\alpha$ -closed in (X, τ) , Therefore gof is $\delta^* g\alpha$ -continuous.

Theorem 4.3 Let $f : (X, \tau) \to (Y, \sigma)$ be $\delta^* g\alpha$ -continuous map (gs - continuous, $g\alpha$ - continuous, g - continuous). If (X, τ) is a ${}_{\alpha\delta}T^{**}_{\frac{3}{4}} g\alpha$ space (${}^{**}_{\delta\alpha}T_{\frac{1}{2}}$ - space, ${}_{\alpha\delta}T^{**}_{c}$ - space ${}_{\delta}T^{**}_{c}$ -space) then f is continuous.

Proof: Let v be a closed set of (Y, σ). Since f is $\delta^* g\alpha$ -continuous map (gs - continuous, $g\alpha$ - continuous, g - continuous), then $f^{-1}(v)$ is $\delta^* g\alpha$ -closed (gs - closed, $g\alpha$ - closed, g - closed). Since (X, τ) is ${}_{\alpha\delta}T^{**}_{\frac{3}{4}} g\alpha$ space (${}^{**}_{\delta\alpha}T_{\frac{1}{2}}$ - space, ${}_{\alpha\delta}T^{**}_{c}$ - space ${}_{\delta}T^{**}_{c}$ -space), then $f^{-1}(v)$ is δ -closed in (X, τ) . Every δ -closed is closed in (X, τ) . Therefore f is continuous.

Theorem 4.4 Let $f : (X, \tau) \to (Y, \sigma)$ be a surjective, $\delta^* g\alpha$ -irresolute and δ -closed map. Then f(A) is $\delta^* g\alpha$ -closed set of (Y, σ) for every $\delta^* g\alpha$ -closed set A of (X, τ) .

Proof: Let A be a $\delta^* g\alpha$ -closed set of (X, τ) . Let U be a $\delta^* g\alpha$ -open set of (Y, σ). Such that $f(A) \subseteq U$. Since f f is surjective and $\delta^* g\alpha$ -irresolute, $f^{-1}(U)$ is $\delta^* g\alpha$ –open set in (X, τ) . Since $A \subseteq f^{-1}(U)$ and A is $\delta^* g\alpha$ -closed set of (X, τ) . $cl_{\delta}(A) \subseteq f^{-1}(U)$. Then $f[cl_{\delta}(A)] \subseteq f[f^{-1}(U)] = U$, since f is δ -closed. $f[cl_{\delta}(A)] = cl_{\delta}[f[cl_{\delta}(A)]]$. This implies $cl_{\delta}[f(A)] \subseteq cl_{\delta}[f[cl_{\delta}(A)] = f[cl_{\delta}(A)] \subseteq U]$, There-fore $f(A)$ is a $\delta^* g\alpha$ -closed set of (Y, σ).

Theorem 4.5 Let $\text{f} : (\text{X}, \tau) \to (\text{Y}, \sigma)$ be a surjective, $\delta^* g\alpha$ -irresolute and δ -closed map. If (X, τ) is an ${}_{\alpha\delta}T^{**}_{\frac{3}{4}} g\alpha$ -space, then (Y, σ) is also an ${}_{\alpha\delta}T^{**}_{\frac{3}{4}} g\alpha$ -space.

Proof: Let A be a $\delta^* g\alpha$ -closed set of (Y, σ). Since f is $\delta^* g\alpha$ -irresolute, $f^{-1}(A)$ is $\delta^* g\alpha$ -closed set in (X, τ) . Since (X, τ) is ${}_{\alpha\delta}T^{**}_{\frac{3}{4}} g\alpha$ -space, $f^{-1}(A)$ is δ -closed of (X, τ) . Since f is δ -closed and surjective, $f[f^{-1}(A)] = A$ is δ -closed in (Y, σ) .Thus A is δ -closed in (Y, σ), therefore (Y, σ) is an ${}_{\alpha\delta}T^{**}_{\frac{3}{4}} g\alpha$ -space.

5: $\delta^* g\alpha$ -Homeomorphism

The authors introduce the following definition.

Definition 5.1 A map $f : (X, \tau) \to (Y, \sigma)$ is said to be a delta star generalized alpha-homeomorphism (briefly $\delta^* g\alpha$ -homeomorphism) if f is bijective, $\delta^* g\alpha$ - continuous and $\delta^* g\alpha$ -open.

Theorem 5.2 Every $\delta^* g\alpha$ -homeomorphism is gs -homeomorphism.

*Proof:*Let $f : (X, \tau) \to (Y, \sigma)$) is $\delta^* g\alpha$ -homeomorphism then f is bijective, $\delta^* g\alpha$ -continuous and $\delta^* g\alpha$ -open. Let V be closed set in (Y, σ), then $f^{-1}(V)$ is $\delta^* g\alpha$ -closed in (X, τ) . Since every $\delta^* g\alpha$ -closed set is gs -closed, then $f^{-1}(V)$ is gs -closed in (X, τ) , Therefore f f is gs -continuous. Let U be open

set in (X,τ), then $f(U)$ is $\delta^* g\alpha$-open in (Y,σ). Since every $\delta^* g\alpha$-open set is gs-open, then $f(U)$ is gs-open in (Y,σ), Therefore f is gs-open map. Hence f is gs-homeomorphism.

Remark 5.3: The converse of the above theorem is not true as shown in the following example.

*Example 5.4*Let $X = \{a, b, c\} = Y$; $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$; $\sigma = \{\phi, X, \{a\}, \{b, c\}\}$

Def. $f : (X,\tau) \to (Y,\sigma)$ where $f(a) = a, f(b) = b, f(c) = c$,

$gsC(X,\tau) = \{\phi, X, \{a\}, \{b\}, \{c\}, \{b, c\}, \{a, c\}\}$;

$gsO(X,\tau) = \{\phi, X, \{a\}, \{b\}, \{c\}, \{a, b\}\{b, c\}, \{a, c\}\}$; $\delta^* g\alpha C(X,\tau) = \{\phi, X, \{c\}, \{a, c\}, \{b, c\}\}$;

$\delta^* g\alpha O(X,\tau) = \{\phi, X, \{a\}, \{b\}, \{c\}, \{a, b\}\{b, c\}, \{a, c\}\}$ Here $f^{-1}[\{a\}] = \{a\}$ is not $\delta^* g\alpha$-closed in (X,τ) therefore it is not $\delta^* g\alpha$-homeomorphism.

Theorem 5.5: For any bijective map $f : (X,\tau) \to (Y,\sigma)$ the following statement are equivalent.

(i) $f^{-1} : (X,\tau) \to (Y,\sigma)$ is $\delta^* g\alpha$-continuous map.

(ii) f is an $\delta^* g\alpha$-open map.

(iii) f is an $\delta^* g\alpha$-closed map.

Proof:

(i) ⇒ (ii) Let U be an open set in (X,τ), then $X - U$ is closed in (X,τ) Since f^{-1} is $\delta^* g\alpha$-continuous, then $[f^{-1}]^{-1}$ is $\delta^* g\alpha$-closed in (Y,σ),that is $f(X-U)$ is $\delta^* g\alpha$-closed in (Y,σ), that is $Y - f(U)$ is $\delta^* g\alpha$-closed in (Y,σ). This implies that $f(U)$ is $\delta^* g\alpha$-open in (Y,σ). Hence f is $\delta^* g\alpha$-open map.

(ii) ⇒ (iii) Let F be a closed set in (X,τ), then $X - F$ is open in (X,τ). Since f is $\delta^* g\alpha$-open, then $f(X-F)$ is $\delta^* g\alpha$-open in (Y,σ). That is $Y - f(F)$ is $\delta^* g\alpha$-open in (Y,σ). This implies that $f(F)$ is $\delta^* g\alpha$-closed in (Y,σ). Hence f is $\delta^* g\alpha$-closed map.

(iii) ⇒ (i) let V be a closed set in (X,τ), since f is $\delta^* g\alpha$-closed map, then $f(V)$ is $\delta^* g\alpha$-closed in (Y,σ). That is $[f^{-1}]^{-1}(V)$ is $\delta^* g\alpha$-closed in (Y,σ). Hence f^{-1} is $\delta^* g\alpha$-continuous.

Theorem 5.6 Let $f : (X,\tau) \to (Y,\sigma)$ be bijective and δ∗gα-continuous map, then the following statement are equivalent.

(i) f is an $\delta^* g\alpha$-open map.

(ii) f is an $\delta^* g\alpha$-homeomorphism.

(iii) f is an $\delta^* g\alpha$-closed map.

Proof:

(i) ⇒ (ii) Let f be an $\delta^* g\alpha$-open map. Since f is bijective and $\delta^* g\alpha$-continuous map, f is $\delta^* g\alpha$-homeomorphism.

(ii) ⇒ (iii) Let f be an $\delta^* g\alpha$-homeomorphism. Then f is $\delta^* g\alpha$-open map, then $f(X-F)$ is $\delta^* g\alpha$-open in (Y,σ). That is $Y - f(F)$ is $\delta^* g\alpha$-open in (Y,σ). This implies that $f(F)$ is $\delta^* g\alpha$-closed in (Y,σ). Hence f is $\delta^* g\alpha$-closed map.

(iii) ⇒ (i) Let U be an open set in (X,τ), Then X-U is closed in (X,τ). Since f is $\delta^* g\alpha$-closed, then $f(X-U)$ is $\delta^* g\alpha$-closed in (Y,σ). That is $Y - f(U)$ is $\delta^* g\alpha$-closed in (Y,σ). Hence $f(U)$ is $\delta^* g\alpha$-open in (Y,σ).

*Theorem 5.7:*The composition of two $\delta^* g\alpha$-homeomorphism is also a $\delta^* g\alpha$-homeomorphism map.

Proof: Let $f : (X,\tau) \to (Y,\sigma)$ and $g : (Y,\sigma) \to (Z,\eta)$ be two $\delta^* g\alpha$-continuous map. Let f be a closed set in (Z,η). Since g is a $\delta^* g\alpha$-continuous map, $g^{-1}(f)$ is $\delta^* g\alpha$-closed in (Y,σ). Since every $\delta^* g\alpha$-closed is closed, $g^{-1}(f)$ is closed in (Y,σ). Since f is a $\delta^* g\alpha$-continuous map, $f^{-1}[g^{-1}(f)] = gof(f)$ is $\delta^* g\alpha$-closed in (X,τ), therefore gof is also $\delta^* g\alpha$-continuous map. Let A be a closed set in (X,τ),

then $X-A$ is an open set in (X,τ). Since f is $\delta^* g\alpha$ -homeomorphism, then $f(X-A)$ is a $\delta^* g\alpha$ -open in (Y,σ), implies $f(A)$ is $\delta^* g\alpha$ - closed in (Y,σ). Since every $\delta^* g\alpha$ -closed is closed, then $f(A)$ is closed in (Y,σ), then $Y-f(A)$ is open in (Y,σ). Since g is $\delta^* g\alpha$ -homeomorphism, $g(Y-f(A))$ is $\delta^* g\alpha$ -open in (Z,η), implies $g(f(A))=gof(A)$ is $\delta^* g\alpha$ -closed in (Z,η) therefore gof is $\delta^* g\alpha$ -continuous and $\delta^* g\alpha$ -open, implies gof is $\delta^* g\alpha$ -homeomorphism.

Theorem 5.9: The composition of two $\delta^* g\alpha$ -homeomorphism is also a $\delta^* g\alpha c$ homeomorphism map.

Proof: Let $f:(X,\tau)\to(Y,\sigma)$ and $g:(Y,\sigma)\to(Z,\eta)$ be two $\delta^* g\alpha c$ homeomorphism. Let f be a $\delta^* g\alpha$ -closed set in (Z,η). Since g is a $\delta^* g\alpha$ -irresolute map, $g^{-1}(f)$ is $\delta^* g\alpha$ -closed in (Y,σ). Since f is $\delta^* g\alpha$ -irresolute, $f^{-1}[g^{-1}(f)]$ is $\delta^* g\alpha$ -closed in (X,τ), that is $(gof)^{-1}(f)$ is $\delta^* g\alpha$ -closed in (X,τ), therefore $gof:(X,\tau)\to(Z,\eta)$ is $\delta^* g\alpha$ -irresolute map.

Let G be a $\delta^* g\alpha$ -closed set in (X,τ), since f^{-1} is a $\delta^* g\alpha$ -irresolute map, $(f^{-1})^{-1}(G)$ is $\delta^* g\alpha$ -closed in (Y,σ), that is $f(G)$ is $\delta^* g\alpha$ -closed in (Y,σ). Since g^{-1} is $\delta^* g\alpha$ -irresolute, $(g^{-1})^{-1}(f(G))$ is $\delta^* g\alpha$ -closed in (Z,η), that is $g(f(G))$ is $\delta^* g\alpha$ closed in (Z,η), therefore $gof(G)$ is $\delta^* g\alpha$ -closed in (Z,η). This implies that $((gof)^{-1})^{-1}(G)$ is a $\delta^* g\alpha$ -closed in (Z,η). This shows that $(gof)^{-1}:(Y,\sigma)\to(Z,\eta)$ is $\delta^* g\alpha$ -irresolute. Hence gof is a $\delta^* g\alpha c$ -homeomorphism.

Theorem 5.10: Every $\delta^* g\alpha$ -homeomorphism from a ${}_{\alpha\delta}T_{\frac{3}{4}}^{**} g\alpha$ -space into another ${}_{\alpha\delta}T_{\frac{3}{4}}^{**} g\alpha$ -space is a homeomorphism

Proof: Let $f:(X,\tau)\to(Y,\sigma)$ be a $\delta^* g\alpha$ -homeomorphism. Then f is bijective, $\delta^* g\alpha$ -open and $\delta^* g\alpha$ -continuous maps. Let U be an open set in (X,τ). Since f is $\delta^* g\alpha$ -open and since (Y,σ) is ${}_{\alpha\delta}T_{\frac{3}{4}}^{**} g\alpha$ -space, $f(U)$ is open set in (Y,σ). This implies that f is open map. Let V be a closed set in (Y,σ), since f is $\delta^* g\alpha$ -continuous map and since (X,τ) is ${}_{\alpha\delta}T_{\frac{3}{4}}^{**} g\alpha$ -space, $f^{-1}(V)$ is closed in (X,τ) Therefore f is continuous. Hence f is a homeomorphism.

Theorem 5.11 Let (Y,σ) be ${}_{\alpha\delta}T_{\frac{3}{4}}^{**} g\alpha$ -space. If $f:(X,\tau)\to(Y,\sigma)$ and $g:(Y,\sigma)\to(Z,\eta)$ are $\delta^* g\alpha$ -homeomorphism then gof is $\delta^* g\alpha$ -homeomorphism.

Proof: Let $f:(X,\tau)\to(Y,\sigma)$ and $g:(Y,\sigma)\to(Z,\eta)$ be two $\delta^* g\alpha$ homeomorphism. Let U be an open set in (X,τ). Since f is $\delta^* g\alpha$ -open map, $f(U)$ is $\delta^* g\alpha$ -open in (Y,σ). Since (Y,σ) is ${}_{\alpha\delta}T_{\frac{3}{4}}^{**} g\alpha$ -space, $f(U)$ is open in (Y,σ). Also since $\delta^* g\alpha$ -open map, $g(f(U))$ is $\delta^* g\alpha$ -open in (Z,η). Hence gof is $\delta^* g\alpha$ open map. Let V be a closed set in (Z,η). Since g is $\delta^* g\alpha$ -continuous and since (Y,σ) is ${}_{\alpha\delta}T_{\frac{3}{4}}^{**} g\alpha$ -space, $g^{-1}(V)$ is closed in (Y,σ). Since f is $\delta^* g\alpha$ -continuous, $f^{-1}(g^{-1}(V))=(gof)^{-1}(V)$ is $\delta^* g\alpha$ -closed in (X,τ). That is gof is $\delta^* g\alpha$ continuous. Hence gof is $\delta^* g\alpha$ -homeomorphism.

Theorem 5.12:Every $\delta^* g\alpha$ -homeomorphism from ${}_{\alpha\delta}T_{\frac{3}{4}}^{**} g\alpha$ -space into another ${}_{\alpha\delta}T_{\frac{3}{4}}^{**} g\alpha$ -space is a $\delta^* g\alpha c$ -homeomorphism.

Proof: Let $f:(X,\tau)\to(Y,\sigma)$ be $\delta^* g\alpha$ -homeomorphism. Let U be $\delta^* g\alpha$ -closed set in (Y,σ). Then U is closed in (Y,σ). Since f is $\delta^* g\alpha$ -continuous, $f^{-1}(U)$ is $\delta^* g\alpha$ -closed in (X,τ). Hence f is a

$\delta^* g\alpha$ -irresolute map. Let V be $\delta^* g\alpha$ -open in (X,τ). Then V is open in (X,τ). Since f is $\delta^* g\alpha$ -open, $f(V)$ is $\delta^* g\alpha$ -open set in (Y,σ). That is $\left(f^{-1}\right)^{-1}(V)$ is $\delta^* g\alpha$ -open set in (Y,σ) and hence f^{-1} is $\delta^* g\alpha$ irresolute. Thus f is $\delta^* g\alpha c$ -homeomorphism.

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GMJ Coding Through a Three Star and Super Mean Labeling

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Abstract: *In this paper, a technique of coding a message using GMJ code on a three star graph* $K_{1,\ell} \cup K_{1,m} \cup K_{1,n}$, $\ell \le m \le n$ *after labeling it with super mean labeling. A few observations and a rule for super mean labeling on the three star is provided. Three illustrations using three different methods for three messages are provided in this paper. Three catchy clues, mathematical and non-mathematical using numbers and words for finding the suitable three star graph and numbering the alphabets are given which are sure to increase the interest in the minds of researchers in coding. The methods maintain a high degree of difficulty in understanding the message. The pictorial coding is also provided to add weightage, secrecy, attraction and attention to the coding techniques.*

Keywords*: spider graph; GMJ coding; RNRO; IPFO; VSCC; star graph*

I. Introduction

Mother Knowledge is very generous. She always patronizes the minds which thirst for her and which go in search of her by providing ideas in abundance. Here is a technique to code a message in three different ways using super mean labeling on a three star graph $K_{1,\ell} \cup K_{1,m} \cup K_{1,n}$, $\ell \le m \le n$. With the dynamic living of the people and the growth of civilization, the need to replace transparency by secrecy increases. In any area, when messages and information need to be communicated only to a group of people or to an individual, one is not able to use a common language. The communication becomes very much limited between the sender and the receiver and not to be understood by others. Cryptography is the science containing methods to transform an intelligible message into one that it is unintelligible and transforming the message back to its original form.

II. Literature Review

Much work is done by researchers on super mean labeling[1-4].Jeyanthi et al., has discussed super mean labeling on several infinite families of graphs[5].The concept of coding labeled trees was introduced by Saverio Caminiti et al.,[6]. Using graph theory and cryptography a paper was presented by R.C.Read[7]. Authenticated key distribution using given set of primes for secret sharing was given by Chandramowliswaran et al.,[8].Motivated by these work we have introduced a few techniques of coding by using super mean labeling on a two star graph[9]. The oldest types of ciphers (algorithm) was developed by Julius Caesar[10]. Motivated by these work, we have struck at a concept of graph labeling on a three star graph and GMJ coding techniques. In this paper, Coding is done on a three star graph.

III. Pre Requisites

A. Definition: Super mean labeling

Let G be a (*p, q*) graph and $f : V(G) \to \{1, 2, 3, \ldots, p+q\}$ be an injection. For each edge $e = uv$, let $f^*(e) = \frac{f(u)+f(v)}{2}$ if $f(u)+f(v)$ is even and $f^*(e) = \frac{f(u)+f(v)+1}{2}$ if $f(u)+f(v)$ is odd. Then f is called super mean labeling if $f(V) \cup \{f^*(e) : e \in E(G)\} = \{1, 2, 3, \ldots, p+q\}$. A graph that admits super mean labeling is called a super mean graph.

B. A rule for labeling

1. Some observations on super mean labeling of $K_{1,\ell} \cup K_{1,m} \cup K_{1,n}$, $\ell \le m \le n$ are listed. Here *p* and *q* represent the number of vertices and edges, $p = 3+\ell+m+n,\ q = \ell+m+n,\ p+q = 3+2\ell+2m+2n.$ The numbers from 1 to $3+2\ell+2m+2n$ must be assigned to the top vertices and the pendant vertices and in the process, the edge values get allotted. Repetition is not permitted. Here $f(u)$, $f(v)$, $f(w)$, $f(u_i)$, $f(v_j)$ and $f(w_k)$ are the numbers assigned to the top vertices and the pendant vertices and $f^*(uu_i)$, $f^*(vv_j)$ and $f^*(ww_k)$ are the numbers assigned to the edges of the first, second and third star respectively. The rule for getting the edge values is $f^*(uu_i) = \dfrac{f(u)+f(u_i)}{2}$ or $f^*(uu_i) = \dfrac{f(u)+f(u_i)+1}{2}$ where the edge connects u and u_i. Note that the edge value can be the actual or adjusted mean.

The average of the largest and the previous number is $\dfrac{(3+2\ell+2m+2n)+(3+2\ell+2m+2n-1)}{2} = 3+2\ell+2m+2n$. As repetition is not allowed this combination is not considered. Hence neither the edge value nor the pendant vertices can exceed $3+2\ell+2m+2n$. So it becomes possible to assign numbers to a three star $K_{1,\ell} \cup K_{1,m} \cup K_{1,n}$, $\ell \le m \le n$ through super mean labeling for all values of ℓ, m and n without omitting any number between 1 to $(p+q)$.

2. If $(f(u)$ and $f(u_i))$ or $(f(v)$ and $f(v_i))$ are both odd or both even, then the edge value is the actual mean. If they are not alike, then the edge value assumes the adjusted mean.

3. When f*(u)* is odd and if $f(u_i) = 2s,\ f(u_{i+1}) = 2s+1,$ they lead to same edge value and hence to be avoided. That is $\dfrac{1+6}{2}$ and $\dfrac{1+7}{2}$ have the same edge value 4 when $f(u) = 1$.

When $f(u_i) = 2s+1$ and $f(u_{i+1}) = 2s+2,$ they give different edge values and hence can be assigned. That is, $\dfrac{1+7}{2} = 4$ and $\dfrac{1+8}{2} = 5$ give different edge values.

Also when *f (u)* is even the situation is reversed.

These to be noted while labeling the numbers to the pendant vertices.

Step 1: Take 1, $\dfrac{p+q+1}{2}$ and $\dfrac{p+q}{2}$ and $p+q$ as $f(u)$, $f(v)$ and $f(w)$ respectively. $f(u_1) \ne 2$, for the edge value becomes 2 when $f(u_1) = 2$. But $f(v_1) = 2$ is permitted as $f(v) \ne 1.$

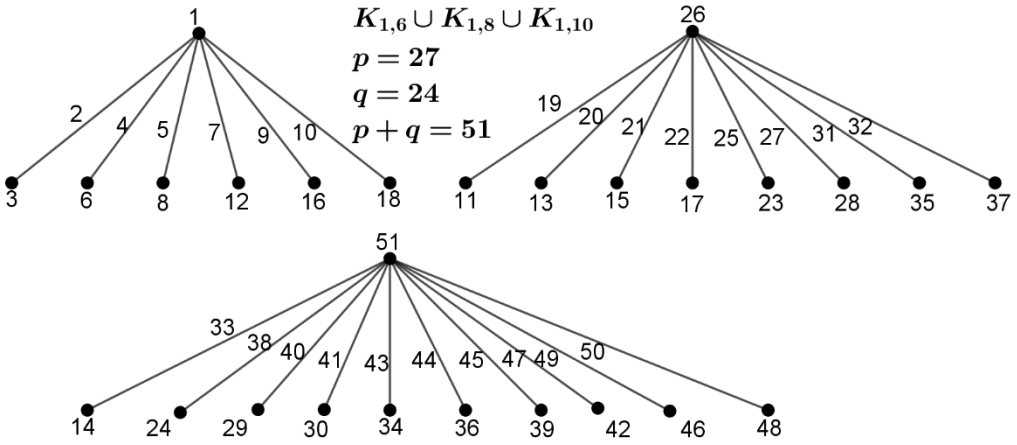
Step 2: If $f(u_1) = 3$, then $f(v_1) = 4$ and if $f(u_1) = 5$, then $f(v_1) = 2.$

Assigning numbers on the first star is done first, the second and third stars are done side by side.

Allot the smallest number omitted in the first star for the first pendant vertex of the second star. Immediately, allot the next least integer to the first pendant vertex of the third star. That is $f(v_1)$ and $f(w_1)$ must be allotted one after another. This process is maintained in allotting numbers to the pendant vertices of the second star and the third star. At times one has to continue in the same star for numbering two pendant vertices, one after another.

An example for super mean labeling on a three star $K_{1,6} \cup K_{1,8} \cup K_{1,10}$ is given below:

Figure 1 $K_{1,6} \cup K_{1,8} \cup K_{1,10}$



C. GMJ coding method

By assigning numbers to the 26 alphabets of English in a different manner, choosing a suitable labeled graph with a given clue mathematical or non-mathematical, finding the number in the graph for each letter of each word of the given message and presenting the letter codes in a unique way in some form, writing it as a horizontal string or in any other way and creating a picture with the codes after shuffling the order of the letters in order to increase the secrecy of the coded message is named as GMJ coding method.GMJ code stands for:

(i) Graph Message Jumble code. A coding technique to communicate a Message through Graphs Jumbling letters is named as GMJ code.

(ii) It also refers to the name of one of the researchers of this research (Gabriel Margaret Joan) who has conceived this method of coding.

D. Procedure for encoding

Step 1: A suitable three star graph has to be taken. Aclue, mathematical or non-mathematical is stated to find the three star graph which is to be used.

Step 2: The super mean labeling is done on the three star graph using the explanation given above.

Step 3: The 26 alphabets of English are divided in some way and the numbers attached to the alphabets are noted down.

Step 4: A coding method for each letter is stated.

Step 5: The message to be coded is written.

Step 6: By using the notations stated in step 4, the coding is written along a someway.

Step 7: Present the coding in a shape desired, shuffling the order of the letters.

Step 8: For decoding the message the instructions given for coding and a knowledge of super mean labeling on any three star graph are required.

Illustration 1

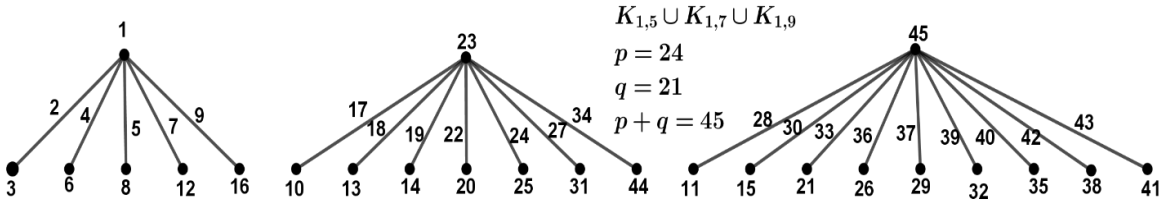
1. Message: Get it from galaxymall and deliver it to Jupiter house in Milky Street.

2. Clue: While four sisters walk hand in hand, one manages to sit on the head of another (The four sisters are 2, 3, 5 and 7 the single digit prime numbers, $3^2 = 9$ and so the three star is understood as $K_{1,5} \cup K_{1,7} \cup K_{1,9}$).

3. Graph: The three star graph $K_{1,5} \cup K_{1,7} \cup K_{1,9}$

4. Labeling: The super mean labeling done for $K_{1,5} \cup K_{1,7} \cup K_{1,9}$ is shown below

Figure 2 $K_{1,5} \cup K_{1,7} \cup K_{1,9}$



5. Numbering of alphabets: RNRO

Clue: LUNAR SOLAR STELLAR
Repeated letters: A, L, R
Not Repeated letters: E,N,O,S,T,U
The repeated letters A,L,R are allotted the numbers 1,2,3, then the not repeated letters E,N,O,S,T,U are given the numbers 4,5,…,9, the rest of the alphabets starting with B are given the numbers from 10 to 26.As $p+q=45,$ the numbers 27 to 52 are given to A to Z in order, thereby allowing 19 letters to be associated with two numbers. The pattern of numbers is shown below and it is named as RNRO (Repeated letters, not repeated letters and others are numbered in the order).

1	10	11	12	4	13	14	15	16	17	18	2	19
A	B	C	D	E	F	G	H	I	J	K	L	M
27	28	29	30	31	32	33	34	35	36	37	38	39

5	6	20	21	3	7	8	9	22	23	24	25	26
N	O	P	Q	R	S	T	U	V	W	X	Y	Z
30	41	42	43	44	45	46	47	48	49	50	51	52

The function for encoding is given below:
Let s,t and u denote the number of repeated letters and the other letters. Here $s=3, t=6, u=19.$

$g(a_i)=i, \quad i=1,2,3,...,s \quad where\ a_i\ is\ a\ repeated\ letter$

$g(a_j)=s+j, \quad j=1,2,3,...,t \quad where\ a_j\ is\ not\ repeated\ letter$

$g(a_k)=(s+t)+k, \quad k=1,2,3,...,(26-(s+t)) \quad where\ a_k \neq a_i, a_j$

Here a_i, a_j and a_k denote the 26 alphabets.

6. Coding a letter

A triplet is used to code a letter. Corresponding to the letter in the message, the number is taken and is located in the graph. If it is in the second star, fifth pendant vertex, then the required triplet is (2,5,0). The top vertices of the three star are denoted by (1,1,1),(2,2,2) and (3,3,3).

7. Coding: (word wise)

$GET-(2,3,0)(1,0,2)(1,3,0)$ $IT-(1,5,0)(1,3,0)$ $FROM-(2,2,0)(1,1,0)(1,2,0)(2,0,3)$
$GALAXY-(2,3,0)(1,1,1)(1,0,1)(2,0,6)(2,0,5)(2,5,0)$ $MALL-(2,0,3)(2,0,6)(1,0,1)(1,9,0)$
$AND-(1,1,1)(1,0,3)(1,4,0)$ $DELIVER-(1,4,0)(1,0,2)(1,0,1)(3,7,0)(2,0,4)(1,6,0)(1,7,0)$
$IT-(3,7,0)(1,3,0)$ $TO-(1,3,0)(3,8,0)$ $JUPITAR-(2,0,1)(1,0,5)(3,0,9)(1,5,0)(1,3,0)(1,6,0)(1,1,0)$
$HOUSE-(1,0,7)(1,2,0)(1,0,5)(3,3,3)(2,6,0)$ $IT-(1,5,0)(1,0,7)$ $MILKY-(2,0,3)(3,7,0)(3,9,0)$
$(2,0,2)(2,5,0)$ $STREET-(1,0,4)(1,3,0)(2,7,0)(1,0,2)(2,6,0)(1,3,0)$

8. Presenting the letter codes

Let the total number of letters be s. Let it be lie between x0 and y0 (the unit place is 0, tenth place is *x* or *y*). If s is close to y0 then consider A.P's with 1 as the first term and y as the common difference, the last term $< (p+q)$ and write them in order. Then written down by the AP with the first term 2 and common difference =y and so on and finally the multiples of y. This way of presenting the code will make to decode. Here there are 58 letters.

$50 < 58 < 60. \quad x = 5, y = 6$

As 58 is close to 60, use 6.

The codes of letters with the positions 1,7,…,55 are written first. Then the letters with the positions 2,8,…,56; followed by 3,9,…,57;6,12,…54; are written in a horizontal string (2,3,0)(1,1,0),(2,0,6)……..

9. Horizontal string

(2, 3, 0)(1, 1, 0)(2, 0, 6)(3, 8, 0)(1, 0, 1)(1, 3, 0)(1, 5, 0)(1, 0, 5)(3, 7, 0)(2, 7, 0)(1, 0, 2)(1, 2, 0)(2, 0, 5)(1, 1, 1)(3, 7, 0)(1, 3, 0)(1, 3, 0)(3, 3, 3)(3, 8, 0)(1, 0, 2)(1, 3, 0)(2, 0, 3)(2, 5, 0)(1, 0, 3)(2, 0, 4)(3, 9, 0)(2, 6, 0)(2, 6, 0)(2, 0, 2)(2, 6, 0)(1, 5, 0)(2, 3, 0)(2, 0, 3)(1, 4, 0)(2, 6, 0)(2, 0, 1) (1, 1, 0)(1, 5, 0)(2, 5, 0)(1, 3, 0)(1, 3, 0)(1, 1, 1)(2, 0, 6)(3, 0, 2)(2, 7, 0)(1, 0, 5)(2, 0, 7)(1, 0, 3) (1, 0, 4)(0, 0, 0)(2, 2, 0)(1, 0, 1)(1, 0, 1)(1, 0, 2)(3, 7, 0)(3, 0, 8)(1, 2, 0)(2, 0, 3)(1, 3, 0)(0, 0, 0).

10. About the picture coding

The crescent, the sun and a star are used for coding. The crescent consists of the first number, the sun the second number and the star the third number of the triplets taken in the clockwise direction. For decoding we reverse the process.

Figure 3 Lunar Solar Stellar

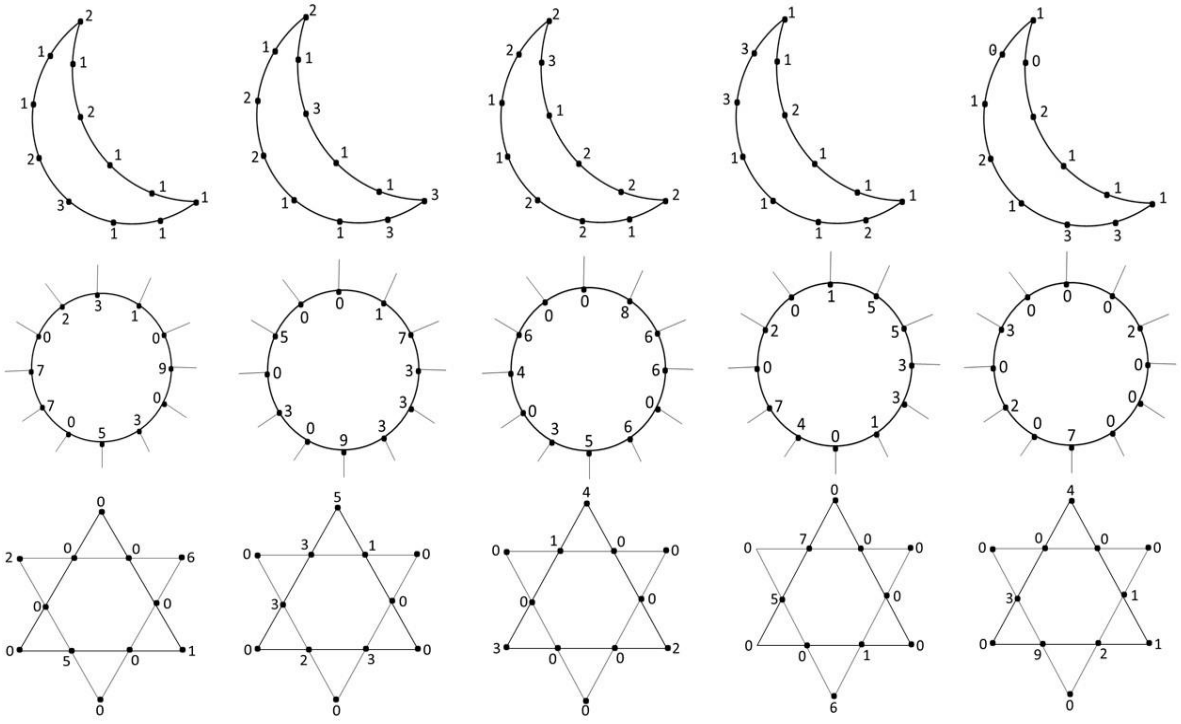


Illustration 2

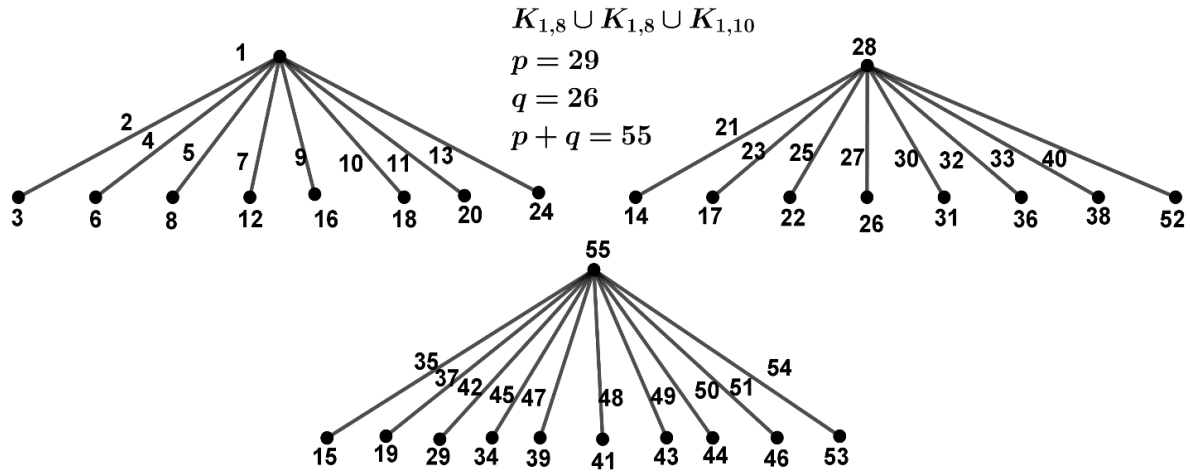
1. Message: Dig the floor under the oil cans in the kitchen

2. Clue : Doubly tasty good food (As the message refers to the kitchen, the Non Mathematical clue is given for guessing three star, 'food' and 'good' contains four letters and 'tasty' contains five letters $K_{1,8} \cup K_{1,8} \cup K_{1,10}$)

3. Graph : The three star graph $K_{1,8} \cup K_{1,8} \cup K_{1,10}$

4. Labeling: The super mean labeling done for $K_{1,8} \cup K_{1,8} \cup K_{1,10}$ is shown in the below

Figure 4 $K_{1,8} \cup K_{1,8} \cup K_{1,10}$



5. Numbering of alphabets: IPFO

Clue: The integral part of $\left[\frac{\ell \times m \times n}{\ell + m + n}\right]$ is given to A, $\left[\frac{\ell \times m \times n}{\ell + m + n}\right] + 1$ to B and so on till $p + q$. As some letters get two different numbers, after $p + q$, numbering is continued from 1 to $\left[\frac{\ell \times m \times n}{\ell + m + n}\right] - 1$. In this process a few letters receive three different numbers.

24	25	26	27	28	29	30	31	32	33	34	35	36
A	B	C	D	E	F	G	H	I	J	K	L	M
50	51	52	53	54	55	1	2	3	4	5	6	7
21	23	23										

37	38	39	40	41	42	43	44	45	46	47	48	49
N	O	P	Q	R	S	T	U	V	W	X	Y	Z
8	9	10	11	12	13	14	15	16	17	18	19	20

The function for encoding is given below

$$g(a_1) = \left[\frac{\ell \times m \times n}{\ell + m + n}\right] = 24 \qquad in\ I\ round$$

$$= 50\ \ (24 + 26) \qquad in\ II\ \ round$$

$$= \ 21 \qquad in\ III\ \ round$$

$$g(a_i) = \left[\frac{\ell \times m \times n}{\ell + m + n}\right] + (i-1) \qquad i = 2\ \ to\ 26\ i.e\ from\ B\ to\ Z$$

$$= 50 + (i-1) \qquad i = 2\ \ to\ 6\ (up\ to\ p + q = 55)$$

$$= 1 + (i-7) \qquad i = 7\ \ to\ 20$$

$$= 21 + (i-1) \qquad i = 2,3$$

6. Coding a letter

As in illustration (1) the number referring to the letter is located on the three star graph and written as a triplet. After finding the components of the triplet, a single number which is the sum of the components of the triplet is found and is used to represent the letter complicating recognition of the letter. They are marked at the vertices of the squares in the clockwise direction. If the triplet is (1,5,0), the number 6 is used to represent it. But 6 = 1+5, 2+4 or 3+3.In order to get a letter, three different possibilities to be checked, whether it is fifth pendant vertex or fifth edge value of the first star, fourth pendant vertex or fourth edge value of the second star or third edge value of the third star. For decoding we reverse the process.

7. Coding: (word wise)

$DIG - (2,0,4)(2,0,6)(2,0,5)$ $THE - (3,7,0)(2,5,0)(2,2,2)$ $FLOOR - (3,3,0)(3,0,1)(2,7,0)$ $(1,5,0)(3,6,0)$ $UNDER - (3,8,0)(3,0,2)(2,0,4)(2,2,2)(3,6,0)$ $OIL - (2,7,0)(2,0,6)(3,0,1)$ $CANS - (2,4,0)(1,8,0)(3,0,2)(3,0,3)$ $IN - (2,0,6)(3,0,2)$ $KITCHEN - (3,4,0)(2,0,6)$ $(2,7,0)(2,4,0)(2,5,0)(2,2,2)(3,0,2)$

8. Presenting the letter code

The single numbers representing letters which are obtained as explained in coding a letter are just written in a horizontal string.

9. Horizontal string

687107664969115669331398469568510713781067650 0.

10. About the picture coding

The vertices of the squares corresponds to the numbers, referring the letters as explained above.

Figure 5 Squares

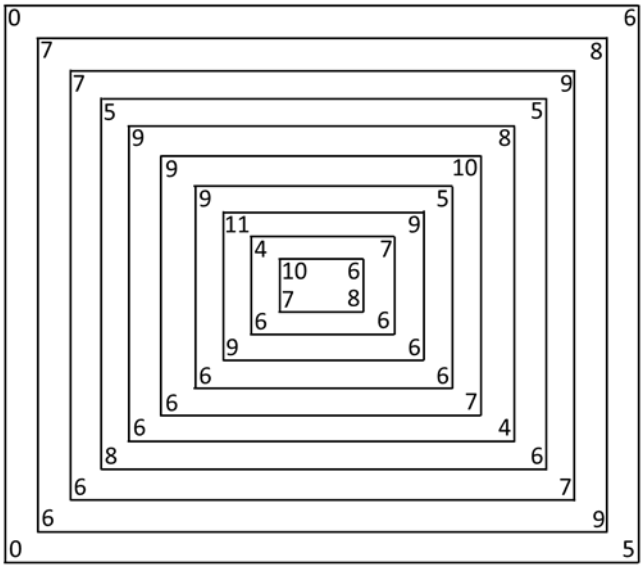


Illustration 3

1. Message: Meet Supreme Spider on the third Friday.

2. Clue : Perfect average. perfect product!

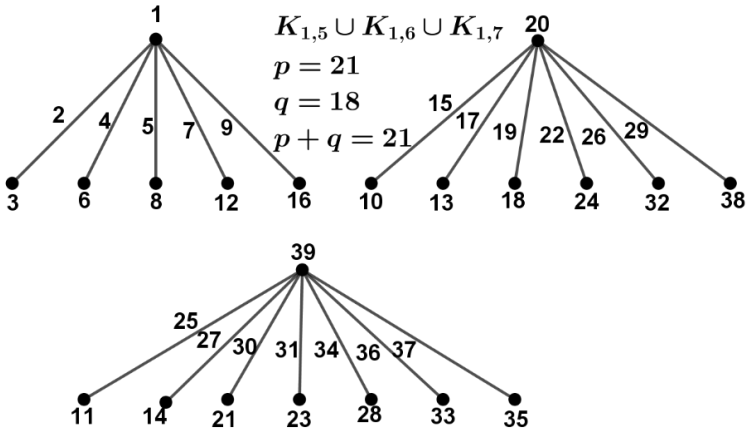
(Here, the average is $\left(\frac{5+6+7}{3}=6\right)$ the product is $\left(5\times6\times7=210=6^3-6\right)$

Expressible in terms of the perfect number 6, the graph $K_{1,5}\cup K_{1,6}\cup K_{1,7}$ to be guessed rightly)

3. Graph : The three star graph $K_{1,5}\cup K_{1,6}\cup K_{1,7}$ is the required graph.

4. Labeling: The super mean labeling done for $K_{1,5}\cup K_{1,6}\cup K_{1,7}$ is shown in the below

Figure 6 $K_{1,5}\cup K_{1,6}\cup K_{1,7}$



5. Numbering of alphabets: VSCC

Clue: VSCC means vowels are allotted square of numbers in order starting with 1 and then numbering of consonants continued.

1	2	3	5	4	6	7	8	9	10	11	12	13
A	B	C	D	E	F	G	H	I	J	K	L	M
27	28	29	30	31	32	33	34	35	36	37	38	39

14	16	15	17	18	19	20	25	21	22	23	24	26
N	O	P	Q	R	S	T	U	V	W	X	Y	Z

The function for encoding is given below

As there are five vowels, let them be represented by $a_i, (a_1 = a), (a_2 = e)$ and so on.

Here $g(a_i) = i,^2 \quad i = 1,2,3,4,5$

The consonants are taken in order and are denoted by b_j, $g(b_j)$ takes value from 1 to 26 omitting 1,4,9,16,25.

6.* Coding *a letter

As in illustrations (1) and (2) the numbering referring to the letter is located on the three star graph and is written as a triplet. For decoding we reverse the process.

7. Coding: (word wise)

$MEET - (2,2,0)(1,0,2)(3,0,4)(2,2,2)$ $SUPREME - (2,0,3)(3,0,1)(2,0,1)(2,3,0)(3,0,4)$
$(3,3,3)(1,0,2)$ $SPIDER - (2,0,3)(2,0,1)(1,0,5)(3,0,3)(1,0,2)(2,3,0)$ $ON - (1,5,0)(3,2,0)$
$THE - (2,2,2)(3,0,5)(3,0,4)$ $THIRD - (2,2,2)(1,3,0)(3,7,0)(2,3,0)(1,0,3)$
$FRIDAY - (2,5,0)(2,3,0)(3,7,0)(3,0,3)(3,0,2)(2,4,0).$

8. Presenting the letter code

$(\alpha_2, 0,1)$ denotes the edge value of the second ray of the first star. If any letter and so the corresponding number is repeated *k* times, we write $k(\alpha_i, 0,1)$ or $k(\alpha_i, 1,0)$. Similarly for second and third stars. The top vertices repeated k times are given by (k,t).

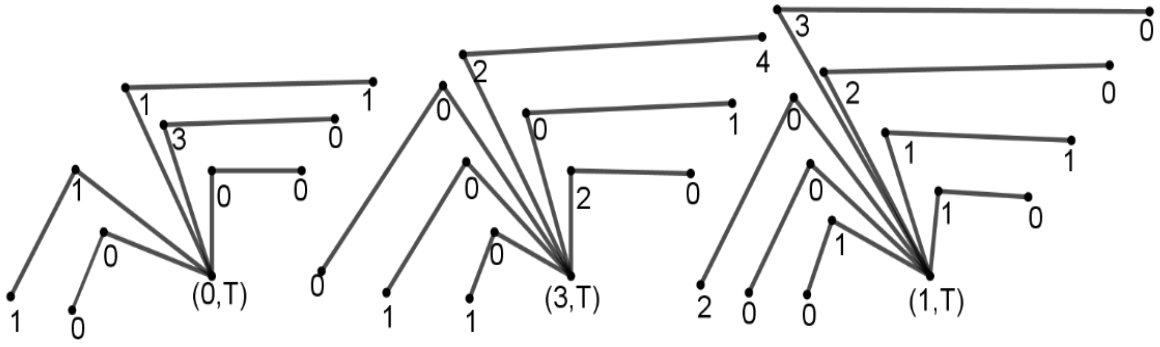
9. Horizontal string

$3(\alpha_2, 0,1)(\alpha_5, 0,1)(\alpha_5, 1,0)(\alpha_3, 1,0)(\alpha_3, 0,1)$
$(\beta_2, 1,0)(\beta_2, 1,1)(\beta_3, 0,1)\, 2(\beta_1, 0,1)\, 4(\beta_3, 1,0)(\beta_2, 0,1)(\beta_5, 1,0)(\beta_4, 1,0)$
$3(\nu_4, 0,1), (\nu_1, 0,1)(\nu_3, 1,1) 2(\nu_3, 0,1)(\nu_2, 1,0)(\nu_5, 0,1) 2(\nu_7, 1,0)(\nu_2, 0,1)$

10. About the picture coding

The spider graph is used for picture coding. A spider graph is a tree with a vertex of atleast degree three and all other vertices of degree two at the most.
(i) The vertices of atleast degree three of the soider graph denote the numbers assigned to the top vertices of the three star graph.
(ii) The vertices of degree two of the spider graph denote the numbers assigned to the edge values of the three star graph.
(iii) The pendant vertices of the spider graph denote the numbers assigned to the pendant vertices of the three star graph.

Figure 7.Coded Message(VSCC)



IV. Conclusion and Future Work

In this paper RNRO,IPFO and VSCC, three methods for numbering the alphabets are used. According to the messages,clues are choosen mathematically or non-mathematically. The researchers intend to work on the combination of super mean labeling and variety of graphs in the future.

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$\mathcal{N}_{\alpha g^{\#}\psi}$-closed sets in neutrosophic topological spaces

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Abstract: *In this paper, we introduce $\mathcal{N}_{\alpha g^{\#}\psi}$-closed sets in neutrosophic topological spaces and study some of its properties in neutrosophic topological spaces.*

Keywords: *Neutrosophic sets, neutrosophic topological spaces and neutrosophic $\alpha g^{\#}\psi$-closed sets.*

1. Introduction

Fuzzy set, proposed through Zadeh[14] in 1965, namely a mold according to come across uncertainty, vagueness then partial truth, represents a quantity on membership because of each feature on the universe over advice according to a subset about it. By including the dimensions of non-membership in accordance with Fuzzy set, Atanassov[2, 3, 4] proposed intuitionistic fuzzy set (IFS) in 1983 which looks more accurately to uncertainty quantification and provides the opportunity to precisely model the problem based on the existing knowledge and observations. Later on fuzzy topology was once delivered with the aid of Chang[5] in 1986. After this, even have been various generalizations on notions about fuzzy set and fuzzy topology. In last little year's quite a number ideas into vague have been prolonged in conformity with intuitionistic fuzzy sets. In 1997, Coker[6] delivered the thinking concerning intuitionistic fuzzy topological spaces. The neutrosophic set was introduced by Smarandache[10, 11] and explained, neutrosophic set is a generalization of intuitionistic fuzzy set. In 2012, Salama et.al[12], brought the thought over neutrosophic topological spaces. They brought neutrosophic topological spaces as like a generalization concerning intuitionistic fuzzy topological space and a neutrosophic set without the degree concerning membership, the degree of indeterminacy and the degree regarding non-membership over each element. Further the fundamental sets like semi-open sets, pre-open sets, α-open sets are introduced in neutrosophic topological spaces then theirs properties are well-read by Ishwarya et.al and Qays Hatem Imran et.al.[7, 9]. Recently Vigneshwaran et.al[13] introduced $\alpha g^{\#}\psi$-closed sets in topological spaces. The purpose of this paper is to introduce a $\mathcal{N}_{\alpha g^{\#}\psi}$-closed sets in neutrosophic topological spaces and study some of its basic properties.

PRELIMINARIES

Definition 2.1. [10]

A neutrosophic set $\mathcal{S}$ is an object of the following form

$$\mathcal{A}=\{\langle x,\ \mathcal{P}_{\mathcal{A}}(s),\ \mathcal{Q}_{\mathcal{A}}(s),\mathcal{R}_{\mathcal{A}}(s): s\epsilon\mathcal{S}\rangle\}$$

where $\mathcal{P}_{\mathcal{A}}(s)$, $\mathcal{Q}_{\mathcal{A}}(s)$ and $\mathcal{R}_{\mathcal{A}}(s)$ denote the degree of membership, the degree of indeterminacy and the degree of nonmembership for each element $s\epsilon\mathcal{S}$ to the set $\mathcal{A}$, respectively.

Definition 2.2. [10]

Let $\mathcal{A}$ and $\mathcal{B}$ be Neutrosophic sets of the form

$\mathcal{A}=\{\langle x,\mathcal{P}_{\mathcal{A}}(s),\ \mathcal{Q}_{\mathcal{A}}(s),\mathcal{R}_{\mathcal{A}}(s): s\epsilon\mathcal{S}\rangle\}$ and
$\mathcal{B}=\{\langle x,\mathcal{P}_{\mathcal{B}}(s),\ \mathcal{Q}_{\mathcal{B}}(s),\mathcal{R}_{\mathcal{B}}(s): s\epsilon\mathcal{S}\rangle\}$. Then

(i) $\mathcal{A}\subseteq\mathcal{B}$ if and only if $\mathcal{P}_{\mathcal{A}}(s)\leq\mathcal{P}_{\mathcal{B}}(s)$, $\mathcal{Q}_{\mathcal{A}}(s)\leq\mathcal{Q}_{\mathcal{B}}(s)$ and $\mathcal{R}_{\mathcal{A}}(s)\geq\mathcal{R}_{\mathcal{B}}(s)$;

(ii) $\bar{\mathcal{A}}=\{\langle\mathcal{R}_{\mathcal{A}}(s),\ \mathcal{Q}_{\mathcal{A}}(s),\mathcal{P}_{\mathcal{A}}(s): s\epsilon\mathcal{S}\rangle\}$;

(iii) $\mathcal{A}\cup\mathcal{B}=\{\langle s,\ \mathcal{P}_{\mathcal{A}}(s)\vee\mathcal{P}_{\mathcal{B}}(s),\ \mathcal{Q}_{\mathcal{A}}(s)\wedge\mathcal{Q}_{\mathcal{B}}(s),\mathcal{R}_{\mathcal{A}}(s)\wedge\mathcal{R}_{\mathcal{B}}(s): s\epsilon\mathcal{S}\rangle\}$;

(iv) $\mathcal{A} \cap \mathcal{B}$={ ⟨s, $\mathcal{P}_{\mathcal{A}}(s) \wedge \mathcal{P}_{\mathcal{B}}(s)$, $\mathcal{Q}_{\mathcal{A}}(s) \vee \mathcal{Q}_{\mathcal{B}}(s), \mathcal{R}_{\mathcal{A}}(s) \vee \mathcal{R}_{\mathcal{B}}(s)$: s$\epsilon\mathcal{S}$⟩.

Definition 2.3. [12]

A neutrosophic topology in a nonempty set $\mathcal{X}$ is a family $\Im$ of neutrosophic sets in $\mathcal{X}$ satisfying the following axioms:

(i) $0_N, 1_N \in \Im$;

(ii) $\mathcal{A} \cap \mathcal{B} \in \Im$ for any $\mathcal{A}, \mathcal{B} \in \Im$;

(iii) $\cup \mathcal{A}_i$ for any arbitrary family $\mathcal{A}_i : i \in J \in \Im$

Definition 2.5. [1]

A subset $\mathcal{S}$ of a neutrosophic space $(\mathcal{X}, \Im)$ is called

(i). a neutrosophic pre-open set if $\mathcal{S} \subseteq \mathcal{N}\text{int}(\mathcal{N}\text{cl}(\mathcal{S}))$, and a neutrosophic pre-closed set if $\mathcal{N}\text{cl}(\mathcal{N}\text{int}(\mathcal{S})) \subseteq \mathcal{S}$,

(ii). a neutrosophic semi-open set if $\mathcal{S} \subseteq \mathcal{N}\text{cl}(\mathcal{N}\text{int}(\mathcal{S}))$, and a neutrosophic semi-closed set if $\mathcal{N}\text{int}(\mathcal{N}\text{cl}(\mathcal{S})) \subseteq \mathcal{S}$,

(iii). a neutrosophic α -open set if $\mathcal{S} \subseteq \mathcal{N}\text{int}(\mathcal{N}\text{cl}(\mathcal{N}\text{int}(\mathcal{S})))$, and a neutrosophic α -closed set if $\mathcal{N}\text{cl}(\mathcal{N}\text{int}(\mathcal{N}\text{cl}(\mathcal{S}))) \subseteq \mathcal{S}$.

Definition 2.6.

A subset $\mathcal{S}$ of a space $(\mathcal{X}, \Im)$ is called

(i) a neutrosophic semi-generalized closed(briefly $\mathcal{N}_{sg}$-closed)[8] set if $\mathcal{N}\text{scl}(\mathcal{S}) \subseteq \mathcal{H}$ whenever $\mathcal{S} \subseteq \mathcal{H}$ and $\mathcal{H}$ is semi-open in $(\mathcal{X}, \Im)$,

(ii) a neutrosophic ψ-closed set (briefly $\mathcal{N}_{\psi}$-closed)[8] if $\mathcal{N}\text{scl}(\mathcal{S}) \subseteq \mathcal{H}$ whenever $\mathcal{S} \subseteq \mathcal{H}$ and $\mathcal{H}$ is $\mathcal{N}_{sg}$-open in $(\mathcal{X}, \Im)$,

(iii) a neutrosophic $g^{\#}\psi$-closed set(briefly $\mathcal{N}_{g^{\#}\psi}$-closed) if $\mathcal{N}\psi cl(\mathcal{S}) \subseteq \mathcal{H}$ whenever $\mathcal{S} \subseteq \mathcal{H}$ and $\mathcal{H}$ is $\mathcal{N}_{\psi}$-open in $(\mathcal{X}, \Im)$.

Throughout this paper neutrosophic $\alpha g^{\#}\psi$-open and neutrosophic $\alpha g^{\#}\psi$-closed set is denoted by $\mathcal{N}_{\alpha g^{\#}\psi}$-open set and $\mathcal{N}_{\alpha g^{\#}\psi}$-closed set respectively.

3. Basic properties of $\mathcal{N}_{\alpha g^{\#}\psi}$-closed sets

Definition 3.1.

A subset $\mathcal{A}$ of $(\mathcal{X}, \Im)$ is called a $\mathcal{N}_{\alpha g^{\#}\psi}$-closed set if $\mathcal{N}\alpha\text{cl}(\mathcal{A}) \subseteq \mathcal{H}$ whenever $\mathcal{A} \subseteq \mathcal{H}$ and $\mathcal{H}$ is $\mathcal{N}_{g^{\#}\psi}$-open in $(\mathcal{X}, \Im)$.

Theorem 3.2.

Every neutrosophic closed set is $\mathcal{N}_{\alpha g^{\#}\psi}$-closed set.

Proof:

Let $\mathcal{A} \subseteq H$ and $\mathcal{H}$ is $\mathcal{N}_{g^{\#}\psi}$-open in $\mathcal{X}$. Since $\mathcal{A}$ is neutrosophic closed set, then $\mathcal{N}\text{cl}(\mathcal{A}) = \mathcal{A}$. But $\mathcal{N}\alpha\text{cl}(\mathcal{A}) \subseteq \mathcal{N}\text{cl}(\mathcal{A})$, then $\mathcal{N}\alpha\text{cl}(\mathcal{A}) \subseteq \mathcal{H}$.

Hence $\mathcal{A}$ is $\mathcal{N}_{\alpha g^{\#}\psi}$-closed.

Theorem 3.3.

Every neutrosophic regular-closed set is $\mathcal{N}_{\alpha g^{\#}\psi}$-closed set.

Proof:

Let $\mathcal{A} \subseteq \mathcal{H}$ and $\mathcal{H}$ is $\mathcal{N}_{g^{\#}\psi}$-open in $\mathcal{X}$. Since $\mathcal{A}$ is neutrosophic regular-closed set, then $\mathcal{N}\text{rcl}(\mathcal{A}) = \mathcal{A}$. But $\mathcal{N}\alpha\text{cl}(\mathcal{A}) \subseteq \mathcal{N}\text{rcl}(\mathcal{A})$, then $\mathcal{N}\alpha\text{cl}(\mathcal{A}) \subseteq \mathcal{H}$. Hence $\mathcal{A}$ is $\mathcal{N}_{\alpha g^{\#}\psi}$-closed.

Theorem 3.4.

Every $\mathcal{N}_{\alpha}$-closed set is $\mathcal{N}_{\alpha g^{\#}\psi}$-closed set.

Proof:

Let $\mathcal{A} \subseteq \mathcal{H}$ and $\mathcal{H}$ is $\mathcal{N}_{g^{\#}\psi}$-open in $\mathcal{X}$. Since $\mathcal{A}$ is $\mathcal{N}_{\alpha}$-closed set, then $\mathcal{N}\alpha cl(\mathcal{A}) = \mathcal{A}$. But $\mathcal{N}\alpha cl(\mathcal{A}) \subseteq \mathcal{H}$. Hence $\mathcal{A}$ is $\mathcal{N}_{\alpha g^{\#}\psi}$-closed.

Theorem 3.5.

Every $\mathcal{N}_{\alpha g^{\#}\psi}$-closed set is $\mathcal{N}_{sg}$-closed set.

Proof:

Let $\mathcal{A} \subseteq \mathcal{H}$ and $\mathcal{H}$ is $\mathcal{N}_s$ in $\mathcal{X}$. Since every $\mathcal{N}_s$ set is $\mathcal{N}_{g^{\#}\psi}$-open, $\mathcal{H}$ is $\mathcal{N}_{g^{\#}\psi}$-open. Since $\mathcal{A}$ is $\mathcal{N}_{\alpha g^{\#}\psi}$-closed, $\mathcal{N}\alpha cl(\mathcal{A}) \subseteq \mathcal{H}$. But $\mathcal{N}scl(\mathcal{A}) \subseteq \mathcal{N}\alpha cl(\mathcal{A})$, then $\mathcal{N}scl(\mathcal{A}) \subseteq \mathcal{H}$. Therefore $\mathcal{A}$ is $\mathcal{N}_{sg}$-closed.

Theorem 3.6.

Every $\mathcal{N}_{\alpha g^{\#}\psi}$-closed set is $\mathcal{N}_{\psi}$-closed set.

Proof:

Let $\mathcal{A} \subseteq \mathcal{H}$ and $\mathcal{H}$ is $\mathcal{N}_{sg}$ -open in $\mathcal{X}$. Since every $\mathcal{N}_{sg}$-open set is $\mathcal{N}_{g^{\#}\psi}$-open, $\mathcal{H}$ is $\mathcal{N}_{g^{\#}\psi}$-open. Since $\mathcal{A}$ is $\mathcal{N}_{\alpha g^{\#}\psi}$-closed, $\mathcal{N}\alpha cl(\mathcal{A}) \subseteq \mathcal{H}$. But $\mathcal{N}scl(\mathcal{A}) \subseteq \mathcal{N}\alpha cl(\mathcal{A})$, then $\mathcal{N}scl(\mathcal{A}) \subseteq \mathcal{H}$. Therefore $\mathcal{A}$ is $\mathcal{N}_{\psi}$-closed.

Theorem 3.7.

Every $\mathcal{N}_{\alpha g^{\#}\psi}$-closed set is $\mathcal{N}_{g^{\#}\psi}$-closed set.

Proof:

Let $\mathcal{A} \subseteq \mathcal{H}$ and $\mathcal{H}$ is $\mathcal{N}_{\psi}$-open in $\mathcal{X}$. Since every $\mathcal{N}_{\psi}$-open set is $\mathcal{N}_{g^{\#}\psi}$-open, $\mathcal{H}$ is $\mathcal{N}_{g^{\#}\psi}$-open. Since $\mathcal{A}$ is $\mathcal{N}_{\alpha g^{\#}\psi}$-closed, $\mathcal{N}\alpha cl(\mathcal{A}) \subseteq \mathcal{H}$. But $\mathcal{N}\psi cl(\mathcal{A}) \subseteq \mathcal{N}\alpha cl(\mathcal{A})$, then $\mathcal{N}\psi cl(\mathcal{A}) \subseteq \mathcal{H}$. Therefore $\mathcal{A}$ is $\mathcal{N}_{\alpha g^{\#}\psi}$-closed.

Remark 3.8. The reverse implication of the above theorems is not true as shown in the following example.

Example 3.9.

Let $\mathcal{S}$={u,v} and Neutrosophic sets $\mathcal{D}_1$ and $\mathcal{D}_2$ are defined by

$\mathcal{D}_1$=⟨s, (0.6,0.5,0.5), (0.4,0.3,0.3), (0.5,0.2,0.2)⟩

D_2= ⟨s, (0.5,0.5,0.5), (0.5,0.4,0.4), (0.6,0.5,0.5)⟩

$\mathcal{N}_{\alpha}$-closed set= $\mathcal{D}_3$=⟨s, (0.6,0.5,0.5), (0.5,0.3,0.3), (0.5,0.5,0.5)⟩

$\mathcal{N}_{sg}$-closed set= $\mathcal{D}_4$=⟨s, (0.5,0.4,0.4), (0.4,0.3,0.3), (0.6,0.5,0.5)⟩

$\mathcal{N}_{\psi}$-closed set= $\mathcal{D}_5$=⟨s, (0.4,0.3,0.3), (0.3,0.2,0.2), (0.6,0.5,0.5)⟩

$\mathcal{N}_{g^{\#}\psi}$-closed set= $\mathcal{D}_6$=⟨s, (0.4,0.3,0.3), (0.2,0.2,0.2), (0.6,0.5,0.5)⟩

$\mathcal{N}_{\alpha g^{\#}\psi}$-closed set= $\mathcal{D}_7$=⟨s, (0.4,0.2,0.2), (0.2,0.2,0.2), (0.5,0.5,0.5)⟩.

Let $\Im = \{0_N, \mathcal{D}_1, \mathcal{D}_2, 1_N\}$. Here $(\mathcal{D}_6)^c$ is $\mathcal{N}_{g^{\#}\psi}$-open set, $\mathcal{N}\alpha cl(\mathcal{D}_7) \subseteq (\mathcal{D}_6)^c$. Then $\mathcal{D}_7$ is $\mathcal{N}_{\alpha g^{\#}\psi}$-closed set in $(\mathcal{X}, \Im)$ but not neutrosophic closed set, $\mathcal{N}_r$-closed set and $\mathcal{N}_{\alpha}$-closed set.

Here $\mathcal{D}_4$, $\mathcal{D}_5$ and $\mathcal{D}_6$ are $\mathcal{N}_{sg}$-closed set, $\mathcal{N}_{\psi}$-closed set and $\mathcal{N}_{g^{\#}\psi}$-closed set respectively. But not $\mathcal{N}_{\alpha g^{\#}\psi}$-closed set because $\mathcal{N}\alpha cl(\mathcal{D}_4) \nsubseteq (\mathcal{D}_6)^c$, $\mathcal{N}\alpha cl(\mathcal{D}_5) \nsubseteq (\mathcal{D}_6)^c$ and $\mathcal{N}\alpha cl(\mathcal{D}_5) \nsubseteq (\mathcal{D}_6)^c$.

Theorem 3.10.

Intersection of two $\mathcal{N}_{\alpha g^{\#}\psi}$-closed sets in X is again in $\mathcal{N}_{\alpha g^{\#}\psi}$-closed set.

Proof:

Let $\mathcal{A}$ and $\mathcal{B}$ be the subsets of $\mathcal{N}_{\alpha g^{\#}\psi}$-closed sets, $\mathcal{A} \subseteq \mathcal{H}$ and $\mathcal{N}\alpha cl(\mathcal{A}) \subseteq \mathcal{H}$, $\mathcal{B} \subseteq H$ and $\mathcal{N}\alpha cl(\mathcal{B}) \subseteq \mathcal{H}$, $\mathcal{H}$ is a $\mathcal{N}_{g^{\#}\psi}$-open. Therefore $\mathcal{A} \cap \mathcal{B} \subseteq \mathcal{A}$ and $\mathcal{N}\alpha cl(\mathcal{A} \cap \mathcal{B}) \subseteq \mathcal{N}\alpha cl(\mathcal{A})$, $\mathcal{A} \cap \mathcal{B} \subseteq \mathcal{B}$ and $\mathcal{N}\alpha cl(\mathcal{A} \cap \mathcal{B}) \subseteq \mathcal{N}\alpha cl(\mathcal{B})$. Hence $\mathcal{N}\alpha cl(\mathcal{A} \cap \mathcal{B}) \subseteq \mathcal{H}$ and $\mathcal{H}$ is a $\mathcal{N}_{g^{\#}\psi}$-open. Thus $\mathcal{A} \cap \mathcal{B}$ is $\mathcal{N}_{g^{\#}\psi}$-closed set.

Theorem 3.11.

Union of two $\mathcal{N}_{\alpha g^{\#}\psi}$-closed sets in $\mathcal{X}$ is again in $\mathcal{N}_{\alpha g^{\#}\psi}$-closed set.

Proof:

Let $\mathcal{A}$ and $\mathcal{B}$ be the subsets of $\mathcal{N}_{\alpha g^{\#}\psi}$-closed sets, $\mathcal{A}\subseteq H$ and $\mathcal{N}\alpha cl(\mathcal{A})\subseteq \mathcal{H}$, $\mathcal{B}\subseteq\mathcal{H}$ and $\mathcal{N}\alpha cl(\mathcal{B})\subseteq \mathcal{H}$, $\mathcal{H}$ is a $\mathcal{N}_{g^{\#}\psi}$-open. Therefore $\mathcal{A}\cup B\subseteq H$ and $\mathcal{N}\alpha cl(\mathcal{A}\cup B)=\mathcal{N}\alpha cl(\mathcal{A})\cup\mathcal{N}\alpha cl(\mathcal{B})\subseteq\mathcal{H}$. That is $\mathcal{N}\alpha cl(\mathcal{A}\cup B)\subseteq H$. Therefore $\mathcal{A}\cup B$ is $\mathcal{N}_{\alpha g^{\#}\psi}$-closed set.

Theorem 3.12.

If a set $\mathcal{A}$ is $\mathcal{N}_{\alpha g^{\#}\psi}$-closed in $\mathcal{X}$ iff $\mathcal{N}\alpha cl(\mathcal{A})-\mathcal{A}$ contains no non-empty $\mathcal{N}_{g^{\#}\psi}$-closed set.

Proof:

Necessity: Let $\mathcal{F}$ be a $\mathcal{N}_{g^{\#}\psi}$- closed in $\mathcal{X}$ such that $\mathcal{F}\subseteq\mathcal{N}\alpha cl(\mathcal{A})-\mathcal{A}$.

Then $\mathcal{F}\subseteq\mathcal{X}-\mathcal{A}$. This implies $\mathcal{A}\subseteq\mathcal{X}-\mathcal{F}$. Now $\mathcal{X}-\mathcal{F}$ is $\mathcal{N}_{g^{\#}\psi}$-open set of (X,τ_N) such that $\mathcal{A}\subseteq X-F$. Since $\mathcal{A}$ is $\mathcal{N}_{\alpha g^{\#}\psi}$-closed set then $\mathcal{N}\alpha cl(\mathcal{A})\subseteq\ \mathcal{X}-\mathcal{F}$. Thus $\mathcal{F}\subseteq\mathcal{X}-\mathcal{N}\alpha cl(\mathcal{A})$. Now $\mathcal{F}\subseteq\mathcal{N}\alpha cl(\mathcal{A})\cap(\mathcal{X}-\mathcal{N}\alpha cl(\mathcal{A}))=0_N$.

Sufficiency: Assume that $\mathcal{N}\alpha cl(\mathcal{A})-\mathcal{A}$ contains no non-empty $\mathcal{N}_{g^{\#}\psi}$-closed set. Let $\mathcal{A}\subseteq H$, $\mathcal{H}$ is $\mathcal{N}_{g^{\#}\psi}$-open set. Suppose that $\mathcal{N}\alpha cl(\mathcal{A})$ is not contained in $\mathcal{H}$ then $\mathcal{N}\alpha cl(\mathcal{A})\cap\mathcal{H}^c$ is a non-empty $\mathcal{N}_{g^{\#}\psi}$-closed set of $\mathcal{N}\alpha cl(\mathcal{A})-\mathcal{A}$, which is a contradiction. Therefore $\mathcal{N}\alpha cl(\mathcal{A})\subseteq H$ and hence $\mathcal{A}$ is $\mathcal{N}_{\alpha g^{\#}\psi}$-closed.

Theorem 3.13.

If a subset $\mathcal{A}$ is $\mathcal{N}_{\alpha g^{\#}\psi}$-closed and $\mathcal{A}\subseteq\mathcal{B}\subseteq\mathcal{N}\alpha cl(\mathcal{A})$, then $\mathcal{B}$ is $\mathcal{N}_{g^{\#}\psi}$-closed set.

Proof:

Let $\mathcal{B}\subseteq H$, $\mathcal{H}$ is a $\mathcal{N}_{g^{\#}\psi}$-open, then $\mathcal{A}\subseteq B$ and $\mathcal{A}\subseteq H$. Since $\mathcal{A}$ is $\mathcal{N}_{\alpha g^{\#}\psi}$, $\mathcal{N}\alpha cl(\mathcal{A})\subseteq\mathcal{H}$ but $\mathcal{B}\subseteq N\alpha cl(\mathcal{A})$ this implies that $\mathcal{N}\alpha cl(\mathcal{B})\subseteq N\alpha cl(\mathcal{A})$. Therefore, $\mathcal{N}\alpha cl(\mathcal{B})\subseteq N\alpha cl(\mathcal{A})\subseteq\mathcal{H}$. Thus $\mathcal{N}\alpha cl(\mathcal{B})\subseteq\mathcal{H}$ and $\mathcal{H}$ is $\mathcal{N}_{g^{\#}\psi}$-open. Hence $\mathcal{B}$ is $\mathcal{N}_{\alpha g^{\#}\psi}$-closed.

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More on Generalized e-closed Sets in Double Fuzzy Topological Spaces

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Abstract: *The purpose of this paper is to introduce and study a new class of sets called (r,s)-generalized fuzzy e-regular weakly closed sets which is lies between the new class of (r,s)-fuzzy regular weakly closed sets and (r,s)-generalized fuzzy e-closed sets in double fuzzy topological spaces. Several fundamental properties are introduced and discueesed. Furthermore, the relationship between the new concepts are introduced and established with some interesting counter examples.*

Keywords: *double fuzzy topology, (r,s)-generalized fuzzy e-regular weakly closed sets.*

I. Introduction

After the development of Zadeh`s theory of fuzzy sets [10], the idea of intuitionistic fuzzy set was first introduced by Atanassov[2], then Coker [3] introduced the notion of intuitionistic fuzzy topological spaces. After the Samanta and Mondal[9], introduced the notion of intuitionistic gradation of openness of fuzzy sets and gave the definition of intuitionistic fuzzy topological spaces as a generalization of smooth topology and the topology of intuitionistic fuzzy sets. Working under the name ``intuitionistic" did not continue and ended in by Gutierrez Garcia and Rodabaugh [6] when the proved that this term is unsuitable in mathematics and applications and they concluded that they work under the name ``double".

II.Preliminaries

Throughout this paper, X will be a non-empty set, $I=[0,1], I_0=(0,1]$ and $I_1=[0,1)$. A fuzzy set A is quasi-coincident with a fuzzy set B (denoted by, AqB) iff there exists $x\in X$ such that $A(x)+B(x)>1$ and they are not quasi-coincident otherwise (denoted by, $A\overline{q}B$). The family of all fuzzy sets on X is denoted by I^X. By $\underline{0}$ and $\underline{1}$, we denote the smallest and the greatest fuzzy sets on X. For a fuzzy set $A\in I^X, \underline{1}-A$ denotes its complement. All other notations are standard notations of fuzzy set theory.

Now, we recall the following definitions which are useful in the sequel.

Definition 1[9] *A double fuzzy topology (T,T^*) on X is a pair of maps $T,T^*: I^X\to I$, which satisfies the following properties:*

- $T(A)\le \underline{1}-T^*(A)$ for each $A\in I^X$.
- $T(A_1\wedge A_2)\ge T(A_1)\wedge T(A_2)$ and $T^*(A_1\wedge A_2)\le T^*(A_1)\vee T^*(A_2)$, for each $A_1,A_2\in I^X$.
- $T(\vee_{i\in\Gamma}A_i)\ge\wedge_{i\in\Gamma}T(A_i)$ and $T^*(\vee_{i\in\Gamma}A_i)\le\vee_{i\in\Gamma}T^*(A_i)$ for each $A_i\in I^X, i\in\Gamma$.

The triplet (X,T,T^*) is called a double fuzzy topological space (briefly, dfts). A fuzzy set A is called an (r,s)-fuzzy open (briefly, (r,s)-fo) if $T(A)\ge r$ and $T^*(A)\le s$. A fuzzy set A is called an (r,s)-fuzzy closed (briefly, (r,s)-fc) set iff $\underline{1}-A$ is an (r,s)-fo set.

Theorem 2[7] *Let (X,T,T^*) be a dfts. Then double fuzzy closure operator and double fuzzy interior operator of $A\in I^X$ are defined by*

$$C_{T,T^*}(A,r,s)=\wedge\ \{B\in I^X | A\le B, T(\underline{1}-B)\ge r, T^*(\underline{1}-B)\le s\}.$$
$$I_{T,T^*}(A,r,s)=\vee\ \{B\in I^X | A\ge B, T(B)\ge r, T^*(B)\le s\}.$$

Where $r\in I_0$ and $s\in I_1$ such that $r+s\le 1$.

Definition 3[1]*Let (X,T,T^*) be a dfts. For each $A \in I^X, r \in I_0$ and $s \in I_1$. A fuzzy set A is called an (r,s)-generalized fuzzy closed (briefly, (r,s)-gfc) if $C_{T,T^{å}}(A,r,s) \leq B, A \leq B, T(B) \geq r$ and $T^*(B) \leq s$. A is called an (r,s)-generalized fuzzy open (briefly (r,s)-gfo) iff $\underline{1} - A$ is (r,s)-gfc set.*

Definition 4[8] *Let (X,T,T^*) be a dfts. Then for each $A \in I^X, r \in I_0$ and $s \in I_1$. A fuzzy set A is called an (r,s)-fuzzy regular open (briefly, (r,s)-fro) if $A = I_{T,T^*}(C_{T,T^*}(A,r,s),r,s)$ and (r,s)-fuzzy regular closed (briefly, (r,s)-frc) if if $A = C_{T,T^*}(I_{T,T^*}(A,r,s),r,s)$.*

Definition 5[4] *Let (X,T,T^*) be a dfts. For each $A \in I^X, r \in I_0$ and $s \in I_1$. A fuzzy set A is called an (r,s)-fuzzy regular semi open (briefly, (r,s)-frso) if there exists an (r,s)-fuzzy regular open set B such that $B \leq A \leq C_{T,T^*}(B,r,s)$.*

Definition 6[5] *Let (X,T,T^*) be a dfts. For each $A,B \in I^X, r \in I_0$ and $s \in I_1$. A fuzzy set A is called an (r,s)-fuzzy regular weakly closed (briefly, (r,s)-frwc) set if $C_{T,T^*}(A,r,s) \leq B$ whenever $A \leq B$ and B is (r,s)-frso. A is called an (r,s)-fuzzy regular weakly open (briefly, (r,s)-frwo) set iff $1 - A$-frwc set.*

Definition 7[8] *Let (X,T,T^*) be a dfts. Then for each $A \in I^X, r \in I_0$ and $s \in I_1$, we define operators $\delta C_{T,T^*}$ and $\delta I_{T,T^*}: I^X \times I_0 \times I_1 \to I^X$ as follows: $\delta C_{T,T^*}(A,r,s) = \wedge \{B \in I^X : A \leq B, Bis(r,s) - frc\}$ and $\delta I_{T,T^*}(A,r,s) = \vee \{B \in I^X : B \leq A, Bis(r,s) - fro\}$.*

Definition 8[8] *Let (X,T,T^*) be a dfts. Then for each $A \in I^X, r \in I_0$ and $s \in I_1$: a fuzzy set A is called an*

• (r,s)-fuzzy δ semiopen (briefly, (r,s)-$f\delta so$) if $A \leq C_{TT^*}(\delta I_{T,T^*}(A,r,s),r,s)$ and (r,s)-fuzzy δ semi closed (briefly, (r,s)-$f\delta sc$) if $A \geq I_{T,T^*}(\delta C_{T,T^*}(A,r,s),r,s)$.

• (r,s)-fuzzy δ pre open (briefly, (r,s)-$f\delta po$) if $A \leq I_{T,T^*}(\delta C_{T,T^*}(A,r,s),r,s)$ and (r,s)-fuzzy δ pre closed (briefly, (r,s)-$f\delta pc$) if $A \geq C_{T,T^*}(\delta I_{T,T^*}(A,r,s),r,s)$.

• (r,s)-fuzzy β open (briefly, (r,s)-$f\beta o$) if $A \leq C_{TT^*}(I_{T,T^*}(C_{T,T^{å}}(A,r,s),r,s)r,s)$ and (r,s)-fuzzy β closed (briefly, (r,s)-$f\beta c$) if $A \geq I_{T,T^*}(C_{T,T^*}(I_{T,T^*}(A,r,s),r,s)r,s)$.

• (r,s)-fuzzy e-open (briefly, (r,s)-feo) if $A \leq C_{T,T^*}(\delta I_{T,T^*}(A,r,s),r,s) \vee I_{T,T^*}(\delta C_{T,T^*}(A,r,s),r,s)$ and (r,s)-fuzzy e-closed (briefly, (r,s)-fec) if $A \geq I_{T,T^*}(\delta C_{T,T^*}(A,r,s),r,s) \wedge C_{T,T^*}(\delta I_{T,T^*}(A,r,s),r,s)$.

Definition 9[8] *Let (X,T,T^*) be a dfts. Then for each $A \in I^X, r \in I_0$ and $s \in I_1$, we define operators eC_{T,T^*} (resp. $\delta SC_{T,T^*}$, $\delta PC_{T,T^*}$ and βC_{TT^*}) and $eI_{T,T^{å}}(resp.\delta SI_{T,T^*}, \delta PI_{T,T^*} and \beta I_{T,T^*}): I^X \times I_0 \times I_1 \to I^X$ as follows: $eC_{T,T^*}(resp.\delta SC_{T,T^*}, \delta PC_{T,T^*} and \beta C_{T,T^*})(A,r,s) = \wedge \{B \in I^X : A \leq B, Bis(r,s) - fec(resp.f\delta sc, f\delta pc and f\beta c)\}$ and $eI_{T,T^*}(resp.\delta SI_{T,T^*}, \delta PI_{T,T^*} and \beta I_{T,T^*})(A,r,s) = \vee \{B \in I^X : B \leq A, Bis(r,s) - feo(resp.f\delta so, f\delta po and f\beta o)\}$.*

Definition 10[8]*Let (X,T,T^*) be a dfts, $A \in I^X, r \in I_0$ and $s \in I_1$, A is called an (r,s)-fuzzy e-Q-neighborhood of $x_t \in P_t(X)$ if there exists an (r,s)-feo set $B \in I^X$ such that $x_t q B$ and $B \leq A$.*

The family of all (r,s)-fuzzy e-Q-neighborhood of x_t denoted by $e\text{-}Q(x_t,r,s)$.

Theorem 11[8] *Let (X,T,T^*) be a dfts. Then for each $A \in I^X, r \in I_0$ and $s \in I_1$, the operator $eC_{T,T^{å}}$ satisfies the following statements:*

• $eC_{T,T^*}(\underline{0},r,s) = \underline{0}$, $eC_{T,T^*}(\underline{1},r,s) = \underline{1}$.
• $A \leq eC_{T,T^*}(A,r,s)$.
• If $A \leq B$, then $eC_{T,T^*}(A,r,s) \leq eC_{T,T^*}(B,r,s)$.
• If A is an (r,s)-fec, then $A = eC_{T,T^*}(A,r,s)$.
• If A is an (r,s)-feo, then BqA iff $BqeC_{T,T^*}(A,r,s)$.
• $eC_{T,T^*}(eC_{T,T^*}(A,r,s),r,s) = eC_{T,T^*}(A,r,s)$.

- $eC_{T,T^*}(A,r,s) \vee eC_{T,T^*}(B,r,s) \leq eC_{T,T^*}(A \vee B,r,s)$.
- $eC_{T,T^*}(A,r,s) \wedge eC_{T,T^*}(B,r,s) \geq eC_{T,T^*}(A \wedge B,r,s)$.

Theorem 12[8] *Let (X,T,T^*) be a dfts. Then for each $A \in I^X, r \in I_0$ and $s \in I_1$, the operator eI_{T,T^*} satisfies the following statements:*

- $eI_{T,T^*}(\underline{1}-A,r,s) = \underline{1} - eC_{T,T^*}(A,r,s)$, $eC_{T,T^*}(\underline{1}-A,r,s) = \underline{1} - eI_{T,T^*}(A,r,s)$.
- $eI_{T,T^*}(\underline{0},r,s) = \underline{0}$, $eI_{T,T^*}(\underline{1},r,s) = \underline{1}$.
- $eI_{T,T^*}(A,r,s) \leq A$.
- If A is an (r,s)-feo, then $A = eI_{T,T^*}(A,r,s)$.
- If $A \leq B$, then $eI_{T,T^*}(A,r,s) \leq eI_{T,T^*}(B,r,s)$.
- $eI_{T,T^*}(eI_{T,T^*}(A,r,s),r,s) = eI_{T,T^*}(A,r,s)$.
- $eI_{T,T^*}(A \vee B,r,s) \geq eI_{T,T^*}(A,r,s) \vee eI_{T,T^*}(B,r,s)$.
- $eI_{T,T^*}(A \vee B,r,s) \leq eI_{T,T^*}(A,r,s) \wedge eI_{T,T^*}(B,r,s)$.

III. An (r,s)-generalized fuzzy e-regular weakly closed set

Definition 13 *Let (X,T,T^*) be a dfts. For each $A,B \in I^X, r \in I_0$ and $s \in I_1$. A fuzzy set A is called*

- An (r,s)-fuzzy e-clopen (briefly,(r,s)-fe-clopen) if A is an (r,s)-feo and an (r,s)-fec set.
- An (r,s)-generalized fuzzy e-regular weakly closed (briefly, (r,s)-gferwc) sets if $eC_{T,T^*}(A,r,s) \leq B$ whenever $A \leq B$ and B is (r,s)-frso, A is called an (r,s)-generalized fuzzy e-regular weakly open (briefly, (r,s)-gferwo) sets iff $1-A$ is (r,s)-gferwc set.

Theorem 14 *Let (X,T,T^*) be a dfts. A fuzy set $A \in I^X$ is (r,s)-gferwo set, $r \in I_0$ and $s \in I_1$ if and only if $B \leq eI_{T,T^*}(A,r,s)$ whenever B is (r,s)-frsc.*

Proof. Suppose that A is an (r,s)-gferwo set in $I^X, r \in I_0$ and $s \in I_1$ and let B is (r,s)-frsc such that $B \leq A$, so by the definition, $\underline{1}-A$ is an (r,s)-gferwc set in I^X. Hence $eC_{T,T^*}(\underline{1}-A,r,s) \leq \underline{1}-B$ and $\underline{1} - eI_{T,T^*}(A,r,s) \leq \underline{1}-B$. So $B \leq eI_{T,T^*}(A,r,s)$.

Conversely, suppose that $B \leq A$, B is (r,s)-frsc, $r \in I_0$ and $s \in I_1$ such that $B \leq eI_{T,T^*}(A,r,s)$. Now $\underline{1} - eI_{T,T^*}(A,r,s) \leq \underline{1}-B$. Thus $eC_{T,T^*}(\underline{1}-A,r,s) \leq \underline{1}-B$. So, $\underline{1}-A$ is an (r,s)-gferwc set and A is an (r,s)-gferwo set.

Theorem 15 *Let (X,T,T^*) be a dfts. $A \in I^X, r \in I_0$ and $s \in I_1$. If A is an (r,s)-gferwc set, then*

- $eC_{T,T^*}(A,r,s) - A$ does not contain any non-zero (r,s)-frso sets.
- A is an (r,s)-ferwc iff $eC_{T,T^*}(A,r,s) - A$ is (r,s)-rfc.
- B is (r,s)-gferwc set for each set $B \in I^X$ such that $A \leq B \leq eC_{T,T^*}(A,r,s)$.
- For each (r,s)-rfo set $B \in I^X$ such that $B \leq A$, B is an (r,s)-gferwc relative to A, B is an (r,s)-gferwc in X.
- $eC_{T,T^*}(A,r,s) - A$ does not contain any non-zero (r,s)-rfo sets.

Proof. (i) Suppose that B is a non-zero (r,s)-frso set of I^X such that $B \leq eC_{T,T^*}(A,r,s) - A$, whenever $A \in I^X$ is an (r,s)-gferwc set, $r \in I_0$ and $s \in I_1$. Hence, $B \leq \underline{1}-A$ or $A \leq \underline{1}-B$. Since $\underline{1}-B$ is an (r,s)-frso set. But, A is an (r,s)-gferwc set, hence $A \leq (\underline{1}-B) \Rightarrow eC_{T,T^*}(A,r,s) \leq (\underline{1}-B) \Rightarrow B \leq (\underline{1} - eC_{T,T^*}(A,r,s)) \Rightarrow B \leq (\underline{1} - eC_{TT^*}(A,r,s)) \wedge (eC_{T,T^*}(A,r,s) - A) = \underline{0}$ and hence $B = \underline{0}$ which is a contradiction. Then $eC_{T,T^*}(A,r,s) - A$ does not contain any non-zero (r,s)-frso sets.

(ii) Suppose A is an (r,s)-gferwc set. So, for each $r \in I_0$ and $s \in I_1$ if A is an (r,s)-ferwc set then, $eC_{T,T^*}(A,r,s) - A = \underline{0}$ which is an (r,s)-rfc set.

Conversely, suppose $eC_{T,T^*}(A,r,s) - A$ is an (r,s)-frso set. Then by (i), $eC_{T,T^*}(A,r,s) - A$ does not contain any non-zero an (r,s)-frso set. But $eC_{T,T^*}(A,r,s) - A$ is an (r,s)-fro set, then $eC_{T,T^*}(A,r,s) - A = \underline{0} \Rightarrow A = eC_{T,T^*}(A,r,s)$. So, A is an (r,s)-rfc set.

(iii) Suppose C is an (r,s)-frso set such that $B \leq C$ and let A be an (r,s)-gferwc set such that $A \leq C, r \in I_0$ and $s \in I_1$. Then $eC_{T,T^*}(A,r,s) \leq C$, hence, $eC_{T,T^*}(B,r,s) \leq eC_{T,T^*}(eC_{T,T^*}(A,r,s),r,s) = eC_{T,T^*}(A,r,s) \leq B$. Therefore B is an (r,s)-gferwc set.

(iv) Suppose that B is an (r,s)-gferwc and D is an (r,s)-frso in I^X such that $B \leq D, r \in I_0, s \in I_1$. But $B \leq A \leq \underline{1}$, therefore $B \leq A$ and $B \leq D$. So $B \leq A \wedge D$. But B is an (r,s)-gferwc relative to A, $A \wedge$

$eC_{T,T^*}(B,r,s) \leq A \wedge D \Rightarrow A \wedge eC_{T,T^*}(B,r,s) \leq D$. Thus $(A \wedge eC_{T,T^*}(B,r,s)) \vee (\underline{1} - eC_{T,T^*}(B,r,s)) \leq D \vee (\underline{1} - eC_{T,T^*}(B,r,s)) \Rightarrow A \vee (\underline{1} - eC_{T,T^*}(B,r,s)) \leq D \vee (\underline{1} - eC_{T,T^*}(B,r,s))$. Since A is an (r,s)-fro and (r,s)-gferwc, then $eC_{T,T^*}(A,r,s) \leq D \vee (\underline{1} - B)$. Also, $B \leq A \Rightarrow eC_{T,T^*}(B,r,s) \leq eC_{T,T^*}(A,r,s)$. Thus $eC_{T,T^*}(B,r,s) \leq eC_{T,T^*}(A,r,s) \leq D \vee (\underline{1} - eC_{T,T^*}(B,r,s))$. Therefore, $eC_{T,T^*}(B,r,s) \leq D$ but, $eC_{T,T^*}(B,r,s)$ is not contained in $(\underline{1} - eC_{T,T^*}(B,r,s))$. Therefore, B is an (r,s)-gferwc relative to X.

(v) Directly from the fact that every (r,s)-rfo set is (r,s)-frso set.

Definition 16 *Let (X,T,T^*) and (Y,S,S^*) be dfts`s. If $A \leq \underline{1}_Y \leq \underline{1}_X$ such that A is an (r,s)-gferwc in $\underline{1}_X, r \in I_0$ and $s \in I_1$, then A is an (r,s)-gferwc relative to $\underline{1}_Y$.*

Proof. Suppose that (X,T,T^*) and (Y,S,S^*) are deft's such that $A \leq \underline{1}_Y \leq \underline{1}_X, r \in I_0, s \in I_1$ and A is an (r,s)-gferwc in I^X. Now, let $A \leq \underline{1}_Y \wedge B$ such that B is an (r,s)-rfo set in I^X. Since A is an (r,s)-gferwc in $I^X, A \leq B \Rightarrow eC_{T,T^*}(A,r,s) \leq B$. So that $\underline{1}_Y \wedge eC_{T,T^*}(A,r,s) \leq \underline{1}_Y \wedge B$. Hence A is an (r,s)-gferwc relative to Y.

Definition 17 *Let (X,T,T^*) be dfts`s. For each A and $B \in I^X, r \in I_0, s \in I_1$,*

- If A and B are (r,s)-gferwc, then $A \wedge B$ is an (r,s)-gferwc.
- If A is an (r,s)-gferwc and $T(1-B) \geq r, T^{å}(1-B) \leq s$, then $A \wedge B$ is an (r,s)-gferwc.

Proof. (i) Suppose that A and B are (r,s)-gferwc sets in I^X such that $A \wedge B \leq D$ for each an (r,s)-frso set $D \in I^X, r \in I_0$ and $s \in I_1$. Since A is an (r,s)-gferwc, $eC_{T,T^*}(A,r,s) \leq D$ for each an (r,s)-frso set $D \in I^X$ and $A \leq D$. Also, B is an (r,s)-gferwc, $eC_{T,T^*}(B,r,s) \leq D$ for each an (r,s)-frso set $D \in I^X$ and $B \leq D$. So, $eC_{T,T^*}(A,r,s) \wedge eC_{T,T^*}(B,r,s) \leq D$, whenever $A \wedge B \leq D$ that is $A \wedge B$ is an (r,s)-gferwc.

(ii) Since every an (r,s)-fc set is an (r,s)-gferwc and from (i) we get the proof.

Proposition 18 *Let (X,T,T^*) be dfts's. For each A and $B \in I^X, r \in I_0$ and $s \in I_1$,*

- If $T(A) \geq r$ and $T^{å}(A) \leq s$ such that A is an (r,s)-gfec set. Then A is an (r,s)-gferwc set.
- If A is both an (r,s)-fro and an (r,s)-gferwc set, then A is an (r,s)-fe-clopen
- If A is both an (r,s)-fro and an (r,s)-rgfc, then A is an (r,s)-gferwc set.
- If A is both an (r,s)-frso and an (r,s)-gferwc set, then A is an (r,s)-fec
- If A is both an (r,s)-frso and an (r,s)-gferwc set, such that $T(\underline{1} - B) \geq r$ and $T^*(\underline{1} - B) \leq s$.

Then $A \wedge B$ is an (r,s)-gferwc set.

Proof. (i) Suppose that $A \leq B$ and B is an (r,s)-frso in I^X such that $r \in I_0$ and $s \in I_1$. Since $T(A) \geq r$ and $T^*(A) \leq s$ such that A is an (r,s)-gfec set and $A \leq A$, then $eC_{T,T^*}(A,r,s) \leq A \Rightarrow eC_{T,T^*}(A,r,s) \leq B$. Hence, A is an (r,s)-gferwc set.

(ii) Suppose that A is an (r,s)-fro and an (r,s)-gferwc set, $r \in I_0$ and $s \in I_1$. Since every (r,s)-fro set is (r,s)-frso set and $A \leq A, eC_{T,T^*}(A,r,s) \leq A$. Also, $A \leq eC_{T,T^*}(A,r,s)$. Therefore, $A = eC_{T,T^*}(A,r,s)$, that is A is A is an (r,s)-fro set, hence, is (r,s)-feo set. Therefore, A is (r,s)-fe-clopen set.

(iii) Suppose that B is an (r,s)-frso in I^X such that $A \leq B, r \in I_0$ and $s \in I_1$. But, A is an (r,s)-fro and an (r,s)-rgfc. Then by (ii), $eC_{T,T^*}(A,r,s) \leq A$. Hence, $eC_{T,T^*}(A,r,s) \leq B$. So, A is an (r,s)-gferwc set.

(iv) Obvious

(v) Suppose that A is an (r,s)-frso and an (r,s)-gferwc set of $I^X, r \in I_0$ and $s \in I_1$. Then by (iv), A is an (r,s)-fec. But, $T(1-B) \geq r$ and $T^*(1-B) \leq s$, so $A \wedge B$ is an (r,s)-fec set. Therefore, $A \wedge B$ is an (r,s)-gferwc set.

IV. Interrelations

The following implication illustrates the relationships between different fuzzy sets:

(r,s)-fc→ (r,s)-frwc → (r,s)-gferwc → (r,s)-gfec

Example 19 *Let $X = \{x,y,z\}$ and let B be fuzzy set defined as follows: $B(x) = 0.7, B(y) = 0.7, B(z) = 0.7$.*

$$T(A) = \begin{cases} 1 \text{ If A} \in \{\underline{0},\underline{1}\} \\ 1/2 \text{ If A} = \text{B} \\ 0 \text{ Otherwise} \end{cases} \quad T^*(A) = \begin{cases} 0 \text{ If A} \in \{\underline{0},\underline{1}\} \\ 1/2 \text{ If A} = \text{B} \\ 1 \text{ Otherwise} \end{cases}$$

Then B is an $(\frac{1}{2},\frac{1}{2})$-frwc set, but not an $(\frac{1}{2},\frac{1}{2})$-fc set.

Example 20 *Let* $X = \{x, y, z\}$ *and let* B *be fuzzy set defined as follows:* $B(x) = 0.3, B(y) = 0.3, B(z) = 0.3$.

$$T(A) = \begin{cases} 1 \text{ If A} \in \{\underline{0}, \underline{1}\} \\ 1/2 \text{ If A} = \text{B} \\ 0 \text{ Otherwise} \end{cases} \quad T^*(A) = \begin{cases} 0 \text{ If A} \in \{\underline{0}, \underline{1}\} \\ 1/2 \text{ If A} = \text{B} \\ 1 \text{ Otherwise} \end{cases}$$

Then B is an $(\frac{1}{2}, \frac{1}{2})$-gferwc set, but not an $(\frac{1}{2}, \frac{1}{2})$-frwc set.

Example 21 *Let* $X = \{x, y, z\}$ *and let* B *and* C *are fuzzy set defined as follows:* $B(x) = 0.1, B(y) = 0.4$*;* $C(x) = 0.3, C(y) = 0.2$*;*$D(x) = 0.1, D(y) = 0.2$*;* $E(x) = 0.3, E(y) = 0.3$*;* $F(x) = 0.7, F(y) = 0.6$*;* $G(x) = 0.4, G(y) = 0.3$

$$T(A) = \begin{cases} 1 \text{ If A} \in \{\underline{0}, \underline{1}\} \\ 1/2 \text{ If A} = \text{B, C, D, E, F} \\ 0 \text{ Otherwise} \end{cases}$$

$$\square^*(\square) = \begin{cases} 0 \text{ If A} \in \{\underline{0}, \underline{1}\} \\ 1/2 \text{ If A} = \text{B, C, D, E, F} \\ 1 \text{ Otherwise} \end{cases}$$

Then □ is an $(\frac{1}{2}, \frac{1}{2})$-gfec set, but not an $(\frac{1}{2}, \frac{1}{2})$-gferwc set.

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Sentiment Analysis in Stock Market-A Survey

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Abstract: *The field of sentiment analysis, within which sentiment is gathered, analyzed, and collective from text, has seen plenty of attention within theprevious couple of years. The corresponding growth of the field has resulted in the emergence of various subareas such as stock market prediction, products' review collection, etc. Sentiment Analysis or opinion mining in a new way of machine learning technique to analyze people's opinions from user generated content. This can be used to predict the mood of the people that have an impact on stock prices which in turn help to predict the actual stock movement. This survey focus on the various methodologies used in the related papers to predict the stock movement.*
Keywords*: sentiment analysis,Baseline Naïve Bayes,Artificial neural network,Support Vector Machine,Natural Language Processing*

I. Introduction

Sentiment Analysis (SA) or Opinion Mining (OM) is the study of people's opinions, attitudes and emotions toward an entity. The entity can represent individuals, events or topics. Ideas and opinions of others have continuously affected our own opinions. Our day to day activities has always been influenced by what people think. The explosion of the internet has led to increased activity in Podcasting, Blogging, Tagging, Contributing to RSS, Social Bookmarking, and Social Networking. As a result there has been an eruption of interest in individuals to mine these immense resources of data for opinions. Sentiment analysis is usually used in opinion mining to recognize a sentiment, affect, subjectivity, and other emotional states in online text.

Mainly there are two approaches that is used in sentiment analysis. One is the computational learning techniques and the other is semantic approach. **Semantic approach** is characterized by using the lexicons(dictionaries of words) with semantic orientation of polarity or opinion. The **computational learning is based** on training a classifier using any supervised learning algorithm from a collection of annotated texts, where each text is usually represented by a vector of words (bag of words), n-grams or skip-grams, in combination with other types of semantic features that attempt to model the syntactic structure of sentences, intensification, negation, subjectivity or irony. Various techniques can be used but the most popular techniques are the classifiers based on Support Vector Machines, Naïve Bayes Classifier and K-Nearest Neighbor. More advanced techniques are used in the most recent investigations, such as Latent Semantic Analysis and Deep Learning.

In Financial sector the stock market and its trends are extremely unstable in nature. It gives lot of opportunities for the researchers to capture the volatility and predicting its next moves. It is a tedious work for an individual to consider all the past and present data for predicting the future trend of a stock

II.Sentiment Analysis

Sentiment analysis is contextual mining of text, like posts and reviews, uploaded by users on microblogging platforms, forums, and electronic businesses, regarding the opinions they have about a product, service, event, person or idea and extracts subjective information in source material, and helping a business to understand the social sentiment of their brand, product or service while monitoring online conversations. Based on the dataset and the reason, Sentiment Classification can be binary (positive or negative) or multi-class (3 or more classes) problem.

In addition, among researchers and stakeholders, the people can find either similar or completely different opinions concerning the relation between emotion detection and sentiment analysis, depending on their perspective.

Sentiment analysis can be applied for the three different levels of scope:

- **Document level** sentiment analysis classifies the opinion as positive, negative or neutral by considering a complete document or paragraph.

- **Sentence level** sentiment analysis classifies the opinion by considering a single sentence.
- **Sub-sentence level** sentiment analysis classifies the sentiment of sub-expressions within a sentence.

III. Methodology

There are three types of techniques for sentiment analysis. They are as follows.

(i).Machine Learning

This approach employs a machine-learning technique to build a classifier that can identify text that expresses sentiment. Nowadays, deep-learning methods are popular because they fit on data learning representations.

(ii).Lexicon-Based

This method uses a sentiment dictionary to determine the general assessment score of a given content. The main advantage of this technique is that it does not need any training data, while its disadvantage is that a large number of words and expressions are not included in sentiment dictionary.

(iii).Hybrid

The combination of machine learning and lexicon-based approaches to address Sentiment Analysis is called Hybrid. This method has high potential to improve the sentiment classification performance.

A. Steps in Sentiment Analysis:

The sentiment analysis is a multifaceted process involving 5 different steps to analyze sentiment data. These steps are:

(i). data collection: The first step is to collect data from various types of user generated content contained in blogs, forums, social networks. These data are unsystematic, expressed in various ways by using different vocabularies, slangs, context of writing etc. It is difficult to do Manual analysis. Therefore, text analytics and natural language processing are used to extract and classify

(ii). text preparation: This process consists of cleaning the extracted data .Irrelevant are recognized and eliminated.

(iii).sentiment detection: The sentences that are extracted from the reviews and opinions are examined. Sentences such as opinions, beliefs and views (subjective communication) are retained and sentences such as facts, factual information (objective communication) are discarded.

(iv). sentiment classification: In this step, subjective sentences are classified in positive, negative, good, bad; like, dislike, but classification can be made by using multiple points.

(v). presentation of output: The main purpose of sentiment analysis is to convert unstructured text into significant information. When the analysis is finished, the text results are displayed visually.

IV. Application and Challenges

Opinion mining or Sentiment analysis are used in various applications such as Online Advertising, Dissatisfaction over online advertising, Movie Review, Online commerce, policy or Government-regulation proposals.

The following are the various challenges in sentiment analysis

(i). Identifying the objects:

It is a difficult task to identify the objectivity of the sentence for classifying identity of the sentence.

Synonyms grouping:

The same feature of the object can be used to refer the same feature of the object. It is difficult to identify the synonym and group together.

(ii). Change of opinions with Time:

Another issue is to examine the opinions that change with the course of time. This helps in observing the improvement of certain product with time, or people change their opinion about a product and get swayed for it with time.

(iii). Document level Sentiment analysis:

It is a tedious work to decide the polarity of the sentence since it contains both positive and negative opinions in the same document.

(iv).Sentences with Mixed Reviews:

A bigger challenge is that when people express positive and negative review in the same sentence. People combine the both positive and negative opinions in the same sentences. Such sentences can be difficult to parse for sentiment analysis.

(v).Misleading Opinions due to spam opinion:

Unfair opinions/reviews affect opinion mining about a product or service. Detecting such opinions is significant for the utilization of opinion mining practically.

(vi).Sentiment analysis in Stock Market:

Nowadays, the era of Internet has changed the way of expressing the views, opinions by the people. This is mostly done through blog posts, online forums, product review websites, social media ,etc. These days, millions of people are using social networking sites to communicate their emotions, opinion and share views about their daily lives.

People mostly depend upon user generated content over online to a great extent for decision making. If someone wants to buy a stock, then they firstly look up its reviews online, discuss about it on social media before taking a decision. As the data generated by stock market increase tremendously every day, it is a tedious process for an individual to consider all the present and past information for predicting future movement of a stock. There are two approaches such as Technical analysis and Fundamental analysis for forecasting market trends. Technical analysis considers past price and volume to forecast the future movement whereas Fundamental analysis of a business involves analyzing its financial data to get some insights.
Stock price trend can be predicted more accurately by using sentiment analysis. Hence Now a days it is an active research area. Therefore, in recent years, considerable efforts have been made in developing models for predicting the future trend of a stock market.
Following is discussion on the previous research work carried on sentiment analysis of text data in predicting the stock market trends.

TABLE 1. VARIOUS MODELS OR ALGORITHMS USED FOR SENTIMENT ANALYSIS

Paper	**Media Source**	**Model/Algorithm Used**	**Summary**	**Performance**
Integrating StockTwits with Sentiment Analysis for better Prediction of Stock Price Movement.[5]	Stock Tweets, Yahoo Finance	Support Vector Machine	In this paper, tweets are extracted from StockTwits through pipeline API, processed them for Natural Language Processing (NLP) and sentiment analysis. After that SVM is applied to predict the sentiment of each tweet. After predicting sentiment extracted historical data is extracted from Yahoo Finance. A model is developed for stock market prediction using stock price data and sentiment score to predict the change in stock market.	Accuracy for training is 75.22% and Test is 76.68%
Stock Market Classification Model Using Sentiment Analysis on Twitter Based on Hybrid Naive Bayes Classifiers.[6]	Twitter data	Baseline Naïve Bayes, Multinomial Naïve Bayes and Semi-Supervised Naïve Bayes	The classification model used in this paper consists of four phases. The first phase l is data collection, and the second involves the filtration and transformation, which are conducted to get only relevant data. The most important phase is labelling, in which polarity of data is determined and negative, positive or neutral values are assigned to people opinion. The last phase is the classification phase in which suitable patterns of the stock market are identified by hybridizing Naïve Bayes Classifiers (NBCs), and the final phase is the performance and evaluation.	It has achieved accuracy equals 90.38%.
News analytics and Dual sentiment analysis for stock market prediction.[7]	Tweets from twitter and historical data from Yahoo finance	Naivesbayes classifier, Support vector machine, Random forest, Maximum entropy modeling	This paper implements the dual sentiment analysis by collecting textual information regarding news of a particular entity which is available in Twitter and historical data from yahoo finance. Then data expansion is done by creating a reverse text for an original text. Sentiment classifier is trained by using original and reversed text. Dual Prediction is made by considering two sides of one text. Sentimental classification process involves measuring the polarity expressed in the sentence, whether a sentence is expressing positive, negative or neutral sentiment towards	High performance canbe achieved

			the subject, and is performed by machine learning models such as SVM, Naivesbayes, Maximum entropy, Random forest. Finally, the results are delivered in a visualization format which determines the accuracy of classifiers and comparison of different classifiers are made.	
Sentiment analysis for online stock market news using RSS feeds.[8]	News	Natural Language Processing	This paper builds a predictive model to predict sentiment around stock news. First the relevant real time RSS stock news has been filtered from ARAB Bank Company and then they have been analyzed to predict the sentiments score values whether it is positive, negative or neutral. Finally sentiment polarity news provides an efficient result to the stock marketers when to buy or sell their stocks.	The proposed model provides 89.5% accuracy for positive ,84% for negative and 81.8% for neutral polarity
Sentiment Analysis for Effective Stock Market Prediction.[9]	RSS News Feeds	Sentence level sentiment score (SSS) algorithm	The system explores the task of automatic identification of news opinions with the help of RSS news feeds and predicts the stock market movement whether goes up or down.	78.75% Accuracy
Using Sentimental Analysis in Prediction of StockMarket Investment. [10]	Apple, Microsoft, Oracle, Google and Facebook in form of comments and tweets from Stock Twits.	Artificial neural network	This paper analyses the data on the basis of polarity indexes Such as happy, up, down and rejected to compute the overall index of the sentiments and to determine whether the sentiments are positive, negative or neutral towards a specific company. The overall index values and the market data of that company became the inputs which are passed to an artificial neural network to train and predict.	Error rate while performing all implementation is reduce to the least
Sentiment Analysis of Twitter Data forPredicting Stock Market Movements. [11]	Tweets	Word2vec and Ngram, for analyzing the public sentiments in tweets. Machine Learning algorithms such as Logistic regression algorithm,LibSVM are used	In this paper sentiment analysis and supervised machine learning principles are applied to the tweets extracted from twitter and correlation between stock market movements of a company and sentiments in tweets are analyzed to predict the stock trend	70 % Accuracy can be obtained in this model
Stock Trend Prediction Relying on Text Mining and Sentiment Analysis withTweets.[12]	Tweets	Support Vector Machine	This paper uses a text-sentiment-based stock trend prediction model. SentiWordNet is used to give the weight to the selected features and then uses the weighted example sets to train a support vector machine (SVM) classification model.	This model yields the best performance of 90.34%.
Stock Prediction Using Event-based Sentiment Analysis.[13]	social media sources	Sentiment Classifiers like Rocchio classifier is used	A novel method for estimating sentiment based on twitter posts and use the sentiment to predict future stock market movement. Training data is generated automatically based on events related with stock markets	A high precision is yielded.
Stock Trend Forecasting Method Based on Sentiment Analysis and System Similarity Model .[14]	Internet Stock News	Bayesian model,System Similarity model	Bayesian model is adopted to classify the stock information, based on system similar model, emotion analysis is used to forecast the price trends in finance boards.	The precision of this system is above 70%

V. Conclusion

Sentiment analysis plays a vital role in predicting the future of the stock trends. Opinions of the people play a vital role in predicting stock market trends. It is a challenging task to measure the sentiment using sentiment

and to classify the positive negative andneutral emotions from those statements. Social media which reflects the opinions has become very important in predicting the stock market returns. The survey of the previous research explicates the results and performance of the various algorithms or the models used in predicting the stock trends.

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Separation Axioms of Semi $^{\#}$Generalized α -Closed Sets in Topological Spaces

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Abstract: *In this Paper, we introduce new type of spaces namely* $\boldsymbol{s^{\#}T_v}$ *space,* $\boldsymbol{s^{\#\#}T_v}$ *space,* $\boldsymbol{s^{\#\#\#}T_v}$ *space,* $\boldsymbol{s^{\#}T_b}$ *space,* $\boldsymbol{s_\alpha T_{\frac{1}{2}}^{**}}$ *space,* $\boldsymbol{s^{*}T_p^{\#}}$ *space and* $\boldsymbol{s^{\#}T_{g\zeta^*}}$ *space via semi* $^{\#}g\boldsymbol{\alpha}$*-closed sets and establish the relation among them in topological spaces.*

Keywords: $\boldsymbol{s^{\#}T_v}$ space, $\boldsymbol{s^{\#\#}T_v}$ space, $\boldsymbol{s^{\#\#\#}T_v}$ space, $\boldsymbol{s^{\#}T_b}$ space, $\boldsymbol{s_\alpha T_{\frac{1}{2}}^{**}}$ space, $\boldsymbol{s^{*}T_p^{\#}}$ space, $s^{\#}\boldsymbol{T_{g\zeta^*}}$ space

I. Introduction

The main objective of this paper is to define some new spaces in topological spaces and investigate their nature. In the year 2013, V.Kokilavani and M.Vivek Prabu [10] introduced semi $^{\#}g\alpha$-closed sets, semi $^{\#}g\alpha$-continuous functions and semi $^{\#}g\alpha$-irresolute functions in topological spaces. In this paper, with the help of semi $^{\#}g\alpha$-closed sets we introduce separation axioms in topological spaces namely $s^{\#}T_v$ space, $s^{\#\#}T_v$ space, $s^{\#\#\#}T_v$ space, $s^{\#}T_b$ space, $s_\alpha T_{\frac{1}{2}}^{**}$ space, $s^{*}T_p^{\#}$ space and $s^{\#}T_{g\zeta^*}$ space. We also establish the relation among them and examine some of their properties.

II. Preliminaries

Definition 2.1: A subset A of X is called

(i) g-closed [12] if cl(A) ⊆ U, whenever A ⊆ U and U is open in X.

(ii) $g^{\#}\alpha$-closed [17] if αcl(A) ⊆ U, whenever A ⊆ U and U is g-open in X.

(iii) $^{\#}g\alpha$-closed [7] if αcl(A) ⊆ U, whenever A ⊆ U and U is $g^{\#}\alpha$-open in X.

(iv) semi $^{\#}g\alpha$-closed [10] if scl(A) ⊆ U, whenever A ⊆ U and U is $^{\#}g\alpha$-open in X.

Definition 2.2: A topological space X is said to be a

(i) α-space [15] if every α-closed set in it is closed.

(ii) semi-space [1] if every semi-closed set in it is closed.

(iii) $T_{\frac{1}{2}}$ space [12] if every g-closed set in it is closed.

(iv) $\alpha T_{\frac{1}{2}}$ space [13] if every gα-closed set in it is closed.

(v) T_b space [3] if every gs-closed set in it is closed.

(vi) αT_b space [6] if every αg-closed set in it is closed.

(vii) $\alpha^{\#}T_{\frac{1}{2}}$ space [17] if every $g^{\#}\alpha$-closed set in it is α-closed.

(viii) $\alpha^{\#\#\#}T_b$ space [7] if every $^{\#}g\alpha$-closed set in it is α-closed.

(ix) T_c^{**} space [20] if every gs-closed set in it is $^{*}g\alpha$-closed.

(x) semi-T_b space [3] if every gs-closed set in it is semi-closed.

(xi) strongly semi *-T_p space [18] if every strongly g^{*}s-closed set in it is closed.

(xii) strongly semi *-$T_{\frac{1}{2}}$ space [18] if every strongly g^{*}s-closed set in it is semi-closed.

(xiii) $\alpha T_{\frac{1}{2}}^{**}$ space [20] if every $^{*}g\alpha$-closed set in it is closed.

(xiv) $\widehat{T_{g\zeta^*}}$ space [9] if every $g\zeta^{*}$-closed set in it is α-closed.

(xv) $T_{g\zeta^*}$ space [9] if every gs-closed set in it is $g\zeta^*$-closed.

III. Separation Axioms Using Semi $^{\#}g\alpha$-Closed Set

In this section, we define some new type of spaces using semi $^{\#}g\alpha$-closed sets and examine their properties. We also analyze the relationship between these spaces and the other existing spaces.

Definition 3.1: A topological space X is said to be a
(i) $s^{\#}T_v$ space if every semi $^{\#}g\alpha$-closed set in it is closed.
(ii) $s^{\#\#}T_v$ space if every semi $^{\#}g\alpha$-closed set in it is α-closed.
(iii) $s^{\#\#\#}T_v$ space if every semi $^{\#}g\alpha$-closed set in it is semi-closed.
(iv) $s^{\#}T_b$ space if every gs-closed set in it is semi $^{\#}g\alpha$-closed.
(v) $s_\alpha T_{\frac{1}{2}}^{**}$ space if every semi $^{\#}g\alpha$-closed set in it is $^{*}g\alpha$-closed.
(vi) $s^{*}T_p^{\#}$ space if every semi $^{\#}g\alpha$-closed set in it is strongly $g^{*}s$-closed.
(vii) $s^{\#}T_{g\zeta^*}$ space if every semi $^{\#}g\alpha$-closed set in it is $g\zeta^*$-closed.

Theorem 3.2: Every T_b space is a $s^{\#}T_v$ space (resp. $s^{\#\#}T_v$ space and $s^{\#\#\#}T_v$ space).
Proof. Let A be a semi $^{\#}g\alpha$-closed set in (X, τ). We know that every semi $^{\#}g\alpha$-closed set is gs-closed [10], A is gs-closed. Since (X, τ) is a T_b space, A is closed (and hence α-closed and semi-closed). Thus (X, τ) is a $s^{\#}T_v$ space (resp. $s^{\#\#}T_v$ space and $s^{\#\#\#}T_v$ space).

Theorem 3.3: Every T_b (resp. T_c^{**} and $s^{\#}T_{g\zeta^*}$) space is a $s^{\#}T_b$ space.
Proof. Let A be a gs-closed set in (X,τ). Since (X,τ) is a T_b (resp. T_c^{**} and $s^{\#}T_{g\zeta^*}$) space, A is closed (resp. $^{*}g\alpha$-closed and $g\zeta^*$-closed). Also since every closed (resp. $^{*}g\alpha$-closed and $g\zeta^*$-closed) set is semi $^{\#}g\alpha$-closed [10], A is semi $^{\#}g\alpha$-closed. Hence (X,τ) is a $s^{\#}T_b$ space.

Theorem 3.4: Every semi-T_b space is a $s^{\#}T_b$ space.
Proof. Let A be a gs-closed set in (X,τ). Since (X,τ) is a semi- T_b space, A is semi-closed. But every semi-closed set is semi $^{\#}g\alpha$-closed [10]. Hence (X,τ) is a $s^{\#}T_b$ space.

Theorem 3.5: Every semi- T_b space is a $s^{\#\#\#}T_v$ space.
Proof. Let A be a semi $^{\#}g\alpha$-closed set in (X,τ). We know that every semi $^{\#}g\alpha$-closed set is a gs-closed set [10]. Since (X,τ) is a semi- T_b space, A is semi-closed. Therefore (X,τ) is a $s^{\#\#\#}T_v$ space.

Theorem 3.6: Every $s^{\#}T_v$ space is a $s^{\#\#}T_v$ space (resp. $s^{\#\#\#}T_v$ space).
Proof. Let A be a semi $^{\#}g\alpha$-closed set in (X,τ). Since (X,τ) is a $s^{\#}T_v$ space, A is closed and hence it is α-closed [16] (and semi-closed [11]). Thus (X,τ) is a $s^{\#\#}T_v$ space (resp. $s^{\#\#\#}T_v$ space).

Theorem 3.7: Every $s^{\#}T_v$ space is a $s_\alpha T_{\frac{1}{2}}^{**}$ space (resp. $s^{*}T_p^{\#}$ space).

Proof. Let A be a semi $^{\#}g\alpha$-closed set in (X,τ). Since (X,τ) is a $s^{\#}T_v$ space, A is closed and hence it is $^{*}g\alpha$-closed [20] (resp. strongly g*s-closed [18]). Thus (X,τ) is a $s_\alpha T_{\frac{1}{2}}^{**}$ space.

Theorem 3.8: Every $s^{\#\#\#}T_v$ space is a $s^{*}T_p^{\#}$ space.
Proof. Let A be a semi $^{\#}g\alpha$-closed set in (X,τ). Since (X,τ) is a $s^{\#\#\#}T_v$ space, A is semi-closed and so it is strongly $g^{*}s$-closed [18]. Thus (X,τ) is a $s^{*}T_p^{\#}$ space.

Theorem 3.9: Every $s^{\#\#}T_v$ space is a $s^{\#\#\#}T_v$ space.
Proof. Let A be a semi $^{\#}g\alpha$-closed set in (X,τ). Since (X,τ) is a $s^{\#\#}T_v$ space, A is α-closed and hence it is semi-closed [11]. Thus(X,τ) is a $s^{\#\#\#}T_v$ space.

The following examples show that the concepts of $s^{\#}T_v$ (resp. $s^{\#\#}T_v$) spaces and $\alpha T_{\frac{1}{2}}$ spaces, $s^{\#}T_v$ (resp. $s^{\#\#}T_v$) spaces and αT_b spaces, $s^{\#\#\#}T_v$ (resp. $s^{\#\#\#}T_v$) spaces and $T_{\frac{1}{2}}$ spaces, $s^{\#}T_v$ (resp. $s^{\#\#}T_v$) spaces and $\alpha^{\#\#\#}T_b$ spaces are independent of each other respectively.

Example 3.10: Let X = {a, b, c}, τ = {∅, X, {a}, {b,c}}, Closed sets in (X,τ) = α-closed sets in (X,τ) = Semi $^{\#}g\alpha$-closed sets in (X,τ) = {∅, X, {a}, {b,c}}, g-closed sets in (X,τ) = $g\alpha$-closed sets in (X,τ) = αg-closed sets in (X,τ) = $^{\#}g\alpha$-closed sets in (X,τ) = P(X). Clearly (X,τ) is $s^{\#}T_v$ and $s^{\#\#}T_v$ spaces but not $T_{\frac{1}{2}}$, $\alpha T_{\frac{1}{2}}$, αT_b and $\alpha^{\#\#\#}T_b$ spaces.

Example 3.11: Let X = {a, b, c}, τ = {∅, X, {a}, {b}, {a,b}}, Closed sets in (X,τ) = g-closed sets in (X,τ) = gα-closed sets in (X,τ) = αg-closed sets in (X,τ) = $^{\#}$gα-closed sets in (X,τ) = {∅, X, {c}, {a,c}, {b,c}}, Semi $^{\#}$gα-closed = {∅, X, {a}, {b}, {c}, {a,c}, {b,c}}. Here (X,τ) is $T_{\frac{1}{2}}$, $\alpha T_{\frac{1}{2}}$, αT_b and $\alpha^{\#\#\#}T_b$ spaces but it is not s$^{\#}$T$_v$ and s$^{\#\#}$T$_v$ spaces.

The concepts of s$^{\#\#}$T$_v$ spaces and $\alpha^{\#}T_{\frac{1}{2}}$ spaces are independent of each other which can be observed from the following two examples.

Example 3.12: Let X = {a, b, c}, τ = {∅, X, {a}, {a,b}}, α-closed sets in (X,τ) = Semi $^{\#}$gα-closed sets in (X,τ) = {∅, X, {b}, {c}, {b,c}}, g$^{\#}\alpha$-closed sets in (X,τ) = {∅, X, {b}, {c}, {a,c}, {b,c}}. Here (X,τ) is a s$^{\#\#}$T$_v$ space but not an $\alpha^{\#}T_{\frac{1}{2}}$ space.

Example 3.13: Let X = {a, b, c}, τ = {∅, X, {a}, {b}, {a,b}}, α-closed sets in (X,τ) = g$^{\#}\alpha$-closed sets in (X,τ) = {∅, X, {c}, {a,c}, {b,c}}, Semi $^{\#}$gα-closed sets in (X,τ) = {∅, X, {a}, {b}, {c}, {a,c}, {b,c}}. Here (X,τ) is an $\alpha^{\#}T_{\frac{1}{2}}$ space but it is not a s$^{\#\#}$T$_v$ space.

Theorem 3.14: In a topological space (X, τ), the following conditions are equivalent.

(i) (X, τ) is both $s_{\alpha}T_{\frac{1}{2}}^{**}$ and $\alpha T_{\frac{1}{2}}^{**}$ (resp. s$^{*}T_p^{\#}$ and strongly semi *-T_p) spaces.
(ii) (X, τ) is a s$^{\#}$T$_v$ space.

Proof. (i) ⟹ (ii): Let A be a semi $^{\#}$gα-closed set in (X, τ). Since (X, τ) is a $s_{\alpha}T_{\frac{1}{2}}^{**}$ (resp. s$^{*}T_p^{\#}$) space, A is *gα-closed (resp. strongly g*s-closed) and since (X, τ) is a $\alpha T_{\frac{1}{2}}^{**}$ (resp. strongly semi *-T_p) space, A is closed. Therefore (X, τ) is a s$^{\#}$T$_v$ space.

(ii) ⟹ (i): Let A be a closed set and B be a semi $^{\#}$gα-closed set in (X, τ). Since every closed set is semi $^{\#}$gα-closed set in (X, τ) [10], we have A ⊆ B. Also since (X, τ) is a s$^{\#}$T$_v$ space, every semi $^{\#}$gα-closed set is closed in (X, τ), (i,e) B ⊆ A. Hence A = B. Let C be a *gα-closed (resp. strongly g*s-closed) set in (X, τ). Since every closed set is *gα-closed [20] (resp. strongly g*s-closed [18]) and every *gα-closed (resp. strongly g*s-closed) set is semi $^{\#}$gα-closed [10], we have A ⊆ C ⊆ B. But A = B implies B ⊆ C. Thus A = B = C. (i,e) every semi $^{\#}$gα-closed set is *gα-closed (resp. strongly g*s-closed) and every *gα-closed (resp. strongly g*s-closed) set is closed in (X, τ). Therefore (X, τ) is both $s_{\alpha}T_{\frac{1}{2}}^{**}$ and $\alpha T_{\frac{1}{2}}^{**}$ (resp. s$^{*}T_p^{\#}$ and strongly semi *-T_p) spaces.

Theorem 3.15: In a topological space (X, τ), the following conditions are equivalent.

(i) (X, τ) is both s$^{\#}T_{g\zeta^{*}}$ and $\widehat{T_{g\zeta^{*}}}$ spaces.
(ii) (X, τ) is a s$^{\#\#}$T$_v$ space.

Proof. (i) ⟹ (ii): Let A be a semi $^{\#}$gα-closed set in (X, τ). Since (X, τ) is a s$^{\#}T_{g\zeta^{*}}$ space, A is $g\zeta^{*}$-closed and since (X, τ) is a $\widehat{T_{g\zeta^{*}}}$ space, A is α-closed. Thus (X, τ) is a s$^{\#\#}$T$_v$ space.

(ii) ⟹ (i): Let A be a α-closed set and B be a semi $^{\#}$gα-closed set in (X, τ). Since every α-closed set is semi $^{\#}$gα-closed set in (X, τ) [10], we have A ⊆ B. Also since (X, τ) is a s$^{\#\#}$T$_v$ space, every semi $^{\#}$gα-closed set is α-closed in (X, τ), (i,e) B ⊆ A. Hence A = B. Let C be a $g\zeta^{*}$-closed set in (X, τ). Since every α-closed set is $g\zeta^{*}$-closed [9] and every $g\zeta^{*}$-closed set is semi $^{\#}$gα-closed [10], we have A ⊆ C ⊆ B. But A = B implies B ⊆ C. Thus A = B = C. (i,e) every semi $^{\#}$gα-closed set is $g\zeta^{*}$-closed and every $g\zeta^{*}$-closed set is α-closed in (X, τ). Therefore (X, τ) is both s$^{\#}T_{g\zeta^{*}}$ and $\widehat{T_{g\zeta^{*}}}$ spaces.

Theorem 3.16: In a topological space (X, τ), the following conditions are equivalent.
(i) (X, τ) is both s$^{*}T_p^{\#}$ and strongly semi *-$T_{\frac{1}{2}}$ spaces.
(ii) (X, τ) is a s$^{\#\#\#}$T$_v$ space.
Proof. (i) ⟹ (ii): Let A be a semi $^{\#}$gα-closed set in (X, τ). Since (X, τ) is a s$^{*}T_p^{\#}$ space, A is strongly g*s-closed and since (X, τ) is a strongly semi *-$T_{\frac{1}{2}}$ space, A is semi-closed. Thus (X, τ) is a s$^{\#\#\#}$T$_v$ space.

(ii) $\Longrightarrow$ (i): Let A be a semi-closed set and B be a semi $^{\#}g\alpha$-closed set in (X, τ). Since every semi-closed set is semi $^{\#}g\alpha$-closed set in (X, τ) [10], we have A $\subseteq$ B. Also since (X, τ) is a $s^{\#\#\#}T_v$ space, every semi $^{\#}g\alpha$-closed set is semi-closed in (X, τ), (i,e) B $\subseteq$ A. Hence A = B. Let C be a strongly g^*s-closed set in (X, τ). Since every semi-closed set is strongly g^*s-closed [18] and every strongly g^*s-closed set is semi $^{\#}g\alpha$-closed [10], we have A $\subseteq$ C $\subseteq$ B. But A = B implies B $\subseteq$ C. Thus A = B = C. (i,e) every semi $^{\#}g\alpha$-closed set is strongly g^*s-closed and every strongly g^*s-closed set is semi-closed in (X, τ). Therefore (X, τ) is both $s^*T_p^{\#}$ and strongly semi *-$T_{\frac{1}{2}}$ spaces.

Theorem 3.17: In a topological space (X, τ), the following conditions are equivalent.
(i) (X, τ) is both $s_\alpha T_{\frac{1}{2}}^{**}$ and T_b spaces.
(ii) (X, τ) is a $s^{\#}T_v$ space.

Proof. (i) $\Longrightarrow$ (ii): Let A be a semi $^{\#}g\alpha$-closed set in (X, τ). Since (X, τ) is a $s_\alpha T_{\frac{1}{2}}^{**}$ space, A is $^*g\alpha$-closed and hence gs-closed. Also since (X, τ) is a T_b space, A is closed. Thus (X, τ) is a $s^{\#}T_v$ space.

(ii) $\Longrightarrow$ (i): Let A be a closed set and B be a semi $^{\#}g\alpha$-closed set in (X, τ). Since every closed set is semi $^{\#}g\alpha$-closed set in (X, τ) [10], we have A $\subseteq$ B. Also since (X, τ) is a $s^{\#}T_v$ space, every semi $^{\#}g\alpha$-closed set is closed in (X, τ), (i,e) B $\subseteq$ A. Hence A = B. Let C be a $^*g\alpha$-closed set in (X, τ). Since every closed set is $^*g\alpha$-closed [20] and every $^*g\alpha$-closed set is semi $^{\#}g\alpha$-closed [10], we have A $\subseteq$ C $\subseteq$ B.
But A = B implies B $\subseteq$ C. Thus A = B = C. (i,e) every semi $^{\#}g\alpha$-closed set is $^*g\alpha$-closed and every $^*g\alpha$-closed set is closed in (X, τ). Therefore (X, τ) is both $s_\alpha T_{\frac{1}{2}}^{**}$ and T_b spaces.

Theorem 3.18: In a topological space (X, τ), the following conditions are equivalent.

(i) (X, τ) is both $s^{\#}T_v$ and $s^{\#\#\#}T_v$ spaces.
(ii) (X, τ) is a semi-T_b space.

Proof. (i) $\Longrightarrow$ (ii): Let A be a gs-closed set in (X, τ). Since (X, τ) is a $s^{\#}T_v$ space, A is semi $^{\#}g\alpha$-closed. Also since (X, τ) is a $s^{\#\#\#}T_v$ space, A is semi-closed. Thus (X, τ) is a semi-T_b space.

(ii) $\Longrightarrow$ (i): Let A be a semi-closed set and B be a gs-closed set in (X, τ). Since every semi $^{\#}g\alpha$-closed set is gs-closed set in (X, τ) [10], we have A $\subseteq$ B. Also since (X, τ) is a semi-T_b space, every gs-closed set is semi-closed in (X, τ), (i,e) B $\subseteq$ A. Hence A = B. Let C be a semi $^{\#}g\alpha$-closed set in (X, τ). Since every semi-closed set is semi $^{\#}g\alpha$-closed and every semi $^{\#}g\alpha$-closed set is gs-closed [10], we have A $\subseteq$ C $\subseteq$ B. But A = B implies B $\subseteq$ C. Thus A = B = C. (i,e) every gs-closed set is semi $^{\#}g\alpha$-closed and every semi $^{\#}g\alpha$-closed set is semi-closed in (X, τ). Therefore (X, τ) is both $s^{\#}T_v$ and $s^{\#\#\#}T_v$ spaces.

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ON NANO g$^{\#}\alpha$ CLOSED SETS IN NANO TOPOLOGICAL SPACES

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Abstract*: The aim of this paper is to introduce and investigate nano* g$^{\#}$ *α-closed sets in nano topological spaces and also study their fundamental properties. We also present Nano* g$^{\#}$ *α-continuous maps with some of their properties.*

Keywords*: Nano* g$^{\#}$ *α-closed sets, Nano* g$^{\#}$ *α-closure, Nano* g$^{\#}$ *α-interior, Nano* g$^{\#}$ *α-neighbourhood, Nano* g$^{\#}$ *α-continuous function.*

I. Introduction and Preliminaries:

Lellis Thivagar et al [5] introduced a nano topological space with respect to a subset X of an universe which is defined in terms of lower approximation, upper approximation and boundary region. K. Nono [8] introduced the concept of g$^{\#}\alpha$ - closed sets to investigate some topological properties. The purpose of this paper is to present the concept of N$g^{\#}\alpha$ – closed sets and study their properties in nano topological space. We likewise present Nano g$^{\#}\alpha$ continuous maps by utilizing nano g$^{\#}\alpha$ closed sets and concentrate their fundamental properties.

Throughout this paper (U, $\tau_R(G)$) and (V, $\sigma_R(H)$) represent nano topological spaces on which no separation axioms are assumed unless otherwise mentioned.

For a subset J of a space ((U, $\tau_R(G)$), Ncl(J) and Nint(J) denote the nano closure of J and the nano interior of J respectively. We recall the following definitions which are useful in the sequel.

Defnition 1.1 ([9]): Let U be a non-empty finite set of objects called the universe and R be an equivalence relation on U named as the indiscernibility relation. Elements belonging to the same equivalence class are said to be indiscernible with one another. The pair (U, R) is said to be the approximation space and Let G ⊆ U. Then
(1). The lower approximation of G with respect to R is the set of all objects, which can be for certain classified as G with respect to R and it is denoted by L_R(G). That is, L_R(G) = ∪{R(a): R(a) ⊆ G, a ∈ U}, where R(a) denotes the equivalence class determined by a.
(2). The upper approximation of G with respect to R is the set of all objects, which can be possibly classified as G with respect to R and it is denoted by U_R(G). That is, U_R(G) = ∪{R(a): R(a) ∩ G ≠ ∅, a ∈ U}.
(3). The boundary region of G with respect to R is the set of all objects, which can be classified neither as G nor as not G with respect to R and it is denoted by B_R(G). That is, B_R(G) = U_R(G) - L_R(G).

Proposition 1.2 ([5]): If (U, R) is an approximation space and G, H ⊆ U. Then
(1). L_R(G) ⊆ G ⊆ U_R(G)
(2). L_R(∅) = U_R(∅) = ∅ and L_R(U) = U = U_R(U)
(3). U_R(G ∪ H) = U_R(G) ∪ U_R(H)
(4). U_R(G ∩ H) = U_R(G) ∩ U_R(H)
(5). L_R(G ∪ H) = L_R(G) ∪ L_R(H)
(6). L_R(G ∩ H) = L_R(G) ∩ L_R(H)
(7). L_R(G) ⊆ L_R(H) and U_R(G) ⊆ U_R(H) whenever G ⊆ H
(8). $U_R(G^c) = (L_R(G))^c$ and $L_R(G^c) = (U_R(G))^c$
(9). $U_R U_R(G) = L_R U_R(G) = U_R(G)$.
(10). $L_R L_R(G) = U_R L_R(G) = L_R(G)$

Defnition 1.3 ([5]): Let U be the universe, R be an equivalence relation on U and $\tau_R(G)$ = {U, $\emptyset$, L_R(G), U_R(G), B_R(G)} where G $\subseteq$ U. Then by the Proportion 1.2, R(G) satisfies the following axioms:
(1). U and $\emptyset \in \tau_R(G)$,
(2). The union of the elements of any sub collection of $\tau_R(G)$ is in $\tau_R(G)$,
(3). The intersection of the elements of any finite subcollection of $\tau_R(G)$ is in $\tau_R(G)$.

That is, $\tau_R(G)$ is a topology on U called the nano topology on U with respect to G. We call (U, $\tau_R(G)$) as the nano topological space. The elements of $\tau_R(G)$ are called as nano open sets and $[\tau_R(G)]^c$ is called as the dual nano topology of $[\tau_R(G)]$.

Remark 1.4 ([5]): If $[\tau_R(G)]$ is the nano topology on U with respect to X, then the set B = {U, $\emptyset$, L_R(G), B_R(G)} is the basis for $\tau_R(G)$.

Definition 1.5 ([5]): If (U, $\tau_R(G)$) is a nano topological space with respect to G and if A $\subseteq$ H, then the nano interior of A is defined as the union of all nano open subsets of A and it is denoted by Nint(B). That is, Nint(B) is the largest nano open subset of A. The nano closure of A is defined as the intersection of all nano closed sets containing A and it is denoted by Ncl(B). That is, Ncl(B) is the smallest nano closed set containing B.

Definition 1.6 ([5]): A subset B of a nano topological space (U, $\tau_R(G)$) is called
(1). nano pre open set if B $\subseteq$ Nint(Ncl(B)).
(2). nano semi open set if B $\subseteq$ Ncl(Nint(B)).
(3). nano α - open set if B $\subseteq$ Nint(Ncl(Nint(B))).

The complements of the above mentioned sets are called their respective closed sets.

Definition 1.7: A subset H of a nano topological space (U, $\tau_R(G)$) is called
(1). nano g-closed [1] if Ncl(H) $\subseteq$ G, whenever H $\subseteq$ G and G is nano open.
(2). nano gs-closed set [2] if Nscl(H) $\subseteq$ G whenever H $\subseteq$ G, G is nano open.
(3). nano αg-closed set [13] if Nαcl(H) $\subseteq$ G whenever H$\subseteq$G and G is nano open.
(4). nano g*-closed set [11] if Ncl(H) $\subseteq$ G whenever H $\subseteq$ G and G is g-open.
(5). nano gαg- closed set [10] if Ncl(H) $\subseteq$ G whenever H $\subseteq$ G and G is αg-open.

The complements of the above mentioned sets are called their respective open sets.

Theorem 1.8 ([6, 14]): In a nano topological space (U, $\tau_R(G)$), then the following statements hold and the converse of each statement is not true:
1. Every nano open set (resp. nano closed set) is a nano α – open set (resp. nano α-closed set).
2. Every nano open set (resp. nano closed set) is a nano g – open set (resp. nano g-closed set).
3. Every nano g - open set (resp. nano g - closed set) is a nano αg – open set (resp. nano αg -closed set).
4. Every nano gα - open set (resp. nano gα - closed set) is a nano αg – open set (resp. nano αg -closed set).

Definition 1.9: Let (U, $\tau_R(G)$) and (V, $\sigma_R(H)$) be nano topological space. Then the map f: (U, $\tau_R(G)$) $\rightarrow$ (V, $\sigma_R(H)$) is called:
1. nano continuous (in short N - continuous) [15] if f^{-1} (j) is a N-open set (resp N-closed set) in (U, $\tau_R(G)$), for each N-open set (resp N-closed set) j in (V, $\sigma_R(H)$).
2. nano α-continuous (in short N α - continuous) [7] if f^{-1} (j) is a N α -open set (resp N α -closed set) in (U, $\tau_R(G)$), for each N-open set (resp N-closed set) j in (V, $\sigma_R(H)$).
3. nano g-continuous (in short N g - continuous) [4] if f^{-1} (j) is a Ng -open set (resp N g -closed set) in (U, $\tau_R(G)$), for each N-open set (resp N-closed set) j in (V, $\sigma_R(H)$).
4. nano αg - continuous (in short N αg - continuous) [7] if f^{-1} (j) is a N αg -open set (resp N αg -closed set) in (U, $\tau_R(G)$), for each N-open set (resp N-closed set) j in (V, $\sigma_R(H)$).
5. nano gs-continuous (in short N$g\ s$ - continuous) [3] if f^{-1} (j) is a N gs -open set (resp N gs -closed set) in (U, $\tau_R(G)$), for each N-open set (resp N-closed set) j in (V, $\sigma_R(H)$).
6. nano gαg-continuous (in short N $g\alpha$g - continuous) [10] if f^{-1} (j) is a N$g\alpha$g -open set (resp Ngαg - closed set) in (U, $\tau_R(G)$), for each N-open set (resp N-closed set) j in (V, $\sigma_R(H)$).
7. nano g*-continuous (in short N g* - continuous) [12] if f^{-1} (j) is a Ng* -open set (resp N g* -closed set) in (U, $\tau_R(G)$), for each N-open set (resp N-closed set) j in (V, $\sigma_R(H)$).

II. On Nano $g^{\#}\alpha$ – closed sets

In this section we present and study the nano $g^{\#}\alpha$ – closed sets and some of their properties.

Definition 2.1: A subset H of (U, $\tau_R(G)$) is called Nano $g^{\#}\alpha$ – closed set (briefly N$g^{\#}\alpha$ – closed) if Nαcl(H) $\subseteq$ V whenever H $\subseteq$ V and V is Nano g – open in (U, $\tau_R(G)$). The complements of Nano $g^{\#}\alpha$ – closed set is Nano $g^{\#}\alpha$ – open set in (U, $\tau_R(G)$).

Theorem 2.2: In a nano topological space (U, $\tau_R(G)$) the following statements are true:

(i) Every nano closed set is a N$g^{\#}\alpha$ – closed set.
(ii) Every nano α - closed set is a N$g^{\#}\alpha$ – closed set.
(iii) Every nano g*-closed set is a N$g^{\#}\alpha$ – closed set.
(iv) Every nano gαg - closed set is a N$g^{\#}\alpha$ – closed set.

Proof: (i) The proof is obvious.
(ii) Let B be a nano α - closed set in a nano topological space (U, $\tau_R(G)$) and let V be any nano g – open set containing B. Then Nαcl(B) = B $\subseteq$ V. Hence B is a N$g^{\#}\alpha$ – closed set.
(iii) Let B be a nano g* - closed set in a nano topological space (U, $\tau_R(G)$) and let V be any nano g – open set containing B. Therefore Ncl(B) $\subseteq$ V . Then Nαcl(B) $\subseteq$ Ncl(B) $\subseteq$ V. Hence B is a N$g^{\#}\alpha$ – closed set.
(iv) Let B be a nano gαg - closed set in a nano topological space (U, $\tau_R(G)$) and let V be any nano g – open set containing B. By theorem 1.8(3), V is Nαg open set in (U, $\tau_R(G)$). Since V is nano gαg - closed set, we have cl(B) $\subseteq$ V. But we have Nαcl(B) $\subseteq$ Ncl(B) $\subseteq$ V. Hence B is a N$g^{\#}\alpha$ – closed set.

The contrary of the above theorems need not be true by the following examples.

Example 2.3: (i) Let U = {t, u, v, w} with U/R = {{t, u}, {v}, {w}} and G = {t, v}. Let $\tau_R(G)$ = {$\emptyset$, {v}, {t, u}, {t, u, v}, U} be a nano topological space. Then the set {t, w} is a N$g^{\#}\alpha$ – closed set but not nano closed set.
(ii) Let U = {p, q, r} with U/R = {{p, q}, {r}} and G = {p, q}. Let $\tau_R(G)$ = {$\emptyset$, {p, q}, U} be a nano topological space. Then the set {p, r} is a N$g^{\#}\alpha$ – closed set but not Nα - closed set.
(iii) Let U = {a, b, c d} with U/R = {{a}, {c}, {b, d}} and G = {a, b}. Let $\tau_R(G)$ = {$\emptyset$, {a}, {b, d}, {a, b, d}, U} be a nano topological space. Then the set {b, d} is N$g^{\#}\alpha$ – closed set but not Ng* - closed set.
(iv) From (i), the set {V} is a N$g^{\#}\alpha$ – closed set but not Ngαg - closed set.

Theorem 2.4: In a nano topological space (U, $\tau_R(G)$), the following statements are true:

(i) Every N$g^{\#}\alpha$ – closed set is Nαg - closed set.
(ii) Every N$g^{\#}\alpha$ – closed set is Ngs - closed set.

Proof: (i) Let H be a N$g^{\#}\alpha$ – closed set in a nano topological space (U, $\tau_R(G)$) and M be any nano open set containing H. We know every nano open set is nano g – open set. Therefore H is nano g – open set in (U, $\tau_R(G)$). Since H is nano $g^{\#}\alpha$ – closed set, we have αcl(H) $\subseteq$ M. Hence H is Nαg – closed set.
(ii) Let H be a N$g^{\#}\alpha$ – closed set in a nano topological space (U, $\tau_R(G)$) and M be any nano open set containing H. We know every nano open set is nano g – open set. Therefore H is nano g – open set in (U, $\tau_R(G)$). Since H is nano $g^{\#}\alpha$ – closed set, we have Nαcl(H) $\subseteq$ M. But we have N$s$cl(H) $\subseteq$ Nαcl(H) $\subseteq$ M. Hence H is Ngs – closed set.

The contrary of the above theorems need not be true by the following examples.

Example 2.5: (i) Let U = {1, 2, 3} with U/R = {{1}, {2,3}} and G = {1, 2}. Let $\tau_R(G)$ = {$\emptyset$, {1}, {2, 3}, U} be a nano topological space. Then the set {1, 3} is Nαg - closed set but not N$g^{\#}\alpha$ – closed set.
(ii) Let U = {e, f, g, h} with U/R = {{e}, {f}, {g, h}} and G = {e, f}. Let $\tau_R(G)$ = {$\emptyset$, {e}, {f, h}, {e, f, h}} be a nano topological space. Then the set {e, f g} is Ngs - closed set but not N$g^{\#}\alpha$ – closed set.

Definition 2.6: For every set H $\subseteq$ U, we define the N$g^{\#}\alpha$ - closure of H to be the intersection of all N$g^{\#}\alpha$ - closed sets containing H. In symbols N$g^{\#}\alpha$ – cl(H) = $\cap${A: A is Nano generalized$^{\#}$ α - closed set and H $\subseteq$ A}.

Definition 2.7: For every set H $\subseteq$ U, we define the N$g^{\#}\alpha$ - Interior of H to be the union of all N$g^{\#}\alpha$ - closed sets contained in H. In symbols N$g^{\#}\alpha$ – Int(H) = $\cup${A: A is Nano generalized$^{\#}$ α - closed set and A $\subseteq$ H}.

Example 2.8: Let U = {t, u, v, w} with U/R = {{t, u}, {v}, {w}} and G = {t, v}. Then $\tau_R(G)$ = {$\emptyset$, {v}, {t, u}, {t, u, v}, U} are nano open sets. The nano closed sets are {$\emptyset$, {w}, {v, w}, {t, u, w}, U}. The nano α - closed

sets are given by {∅, {v}, {w}, {v, w}, {t, u, w}, U}. Hence Nano $g^{\#}\alpha$ – closed sets are {∅, {v}, {w}, {t, u},{v, w}, {t, w}{u, w}, {u, v, w}, {t, v, w}, {t, u, w}}.

Remark 2.9: For any set M ⊆ U, M ⊆ N$g^{\#}\alpha$cl(M) ⊆ Nαcl(M) ⊆ Ncl(M). In the above example 2.8, Let M = {u, v, w}⊆ U. Now N$g^{\#}\alpha$cl(M) = {u, v, w} and Nαcl(M) = {t, u, v, w} and Ncl(M) = {t, u, v, w}. It follows that M ⊆ N$g^{\#}\alpha$cl(M) ⊆ Nαcl(M) ⊆ Ncl(M).

Remark 2.10: For any subset J of a nano topological space (U, $\tau_R(G)$), the following properties are true:
(i) Ng$^{\#}\alpha$cl (U - J) = U - N$g^{\#}\alpha$Int(J)
(ii) Ng$^{\#}\alpha$ Int(U - J) = U - Ng$^{\#}\alpha$ Cl(J)

Example 2.11: In Example 2.8, let J ⊆ U be such that J = {t, w} and thus U – J = {u, v}
(i) Now Ng$^{\#}\alpha$ Cl(U - J) = Ng$^{\#}\alpha$ Cl({u, v}) = {u, v, w} and Ng$^{\#}\alpha$ Int(J) = Ng$^{\#}\alpha$ Int({t, w}) = {t}. Further, U - Ng$^{\#}\alpha$ Int(J) = U – {t} = {u, v, w}. Hence we have Ng$^{\#}\alpha$ Cl(U - J) = U - N$g^{\#}\alpha$Int(J).
(ii) Now Ng$^{\#}\alpha$ Int(U - J) = Ng$^{\#}\alpha$ Int({u, v}) = {u, v} and Ng$^{\#}\alpha$Cl(J) = Ng$^{\#}\alpha$ Cl({t, w}) = {t, w}. It follows that U - Ng$^{\#}\alpha$ Cl(J) = U – {t, w} = {u, v}. Hence Ng$^{\#}\alpha$ Int(U - J) = U - Ng$^{\#}\alpha$ Cl(J).

Definition 2.12: A subset A of a nano topological space (U, $\tau_R(G)$) is called Ng$^{\#}\alpha$ - neighbourhood (briefly Ng$^{\#}\alpha$ -nbd) of a point x of U if there exists Ng$^{\#}\alpha$ -open set F containing x such that x ∈ F ⊂ A.

Theorem 2.13: A subset of a nano topological space (U, $\tau_R(G)$) is Ng$^{\#}\alpha$ –open if and only if it is a Ng$^{\#}\alpha$ -nbd of each of its points.

Proof: Let F be a subset of (U, $\tau_R(G)$). Let F be a Ng$^{\#}\alpha$ - open set and let x ∈ F . Now F is a Ng$^{\#}\alpha$ -open set containing a point x. Hence F is a Ng$^{\#}\alpha$ -nbd of each of its points. Conversely, let B be a Ng$^{\#}\alpha$ -nbd of each of its points. Hence B is Ng$^{\#}\alpha$ - open set containing each of its points and B is Ng$^{\#}\alpha$ -open.

Definition 2.14: A point x of U is called a Ng$^{\#}\alpha$ - interior point of A ⊂ U if and only if x ∈ A.

Theorem 2.15 Let A be a subset of a nano topological space (U, $\tau_R(G)$) and let x ∈ U. Then x is a Ng$^{\#}\alpha$ - interior point of A if and only if A is a Ng$^{\#}\alpha$ - nbd of x.

Proof: Let A ⊂ U and x ∈ U. Let x be the Ng$^{\#}\alpha$ - interior point of A. Hence x ∈ A and by the given hypothesis, there exists a Ng$^{\#}\alpha$ - open set F containing x such that x ∈ F ⊂ A. Hence A is the Ng$^{\#}\alpha$ - nbd of x.
Conversely, let A be a Ng$^{\#}\alpha$ - nbd of x . Given A ⊂ U, x ∈ U and hence there exists a Ng$^{\#}\alpha$ -open set F containing x such that x ∈ A ⊂ F. Since x ∈ F, a Ng$^{\#}\alpha$ -open set such that A ⊂ F, x ∈ A, it follows that x is the Ng$^{\#}\alpha$ -interior point of A.

Theorem 2.16: If M be a subset of U. Then Ng$^{\#}\alpha$ - Int(A) = U{A : A is Nano $g^{\#}\alpha$ - open set and A ⊂ M}.

Proof: Let M be a subset of U. x ∈ Ng$^{\#}\alpha$ - Int(M) ⟺ x is a Ng$^{\#}\alpha$ -Interior point of M.
⟺M is a Ng$^{\#}\alpha$ -neighbourhood of point x.
⟺ there exists Ng$^{\#}\alpha$ -open set A such that x ∈ A ⊂ M.
⟺ x ∈ U{A : A is Ng$^{\#}\alpha$ -open set and A ⊂ M}.
Hence, Ng$^{\#}\alpha$ - Int(M) = U{A : A is Nano generalized α - open set and A ⊂ M}.

Theorem 2.17: Let E and F be subsets of U. Then
(i) Ng$^{\#}\alpha$ -Int(U) = U and Ng$^{\#}\alpha$ - Int(ϕ) = ϕ.
(ii) Ng$^{\#}\alpha$ -Int(E) ⊂ E.
(iii) If F is any Ng$^{\#}\alpha$ -open set contained in E, then F ⊂ Ng$^{\#}\alpha$ -Int(E).
(iv) If E ⊂ F, then Ng$^{\#}\alpha$ - Int(E) ⊂ Ng$^{\#}\alpha$ - Int(F).
(v) Ng$^{\#}\alpha$ -Int(Ng$^{\#}\alpha$ -Int(E)) = Ng$^{\#}\alpha$ -Int(E).

Proof: (i) Since U and ϕ are Ng$^{\#}\alpha$ - open sets, by theorem 2.16, Ng$^{\#}\alpha$ - Int(U) = U{F : F is Ng$^{\#}\alpha$ -open and F ⊂ U} = U{E : E is a Ng$^{\#}\alpha$ -open set} = U. That is, Ng$^{\#}\alpha$ - Int(U) = U. Since, ϕ is the only Ng$^{\#}\alpha$ -open set contained in ϕ, Ng$^{\#}\alpha$ -Int(ϕ) = ϕ.

(ii) Let $x \in Ng^{\#}\alpha$ - Int(E) $\Longrightarrow$ x is a $Ng^{\#}\alpha$ - interior point of E. $\Longrightarrow$ E is a $Ng^{\#}\alpha$ neighbourhood of x $\Longrightarrow$ $x \in A$. Thus, $x \in Ng^{\#}\alpha$ -Int(E) $\subset$ E.
(iii) Let F be any $Ng^{\#}\alpha$-open sets such that $F \subset E$. Let $x \in F$, then since, F is a $Ng^{\#}\alpha$ - open set contained in E, x is a $Ng^{\#}\alpha$ - interior point of E. That is F is a $Ng^{\#}\alpha$ - Int(E). Hence, $F \subset Ng^{\#}\alpha$ - Int(E).
(iv) Let E and F be subsets of U such that $E \subset F$. Let $x \in Ng^{\#}\alpha$ - Int(E). Then x is a $Ng^{\#}\alpha$ - interior point of E and so E is $Ng^{\#}\alpha$ - neighbourhood of x. This implies that $x \in Ng^{\#}\alpha$ - Int(F). Thus we have shown that $x \in Ng^{\#}\alpha$ -Int(F). Hence, $Ng^{\#}\alpha$ -Int(E) $\subset Ng^{\#}\alpha$ -Int(F).
(v) Let E be any subset of U. By definition of $Ng^{\#}\alpha$ – interior, $Ng^{\#}\alpha$ - Int(E) = $\cap\{E \subset A \in g^{\#}\alpha C(U)\}$, if $E \subset A \in Ng^{\#}\alpha C(U)$, then $Ng^{\#}\alpha$ -Int(E) $\subset$ A. Since A is a $Ng^{\#}\alpha$ - closed set containing $Ng^{\#}\alpha$ - Int(E). By (iii), $Ng^{\#}\alpha$ - Int($Ng^{\#}\alpha$ - Int(E)) $\subset$ E. Hence, $Ng^{\#}\alpha$ - Int($Ng^{\#}\alpha$ -Int(E)) $\subset \cap\{E \subset A \in Ng^{\#}\alpha C(U)\} = Ng^{\#}\alpha Cl(E)$. That is, $Ng^{\#}\alpha$ - Int($Ng^{\#}\alpha$ -Int(E)) = $Ng^{\#}\alpha$ - Int(E).

Theorem 2.18: Let F be a subset of $(U, \tau_R(G))$. Then $x \in Ng^{\#}\alpha Cl(F)$ if and only if for any $Ng^{\#}\alpha$ - nbd N_X of x in $(U, \tau_R(G))$, $F \cap N_x \neq \emptyset$.

Proof: Assume $x \in Ng^{\#}\alpha Cl(F)$. Let N_X be a $Ng^{\#}\alpha$ - nbd of the point x in $(U, \tau_R(G))$ such that $F \cap N_x \neq \emptyset$. Since N_X is a $Ng^{\#}\alpha$ - nbd of x in $(U, \tau_R(G))$, there exists a $Ng^{\#}\alpha$ - open set B_X such that $x \in B_X \subseteq N_x$. Hence, it follows that $B_X \cap F = \emptyset$ and so $F \subseteq (B_X)^c$. Since $(B_X)^c$ is a $Ng^{\#}\alpha$ - closed set containing F, $Ng^{\#}\alpha Cl(F) \subseteq (B_X)^c$ and hence $x \notin Ng^{\#}\alpha Cl(F)$ which is a contradiction.

Conversely, let N_x of x be a $Ng^{\#}\alpha$ - nbd of the nano topological space $(U, \tau_R(G))$ such that $F \cap N_x \neq \emptyset$. Let us prove by the method of contradiction. Suppose that $x \notin Ng^{\#}\alpha Cl(F)$, then there exists a $Ng^{\#}\alpha$ - closed set F of $(U, \tau_R(G))$ such that $F \subseteq A$ and $x \notin A$. Thus $x \in A^c$ and A^c is $Ng^{\#}\alpha$ - open in $(U, \tau_R(G))$ and hence A^c is a $Ng^{\#}\alpha$ - nbd of x in $(U, \tau_R(G))$. But $F \cap A^c = \emptyset$ which is a contradiction. Thus $x \in Ng^{\#}\alpha Cl(F)$.

Theorem 2.19: If a subset J of a space U is $Ng^{\#}\alpha$ - open then $Ng^{\#}\alpha$ - Int(J) = J.

Proof: Let J be $Ng^{\#}\alpha$ - open subset of U. We know that $Ng^{\#}\alpha$ - Int(J) $\subset$ J. Also J is $Ng^{\#}\alpha$ - open set contained in J. From theorem 2.17(iii), J $\subset Ng^{\#}\alpha$ - Int(J). Hence, $Ng^{\#}\alpha$ - Int(J) = J.

Theorem 2.20: If A and B are subsets of U, then $Ng^{\#}\alpha$ - Int(A) U $Ng^{\#}\alpha$ - Int(B) $\subset Ng^{\#}\alpha$ - Int(AUB).

Proof: We know that $A \subset A \cup B$ and $B \subset A \cup B$. We have by theorem 2.17 (iv), $Ng^{\#}\alpha$ -Int(A) $\subset Ng^{\#}\alpha$ - Int(AUB) and $Ng^{\#}\alpha$ - Int(B) $\subset Ng^{\#}\alpha$ - Int(AUB). This implies that $Ng^{\#}\alpha$ - Int(A) U $Ng^{\#}\alpha$ - Int(B) $\subset Ng^{\#}\alpha$ - Int(AUB).

Theorem 2.21: If E and F are subsets of space U, then $Ng^{\#}\alpha$ - Int(E∩F) = $Ng^{\#}\alpha$ -Int(E) ∩ $Ng^{\#}\alpha$ -Int(F).

Proof: We know that $E \cap F \subset E$ and $E \cap F \subset F$. We have, by theorem 2.17 (iv), $Ng^{\#}\alpha$ - Int(E∩F) $\subset Ng^{\#}\alpha$ - Int(E) and $Ng^{\#}\alpha$ - Int(E∩F) $\subset Ng^{\#}\alpha$ –Int(F). This implies that $Ng^{\#}\alpha$ - Int(E∩F) $\subset Ng^{\#}\alpha$ -Int(E) ∩ $Ng^{\#}\alpha$ - Int(F).-----(1). Again, let $x \in Ng^{\#}\alpha$ - Int(E) ∩ $Ng^{\#}\alpha$ - Int(F). Then $x \in Ng^{\#}\alpha$ -Int(E) and $x \in Ng^{\#}\alpha$ - Int(F). Hence, x is a $Ng^{\#}\alpha$ - interior point of each sets E and F. It follows that E and F is $Ng^{\#}\alpha$ neighbourhood of x, so that their intersection E∩F is also $Ng^{\#}\alpha$ neighbourhood of x. Hence, $x \in Ng^{\#}\alpha$ - Int(E∩F). Therefore, $Ng^{\#}\alpha$ - Int(E) ∩ $Ng^{\#}\alpha$ - Int(F) $\subset Ng^{\#}\alpha$ - Int(E∩F)-------(2). From (1) & (2), we get $Ng^{\#}\alpha$ - Int(E∩F) = $Ng^{\#}\alpha$ - Int(E) ∩ $Ng^{\#}\alpha$ - Int(F).

Theorem 2.22: A subset J of $(U, \tau_R(G))$ is $Ng^{\#}\alpha$ - closed if and only if NαCl(J) - J contains no non–empty nano closed set.

Proof: Let a $Ng^{\#}\alpha$ - closed set be taken as J and let F be a nano closed subset of NαCl(J) - J. Then it follows that $J \subseteq F^c$. Then the nano open set F^c is nano α - open. Since J is $Ng^{\#}\alpha$ -closed, then it implies that $N\alpha Cl(J) \subseteq F^c$. Consequently $F \subseteq (N\alpha Cl(J))^c$ and also $F \subseteq N\alpha Cl(J)$. Hence by the implication that $F \subseteq N\alpha Cl(J) \cap (N\alpha Cl(J))^c$, it follows that the nano closed subset F is empty.

Conversely, let NαCl(J) - J contains no non-empty nano closed set. Let $J \subseteq F$ and F be a nano open set. Suppose that $N\alpha Cl(J) \nsubseteq F$ (by contradiction), then $N\alpha Cl(J) \cap F^c \neq \emptyset$. Since NαCl(J) is a nano closed set in $(U, \tau_R(G))$. Therefore $\emptyset \neq N\alpha Cl(J) \cap F^c \subseteq N\alpha Cl(J)$ - J and so NαCl(J) - J contains a non-empty nano closed set which is a contradiction to the given hypothesis. Hence it follows that $N\alpha Cl(J) \subseteq F$. Thus J is $Ng^{\#}\alpha$ -closed.

Theorem 2.23: If L and M are $Ng^{\#}\alpha$ – closed set in a nano topological space $(U, \tau_R(G))$ then $L \cup M$ is a $Ng^{\#}\alpha$ – closed set.

Proof: Let L and M be $Ng^{\#}\alpha$ – closed set in a nano topological space $(U, \tau_R(G))$ and let D be any Ng – open set containing L & M. Then $L \cup M \subseteq D$. Then $L \subseteq D$ and $M \subseteq D$. Since L and M are $Ng^{\#}\alpha$ - closed set, $Ncl(L) \subseteq D$ and $Ncl(M) \subseteq D$. Now, $Ncl(L \cup M) = Ncl(L) \cup Ncl(M) \subseteq D$ and so $Ncl(L \cup M) \subseteq D$. Hence $L \cup M$ is a $Ng^{\#}\alpha$ – closed set.

Theorem 2.24: If L and M are $Ng^{\#}\alpha$ – open set in a nano topological space $(U, \tau_R(G))$, then $L \cap M$ is a $Ng^{\#}\alpha$ – open set.

Proof: Let L and M be $Ng^{\#}\alpha$ – open set in a nano topological space $(U, \tau_R(G))$. Then U – L and U – M are $Ng^{\#}\alpha$ – closed set. By above theorem 2.23, $(U - L) \cup (U - M)$ is a $Ng^{\#}\alpha$ – closed set. Since $(U - L) \cup (U - M) = U - (L \cap M)$. Hence $L \cap M$ is a $Ng^{\#}\alpha$ – open set.

III. NANO $g^{\#}\alpha$ - CONTINUOUS MAPS

In this section we present the nano $g^{\#}\alpha$ – continuous maps and study some of their properties.

Definition 3.1: A map f: $(U, \tau_R(G)) \to (V, \sigma_R(H))$ is said to be a nano $g^{\#}\alpha$ – continuous if $f^{-1}(h)$ is a $Ng^{\#}\alpha$ – closed set in $(U, \tau_R(G))$ for every nano closed set h in $(V, \sigma_R(H))$.

Theorem 3.2: Let $(U, \tau_R(G))$ and $(V, \sigma_R(H))$ be nano topological space, and j: $(U, \tau_R(G)) \to (V, \sigma_R(H))$ be a map. Then j is a $Ng^{\#}\alpha$ – continuous map if and only if $j^{-1}(h)$ is a $Ng^{\#}\alpha$ – open set in $(U, \tau_R(G))$ for every nano open set h in $(V, \sigma_R(H))$.

Proof: Let h be a nano open set in $(V, \sigma_R(H))$. Then h^c is a nano closed set in $(V, \sigma_R(H))$, so $j^{-1}(h^c) = (j^{-1}(h))^c$ is a $Ng^{\#}\alpha$ – closed set in $(U, \tau_R(G))$. Thus $j^{-1}(h)$ is a $Ng^{\#}\alpha$ – open set in $(U, \tau_R(G))$.

Theorem 3.3: Every $N\alpha$ – continuous map is a $Ng^{\#}\alpha$ – continuous map.

Proof: The proof is obvious.

The contrary need not be true by the following example.

Example 3.4: Let U = {a, b, c, d} with U/R = {{a, b}, {c}, {d}} and G = {a, c}. $\tau_R(G)$ = {∅, {c}, {a, b}, {a, b, c}, U} and $\tau_R{}^c(G)$ = {∅, {d}, {c, d}, {a, b, d}, U} be the nano topological space. Let V = {p, q, r, s} with V/R = {∅, {p, q}, {r}, {s}, U} and H = {p, r, s}. Let $\sigma_R(H)$ = {∅, {p, q}, {r, s}} be the nano topological space. A map f: $(U, \tau_R(G)) \to (V, \sigma_R(H))$ be defined by f(a) = p, f(b) =q, f(c) = r, f(d) = s. Then f is $Ng^{\#}\alpha$ – continuous map but not $N\alpha$ – continuous map.

Theorem 3.5: Every $Ng\alpha g$ – continuous map is a $Ng^{\#}\alpha$ – continuous map.

Proof: Let f: $(U, \tau_R(G)) \to (V, \sigma_R(H))$ be a $Ng\alpha g$ – continuous map and h be a nano closed set in $(V, \sigma_R(H))$. Since f is $Ng\alpha g$ – continuous map, $f^{-1}(h)$ is a $Ng\alpha g$ – closed set in $(U, \tau_R(G))$. By theorem 2.2 (part iv), $f^{-1}(h)$ is a $Ng^{\#}\alpha$ – closed set in $(U, \tau_R(G))$. Thus f is $Ng^{\#}\alpha$ – continuous map.

The contrary of the above theorem need not be true by the following example.

Example 3.6: Let U = {t, u, v, w} with U/R = {{t,u}, {v}, {w}} and G = {t, v}. Let $\tau_R(G)$ = {∅, {v}, {t, u}, {t, u, v}, U} and $\tau_R{}^c(G)$ = {∅, {w}, {v, w}, {t, u, w}, U} be the nano topological space. Let V= {a, b, c, d} with V/R = {{a, b}, {c}, {d}} and H = {a, c, d}. Let $\sigma_R(H)$ = {∅, {a, b}, {c, d}, U} be the nano topological space. Define a map such that f: $(U, \tau_R(G)) \to (V, \sigma_R(H))$ as f(t) = a, f(u) = b, f(v) = c, f(w) = d. Then f is $Ng^{\#}\alpha$ – continuous map but not $Ng\alpha g$ – continuous map.

Theorem 3.7: Every $Ng^{\#}\alpha$ – continuous map is a $N\alpha g$ – continuous map.

Proof: Let f: $(U, \tau_R(G)) \to (V, \sigma_R(H))$ be a $Ng^{\#}\alpha$ – continuous map and h be a nano closed set in $(V, \sigma_R(H))$. Since f is $Ng^{\#}\alpha$ – continuous map, $f^{-1}(h)$ is a $Ng^{\#}\alpha$ – closed set in $(U, \tau_R(G))$. But we have every $Ng^{\#}\alpha$ – closed set is $N\alpha g$ – closed set in $(U, \tau_R(G))$. Thus f is $N\alpha g$ – continuous map.

The converse of the above theorem need not be true by the following example.

Example 3.8: Let U = {e, f, g} with U/R = {{e}, {f, g}} and G = {e, f}. Let $\tau_R(G)$ = {∅, {e}, {f, g}, U} be the nano topological space. Let V = {p, q, r} with V/R = {{p, q}, r} and H = {p, q}. Let $\sigma_R(H)$ = {∅, {p, q}, U} and $\sigma_R{}^c(H)$ = {∅, {r}, U} be the nano topological space. Define a map such that f: (U, $\tau_R(G)$) → (V, $\sigma_R(H)$) as f(e) = p, f(f) = q, f(g) = r. Then f is Nαg – continuous map but not N$g^{\#}\alpha$ – continuous map.

Theorem 3.9: Every N$g^{\#}\alpha$ – continuous map is a Ngs – continuous map.

Proof: Let f: (U, $\tau_R(G)$) → (V, $\sigma_R(H)$) be a N$g^{\#}\alpha$ – continuous map and h be a nano closed set in (V, $\sigma_R(H)$). Since f is N$g^{\#}\alpha$ – continuous map, f^{-1}(h) is a N$g^{\#}\alpha$ – closed set in (U, $\tau_R(G)$). By theorem 2.4 (part (ii)), f^{-1}(h) is a Ngs – closed set in (U, $\tau_R(G)$) . Thus f is Ngs – continuous map.

The converse of the above theorem need not be true by the following example.

Example 3.10: In example 3.8, f is Ngs – continuous map but not N$g^{\#}\alpha$ – continuous map.

Theorem 3.11: Let f: (U, $\tau_R(G)$) → (V, $\sigma_R(H)$) is a N$g^{\#}\alpha$ – continuous map then for each point c of U and E $\in$ $\sigma_R(H)$ such that f(c) $\in$ E there exists a N$g^{\#}\alpha$ – open set C of (U, $\tau_R(G)$) such that c $\in$ C and h(C) ⊆ E.

Proof: Let c be a point of U and E $\in$ $\sigma_R(H)$ such that f(c) $\in$ E. Take C = f^{-1}(E). Since V – E is a nano closed set in V and f is a N$g^{\#}\alpha$ – continuous map, we have f^{-1}(V - E) = U - f^{-1}(E) is a N$g^{\#}\alpha$ – closed set in (U, $\tau_R(G)$). This gives C = f^{-1}(E) is a N$g^{\#}\alpha$ – open set in (U, $\tau_R(G)$) and c $\in$ C and f(c) = f(f^{-1}(E)) ⊆ E.

Theorem 3.12: Let h: (U, $\tau_R(G)$) → (V, $\sigma_R(H)$) be a function. Then the following are equivalent.
(i) The function is N$g^{\#}\alpha$ – continuous.
(ii) The inverse of each Nano open set in (V, $\sigma_R(H)$) is N$g^{\#}\alpha$ – open in (U, $\tau_R(G)$).
(iii) For each x in U, the inverse of every neighborhood of h(x) is a N$g^{\#}\alpha$ -neighborhood of x.
(iv) For each x in U and each neighborhood E of h(x), there is a N$g^{\#}\alpha$ - neighborhood D of x such that h(D) ⊆ E.
(v) For each subset A of (U, $\tau_R(G)$), h(N$g^{\#}\alpha$ Cl(A)) ⊆ NCl(h(A)).
(vi) For each subset B of (V, $\sigma_R(H)$), N$g^{\#}\alpha$ cl($h^{-1}(B)$) ⊆ h^{-1} (N$g^{\#}\alpha$ cl(B))

Proof: (i) ⇔ (ii): Let h: (U, $\tau_R(G)$) → (V, $\sigma_R(H)$) is N$g^{\#}\alpha$ - continuous and M be an Nano open set in (V, $\sigma_R(H)$). Then M^c is Nano closed in V. Hence by the definition of N$g^{\#}\alpha$ - continuous function, $h^{-1}(M^c)$ = $(h^{-1}(M))^C$ is N$g^{\#}\alpha$ - closed set in U. Thus h^{-1} (M) is N$g^{\#}\alpha$ - open in U.

Conversely, assume that h^{-1} (M) is N$g^{\#}\alpha$ - open in U for each Nano open set M in V. Let F be a Nano closed set in (V, $\sigma_R(H)$). Then F^C is a Nano open in (V, $\sigma_R(H)$) and by assumption $h^{-1}(F^C)$ is N$g^{\#}\alpha$ - open in U. Also $h^{-1}(F^C)$ = $(h^{-1}(F))^c$, we have h^{-1}(F) is Nano $g^{\#}\alpha$ - closed in U. So h is N$g^{\#}\alpha$ -continuous.

(ii) ⇒ (iii): For x ∈ (U, $\tau_R(G)$), let E be the neighborhood of h(x). Then there exist an Nano open set M in (V, $\sigma_R(H)$) such that h(x) ∈ M ⊆ E. Consequently, h^{-1}(M) is N$g^{\#}\alpha$ - open set in (U, $\tau_R(G)$) and x ∈ h^{-1}(M) ⊆ h^{-1}(E). Thus h^{-1}(E) is N$g^{\#}\alpha$ - neighborhood in (U, $\tau_R(G)$).

(iii) ⇒ (iv): Let x ∈ U and E be the neighborhood of h(x). Then by assumption, D = h^{-1}(E) is a N$g^{\#}\alpha$ - neighborhood of x and h(D) = h(h^{-1}(E)) ⊆ E.

(iv) ⇔ (v): Suppose (iv) holds and let j ∈ h(N$g^{\#}\alpha$ Cl(A)) and let E be the neighborhood of j. Then there exists x ∈ U and a N$g^{\#}\alpha$ - neighborhood D of x such that h(x) = j, x ∈ D, x ∈ N$g^{\#}\alpha$ Cl (A) and h(D) ⊆ E. Since x ∈ N$g^{\#}\alpha$ Cl(A), by theorem 2.18, (D ∩ A) ≠ ∅ and hence (h(A) ∩ E) ≠ ∅. Hence j = h(x) ∈ Ncl(h(A)). That is h(N$g^{\#}\alpha$ cl(A)) ⊆ Ncl(h(A). Conversely, suppose that (v) holds and let x ∈ U and E be the neighborhood of h(x). Let A = h^{-1}(V/E). Since h(N$g^{\#}\alpha$ cl(A)) ⊆ Ncl(h(A) ⊆ V/E, We have N$g^{\#}\alpha$ cl(A) = A. Since x does not belong to N$g^{\#}\alpha$ Cl(A), there exist a N$g^{\#}\alpha$ neighborhood D of x such that (D ∩ A) ≠ ∅ and hence h(D) ⊆ h(U/A) ⊆ E.

(v)⇔ (vi): Suppose that (v) holds and B be any subset of (V, $\sigma_R(H)$). Then replacing A by h^{-1}(B) in (v), we have h(N$g^{\#}\alpha$ cl(h^{-1}(B))) ⊆ Ncl(h(h^{-1}(B)) ⊆ Ncl(B). That is N$g^{\#}\alpha$ cl(h^{-1}(B)) ⊆ h^{-1}(Ncl(B)). Conversely, suppose that (vi) holds and let B = h(A) where A is the subset of (U, $\tau_R(G)$). Then we have N$g^{\#}\alpha$ cl(A) ⊆ N$g^{\#}\alpha$ cl(h^{-1}(B)) ⊆ h^{-1}(Ncl(B)) ⊆ h^{-1}(Ncl(h(A)) which implies h(N$g^{\#}\alpha$ cl(A)) ⊆ Ncl(h(A)). This completes the proof of the theorem.

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On Nano Generalized$^{\#}\alpha$-Closed Sets in Nano Hausdorff and Nano Lindelof Spaces

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***Abstract*:** *The main objective of this paper is to define and study a new class of maps namely Ng$^{\#}\alpha$ irresolute maps, strongly Ng$^{\#}\alpha$ continuous maps in Nano Hausdorff space and Nano Lindelof space. We also analyze and study some of the properties of these mappings via Ng$^{\#}\alpha$-closed sets. We also introduce the notion of Ng$^{\#}\alpha$ Lindelof space.*

Keywords: *Ng$^{\#}\alpha$-closed set, Ng$^{\#}\alpha$ irresolute maps, strongly Ng$^{\#}\alpha$ continuous map, Nano Hausdorff space, Nano Lindelof space, Ng$^{\#}\alpha$ open cover, Ng$^{\#}\alpha$ Lindelof.*

I. Introduction

The concept of nano topology was introduced by Lellis Thivagar and Carmel Richard [5] and was defined in terms of approximations and boundary region of a subset of an universe using an equivalence relation in it. K.Nono et al.,[6] introduced the concept of g$^{\#}\alpha$-closed sets and investigated some of their topological properties. V.Kokilavani and S.Visagapriya introduced the notion of Ng$^{\#}\alpha$-closed sets[4] and studied their properties in nano topological space. In this paper, we investigate the properties of Ng$^{\#}\alpha$-closed set in Nano Hausdorff space and Nano Lindelof space.

II. Preliminaries

Definition 2.1[5] Let U be a non-empty finite set of objects called the universe and R be an equivalence relation on U named as the indiscernibility relation. Elements belonging to the same equivalence class are said to be indiscernible with one another.

Let X⊆U. Then,

1. The lower approximation of X with respect to R is the set of all objects, which can be for certain classified as X with respect to R and it is denoted by $L_R(X)$. That is,

$L_R(X) = \bigcup_{x\in U}\{R(x): R(x) \subseteq X\}$, where R(x) denotes the equivalence class determined by x.

2. The upper approximation of X with respect to R is the set of all objects, which can be possibly classified as X with respect to R and it is denoted by $U_R(X)$. That is,

$U_R(X) = \bigcup_{x\in U}\{R(x): R(x)\cap X \neq \emptyset\}$, where R(x) denotes the equivalence class determined by x.

3. The boundary region of X with respect to R is the set of all objects, which can be classified neither as X nor as not X with respect to R and it is denoted by $B_R(X)$. That is,

$B_R(X) = U_R(X) - L_R(X)$

Preposition:2.2[5] If (U,R) is an approximation space and X, Y⊆U, then

1. $L_R(X) \subseteq X \subseteq U_R(X)$
2. $L_R(\emptyset) = U_R(\emptyset) = \emptyset$ and $L_R(U) = U_R(U) = U$
3. $U_R(X \cup Y) = U_R(X) \cup U_R(Y)$
4. $U_R(X \cap Y) \subseteq U_R(X) \cap U_R(Y)$
5. $L_R(X \cup Y) \supseteq L_R(X) \cup L_R(Y)$
6. $L_R(X \cap Y) \subseteq L_R(X) \cap L_R(Y)$
7. $L_R(X) \subseteq L_R(Y)$ and $U_R(X) \subseteq U_R(Y)$, whenever X⊆ Y

8. $U_R(X^c) = [L_R(X)]^c$ and $L_R(X^c) = [U_R(X)]^c$

9. $U_R U_R(X) = L_R U_R(X) = U_R(X)$

10. $L_R L_R(X) = U_R L_R(X) = L_R(X)$

Definition: 2.3[5] Let U be the universe, R be an equivalence relation on U and $\tau_R(X) = \{U, \emptyset, L_R(X), U_R(X), B_R(X)\}$, where X⊆U. Then by preposition 2.2, R(X) satisfies the following axioms:

1. U and $\emptyset \in \tau_R(X)$.
2. The union of the elements of any sub collection of $\tau_R(X)$ is in $\tau_R(X)$.
3. The intersection of the elements of any finite sub-collection of $\tau_R(X)$ is in $\tau_R(X)$.

That is, $\tau_R(X)$ is a topology on U called the nano topology on U with respect to X. We call $(U, \tau_R(X))$ as the nano topological space. The elements of $\tau_R(X)$ are called as nano open sets and $[\tau_R(X)]^c$ is called as the dual nano topology of $\tau_R(X)$.

Remark: 2.4 [5] If $\tau_R(X)$ is the nano topology on U with respect to X, then the set $B = \{U, \emptyset, L_R(X), B_R(X)\}$ is the basis for $\tau_R(X)$.
Definition: 2.5[5]If $(U, \tau_R(X))$ is a nano topological space with respect to X and if A⊆U, then the nano interior of A is defined as the union of all nano open subsets of A and it is denoted by Nint(A).
That is, Nint(A) is the largest nano open subset of A.

The nano closure of A is defined as the intersection of all nano closed sets containing A and it is denoted by Ncl(A).
That is, Ncl(A) is the smallest nano closed set containing A.
Definition: 2.6[1] Let $(U, \tau_R(X))$ be a nano topological space and A⊆U. Then A is said to be,

- Nano semi open, if A⊆Ncl(Nint(A))
- Nano semi closed, if A⊇Nint(Ncl(A))
- Nano α-open, if A⊆Nint(Ncl(Nint(A)))
- Nano α-closed, if A⊇Ncl(Nint(Ncl(A)))

Definition: 2.7[1] A subset A of a nano topological space $(U, \tau_R(X))$ is called nano generalized closed set (briefly Ng closed), if Ncl(A)⊆V, whenever A⊆V and V is nano open in $(U, \tau_R(X))$.

Definition: 2.8[2] Let $(U, \tau_R(X))$ be a nano topological space and A⊆U. Then A is said to be,

- Nano generalized α-closed set (briefly Ngα-closed), if Nαcl(A)⊆V, whenever A⊆V, V is nano open in U.
- Nano α-generalized closed set (briefly Nαg-closed), if Nαcl(A)⊆V, whenever A⊆V, V is nano α-open in U.

Definition: 2.9[4] A subset A of a nano topological space $(U, \tau_R(X))$ is called Nano generalized$^{\#}\alpha$ -closed set (briefly Ng$^{\#}\alpha$-closed) if Nαcl(A)⊆V, whenever A⊆V and V is Nano g open in $(U, \tau_R(X))$.

Definition: 2.10[4] A map $f : (U, \tau_R(X)) \rightarrow (V, \tau_{R^1}(Y))$ is said to be nano generalized $^{\#}\alpha$ continuous if $f^{-1}(V)$ is a Ng$^{\#}\alpha$ closed set in $(U, \tau_R(X))$ for every nano closed set V in $(V, \tau_{R^1}(Y))$.

III.Ng$^{\#}\alpha$-CLOSED SET IN NANO HAUSDORFF SPACE

Definition: 3.1 Let $(U, \tau_R(X))$ be a nano topological space and x,y∈U, such that x and y are disjoint. $(U, \tau_R(X))$ is called Nano Hausdorff space, if there exist nano open sets A and B such that x∈A, y∈B and $A \cap B = \emptyset$.

Theorem: 3.2 Let $(U, \tau_R(X))$ be a nano topological space and (Y, σ) be a Hausdorff space.
If $f : (U, \tau_R(X)) \rightarrow (Y, \sigma)$ is injective and continuous, then $(U, \tau_R(X))$ is a Nano Hausdorff space.
Proof: Let x and y be any two distinct points of $(U, \tau_R(X))$.

It is given that,
$f : (U, \tau_R(X)) \rightarrow (Y,\sigma)$ is injective and continuous.
Then f(x) and f(y) are different points of Y. Since Y is Hausdorff, there exist disjoint open sets U and V in Y containing f(x) and f(y) respectively.
Since f is continuous and $U \cap V = \emptyset$, $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint open sets in $(U, \tau_R(X))$ such that $x \in f^{-1}(U)$ and $y \in f^{-1}(V)$.
$\Rightarrow f^{-1}(U) \cap f^{-1}(V) = \emptyset$
$\Rightarrow (U, \tau_R(X))$ is a Nano Hausdorff space

Example: 3.3 Let (Y,σ) be a Hausdorff space. $Y= \{a, b, c\}$
$\tau = \{ \emptyset, Y, \{a\}, \{b,c\} \}$ a,b ∈ Y , a ≠ b.
a ∈ {a}, b ∈ {b,c}, $\{a\} \cap \{b,c\} = \emptyset$
Let $(U, \tau_R(X))$ be a nano topological space.
$U= \{x, y, z\}$ with U/R= $\{\{x\}, \{y,z\}\}$ and $X= \{x, y\}$. Then,
$\tau_R(X) = \{U, \emptyset, \{x\}, \{y, z\}\}$
$f : (U, \tau_R(X)) \rightarrow (Y, \sigma)$ is defined by f(x)=a, f(y)=b, f(z)=c.
Since f is continuous and injective, x is open and f(x) = a is open.
$\Rightarrow f^{-1}\{a\}$ is a neighbourhood of x. Similarly,
$f^{-1}\{b,c\}$ is a neighbourhood of y and $f^{-1}\{a\} \cap f^{-1}\{b,c\} = \emptyset$.
Therefore, $(U, \tau_R(X))$ is a Nano Hausdorff space.

Definition: 3.4 A map $f : (U, \tau_R(X)) \rightarrow (V,\tau_{R^1}(Y))$ is nano irresolute in a Nano Hausdorff space, if the inverse images $f^{-1}(A)$ and $f^{-1}(B)$ of every disjoint non-empty nano semi-open sets A and B respectively in V are disjoint non-empty nano semi-open sets in U containing distinct points respectively.

Definition: 3.5 A map $f : (U, \tau_R(X)) \rightarrow (V, \tau_{R^1}(Y))$ is $Ng^{\#}\alpha$ irresolute in a Nano Hausdorff space, if the inverse images $f^{-1}(A)$ and $f^{-1}(B)$ of every disjoint non-empty $Ng^{\#}\alpha$ open sets A and B respectively in V are disjoint non-empty $Ng^{\#}\alpha$-open sets in U containing distinct points respectively.

Theorem:3.6 Let $(U, \tau_R(X)),(V, \tau_{R^1}(Y)),(W, \tau_{R^{11}} Z))$ be Nano Hausdorff spaces. If the function $f : (U, \tau_R(X)) \rightarrow (V, \tau_{R^1}(Y))$ and $g : (V, \tau_{R^1}(Y)) \rightarrow (W, \tau_{R^{11}}(Z))$ both are $Ng^{\#}\alpha$ irresolute, then $g \circ f : (U,\tau_R(X)) \rightarrow (W, \tau_{R^{11}}(Z))$ is $Ng^{\#}\alpha$ irresolute function.

Proof: It is given that, $g : (V, \tau_{R^1}(Y)) \rightarrow (W, \tau_{R^{11}}(Z))$ is $Ng^{\#}\alpha$ irresolute.

Let A and B be two disjoint $Ng^{\#}\alpha$ open sets in W having distinct points respectively. Therefore by definition, the inverse images $g^{-1}(A)$ and $g^{-1}(B)$ of every disjoint $Ng^{\#}\alpha$ open sets A and B in W respectively are disjoint $Ng^{\#}\alpha$ open sets in V.
Hence, $g^{-1}(A)$ and $g^{-1}(B)$ are disjoint $Ng^{\#}\alpha$ open sets in V.
It is given that $f : (U, \tau_R(X)) \rightarrow (V, \tau_{R^1}(Y))$ is $Ng^{\#}\alpha$ irresolute.
Since $g^{-1}(A)$ and $g^{-1}(B)$ are two arbitrary disjoint $Ng^{\#}\alpha$ open sets in V.
Therefore, their inverse images $f^{-1}(g^{-1}(A))$ and $f^{-1}(g^{-1}(B))$ are disjoint $Ng^{\#}\alpha$ open sets in U.
Thus, $f^{-1}(g^{-1}(A))$ and $f^{-1}(g^{-1}(B))$ are disjoint $Ng^{\#}\alpha$ open sets in U.
We know that, $f^{-1}(g^{-1}(A)) = (g \circ f)^{-1}(A)$ and $f^{-1}(g^{-1}(B)) = (g \circ f)^{-1}(B)$.
$\Rightarrow (g \circ f)^{-1}(A)$ and $(g \circ f)^{-1}(B)$ are disjoint $Ng^{\#}\alpha$ open sets in U.
Hence, $g \circ f : (U,\tau_R(X)) \rightarrow (W, \tau_{R^{11}}(Z))$ is also $Ng^{\#}\alpha$ irresolute.

Definition: 3.7 Let $(U, \tau_R(X))$ and $(V, \tau_{R^1}(Y))$ be two Nano Hausdorff spaces. Then a mapping $f : (U, \tau_R(X)) \rightarrow (V, \tau_{R^1}(Y))$ is said to be,

- Nano continuous, if the inverse images of every disjoint nano open sets having distinct points in V respectively are disjoint nano open sets respectively in U
- $Ng^{\#}\alpha$ continuous, if the inverse images of every disjoint nano open sets having distinct points in V respectively are disjoint $Ng^{\#}\alpha$ open sets respectively in U

Theorem: 3.8 Let $(U, \tau_R(X)),(V, \tau_{R^1}(Y)),(W, \tau_{R^{11}}(Z))$ be Nano Hausdorff spaces. For any $Ng^{\#}\alpha$ irresolute map $f : (U, \tau_R(X)) \rightarrow (V, \tau_{R^1}(Y))$ and any $Ng^{\#}\alpha$-continuous map $g : (V, \tau_{R^1}(Y)) \rightarrow (W, \tau_{R^{11}}(Z))$, their composition $g \circ f : (U, \tau_R(X)) \rightarrow (W, \tau_{R^{11}}(Z))$ is $Ng^{\#}\alpha$-continuous.

Proof: It is given that, $g : (V, \tau_{R^1}(Y)) \rightarrow (W, \tau_{R^{11}}(Z))$ is $Ng^{\#}\alpha$-continuous.
Let A and B be two disjoint nano open sets having distinct points respectively in W.
Therefore by definition, the inverse images $g^{-1}(A)$ and $g^{-1}(B)$ of every disjoint nano open sets A and B respectively in W are disjoint $Ng^{\#}\alpha$ open sets in V respectively.
Therefore, $g^{-1}(A)$ and $g^{-1}(B)$ are disjoint $Ng^{\#}\alpha$ open sets in V.
It is given that, $f : (U, \tau_R(X)) \rightarrow (V, \tau_{R^1}(Y))$ is $Ng^{\#}\alpha$ irresolute.
Since $g^{-1}(A)$ and $g^{-1}(B)$ are two arbitrary disjoint $Ng^{\#}\alpha$ open sets in V, their inverse images $f^{-1}(g^{-1}(A))$ and $f^{-1}(g^{-1}(B))$ are disjoint $Ng^{\#}\alpha$ open sets in U.
Thus, $f^{-1}(g^{-1}(A))$ and $f^{-1}(g^{-1}(B))$ are disjoint $Ng^{\#}\alpha$ open sets in U.
We know that, $f^{-1}(g^{-1}(A)) = (g \circ f)^{-1}(A)$ and $f^{-1}(g^{-1}(B)) = (g \circ f)^{-1}(B)$.
$\Rightarrow (g \circ f)^{-1}(A)$ and $(g \circ f)^{-1}(B)$ are disjoint $Ng^{\#}\alpha$ open sets in U for every disjoint nano open sets A and B in W respectively. Hence, $g \circ f : (U, \tau_R(X)) \rightarrow (W, \tau_{R^{11}}(Z))$ is $Ng^{\#}\alpha$-continuous.

Theorem: 3.9 Let $(U, \tau_R(X)),(V, \tau_{R^1}(Y)),(W, \tau_{R^{11}}(Z))$ be Nano Hausdorff spaces. If $f : (U, \tau_R(X)) \rightarrow (V, \tau_{R^1}(Y))$ is $Ng^{\#}\alpha$-continuous and any $g : (V, \tau_{R^1}(Y)) \rightarrow (W, \tau_{R^{11}}(Z))$ is nano-continuous, then the composition $(g \circ f) : (U, \tau_R(X)) \rightarrow (W, \tau_{R^{11}}(Z))$ is $Ng^{\#}\alpha$-continuous.
Proof: It is given that, $g : (V, \tau_{R^1}(Y)) \rightarrow (W, \tau_{R^{11}}(Z))$ is nano continuous.
Let A and B be two disjoint nano open sets having distinct points respectively in W.
Therefore by definition, the inverse images $g^{-1}(A)$ and $g^{-1}(B)$ of every disjoint nano open sets A and B respectively in W are disjoint nano open sets in V respectively.
It is given that, $f : (U, \tau_R(X)) \rightarrow (V, \tau_{R^1}(Y))$ is $Ng^{\#}\alpha$ continuous.
Since $g^{-1}(A)$ and $g^{-1}(B)$ are two arbitrary disjoint nano open sets in V, their inverse images $f^{-1}(g^{-1}(A))$ and $f^{-1}(g^{-1}(B))$ are disjoint $Ng^{\#}\alpha$-open sets in U.
Thus, $f^{-1}(g^{-1}(A))$ and $f^{-1}(g^{-1}(B))$ are disjoint $Ng^{\#}\alpha$ open sets in U.
We know that, $f^{-1}(g^{-1}(A)) = (g \circ f)^{-1}(A)$ and $f^{-1}(g^{-1}(B)) = (g \circ f)^{-1}(B)$.
$\Rightarrow (g \circ f)^{-1}(A)$ and $(g \circ f)^{-1}(B)$ are disjoint $Ng^{\#}\alpha$ open sets in U for every disjoint nano open sets A and B in W respectively.
Hence, $g \circ f : (U, \tau_R(X)) \rightarrow (W, \tau_{R^{11}}(Z))$ is $Ng^{\#}\alpha$-continuous.

Definition: 3.10 Let $(U, \tau_R(X)),(V, \tau_{R^1}(Y))$ be Nano Hausdorff spaces. A map $f : (U, \tau_R(X)) \rightarrow (V, \tau_{R^1}(Y))$ is strongly $Ng^{\#}\alpha$-continuous, if the inverse images $f^{-1}(A)$ and $f^{-1}(B)$ of every disjoint non-empty $Ng^{\#}\alpha$ open sets A and B respectively in V are disjoint non-empty nano open sets in U containing distinct points among them.

Theorem: 3.11 Let $(U, \tau_R(X)),(V, \tau_{R^1}(Y))$ be Nano Hausdorff spaces. If the map $f : (U, \tau_R(X)) \rightarrow (V, \tau_{R^1}(Y))$ is strongly $Ng^{\#}\alpha$-continuous, then it is nano continuous.
Proof: Let A and B be two disjoint nano open sets having distinct points among them in V.
We know that, Every nano open set is $Ng^{\#}\alpha$ open.
$\Rightarrow$ A and B are disjoint $Ng^{\#}\alpha$ open sets in V.
By the given hypothesis, $f : (U, \tau_R(X)) \rightarrow (V, \tau_{R^1}(Y))$ is strongly $Ng^{\#}\alpha$-continuous.
Hence the inverse images of disjoint $Ng^{\#}\alpha$ open sets A and B are disjoint nano open sets in U.
That is, $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint nano open sets in U.
Therefore, $f : (U, \tau_R(X)) \rightarrow (V, \tau_{R^1}(Y))$ is nano continuous.

IV. $Ng^{\#}\alpha$ –CLOSED SET IN NANO LINDELOF SPACE

Definition: 4.1 A nano topological space $(U, \tau_R(X))$ is said to be a Nano Lindelof space, if for any collection B of nano open sets such that $U = \bigcup_{B \in \mathcal{B}} B$, there exists a countable sub-collection $C \subseteq B$ such that $U = \bigcup_{C \in \mathcal{C}} C$.

That is, A nano topological space $(U, \tau_R(X))$ is said to be Nano Lindelof space, iff every nano open cover of X has a countable subcover for X.
Definition: 4.2 A collection B of $Ng^{\#}\alpha$ open sets in a nano topological space $(U, \tau_R(X))$ is called $Ng^{\#}\alpha$ open cover of a subset A of U, if $A \subseteq \bigcup_{B_i \in \mathcal{B}} B_i$.

Theorem: 4.3 Let $(U, \tau_R(X))$ be a nano Lindelof space and suppose that A is $Ng^{\#}\alpha$-closed subset of U, then A is Nano Lindelof.

Proof: Let B be a subcollection of nano open sets in U.

Let $\cup_{B_i \in \mathcal{B}} B_i$ be a countable nano open cover of A.

Since every nano open set is $Ng^{\#}\alpha$ open, $\cup_{B_i \in \mathcal{B}} B_i$ is a countable $Ng^{\#}\alpha$ open cover of A.

Hence $\cup B_i$ is $Ng^{\#}\alpha$ open. Then $N\alpha cl(A) \subseteq \cup B_i$, because A is $Ng^{\#}\alpha$ closed. Since a closed subset of a Lindelof space is Lindelof, $N\alpha cl(A)$ is Lindelof.

Therefore, $N\alpha cl(A)$ has a countable sub-cover $B_1, B_2, ..., B_n$ and it follows,

$$A \subseteq N\alpha cl(A) \subseteq B_1 \cup B_2 \cup ... \cup B_n.$$

$$A \subseteq N\alpha cl(A) \subseteq \cup B_i$$

$$A \subseteq \cup B_i$$

Therefore, A is Nano Lindelof.

Definition: 4.4 A nano topological space $(U, \tau_R(X))$ is called $Ng^{\#}\alpha$ Lindelof, if every $Ng^{\#}\alpha$ open cover of U has a countable sub-cover.

Definition: 4.5 A subset A of nano topological space $(U, \tau_R(X))$ is said to be $Ng^{\#}\alpha$ Lindelof, if A is $Ng^{\#}\alpha$ Lindelof as a subspace of U.

Definition: 4.6 A subset A of nano topological space $(U, \tau_R(X))$ is said to be $Ng^{\#}\alpha$ Lindelof relative to U, if for every collection $\mathcal{B}$ of $Ng^{\#}\alpha$ open subset of U such that $A \subseteq \cup_{B_i \in \mathcal{B}} B_i$, there exits a finite subset $\mathcal{B}_1$ of B such that, $A \subseteq \cup_{B_i \in \mathcal{B}_1} B_i$.

Theorem: 4.7 Every $Ng^{\#}\alpha$ closed subset of a $Ng^{\#}\alpha$- Lindelof space U is $Ng^{\#}\alpha$ Lindelof relative to U.

Proof: Let A be a $Ng^{\#}\alpha$-closed subset of U and $\mathcal{B}$ be a cover of A by $Ng^{\#}\alpha$ open subsets in U. Then $\mathcal{B}_1$ = B $\cup$ {U – A} is a $Ng^{\#}\alpha$-open cover of U.

Since U is $Ng^{\#}\alpha$-Lindelof, $\mathcal{B}_1$ has a countable subcover $\mathcal{B}_2$ for U. Now, $\mathcal{B}_3 = \mathcal{B}_2 - \{U - A\}$ is a countable subcover of B for A.

Therefore, A is $Ng^{\#}\alpha$-Lindelof relative to U.

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A note on the nonoscillatory properties of certain third order nonlinear difference equations

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Abstract: *New nonoscillatory criteria are obtained for certain third order nonlinear neutral delay difference equations with examples.*

Keywords: *difference equation; oscillatory; third order; neutral delay; nonlinear.*

I. Introduction

Several physical problems are modelled into nonlinear differential equations. Existence of the solution of the all nonlinear differential equations are impossible, solution exists only for certain kind of differential equations. Difference equation is a discrete analogue of differential equations. Numerical methods help us to solve certain initial and boundary value problems. In an equation, the coefficients determine the behavior of its solutions. The lack of existence of solutions of certain nonlinear difference equations, we focused on its qualitative behavior of solutions, in particular, asymptotic and oscillatory behavior of its solutions. Many authors discussed the qualitative behavior of certain first, second, third and higher order nonlinear difference equations with some conditions on its coefficients. Sometimes two same order difference equations with slight changes in its same coefficients will be in different behavior. From these observations, we focused on same difference equations with different conditions on its coefficients. We extended the previous results with the different conditions imposed on the coefficients. In[4], we have studied the oscillatory and nonoscillatory behavior of certain second order difference equations of the form

$$\Delta^2\left(x_n + p_n x_{n-\tau} - q_n x_{n-\sigma}\right) + r_n f^{\nu}\left(x_{n-l}\right) = 0, \quad n \in N\left(n_0\right) \tag{1}$$

In [3], we have considered the oscillatory properties of third order difference equations of the form

$$\Delta^2\left(p_n\left(\Delta x_n\right)^{\alpha}\right) - q_n\left(\Delta x_n\right)^{\alpha} + r_n f\left(x_n\right) = 0 \ , n = 0,1,2,3,... \tag{2}$$

In [6], we have considered the third order difference equation of the form

$$\Delta\left(a_n\left(\Delta\left(b_n \Delta x_n\right)\right)\right) + c_n f^{\nu}\left(x_{n+1-\tau}\right) = 0 \ , n \in N\left(a\right) \tag{3}$$

In [5], we have considered the following difference equations:

$$\Delta\left(a_n\left(\Delta\left(x_n - p_n x_{n-1}\right)\right)\right) + q_n f\left(x_{n-\tau}\right) = 0 \ , \ n \in N\left(a\right) \tag{4}$$

and

$$\Delta\left(a_n\left(\Delta\left(b_n \Delta y_n\right)\right)\right) + q_n^{*} g\left(x_{n+1-\tau}\right) = 0 \ , n \in N\left(a\right) \tag{5}$$

Motivated by these observations, in this article, we considered the third order difference equations of the form

$$\Delta^2\left(a_n\left(\Delta\left(x_n + p_n x_{n-k} - q_{n-l}\right)\right)^{\alpha}\right) + r_n f\left(x_n\right) = 0, \quad \text{for } n \in N\left(n_0\right), \tag{6}$$

where $k \geq 1, l \geq 1, n = 0,1,2,\ldots$, $\alpha = \frac{u}{v}$, ($u \geq v$), u and v are odd positive integers, and Δ is the forward difference operator defined by $\Delta x_n = x_{n+1} - x_n$. Further, it is assumed that

(C1) $\{a_n\}, \{p_n\}, \{q_n\}$ are real positive sequences such that $\sum_{n=n_0}^{\infty} \frac{1}{\frac{1}{a_n^{\alpha}}} = \infty$, $o < q_n < p_n < 1$, for all $n \in N(n_o)$,

(C2) $\{r_n\}$ is a sequence of real numbers, for all $n \in N(n_o)$.

(C3) f: P→P is a continuous function such that $xf(x) > 0$, for all $x \in$ P and $x \neq 0$.

A real sequence $\{x_n\}$, n = 0, 1, 2,... is called a solution of (6) if it satisfies (6) for all $n \geq n_0$, $n_0 \geq 0$. A nontrivial solution of (6) is known as oscillatory if it is either eventually positive or eventually negative; otherwise it is said to be nonoscillatory. Moreover, $\{x_n\}$ is bounded if it is both bounded below and bounded above. A nontrivial solution of (6) is bounded below if for every $v > 0$ there exists an integer $n \geq v$ such that $|x_n| > v$. A nontrivial solution of (6) bounded above if for every $V > 0$ there exists an integer $n \geq V$ such that $|x_n| < V$. Throughout this paper we denote $N(n_o) = \{n_0, n_0 + 1, n_0 + 2, \ldots\}$, $n_0 \geq 0$.

II. Main Results

We discuss the new criteria for the nonoscillatory behavior of (6). Throughout this section we denote $z_n = x_n + p_n x_{n-k} - q_n x_{n-l}$.

Theorem 2.1. *If $\{x_n\}$ is an eventually positive, unbounded solution of (6) such that the conditions (C1), (C2), (C3) are satisfied. then the following inequalities hold:*

$x_n > 0, z_n > 0, (\Delta z_n)^{\alpha} > 0, a_n(\Delta z_n)^{\alpha} > 0, \Delta\left(a_n(\Delta z_n)^{\alpha}\right) > 0$ *for large $n \in N(n_0)$.*

Proof. Suppose $\{x_n\}$ be an eventually positive and unbounded solution of (6). Then there is $n_1 \geq n_0$ so that $x_{n-k} > 0$ for $n \in N(n_1)$. From the definition of z_n, we see that $z_n > 0$ for $n \in N(n_1)$. In view of (C1), we have $(\Delta z_n)^{\alpha} > 0, a_n(\Delta z_n)^{\alpha} > 0$ for $n \geq n_1$. We claim $\Delta\left(a_n(\Delta z_n)^{\alpha}\right) > 0$ for large $n \in N(n_0)$.

If $\Delta\left(a_n(\Delta z_n)^{\alpha}\right) \leq 0$ for large $n \in N(n_0)$, we find that

$$\Delta\left(a_n(\Delta z_n)^{\alpha}\right) < -K \text{ for some } K > 0;\ n_1 \geq n_0; \text{ where } n \in N(n_1) \tag{7}$$

This implies that

$$a_n(\Delta z_n)^{\alpha} - a_{n_1}\left(\Delta z_{n_1}\right)^{\alpha} < -K(n - n_1), \text{ for} n \in N(n_1) \tag{8}$$

Define $\delta = a_{n_1}\left(\Delta z_{n_1}\right)^{\alpha}$. Then the inequality (8) becomes

$$a_n(\Delta z_n)^{\alpha} < \delta - K(n - n_1), \text{ for} n \in N(n_1) \tag{9}$$

Consequently, $a_n(\Delta z_n)^{\alpha} < -\infty$ as $n \to \infty$. We attain the claim.

Theorem 2.2. *If $\{x_n\}$ is an eventually positive, bounded solution of (6) along with (C1), (C3), then either of the following cases are true for sufficiently large n.*

(i) $x_n > 0, z_n > 0, (\Delta z_n)^{\alpha} > 0, a_n(\Delta z_n)^{\alpha} > 0, \Delta\left(a_n(\Delta z_n)^{\alpha}\right) < 0$,

(ii) $x_n > 0, z_n > 0, (\Delta z_n)^{\alpha} < 0, a_n(\Delta z_n)^{\alpha} < 0, \Delta\left(a_n(\Delta z_n)^{\alpha}\right) > 0$.

Proof. Since $\{x_n\}$ is eventually positive and unbounded solution of (6), there is $n_1 \geq n_0$, in which $x_{n-k} > 0$ for $n \in N(n_1)$. It follows that $z_n > 0$. But then (C1) implies that $\{\Delta z_n\}$, $\{(\Delta z_n)^\alpha\}$, $\{a_n(\Delta z_n)^\alpha\}$ are of in one sign for $n \in N(n_1)$.

Case(I): If $\Delta z_n > 0$, $(\Delta z_n)^\alpha > 0$, $a_n(\Delta z_n)^\alpha > 0$, for $n \geq n_1$. Claim that $\Delta\left(a_n(\Delta z_n)^\alpha\right) < 0$ for $n \geq n_1$. In view of (C1) and (C3), we obtain from (6) that

$$\Delta^2\left(a_n(\Delta z_n)^\alpha\right) < 0, \text{ for } n \in N(n_1).$$

It follows that

$$\Delta^2\left(a_n(\Delta z_n)^\alpha\right) < -K_1, \text{ for } n_2 \geq n_1 \text{ and a real number } K_1 > 0. \quad (10)$$

Hence $\Delta\left(a_n(\Delta z_n)^\alpha\right) \leq -K_1(n-n_2)$, for $n \in N(n_2)$. This proves the claim.

Case(II): Without loss of generality we assume that $\Delta z_n < 0$, $(\Delta z_n)^\alpha < 0$, $a_n(\Delta z_n)^\alpha < 0$, for $n \in N(n_1)$. It is enough to prove $\Delta\left(a_n(\Delta z_n)^\alpha\right) > 0$, for large $n \geq n_1$. Suppose that $\Delta\left(a_n(\Delta z_n)^\alpha\right) \leq 0$ for $n \geq n_1$. From (C1), (C3) and (6), we have $\Delta^2\left(a_n(\Delta z_n)^\alpha\right) < 0$. Thus

$$\Delta^2\left(a_n(\Delta z_n)^\alpha\right) < -K_2, \text{ for some } n \geq n_2 \text{ where } K_2 > 0 \text{ is a real number.} \quad (11)$$

Summing (11) from n_2 to $n-1$, we have

$$\Delta\left(a_n(\Delta z_n)^\alpha\right) - \Delta\left(a_{n_2}(\Delta z_{n_2})^\alpha\right) < -K_2(n-n_2) \text{ for } n \in N(n_2).$$

Then we can find $n_3 \geq n_2$ such that

$$\Delta\left(a_n(\Delta z_n)^\alpha\right) \leq -K_2 n_3. \quad (12)$$

Summing the inequality (12) from n_3 to $n-1$, we obtain

$$a_n(\Delta z_n)^\alpha - a_{n_3}(\Delta z_{n_3})^\alpha \leq -K_2 n_3(n-n_3) \text{ for } n \in N(n_3).$$

That is,

$$\Delta z_n \leq -(K_2 n_4)^{\frac{1}{\alpha}} \frac{1}{a_n^{\frac{1}{\alpha}}} \text{ for } n \geq n_3 \geq n_4. \quad (13)$$

Hence $z_n \leq -(K_2 n_3)^{\frac{1}{\alpha}} \sum_{s=n_3}^{n-1} \frac{1}{a_s^{\frac{1}{\alpha}}}$, for $n \in N(n_4)$.

III. Examples

Example 3.1. We consider

$$\Delta^2\left(\frac{1}{n^3}\left(\Delta\left(x_n + \frac{3}{4}x_{n-1} - \frac{1}{2}x_{n-2}\right)\right)^3\right) + \left(\frac{-14n^6 - 78n^5 - 62n^4 - 50n^3 + 66n^2 + 36n + 8}{n^3(n+1)^3(n+2)^3}\right){x_n}^3 = 0, \quad n \geq 1. \quad (14)$$

Here $a_n = \frac{1}{n^3}$, $p_n = \frac{3}{4}$, $q_n = \frac{1}{2}$, $r_n = -\left(\frac{14n^6 + 78n^5 + 62n^4 + 50n^3 - 66n^2 - 36n - 8}{n^3(n+1)^3(n+2)^3}\right)$,

$k = 1$, $l = 2$, $\alpha = 3$ and $f(x_n) = {x_n}^3$.

We note that (14) satisfies the Theorem 2.1. and $\{x_n\} = \{2^n\}$ is one of the solution of (14).

Example 3.1. We consider

$$\Delta^2\left(\frac{1}{n}\left(\Delta\left(x_n+\frac{3}{4}x_{n-1}-\frac{1}{4}x_{n-1}\right)\right)^3\right)+$$

$$\left(\frac{18n^{11}+154n^{10}+930n^9+3362n^8+6294n^7+5308n^6+5526n^5+18416n^4+28904n^3+16920n^2-1296n-3024}{(n-1)^3(n+1)^4(n+2)^4(n+3)^3}\right)x_n{}^4=0,$$

$$n\geq 2. \quad (15)$$

Here $a_n=\frac{1}{n},\ p_n=\frac{3}{4},\ q_n=\frac{1}{4},$

$$r_n=\frac{18n^{11}+154n^{10}+930n^9+3362n^8+6294n^7+5308n^6+5526n^5+18416n^4+28904n^3+16920n^2-1296n-3024}{(n-1)^3(n+1)^4(n+2)^4(n+3)^3},$$

$k=1,\ l=1,\ \alpha=3$ and $f(x_n)=x_n{}^4.$

Here (15) satisfies the Theorem 2.2. Thus all solutions of (15) are bounded and nonoscillatory. In fact, $\{x_n\}=\left\{\frac{1}{n}\right\}$ is one such solution of (15).

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Weak Alpha-Cuts in Fuzzy Sets

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Abstract: *Zadeh introduced the concept fuzzy sets as a generalization of classical sets in 1965. One of the most important concept of fuzzy sets is the concept of an Alpha-cut and its variant strong Alpha-cut. Recently the author investigated the properties of Alpha-cuts, strong Alpha-cuts and the topologies generated by these Alpha-cuts and strong Alpha-cuts . The purpose of this paper is to further investigate such properties. In this direction the concept of weak Alpha-cuts are introduced and its properties are investigated.*

Keywords*: Fuzzy sets;Alpha-cuts; Weak Alpha-cuts;support ; generated topologies:*

I. Introduction

Theory of fuzzy mathematics is based on the notion of relative graded membership as inspired by the process of human perception and cognition. Zadeh [12] published a paper on fuzzy sets in 1965 as the generalization of ordinary sets. Fuzzy set theory have applications in many directions. For example, Ananthanarayanan et.al. [1] discussed about the Fuzzy Graph Coloring using α-Cuts, Isabel et.al. [4] studied the applications of Topology to DNA and Sohrab Kordrostami et.al. [11] discussed the application of α cuts for interval data in DEA.Recently the author investigated [6]- [10] the properties of Alpha-cuts, strong Alpha-cuts and the topologies generated by these Alpha-cuts and strong Alpha-cuts. In the same way in this paper the author introduced a new concept of Weak Alpha-cuts in fuzzy set theory and investigated its properties.

II. Preliminaries

2.1 Definition

Let X be the universal set. Then a fuzzy subset A of X is defined by its membership function $\mu_A : X \to [0,1]$ which assigns a unique real number $\mu_A(x)$ in the interval [0,1] to each element $x \in X$ where the value of μ_A at x shows the membership grade of x in A.

2.2 Definition

Let $0 \leq \alpha \leq 1$ and A be a fuzzy subset of X. The Alpha-cut of A denoted by ${}^{\alpha}A$, is the crisp set ${}^{\alpha}A = \{ x \in X : A(x) \geq \alpha \}$. The strong Alpha-cut of A denoted by ${}^{\alpha+}A$, is the crisp set ${}^{\alpha+}A = \{ x \in X : A(x) > \alpha \}$.

2.3 Definition

The support of a fuzzy set A within the Universal set X is the crisp set that contains all the elements of X that have non-zero membership grades in A. It is denoted by Supp(A). Clearly the support of A is exactly the same as the strong Alpha-cut of A for $\alpha = 0$.

III. Topologies Generated by the Alpha-cuts and strong Alpha-cuts

Topologies generated by the Alpha-cuts and strong Alpha-cuts are already investigated by the author [6]-[10]. The following lemma will be useful in sequel.

3.1 Lemma [2], [5]

Let Ɓ and Ɓ' be bases for the topologies τ and τ ' respectively on X. Then the following are equivalent.
(i) τ ' is finer than τ .
(ii) For each $x \in$ X and each basis element B $\in$ Ɓ containing x , there is a basis element B' $\in$ Ɓ ' such that $x \in$ B' $\subset$ B.

The next theorem establishes that the collection T(A) of Alpha-cuts and the collection $T^+(A)$ of strong Alpha-cuts of a fuzzy subset A of X may generate the same topology on X. Let τ (T(A)) and τ ($T^+(A)$) denote the topologies generated by T(A) and $T^+(A)$ respectively.

.

3.2 Theorem

Let A be a fuzzy subset of a finite set X and the support of A be X. Then T(A) and $T^+(A)$ generate the same topology on X.

Proof:

Let A be a fuzzy subset of X with Supp(A) = X. To prove that T(A) and $T^+(A)$ generate the same topology on X. First let us prove that for each $\alpha \in$ [0, 1]and for each $x \in {}^{\alpha}$A there is a $\beta \in$ [0,1] such that $x \in {}^{\beta+}$A $\subseteq {}^{\alpha}$ A. Fix $x \in$ X. Let $\alpha \in$[0, 1] with $x \in {}^{\alpha}$A. If A (x) $>\alpha$ then $x \in {}^{\alpha+}$A and ${}^{\alpha+}$A $\subseteq {}^{\alpha}$A that implies $x \in {}^{\beta+}$A $\subseteq {}^{\alpha}$A. Let A(x) = α . Choose $\varepsilon > 0$ such that A(x) $> \alpha - \varepsilon = \beta$ that implies $x \in {}^{\beta+}$A. Now $\alpha > \beta$. Therefore ${}^{\beta}$A $\subseteq {}^{\alpha}$A which shows ${}^{\beta+}$A $\subseteq {}^{\alpha}$A that implies $x \in {}^{\beta+}$A $\subseteq {}^{\alpha}$A. Therefore τ ($T^+(A)$) is finer than τ (T(A)).

Now let us prove that for each $\alpha \in$ [0, 1] and for each $x \in {}^{\alpha+}$A there is a $\beta \in$ [0,1] such that $x \in {}^{\beta}$A $\subseteq {}^{\alpha+}$A. Fix $x \in$ X. Let $\alpha \in$ [0, 1] with $x \in {}^{\alpha+}$A. Then A (x)$>\alpha$. Choose $\varepsilon > 0$ such that A(x) $\geq \alpha +$ $\varepsilon = \beta$ that implies $x \in {}^{\beta}$A. Let y $\in {}^{\beta}$A. Then A (y) $\geq \beta > \alpha$ which implies y $\in {}^{\alpha+}$A. Hence y $\in {}^{\beta}$A $\subseteq {}^{\alpha+}$ A. By using Lemma 3.1, T(A) and $T^+(A)$ generate the same topology on X.

3.3 Remark

If the support of A is a proper subset of X then the two topologies generated by T(A) and $T^+(A)$ need not be the same as shown in the following example.

3.4 Example

Let X = {a, b, c, d} and A $= \frac{0.7}{a} + \frac{0.2}{b} + \frac{0.4}{d}$

$$\text{Then } {}^{\alpha}A = \begin{cases} X & \text{for } \alpha = 0 \\ \{a, b, d\} & \text{for } 0 < \alpha \leq 0.2 \\ \{a, d\} & \text{for } 0.2 < \alpha \leq 0.4 \\ \{a\} & \text{for } 0.4 < \alpha \leq 0.7 \\ \varphi & \text{for } 0.7 < \alpha \leq 1 \end{cases}$$

Therefore T (A) = { $\varnothing$, X, {a, b, d}, { a, d }, {a}}.

$$\text{Also } {}^{\alpha+}A = \begin{cases} \{a, b, d\} & \text{for } 0 \leq \alpha < 0.2 \\ \{a, d\} & \text{for } 0.2 \leq \alpha < 0.4 \\ \{a\} & \text{for } 0.4 \leq \alpha < 0.7 \\ \varphi & \text{for } 0.7 \leq \alpha \leq 1 \end{cases}$$

Therefore T^+ (A) = { $\varnothing$, { a, d }, {a, b, d }, {a}}.

We have seen that if the support of a fuzzy subset A is X then T(A) and $T^+(A)$ generate the same Topology on X.

IV. Weak Alpha-cut

4.1 Definition

Let X be a non-empty set and A be the fuzzy subset of X. Then ${}^{\alpha-}A = \{ x \in X : A(x) < \alpha\} = X - \{ x \in X : A(x) \geq \alpha \} = X - {}^{\alpha}A$ is called a weak Alpha-cut of A.

The following Lemma and Proposition will be useful to investigate the properties of Weak Alpha-cuts.

4.2 Lemma

Let A, B be the fuzzy subsets of X and $\alpha, \beta \in [0, 1]$. Then

(i) ${}^{\alpha+}A \subseteq {}^{\alpha}A$.

(ii) $\alpha \leq \beta$ implies ${}^{\alpha}A \supseteq {}^{\beta}A$ and ${}^{\alpha+}A \supseteq {}^{\beta+}A$.

(iii) ${}^{\alpha}(A \cap B) = {}^{\alpha}A \cap {}^{\alpha}B$ and ${}^{\alpha}(A \cup B) = {}^{\alpha}A \cup {}^{\alpha}B$.

(iv) ${}^{\alpha+}(A \cap B) = {}^{\alpha+}A \cap {}^{\alpha+}B$ and ${}^{\alpha+}(A \cup B) = {}^{\alpha+}A \cup {}^{\alpha+}B$.

(v) ${}^{\alpha}(A') = X - {}^{(1-\alpha)+}A =$ the crisp complement of ${}^{(1-\alpha)+}A$

(vi) $A \subseteq B$ if and only if ${}^{\alpha}A \subseteq {}^{\alpha}B$ and $A \subseteq B$ if and only if ${}^{\alpha+}A \subseteq {}^{\alpha+}B$.

(vii) $A = B$ if and only if ${}^{\alpha}A = {}^{\alpha}B$ and $A = B$ if and only if ${}^{\alpha+}A = {}^{\alpha+}B$.

(viii) ${}^{\alpha}A = \bigcap_{\beta \prec \alpha} {}^{\beta}A = \bigcap_{\beta \prec \alpha} {}^{\beta+}A$.

(ix) ${}^{\alpha+}A = \bigcup_{\alpha \prec \beta} {}^{\beta}A = \bigcup_{\alpha \prec \beta} {}^{\beta+}A$.

Let A_i be the family of fuzzy subsets of X and for all $i \in I$, where I is an index set. Then

(x) $\bigcup_{i \in I} {}^{\alpha}A_i \subseteq {}^{\alpha}\left(\bigcup_{i \in I} Ai \right)$ and $\bigcap_{i \in I} {}^{\alpha}A_i = {}^{\alpha}\left(\bigcap_{i \in I} A_i \right)$.

(xi) $\bigcup_{i \in I} {}^{\alpha+}A_i = {}^{\alpha+}\left(\bigcup_{i \in I} Ai \right)$ and $\bigcap_{i \in I} {}^{\alpha+}A_i \supseteq {}^{\alpha+}\left(\bigcap_{i \in I} A_i \right)$.

4.3 Proposition

Let A be a fuzzy subset of X and $0 \leq \alpha_1, \alpha_2, \ldots, \alpha_n \leq 1$. Then

(i) ${}^{\alpha_1}A \cup {}^{\alpha_2}A \cup {}^{\alpha_3}A \cup \ldots \cup {}^{\alpha_n}A = {}^{\beta}A$ Where $\beta = \min \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ and

(ii) ${}^{\alpha_1}A \cap {}^{\alpha_2}A \cap {}^{\alpha_3}A \cap \ldots \cap {}^{\alpha_n}A = {}^{\gamma}A$ Where $\gamma = \max \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$.

4.4 Properties

Let A and B be the fuzzy subsets of X and $\alpha, \beta \in [0,1]$ then

(i) ${}^{0-}A = \varnothing$.

(ii) ${}^{1-}A = X - \{ x : A(x) = 1 \}$

(iii) ${}^{\alpha-}A = X - {}^{\alpha}A$

(iv) ${}^{\alpha-}A \cap {}^{\alpha}A = \varnothing$

(v) If $\alpha \leq \beta$ then ${}^{\alpha-}A \subseteq {}^{\beta-}A$

(vi) ${}^{\alpha-}(A \cap B) = {}^{\alpha-}A \cup {}^{\beta-}A$

(vii) ${}^{\alpha-}(A \cup B) = {}^{\alpha-}A \cap {}^{\beta-}A$

(viii) ${}^{\alpha-}\left(\bigcup_{i\in I} Ai\right) \subseteq \bigcap_{i\in I} {}^{\alpha-}A_i$

(ix) $\bigcup_{i\in I} {}^{\alpha-}A_i = {}^{\alpha-}\left(\bigcap_{i\in I} A_i\right)$.

(x) $A \subseteq B$ if and only if ${}^{\alpha-}B \subseteq {}^{\alpha-}A$

(xi) $A = B$ if and only if ${}^{\alpha-}A = {}^{\alpha-}B$

(xii) ${}^{\alpha-}A = \bigcup_{\beta\prec\alpha} {}^{\beta-}A$

(xiii) ${}^{\alpha-}A \cup {}^{\beta-}A = \begin{cases} {}^{\beta-}A & if\ \alpha \leq \beta \\ {}^{\alpha-}A & otherwise \end{cases}$

(xiv) ${}^{\alpha-}A \cap {}^{\beta-}A = \begin{cases} {}^{\alpha-}A & if\ \alpha \leq \beta \\ {}^{\beta-}A & otherwise \end{cases}$

Let A be a fuzzy subset of X and $0 \leq \alpha_1, \alpha_2, \ldots, \alpha_n \leq 1$. Then

(xv) ${}^{\alpha_1-}A \cap {}^{\alpha_2-}A \cap {}^{\alpha_3-}A \cap \ldots\ldots \cap {}^{\alpha_n-}A = {}^{\beta-}A$ Where $\beta = \min\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ and

(xvi) ${}^{\alpha_1-}A \cup {}^{\alpha_2-}A \cup {}^{\alpha_3-}A \cup \ldots\ldots\ldots \cup {}^{\alpha_n-}A = {}^{\gamma-}A$ Where $\gamma = \max\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$.

Proof :

Let A and B be the fuzzy subsets of X and $\alpha, \beta \in [0,1]$ then

${}^{0-}A = \{x : A(x) < 0\} = \varnothing$. This proves (i).

${}^{1-}A = \{x : A(x) < 1\}$
$= X - \{x : A(x) \geq 1\}$
$= X - \{x : A(x) = 1\}$. This proves (ii).

${}^{\alpha-}A = \{x : A(x) < \alpha\}$
$= X - \{x : A(x) \geq \alpha\}$
$= X - {}^{\alpha}A$. This proves (iii).

If $x \in {}^{\alpha-}A \cap {}^{\alpha}A$ then $x \in {}^{\alpha-}A$ and $x \in {}^{\alpha}A$ that implies $A(x) < \alpha$ and $A(x) \geq \alpha$ which is not possible. That implies ${}^{\alpha-}A \cap {}^{\alpha}A = \varnothing$. This proves (iv).

If $\alpha \leq \beta$ then by using Lemma 4.2 (ii), ${}^{\beta}A \subseteq {}^{\alpha}A$ that implies $X - {}^{\beta}A \supseteq X - {}^{\alpha}A$ gives ${}^{\beta-}A \supseteq {}^{\alpha-}A$. Hence ${}^{\alpha-}A \subseteq {}^{\beta-}A$. This proves (v).

By using Lemma 4.2 (iii),

${}^{\alpha}(A \cap B) = {}^{\alpha}A \cap {}^{\alpha}B \Leftrightarrow X - {}^{\alpha}(A \cap B) = X - {}^{\alpha}A \cap {}^{\alpha}B$
$\Leftrightarrow {}^{\alpha-}(A \cap B) = (X - {}^{\alpha}A) \cup (X - {}^{\alpha}B)$
$\Leftrightarrow {}^{\alpha-}(A \cap B) = {}^{\alpha-}A \cup {}^{\alpha-}B$. This proves (vi).

By using Lemma 4.2 (iii),

${}^{\alpha}(A \cup B) = {}^{\alpha}A \cup {}^{\alpha}B \Leftrightarrow X - {}^{\alpha}(A \cup B) = X - {}^{\alpha}A \cup {}^{\alpha}B$
$\Leftrightarrow {}^{\alpha-}(A \cup B) = (X - {}^{\alpha}A) \cap (X - {}^{\alpha}B)$
$\Leftrightarrow {}^{\alpha-}(A \cup B) = {}^{\alpha-}A \cap {}^{\alpha-}B$. This proves (vii).

By using Lemma 4.2 (x),

$$\bigcup_{i\in I}{}^{\alpha}A_i \subseteq {}^{\alpha}\left(\bigcup_{i\in I} Ai\right) \Leftrightarrow X - \bigcup_{i\in I}{}^{\alpha}A_i \supseteq X - {}^{\alpha}\left(\bigcup_{i\in I} Ai\right)$$

$$\Leftrightarrow \bigcap_{i\in I}(X - {}^{\alpha}A_i) \supseteq {}^{\alpha-}\left(\bigcup_{i\in I} Ai\right)$$

$$\Leftrightarrow \bigcap_{i\in I}{}^{\alpha-}A_i \supseteq {}^{\alpha-}\left(\bigcup_{i\in I} Ai\right)$$

$$\Leftrightarrow {}^{\alpha-}\left(\bigcup_{i\in I} Ai\right) \subseteq \bigcap_{i\in I}{}^{\alpha-}A_i$$

. This proves (viii).

By using Lemma 4.2 (x),

$$\bigcap_{i\in I}{}^{\alpha}A_i = {}^{\alpha}\left(\bigcap_{i\in I}A_i\right) \Leftrightarrow X - \bigcap_{i\in I}{}^{\alpha}A_i = X - {}^{\alpha}\left(\bigcap_{i\in I}A_i\right)$$

$$\Leftrightarrow \bigcup_{i\in I}(X - {}^{\alpha}A_i) = {}^{\alpha-}\left(\bigcap_{i\in I}A_i\right)$$

$$\Leftrightarrow \bigcup_{i\in I}{}^{\alpha-}A_i = {}^{\alpha-}\left(\bigcap_{i\in I}A_i\right)$$

. This proves (ix).

By using Lemma 4.2 (vi),

$A \subseteq B \Leftrightarrow {}^{\alpha}A \subseteq {}^{\alpha}B$

$\Leftrightarrow X - {}^{\alpha}A \supseteq X - {}^{\alpha}B$

$\Leftrightarrow {}^{\alpha-}A \supseteq {}^{\alpha-}B$. This proves (x).

By using Lemma 4.2 (vii),

$A = B \Leftrightarrow {}^{\alpha}A = {}^{\alpha}B$

$\Leftrightarrow X - {}^{\alpha}A = X - {}^{\alpha}B$

$\Leftrightarrow {}^{\alpha-}A = {}^{\alpha-}B$. This proves (xi).

By using Lemma 4.2 (viii),

$${}^{\alpha}A = \bigcap_{\beta\prec\alpha}{}^{\beta}A \Leftrightarrow X - {}^{\alpha}A = X - \bigcap_{\beta\prec\alpha}{}^{\beta}A$$

$$\Leftrightarrow {}^{\alpha-}A = \bigcup_{\alpha\prec\beta}(X - {}^{\beta}A)$$

$$\Leftrightarrow {}^{\alpha-}A = \bigcup_{\alpha\prec\beta}{}^{\beta-}A$$

. This proves (xii).

Since $\alpha \leq \beta$ by using (v), ${}^{\alpha-}A \subseteq {}^{\beta-}A$ that implies ${}^{\alpha-}A \cup {}^{\beta-}A = {}^{\beta-}A$.. If $\alpha > \beta$ then ${}^{\alpha-}A \cup {}^{\beta-}A = {}^{\alpha-}A$. This proves (xiii).

Also if ${}^{\alpha-}A \subseteq {}^{\beta-}A$ then ${}^{\alpha-}A \cap {}^{\beta-}A = {}^{\alpha-}A$. If $\alpha > \beta$ then ${}^{\alpha-}A \cup {}^{\beta-}A = {}^{\beta-}A$. This proves (xiv).

By using Proposition 4.3 (i), ${}^{\alpha_1}A \cup {}^{\alpha_2}A \cup {}^{\alpha_3}A \cup ... \cup {}^{\alpha_n}A = {}^{\beta}A$ where $\beta = \min\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ that implies $X - ({}^{\alpha_1}A \cup {}^{\alpha_2}A \cup {}^{\alpha_3}A \cup ... \cup {}^{\alpha_n}A) = X - {}^{\beta}A$ which implies ${}^{\alpha_1-}A \cap {}^{\alpha_2-}A \cap {}^{\alpha_3-}A \cap ... \cap {}^{\alpha_n-}A = {}^{\beta-}A$ where $\beta = \min\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$. This proves (xv).

By using Proposition 4.3 (ii), ${}^{\alpha_1}A \cap {}^{\alpha_2}A \cap {}^{\alpha_3}A \cap ... \cap {}^{\alpha_n}A = {}^{\gamma}A$ where $\gamma = \max\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ that implies $X - ({}^{\alpha_1}A \cap {}^{\alpha_2}A \cap {}^{\alpha_3}A \cap ... \cap {}^{\alpha_n}A) = X - {}^{\gamma}A$ which implies ${}^{\alpha_1-}A \cup {}^{\alpha_2-}A \cup {}^{\alpha_3-}A \cup ... \cup {}^{\alpha_n-}A = {}^{\gamma-}A$ where $\gamma = \max\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$. This proves (xvi).

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