



On Complex Valued Rectangular b -metric Spaces

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In this paper, we have discussed the concept of complex valued rectangular b -metric space in more general approach. Ozgur Ege introduced complex valued rectangular b -metric space and also proved Banach contraction principle in this space. We have extended some results using complex valued rectangular b -metric space.

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1. Introduction

The Metric fixed point theory is one of the useful branch based on the use of fixed point theorem to establish existence and uniqueness of a solutions. The Banach contraction principle have been widely generalized theorem by many authors in this field according to their requirement by changing or imposing some new conditions on contraction. In this process, Branciari introduced the concept of rectangular metric space which contains change in the triangular inequality condition of definition of metric space by four point expression. This resulted into many generalizations of the different theorems in Rectangular metric space [RMS]. [See [3], [8] and references therein] In 1989, Bakhtin had introduced the concept of b -metric space. Which has different condition on the constant α . In this b -metric space, many existing results then have been generalized and proved.(see [3], [5], [6], [7], [8], [9], [10], [11], [12] and references therein). In 2015, R George Et. Al. introduced the mixed concept of b -metric space and rectangular metric

2 Preliminaries

In this section, we discuss some theorems, definitions and examples which explains the required background of the main results.

Definition 2.1. Let M be the non empty set and d be the mapping from $X \times X$ to $[0, \infty)$.

Then d is called metric on M if;

(d1): $d(x, y) = 0 \iff x = y$, for all $(x, y) \in X \times X$

(d2): $d(x, y) = d(y, x)$, for all $(x, y) \in X \times X$

(d3): $d(x, y) \leq [d(x, z) + d(z, y)]$ for all $(x, y, z) \in X \times X \times X$

Definition 2.2. Let X be a non empty set and $d : X \times X \rightarrow [0, \infty)$ be a mapping. Then the mapping d is called b -metric space if following conditions are satisfied:

(b1): $d(x, y) = 0$ iff $x = y$ for all $x, y \in X$;

(b2): $d(x, y) = d(y, x)$ for all $x, y \in X$;

(b3): there exists a real number $s \geq 1$ such that $d(x, y) \leq s[d(x, z) + d(z, y)]$, here (X, d) is called b -metric space. [bMS]

Definition 2.3. Let X be a non empty set and $d : X \times X \rightarrow [0, \infty)$ be a mapping. Then the mapping d is called rectangular metric if following conditions are satisfied:

(R1): $d(x, y) = 0$ iff $x = y$ for all $x, y \in X$;

(R2): $d(x, y) = d(y, x)$ for all $x, y \in X$;

(R3): $d(x, y) \leq [d(x, u) + d(u, v) + d(v, y)]$, for all $x, y \in X$ and all distinct $u, v \in X \setminus \{x, y\}$, here (X, d) is called rectangular metric space. [RMS]

Definition 2.4. Let X be a non empty set and $d : X \times X \rightarrow [0, \infty)$ be a mapping. Then the mapping d is called rectangular b-metric if following conditions are satisfied:

(Rb1): $d(x, y) = 0$ iff $x = y$ for all $x, y \in X$;

(Rb2): $d(x, y) = d(y, x)$ for all $x, y \in X$;

(Rb3): there exists a real number $s \geq 1$ such that $d(x, y) \leq s[d(x, u) + d(u, v) + d(v, y)]$, for all $x, y \in X$ and all distinct $u, v \in X \setminus \{x, y\}$,

here (X, d) is called rectangular b-metric space. [RbMS]

Example 2.5. Let $X = \mathbb{N}$, define $d : X \times X \rightarrow X$ as follows;

$$d(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 3\alpha & \text{if } x, y \in \{1, 3\}, x \neq y \\ \alpha/2 & \text{if } x \text{ or } y \notin \{1, 3\}, x \neq y \end{cases}$$

Here (X, d) is a rectangular b-metric space with coefficient $s = 2 > 1$. $\alpha > 0$ is a constant.

But not a rectangular metric space, since $d(1, 3) = 3\alpha$,

$$\begin{aligned} d(1, 3) &\leq d(1, 4) + d(4, 5) + d(5, 3) \\ &\leq \alpha/2 + \alpha/2 + \alpha/2 \\ &\leq 3\alpha/2 < 3\alpha. \end{aligned}$$

Here we note that every metric space is a b-metric space, every b-metric space is a rectangular b-metric space but converse is not true. See example 1.5 and 1.7 of [1].

Definition 2.6. [2] Let (X, d) be a rectangular metric space, $\{x_n\}$ be a sequence in X and let $x \in X$

- The sequence $\{x_n\}$ is said to be a convergent in (X, d) and converges to x if for every $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x) < \epsilon$ for all $n > n_0$, can also be written as $x_n \rightarrow x$ as $n \rightarrow \infty$.
- If for every $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x_{n+p}) < \epsilon$ for all $n > n_0$, $p > 0$. i.e $\lim_{n \rightarrow \infty} d(x_n, x_{n+p}) = 0$ for all $p > 0$, then the sequence $\{x_n\}$ is called Cauchy sequence.
- The metric space (X, d) is said to be a complete rectangular b-metric space if every Cauchy sequence in X converges to some $x \in X$.

In 2011, the concept of complex valued metric space was introduced by Azam, Fisher and Khan [3]. According to Azam et al.

Let $\mathbb{C}$ be the set of complex numbers and $u, v \in \mathbb{C}$. Define a partial order $\preceq$ on $\mathbb{C}$ as follows ;

We have $u \preceq v$ if and only if $Re(u) \leq Re(v)$ and $Im(u) \leq Im(v)$.

It follows that, $u \preceq v$ if one of the following conditions is satisfied;

- (P1) $Re(u) = Re(v), Im(u) < Im(v)$,
- (P2) $Re(u) < Re(v), Im(u) = Im(v)$,
- (P3) $Re(u) < Re(v), Im(u) < Im(v)$,
- (P4) $Re(u) = Re(v), Im(u) = Im(v)$

In particular, we will write $u \prec v$ if $u \neq v$ and one of (P1), (P2), and (P3) is satisfied and we will write $u \preceq v$ if only (P3) is satisfied.

Definition 2.7. [2] Let (X, d) be a complex valued rectangular b-metric space and let $\{x_n\}$ be the sequence in X converges to $x \in X$. Then;

- The sequence $\{x_n\}$ is said to be a complex valued convergent in (X, d) and converges to x if for every $\epsilon \succ 0$, there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x) \succ \epsilon$ for all $n > n_0$, can also be written as $x_n \rightarrow x$ as $n \rightarrow \infty$.
- If for every $p \succ 0$, $\lim_{n \rightarrow \infty} d(x_n, x_{n+p}) = 0$, the sequence $\{x_n\}$ is called complex valued Cauchy sequence in (X, d) .
- The metric space (X, d) is said to be a complex valued complete rectangular b-metric space if every complex valued Cauchy sequence in X converges to some $x \in X$.

3 Main Results

Ozgur Ege [2] applied this concept to prove the well known Banach contraction principle in complex valued rectangular b-metric space (X, d) . In the theorem 3.6 [2], Banach contraction principle is proved by taking $T : X \rightarrow X$ as a mapping over the complex valued complete rectangular b-metric space (X, d) which satisfies

$d(Tx, Ty) \lesssim \alpha d(x, y)$ with positive constant $s > 1$ and $\alpha \in [0, \frac{1}{s}]$, Then T has unique fixed point. Due to this result many researchers generalized the existing results in complex valued rectangular b-metric space. and also generated new results.

Theorem 3.1. *Let (X, d) be the complex valued complete rectangular b-metric space with $p > 1$ and $T : X \rightarrow X$ be the mapping satisfying;*

$$d(Tx, Ty) \lesssim \alpha [d(x, Tx) + d(y, Ty)] \quad (3.1)$$

for all $x, y \in X$ where $\alpha \in [0, \frac{1}{p+1}]$, Then T has unique fixed point.

Proof. Let $x_0 \in X$ and T satisfies 3.1. Now define the sequence $\{x_n\}$ by $Tx_n = x_{n+1}$ for all $n \geq 0$. Now our work is to prove $\{x_n\}$ is cauchy's sequence. If $x_n = x_{n+1}$ which implies $Tx_n = x_n$ then $\{x_n\}$ is a fixed point and we are done.

Let $x_n \neq x_{n+1}$, for all $n \geq 0$, then from 3.1 we have,

$$\begin{aligned} d(x_n, x_{n+1}) &\lesssim d(Tx_{n-1}, Tx_n) \\ &\lesssim \alpha [d(x_{n-1}, Tx_{n-1}) + d(x_n, Tx_n)] \\ &\lesssim \alpha [d(x_{n-1}, x_n) + d(x_n, x_{n+1})] \\ (1 - \alpha)d(x_n, x_{n+1}) &\lesssim \alpha d(x_{n-1}, x_n) \\ d(x_n, x_{n+1}) &\lesssim \frac{\alpha}{(1 - \alpha)} d(x_{n-1}, x_n) \\ d(x_n, x_{n+1}) &\lesssim \beta d(x_{n-1}, x_n) \end{aligned} \quad (3.2)$$

$$\text{where } \beta = \frac{\alpha}{(1-\alpha)} < \frac{1}{p} \implies p\alpha < 1 - \alpha \implies \alpha < \frac{1}{p+1}$$

On successive repetition on 3.2, we get,

$$d(x_n, x_{n+1}) \lesssim \beta^n d(x_0, x_1) \quad (3.3)$$

Obviously if x_0 is a periodic point of T then we have, $x_0 = x_n$

Then from 3.3, we have,

$$\begin{aligned} d(x_0, Tx_0) &= d(x_n, Tx_n) \\ d(x_0, x_1) &= d(x_n, x_{n+1}) \\ &\lesssim \beta^n d(x_0, x_1) \end{aligned}$$

$$\implies d(x_0, x_1) = 0 \implies x_0 = x_1 \text{ Therefore } x_0 \text{ is a fixed point of } T.$$

Secondly, for any $n \in \mathbb{N}$ and all $n \neq m \in \mathbb{N}$, we have from 3.1 and 3.3,

$$\begin{aligned} d(x_n, x_{n+2}) &= d(Tx_{n-1}, Tx_{n+1}) \\ &\lesssim \alpha [d(x_{n-1}, Tx_{n-1}) + d(x_{n+1}, Tx_{n+1})] \\ &\lesssim \alpha [d(x_{n-1}, x_n) + d(x_{n+1}, x_{n+2})] \\ &\lesssim \alpha [\beta^{n-1} d(x_0, x_1) + \beta^{n+1} d(x_0, x_1)] \\ &\lesssim \alpha \beta^{n-1} [1 + \beta^2] d(x_0, x_1) \\ d(x_n, x_{n+2}) &\lesssim \gamma \beta^{n-1} d(x_0, x_1), \text{ where } \gamma = \alpha [1 + \beta^2] \end{aligned} \quad (3.4)$$

So we proved, $d(x_n, x_{n+1}) \lesssim \beta^n d(x_0, x_1)$

$$d(x_n, x_{n+2}) \lesssim \gamma \beta^{n-1} d(x_0, x_1)$$

Now consider in general $d(x_n, x_{n+k})$ for even, odd k .

Case (i) : Let $k=2m+1$ be an odd, then by 3.2,

$$\begin{aligned}
 d(x_n, x_{n+k}) &= d(x_n, x_{n+2m+1}) \\
 &\preceq \alpha[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+2m+1})] \\
 &\preceq \alpha[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})] + \alpha^2[d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4}) + d(x_{n+4}, x_{n+2m+1})] \\
 &\vdots \\
 &\preceq \alpha[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})] + \alpha^2[d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4})] \\
 &\quad + \alpha^3[d(x_{n+4}, x_{n+5}) + d(x_{n+5}, x_{n+6})] + \cdots + \alpha^n[d(x_{n+2m}, x_{n+2m+1})] \\
 &\preceq \alpha[\beta^n d(x_0, x_1) + \beta^{n+1} d(x_0, x_1)] + \alpha^2[\beta^{n+2} d(x_0, x_1) + \beta^{n+3} d(x_0, x_1)] + \cdots + \alpha^n \beta^{n+2m} d(x_0, x_1) \\
 &\preceq \alpha \beta^n [1 + \beta + \alpha \beta^2 + \alpha \beta^3 + \alpha^2 \beta^4 + \alpha^2 \beta^5 + \cdots + \alpha^{n-1} \beta^{2m}] d(x_0, x_1) \\
 &\preceq \alpha \beta^n d(x_0, x_1) [(1 + \beta) + \alpha \beta^2 (1 + \beta) + \alpha^2 \beta^4 (1 + \beta) + \cdots] \\
 &\preceq \frac{\alpha \beta^n (1 + \beta)}{1 - \alpha \beta^2} d(x_0, x_1), \quad \text{here, } \alpha \beta^2 < 1
 \end{aligned}$$

$\therefore$ We get,

$$d(x_n, x_{n+k}) \preceq \frac{\alpha \beta^n (1 + \beta)}{1 - \alpha \beta^2} d(x_0, x_1), \quad (3.5)$$

Where $k = 2m + 1$ an odd number.

Case (ii) : If $k = 2m$, an even number.

Then we have from 3.3 and 3.4,

$$\begin{aligned}
 d(x_n, x_{n+k}) &= d(x_n, x_{n+2m}) \\
 &\lesssim \alpha [d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+2m})] \\
 &\lesssim \alpha [d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})] + \alpha^2 [d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4}) + d(x_{n+4}, x_{n+2m})] \\
 &\lesssim \alpha [d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})] + \alpha^2 [d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4})] \\
 &\quad + \alpha^3 [d(x_{n+4}, x_{n+5}) + d(x_{n+5}, x_{n+6})] + \dots + \alpha^{m-1} [d(x_{2m-4}, x_{2m-3}) + d(x_{2m-3}, x_{2m-2})] \\
 &\quad + \alpha^{m-1} [d(x_{n+2m-2}, x_{n+2m})] \\
 &\lesssim \alpha [\beta^n d(x_0, x_1) + \beta^{n+1} d(x_0, x_1)] + \alpha^2 [\beta^{n+2} d(x_0, x_1) + \beta^{n+3} d(x_0, x_1)] + \dots \\
 &\quad + \alpha^{m-1} [\beta^{2m-4} d(x_0, x_1) + \beta^{2m-3} d(x_0, x_1)] + \alpha^{m-1} [\gamma \beta^{n+2m-3} d(x_0, x_1)], \text{ from [3.4]} \\
 &\lesssim \alpha \beta^n d(x_0, x_1) [1 + \beta + \alpha \beta^2 + \alpha \beta^3 + \dots + \alpha^{m-2} \beta^{2m-n-4} + \alpha^{m-2} \beta^{2m-n-3}] \\
 &\quad + \alpha^{m-1} \gamma \beta^{2m+n-3} d(x_0, x_1) \\
 &\lesssim \alpha \beta^n d(x_0, x_1) [(1 + \beta) + \alpha \beta^2 (1 + \beta) + \dots] + \alpha^{m-1} \gamma \beta^{2m+n-3} d(x_0, x_1) \\
 &\lesssim \frac{\alpha \beta^n d(x_0, x_1) (1 + \beta)}{1 - \alpha \beta^2} + \alpha^{m-1} \gamma \beta^{2m+n-3} d(x_0, x_1)
 \end{aligned}$$

$$\therefore \text{As } n \rightarrow \infty, d(x_n, x_{n+2m}) = 0 \quad (3.6)$$

$$\text{In general, } \lim_{n \rightarrow \infty} d(x_n, x_{n+k}) = 0, \text{ for all } k > 0 \quad (3.7)$$

$\Rightarrow x_n$ is a Cauchy's sequence in X

Since (X, d) is a complex valued complete rectangular b-metric space, $\exists x^* \in X$ such that,

$$\lim_{n \rightarrow \infty} x_n = x^* \quad (3.8)$$

Now we claim that x^* is a fixed point of T

Let for any $n \in \mathbb{N}$

$$\begin{aligned}
 d(x^*, Tx^*) &\lesssim p[d(x^*, x_n) + d(x_n, x_{n+1}) + d(x_{n+1}, Tx^*)] \\
 &\lesssim p[d(x^*, x_n) + d(x_n, x_{n+1}) + d(Tx_n, Tx^*)] \\
 &\lesssim p[d(x^*, x_n) + d(x_n, x_{n+1}) + \alpha[d(x_n, Tx_n) + d(x^*, Tx^*)]] \\
 &\lesssim p[d(x^*, x_n) + d(x_n, x_{n+1})] + p\alpha[d(x_n, Tx_n)] + p\alpha d(x^*, Tx^*). \\
 (1 - p\alpha)d(x^*, Tx^*) &\lesssim p[d(x^*, x_n) + d(x_n, x_{n+1})] + p\alpha[d(x_n, Tx_n)]
 \end{aligned}$$

$$d(x_n, x_{n+1}) = 0 \text{ as } n \rightarrow \infty \text{ from 3.7}$$

$$d(x_n, x^*) = 0 \text{ as } n \rightarrow \infty \text{ from 3.8}$$

$$\implies d(x^*, Tx^*) = 0$$

$$\implies x^* = Tx^*, \text{ therefore } x^* \text{ is a fixed point.}$$

Finally to show the uniqueness of a fixed point, Let y^* be the another fixed point such that $Ty^* = y^*$.

Then,

$$\begin{aligned} d(x^*, y^*) &= d(Tx^*, Ty^*) \preceq \alpha [d(x^*, Tx^*) + d(y^*, Ty^*)] \\ &\preceq \alpha [d(x^*, x^*) + d(y^*, y^*)] \\ &\preceq 0 \end{aligned}$$

$$\implies d(x^*, y^*) = 0$$

$$\implies x^* = y^*, \text{ proves that fixed point } x^* \text{ is unique.}$$

□

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