



A Common Fixed point Theorem for Three Self-Mappings in Cone Metric Space

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Abstract: The aim of this paper is to prove a coincidence and common fixed point theorem of three self-mappings satisfying contractive type condition (A) in cone metric space.

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I. Introduction and Preliminaries

Huang & Zhang [2] familiarized the idea of cone metric space and prove some fixed point theorems for contractive type mappings in a normal cone metric space. Subsequently, some other authors [1, 3 to 6] studied the existence of fixed points of self-mappings satisfying a contractive type condition. Here, we obtain points of coincidence and common fixed points for three self-mappings satisfying condition (A) in a complete cone metric space.

Definition 1.1 (see [1]): A subset P of a real Banach space E is called a cone if it has the following properties:

1. P is non-empty, closed and $P \neq \{0\}$
2. $0 \leq a, b \in \mathbb{R}$ and $u, v \in P \Rightarrow au + bv \in P$;
3. $u \in P$ and $-u \in P \Rightarrow u = 0 \Leftrightarrow P \cap (-P) = \{0\}$.

Definition 1.2 (see [1]): For a given cone $P \subseteq E$, we can define a partial ordering $\leq$ on E with respect to P by $u \leq v$ if and only if $v - u \in P$. We shall write $u < v$ if $u \leq v$ while $u \ll v$ stands for $v - u \in P^0$ where P^0 denotes the interior of P . The cone P is called normal if for some $K > 0$ for $u, v \in E$,

$$0 \leq u \leq v \Rightarrow \|u\| \leq K\|v\| \quad (1.1)$$

The least positive number K satisfying (1.1) is called the normal constant of P .

In the following, we always suppose that E is a real Banach space and P is a cone in E with $\text{int}P \neq \emptyset$ and $\leq$ is a partial ordering with respect to P .

Proposition 1.3 (see [7]): Let P be a cone in a real Banach space E . If for $b \in P$ and $b \leq \gamma b$, for some $\gamma \in [0, 1)$ then $b = 0$.

Proposition 1.4 (see [7]): Let P be a cone in a real Banach space E with non-empty interior. If for $b \in E$ and $b \ll c$, for all $c \in P^0$, then $b = 0$.

Definition 1.5 (see [1]): Let X be a nonempty set. Suppose that the mapping $d: X \times X \rightarrow E$ satisfies

1. $0 \leq d(u, v), \forall u, v \in X$ and $d(u, v) = 0 \Leftrightarrow u = v$;
2. $d(u, v) = d(v, u), \forall u, v \in X$;
3. $d(u, v) \leq d(u, w) + d(w, v), \forall u, v, w \in X$.

Then d is called a cone metric on X , and (X, d) is called a cone metric space.

Example 1.6 (see [1]): Let $E = \mathbb{R}^2$, $P = \{(u, v) \in E : u, v \geq 0\} \subset \mathbb{R}^2$, $X = \mathbb{R}$ and $d: X \times X \rightarrow E$ such that $d(u, v) = (|u - v|, \alpha|u - v|)$, where $\alpha \geq 0$ is a constant. Then (X, d) is cone metric space.

Definition 1.7 (see [1]): Let $\{u_n\}$ be a sequence in X and $u \in X$. If for each $0 \ll c$, $\exists n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d(u_n, u) \ll c$ then $\{u_n\}$ is said to be convergent (or $\{u_n\}$ converges) to u and u is called the limit of $\{u_n\}$. We denote this by $\lim_{n \rightarrow \infty} u_n = u$ or $u_n \rightarrow u$ as $n \rightarrow \infty$. If for each $0 \ll c \exists n_0 \in \mathbb{N}$ such that for all $n, m > n_0$, $d(u_n, u_m) \ll c$, then $\{u_n\}$ is called a Cauchy sequence in X . If every Cauchy sequence is convergent in X , then X is called a complete cone metric space.

Definition 1.8 (see [1]): A pair (f, T) of self-mappings on X one said to be weakly compatible if they commute at their coincidence point i.e. $fTu = Tfu$ whenever $fu = Tu$.

Definition 1.9 (see [1]): A point $v \in X$ is called a point of coincidence of f and T if $\exists$ a point $u \in X$ such that $v = fu = Tu$.

Condition (A): Let (X, d) be a cone metric space, P be a normal cone with normal constant K and $S, T, f: X \rightarrow X$ are three self-mappings. Then S, T, f are said to satisfy condition (A) if

$$d(Su, Tv) \leq ad(fv, Sv) \left[\frac{1+d(fu, Tu)}{1+d(fu, fv)} \right] + b[d(fu, Su) + d(fv, Tv)] + cd(fu, fv)$$

for $u, v \in X$ where $0 \leq a, b, c < 1$ with $a + 2b + c < 1$.

Proposition 1.10 (see [6]): Let (X, d) be a cone metric space and P be a cone in a real Banach space E . If $u \leq v, v \ll w$ then $u \ll w$.

II. Main Results

Theorem 2.1: Let (X, d) be a cone metric space, P be a normal cone with normal constant K . Suppose the mapping $S, T, f: X \rightarrow X$ satisfying condition (A). If $S(X) \cup T(X) \subseteq f(X)$ and $f(X)$ is a complete subspace of X , then S, T and f have a unique point of coincidence. Moreover, if (S, f) and (T, f) are weakly compatible, then S, T and f have a unique common fixed point.

Proof: Let u_0 be arbitrary point in X . Choose a point u_1 in X such that $fu_0 = Su_1$. Similarly, choose a point u_2 in X such that $fu_1 = Tu_2$. Continuing in this way choose u_k in X to obtain u_{k+1} in X such that

$$fu_{2k+1} = Su_{2k}, \quad fu_{2k+2} = Tu_{2k+1}, \quad (k \geq 0)$$

Then,

$$\begin{aligned} d(fu_{2k+1}, fu_{2k+2}) &= d(Su_{2k}, Tu_{2k+1}) \\ &\leq ad(fu_{2k+1}, Su_{2k+1}) \left[\frac{1+d(fu_{2k}, Tu_{2k})}{1+d(fu_{2k}, fu_{2k+1})} \right] \\ &\quad + b[d(fu_{2k}, Su_{2k}) + d(fu_{2k+1}, Tu_{2k+1})] + cd(fu_{2k}, fu_{2k+1}) \\ &\leq ad(fu_{2k+1}, fu_{2k+2}) \left[\frac{1+d(fu_{2k}, fu_{2k+1})}{1+d(fu_{2k}, fu_{2k+1})} \right] \\ &\quad + b[d(fu_{2k}, fu_{2k+1}) + d(fu_{2k+1}, fu_{2k+2})] + cd(fu_{2k}, fu_{2k+1}) \end{aligned}$$

This $\Rightarrow$

$$d(fu_{2k+1}, fu_{2k+2}) \leq \left[\frac{b+c}{1-(a+b)} \right] d(fu_{2k}, fu_{2k+1})$$

Similarly,

$$\begin{aligned}
 d(fu_{2k+2}, fu_{2k+3}) &= d(Su_{2k+1}, Tu_{2k+2}) \\
 &\leq ad(fu_{2k+2}, Su_{2k+2}) \left[\frac{1+d(fu_{2k+1}, Tu_{2k+1})}{1+d(fu_{2k+1}, fu_{2k+2})} \right] \\
 &\quad + b[d(fu_{2k+1}, Su_{2k+1}) + d(fu_{2k+2}, Tu_{2k+2})] + cd(fu_{2k+1}, fu_{2k+2}) \\
 &\leq ad(fu_{2k+2}, fu_{2k+3}) \left[\frac{1+d(fu_{2k+1}, fu_{2k+2})}{1+d(fu_{2k+1}, fu_{2k+2})} \right] \\
 &\quad + b[d(fu_{2k+1}, fu_{2k+2}) + d(fu_{2k+2}, fu_{2k+3})] + cd(fu_{2k+1}, fu_{2k+2})
 \end{aligned}$$

This $\Rightarrow$

$$d(fu_{2k+2}, fu_{2k+3}) \leq \left[\frac{b+c}{1-(a+b)} \right] d(fu_{2k+1}, fu_{2k+2})$$

Now by induction we arrive at

$$\begin{aligned}
 d(fu_{2k+1}, fu_{2k+2}) &\leq \left[\frac{b+c}{1-(a+b)} \right] d(fu_{2k}, fu_{2k+1}) \\
 &\leq \left[\frac{b+c}{1-(a+b)} \right]^2 d(fu_{2k-1}, fu_{2k}) \\
 &\leq \dots \leq \left[\frac{b+c}{1-(a+b)} \right]^k d(fu_0, fu_1)
 \end{aligned}$$

And

$$\begin{aligned}
 d(fu_{2k+2}, fu_{2k+3}) &\leq \left[\frac{b+c}{1-(a+b)} \right] d(fu_{2k+1}, fu_{2k+2}) \\
 &\leq \left[\frac{b+c}{1-(a+b)} \right]^2 d(fu_{2k}, fu_{2k+1}) \\
 &\leq \dots \leq \left[\frac{b+c}{1-(a+b)} \right]^{k+1} d(fu_0, fu_1)
 \end{aligned}$$

for each $k \geq 0$.

Let $\lambda = \frac{b+c}{1-(a+b)}$, then $\lambda < 1$.

Hence in general we can write

$$d(fu_k, fu_{k+1}) \leq \lambda^k d(fu_0, fu_1)$$

Now for positive integer p , we have

$$\begin{aligned}
 d(fu_k, fu_{k+p}) &\leq d(fu_k, fu_{k+1}) + d(fu_{k+1}, fu_{k+2}) + \dots + d(fu_{k+p-1}, fu_{k+p}) \\
 &\leq [\lambda^k + \lambda^{k+1} + \lambda^{k+2} + \dots + \lambda^{k+p-1}] d(fu_0, fu_1) \\
 &\leq \lambda^k \left[\frac{1-\lambda^{p-1}}{1-\lambda} \right] d(fu_0, fu_1) \\
 &\leq \lambda^k \left[\frac{1}{1-\lambda} \right] d(fu_0, fu_1)
 \end{aligned}$$

Now for $c \in P^0$, $\exists r > 0$ such that $c - v \in P^0$ if $\|v\| < r$. Choose a positive integer n_0 such that for all $k \geq n_0$, $\left\| \frac{\lambda^k}{1-\lambda} d(fu_0, fu_1) \right\| < r$, which implies that,

$$c - \frac{\lambda^k}{1-\lambda} d(fu_0, fu_1) \in P^0 \text{ and } \frac{\lambda^k}{1-\lambda} d(fu_0, fu_1) - d(fu_k, fu_{k+p}) \in P^0$$

$k > N_c$ and for all P , by Proposition 1.10, $d(fu_k, fu_{k+p}) \ll c$ for all $k > N_c$ and for all P . Hence $\{fu_k\}$ is a Cauchy sequence in $f(X)$. Since $f(X)$ is complete, there exist $x, y \in X$ such that $fu_k \rightarrow y = fx$.

Since

$$\begin{aligned} d(fx, Sx) &\leq d(fx, fu_{2k}) + d(fu_{2k}, Sx) \\ &\leq d(y, fu_{2k}) + d(Tu_{2k-1}, Sx) \\ &\leq d(y, fu_{2k}) + d(Sx, Tu_{2k-1}) \\ &\leq d(y, fu_{2k}) + ad(fu_{2k-1}, Su_{2k-1}) \left[\frac{1+d(fx, Tx)}{1+d(fx, fu_{2k-1})} \right] \\ &\quad + b[d(fx, Sx) + d(fu_{2k-1}, Tu_{2k-1})] + cd(fx, fu_{2k-1}) \\ &= d(y, fu_{2k}) + ad(fu_{2k-1}, fu_{2k}) \left[\frac{1+d(fx, Tx)}{1+d(fx, fu_{2k-1})} \right] \\ &\quad + b[d(fx, Sx) + d(fu_{2k-1}, fu_{2k})] + cd(fx, fu_{2k-1}) \end{aligned}$$

This $\Rightarrow$

$$d(fx, Sx) \leq \frac{1}{1-b} \left[d(y, fu_{2k}) + ad(fu_{2k-1}, fu_{2k}) \left(\frac{1+d(fx, Tx)}{1+d(fx, fu_{2k-1})} \right) + bd(fu_{2k-1}, fu_{2k}) + cd(fx, fu_{2k-1}) \right]$$

Hence, it concludes that

$$\|d(fx, Sx)\| \leq \frac{K}{1-b} \left\| d(y, fu_{2k}) + ad(fu_{2k-1}, fu_{2k}) \left(\frac{1+d(fx, Tx)}{1+d(fx, fu_{2k-1})} \right) + bd(fu_{2k-1}, fu_{2k}) + cd(fx, fu_{2k-1}) \right\|$$

where K is a normal constant. If $n \rightarrow \infty$, then we arrive at $\|d(fx, Sx)\| = 0$. Hence $fx = Sx$.

Similarly by using the inequality

$$d(fx, Tx) \leq d(fx, fu_{2k+1}) + d(fu_{2k+1}, Tx)$$

We can show that $fx = Tx$, implying that y is a common point of coincidence of S, T and f ; i.e. $y = fx = Sx = Tx$. Now we show that S, T and f have unique point of coincidence. For this, assume that there is another point y^* in X such that $y^* = fx^* = Sx^* = Tx^*$ for some $x^* \in X$. Now

$$\begin{aligned} d(y, y^*) &= d(Sx, Tx^*) \\ &\leq ad(fx^*, Sx^*) \left[\frac{1+d(fx, Tx)}{1+d(fx, fx^*)} \right] + b[d(fx, Sx) + d(fx^*, Tx^*)] + cd(fx, fx^*) \\ &= ad(fx^*, fx^*) \left[\frac{1+d(fx, fx)}{1+d(fx, fx^*)} \right] + b[d(fx, fx) + d(fx^*, fx^*)] + cd(fx, fx^*) \end{aligned}$$

The last inequality gives

$$d(y, y^*) \leq cd(y, y^*)$$

This is possible only when $y = y^*$.

If (S, f) and (T, f) are weakly compatible, then

$$Sy = Sfx = fSx = fy \text{ and } Ty = Tfx = fTx = fy$$

It implies that $Sy = Ty = fy = z$ (say). Hence, z is a point of coincidence of S, T and f and so $y = z$ by uniqueness. Thus y is the unique common fixed point of S, T and f .

III. Competing Interests

Authors have declared that no competing interests exist.

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