



Coincidences and Fixed Point in Fuzzy Metric Space

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Abstract: The idea of perfect mappings of sort A given by Jungck and demonstrated regular fixed-point hypothesis in metric space. Further utilizing this thought of perfect mappings of sort A some other creator demonstrated a few normal fixed-point hypotheses on metric space just as non-Archimedean Menger probabilistic measurement space. However, when vulnerability is because of fluffiness instead of arbitrariness as some time in the estimation of conventional length, it appears that the idea of fluffy measurement space, would be progressively appropriate decision. Singh and Jain presented the idea of semi similarity in fluffy measurement space and it has been connected to demonstrate exceptional normal fixed-point hypothesis of four self-mappings on fluffy measurement space. In this paper we improve the consequences of Singh and Jain.

Keywords: Fuzzy metric space, compatible mapping, coincidences, common fixed point

I. Introduction

S. Banach introduced and studied a very important concept of contraction mapping in metric spaces. A mapping f defined on a metric space (X, d) is called contraction if,

$$d(fp, fq) \leq kd(p, q), \text{ for all } p, q \in X, 0 \leq k < 1$$

It is realized that each contraction mapping is consistent yet talk might possibly be valid. Further, Banach demonstrated the hypothesis regularly known as Banach's contraction standard which expresses that a compression mapping f of a total measurement space (X, d) to itself has an interesting fixed point z in X . This hypothesis seemed extremely helpful in the presence and uniqueness speculations and has assumed an indispensable job in the advancement of fixed point and estimation hypothesis. Banach's withdrawal guideline is the primary wellspring of motivation to mathematicians working in the field of fixed-direct hypothesis toward present various compression type mapping on different spaces. A portion of the mathematicians who have expand the Banach's contraction rule are Edelstein. A large portion of the papers managing fixed focuses for grouping of maps can be categorized as one of three classifications. The primary classification accept that each pair fulfills the equivalent contractive condition and infers that has a typical fixed point. The second class expect that each fulfills the equivalent contractive condition and joins guide insightful toward a point of confinement work f . The end is that f has a fixed-point p , which is the farthest point of every one of the fixed purposes of. The third kind accept that every function satisfying the contraction property as above has a fixed point, and that joins consistently to a capacity which fulfills a specific contractive condition, with p is a fixed purpose of f . The end is that. Rhoades broadened a hypothesis for an arrangement of maps. In this the capacity fulfill pair astute withdrawal standard, yet with various contractive constants. The end is that the succession has a typical interesting fixed point. There have been countless of metric space. One such speculation is Menger probabilistic space presented in 1942 by K. Menger. A probabilistic measurement space (PM space) is an arranged pair (X, F) , X is a nonempty set and $F : X \times X \rightarrow L$ is mapping s.t.

$$(I) \quad F_{p,q}(x) = 1 \quad \forall x > 0 \text{ iff } p = q$$

$$(II) \quad F_{p,q}(0) = 0$$

$$(III) \quad F_{p,q} = F_{q,p}$$

$$(IV) \quad F_{p,q}(x) = 1, F_{q,r}(y) = 1 \Rightarrow F_{p,r}(x+y) = 1$$

Where $t : [0,1] \times [0,1] \rightarrow [0,1]$ is a t-norm function such that t is not decreasing as well as commutative and associative and $t(a,1) = a \quad \forall a \in [0,1]$. A Menger PM space is a triple $(X, F; t)$ where (X, F) is a PM space and t is t-norm such that $F_{p,r}(x+y) \geq t(F_{p,q}(x), F_{q,r}(y)) \quad \forall x, y \geq 0$. In 1960, B. Schweizer and A. Sklar have been studied these spaces in depth. These spaces have also been considered by several other authors. The first result for a contractive self-mapping on a Menger PM space was obtained by Sehgal and Bharucha Reid. Suppose that (X, F) be a Probabilistic Metric space and $f : X \rightarrow X$ be a mapping. Then f is called as contraction if $\exists k \in (0,1)$ s.t. $\forall p, q \in X$, $F_{f(p)f(q)}(kx) \geq F_{pq}(x)$, $x > 0$

In 1983 B. Schweizer and A. Sklar introduced the probabilistic semi metric space (PSM space). The pair (X, F) is called PSM space if

$$(I) \quad F_{p,q}(x) = \varepsilon \quad \forall x > 0 \text{ iff } p = q$$

$$(II) \quad F_{p,q}(0) = 0$$

$$(III) \quad F_{p,q} = F_{q,p}$$

$$(IV) \quad F_{p,q}(x) = 1, F_{q,r}(y) = 1 \Rightarrow F_{p,r}(x+y) = 1$$

$$(V) \quad F_{p,q}(x+y) \geq t(F_{p,r}(x), F_{r,q}(y))$$

In 1983, T.L. Hicks introducing the notion of C- contraction in PSM space as. Let (X, F) be a PSM space, a mapping $f : X \rightarrow X$ is called C- contraction if $\exists k \in (0,1)$ such that $F_{p,q}(x) > 1-x \Rightarrow F_{fp,fq}(kx) > 1-kx$, $x > 0$. Using this C- contraction Hicks proved if $(X, F; t)$ is complete Menger space then every C- contraction on X has a unique fixed point p and if $p_0 \in X$, then $\lim_{n \rightarrow \infty} f^n p_0 = p$.

In 2002 D. Mihet defined weak Hicks contraction (shortly wH- contraction) same as C- contraction defined as above $\forall p, q \in X$ and $\forall t \in (0,1)$. The class of wH- contraction is larger than that of C- contraction. Using the definition of wH- contraction D. Mihet proved that every wH- contraction has a fixed point.

In 2002, V. Radu defined the contraction of type r in PM space for self mapping f on a Menger space X as follows:

$$F_{fp,fq}(L^r x) \geq \frac{F_{p,q}(x)}{F_{p,q}(x) + L^{1-r}(1 - F_{p,q}(x))}, \quad \forall x > 0 \text{ and } \forall p, q \in X.$$

For $r = 1$ the contraction is of type Sehgal on Menger space. Using the above definition contraction of type r Radu proved every contraction of type r has a fixed point iff there exist $p \in X$ such that $\lim_{x \rightarrow \infty} \inf_{n \geq 0} F_{p,f^n p}(x) > 0$

In 2005, Singh and Jain [5] presented the idea of semi similarity in fluffy measurement space and it has been connected to demonstrate extraordinary normal fixed-point hypothesis of four self-mappings on fluffy measurement space. In this part we improve the aftereffects of Singh and Jain [5].

DEFINITION 1.1. The binary operation $* : [0,1] \times [0,1] \rightarrow [0,1]$ is continuous t-norm (for details, please refer [4]).

LEMMA 1.1 [3]. For all x, y in X , $M(x, y, \cdot)$ is non decreasing function.

DEFINITION 1.2 [3]. Let $(X, M, *)$ be a fuzzy metric space. A sequence $\{x_n\}$ in X is said to be convergent to a point x in X if

$$\lim_{n \rightarrow \infty} M(x_n, x, t) = 1 \quad \text{For all } t > 0.$$

Further the sequence $\{x_n\}$ is said to be Cauchy sequence in X if

$$\lim_{n \rightarrow \infty} M(x_n, x_{n+p}, t) = 1 \text{ for all } t > 0 \text{ and } p > 0.$$

The space $(X, M, *)$ is said to be complete if every Cauchy sequence in it converges to a point of it.

In this chapter, $(X, M, *)$ is considered to be the fuzzy metric space with the condition,

$$\lim_{n \rightarrow \infty} M(x, y, t) = 1 \text{ for all } x, y \in X. \quad (1.1)$$

LEMMA 1.2 [1]. Let $\{y_n\}$ be a sequence in fuzzy metric space $(X, M, *)$ with the condition (1.1). Suppose there exists a number $k \in (0, 1)$ such that, $M(y_{n+2}, y_{n+1}, kt) \geq M(y_{n+1}, y_n, t)$, for all $t > 0$.

Then $\{y_n\}$ is a Cauchy sequence.

DEFINITION 1.3. Let f and g be mappings from a fuzzy metric space $(X, M, *)$ to itself. The mappings are said to be weak compatible if they commute at their coincidence points, that is,

$$fx = gx \text{ implies that } fgx = gfx.$$

DEFINITION 1.4. Let f and g be mappings from a fuzzy metric space $(X, M, *)$ to itself. Then the mappings are said to be compatible if, $\lim_{n \rightarrow \infty} M(fgy_n, gfy_n, t) = 1$, $\forall t > 0$, whenever $\{y_n\}$ is a sequence in X such that, $\lim_{n \rightarrow \infty} fy_n = \lim_{n \rightarrow \infty} gy_n = x \in X$.

PROPOSITION 1.1 [6]. Self-mappings f and g of a metric space $(X, M, *)$ are compatible, then they are weak compatible. The converse is not true.

EXAMPLE 1.2. Let $(X, M, *)$ be a fuzzy metric space, where $X = [0, 2]$, with t -norm defined by $a * b = \min\{a, b\}$ for all a, b in $[0, 1]$,

Also we define

$$M(x, y, t) = \frac{t}{t + d(x, y)}, \quad \forall t > 0, \text{ and } M(x, y, 0) = 0,$$

for all x, y in X .

Define self-maps f and g on X as follows:

$$fx = \begin{cases} 2 & \text{if } 0 \leq x \leq 1, \\ \frac{x}{2} & \text{if } 1 \leq x \leq 2, \end{cases}$$

and

$$gx = \begin{cases} 2 & \text{if } x = 1, \\ \frac{x+3}{2} & \text{otherwise.} \end{cases}$$

Let $y_n = 2 - \frac{1}{2n}$. Then we have $g(1) = f(1) = 2$ and $g(2) = f(2) = 1$.

Also $gf(1) = fg(1) = 1$ and $gf(2) = fg(2) = 2$. Thus f and g are weak compatible. Again,

$$fy_n = 1 - \frac{1}{4n}, \quad gy_n = 1 - \frac{1}{10n}.$$

Thus

$$fy_n \rightarrow 1, \quad gy_n \rightarrow 1.$$

Further,

$$gfy_n = \frac{4}{5} - \frac{1}{20n}, \quad fg y_n = 2.$$

Now,

$$\lim_{n \rightarrow \infty} M(fg y_n, g(1), t) = M(2, 2, t) = 1,$$

$$\text{and } \lim_{n \rightarrow \infty} M(fg y_n, g y_n, t) = \lim_{n \rightarrow \infty} M\left(2, \frac{4}{5} - \frac{1}{20n}, t\right) = \frac{t}{t + \frac{6}{5}} < 1, \quad \forall t > 0.$$

Hence f and g are weak compatible but not compatible.

DEFINITION 1.5 [5]. Let f and g be mappings from a fuzzy metric space $(X, M, *)$ to itself. Then the mappings are said to be semi compatible if $\lim_{n \rightarrow \infty} M(fg y_n, g x, t) = 1, \quad \forall t > 0,$

whenever $\{y_n\}$ is a sequence in X such that, $\lim_{n \rightarrow \infty} f y_n = \lim_{n \rightarrow \infty} g y_n = x \in X$.

Singh and Jain [5] proved that if the mappings f and g are semi compatible, then they are weak compatible without the converse being true (see Example 1.2).

Recently Piyush Tripathi and Alok Agrawal [7] using the property of dual contraction proved fixed point theorem in probabilistic metric space. We are extending the result of this paper in Fuzzy Metric Space.

II. Main Results

THEOREM 2.1. Let $(X, M, *)$ be a fuzzy metric space and Y is an arbitrary set. Suppose $k \in (0, 1)$ and $f, g, h, : Y \rightarrow X$ are mappings such that,

$$(i) M(fx, gy, kt) \leq \max\{M(hx, hy, t), M(fx, hx, t)\} \quad \forall x, y \in Y \text{ and } t > 0,$$

$$(ii) f(Y) \cup g(Y) \subset h(Y),$$

and (iii) one of $f(Y), g(Y), h(Y)$ is complete.

Then f, g and h have coincidence point.

PROOF. For $p_0 \in Y$ there exist $p_1, p_2 \in Y$ such that $fp_0 = hp_1, gp_1 = hp_2$ (because $f(Y) \cup g(Y) \subset h(Y)$). Inductively we can construct a sequence $\{p_n\}$ such that $fp_{2n} = hp_{2n+1}, gp_{2n+1} = hp_{2n+2}$.

Putting $x = p_{2n}$ and $y = p_{2n+1}$ in (i), we have

$$M(fp_{2n}, gp_{2n+1}, kt) \leq \max\{M(hp_{2n}, hp_{2n+1}, t), M(fp_{2n}, hp_{2n}, t)\}$$

$$\text{i.e. } M(hp_{2n+1}, gp_{2n+2}, kt) \leq M(hp_{2n+1}, gp_{2n}, t).$$

So by Lemma 1.1 $\{hp_n\}$ is a Cauchy sequence. Suppose $h(Y)$ is complete. Then $\{hp_n\} \rightarrow p \in h(Y)$.

Also then there exists $u \in Y$ such that $hu = p$.

Putting $x = u, y = p_{2n+1}$ in (i), we have,

$$M(fu, gp_{2n+1}, kt) \leq \max\{M(hu, hp_{2n+1}, t), M(fu, hu, t)\}.$$

Therefore in limiting case as $n \rightarrow \infty$,

$$M(fu, p, kt) \leq \max\{M(p, p, t), M(fu, hu, t)\}$$

$$\text{i.e. } M(fu, p, kt) \leq M(fu, p, t).$$

Hence $f(u) = p = h(u)$.

Lastly, putting $x = p_{2n+1}, y = u$ in (i), we have,

$$M(fp_{2n+1}, gu, kt) \leq \max\{M(hp_{2n+1}, hu, t), M(fp_{2n+1}, hp_{2n+1}, t)\}$$

$$\text{i.e. } M(p, gu, kt) \leq M(p, p, t).$$

Therefore $p = gu = f(u) = h(u)$.

Hence u is coincidence point of f, g and h .

THEOREM 2.2. Let $(X, M, *)$ be a fuzzy metric space. Suppose $k \in (0, 1)$ and $f, g, h, : X \rightarrow X$ are mappings such that,

$$(i) M(fx, gy, kt) \leq \max\{M(hx, hy, t), M(fx, hx, t) \quad \forall \quad x, y \in X \text{ and } t > 0,$$

$$(ii) f(X) \cup g(X) \subset h(X),$$

$$(iii) \text{ one of } f(X), g(X), h(X) \text{ is complete,}$$

$$\text{and } (iv) f \text{ and } h \text{ are coincidently commuting.}$$

Then f, g and h have a unique common fixed point.

PROOF. In the theorem 2.1 if we take $Y = X$ then we get $p = gu = f(u) = h(u)$.

Since f and g are coincidently commuting so,

$$fhu = hfu \text{ and } fp = hp.$$

Putting $x = fu, y = p_{2n+1}$ in (i), we have,

$$M(ffu, gp_{2n+1}, kt) \leq \max\{M(hfu, hp_{2n+1}, t), M(ffu, hfu, t).$$

Taking $n \rightarrow \infty$, we have,

$$M(ffu, p, kt) \leq M(fp, p, t).$$

Therefore $fp = hp = p$.

Again putting $x = p_{2n}, y = p$ in (i) and taking limit as $n \rightarrow \infty$ we get,

$$gp = fp = hp = p.$$

Hence p is a common fixed point of f, g and h .

For uniqueness suppose p and q are common fixed points of f, g and h .

Then by putting $x = p, y = q$ in (i) we get $p = q$. This proves the uniqueness.

References

- [1]. Cho Y. J. Fixed point in fuzzy metric spaces, J. Fuzzy Math. 4 (1997), 949-962. MR 98h: 54011.
- [2]. George A. and Veeramani P. On some results in fuzzy metric spaces, Fuzzy sets and Systems 64 (1994), 395-399.
- [3]. Grabiec M. Fixed points in fuzzy metric spaces, Fuzzy Sets and Systems 27 (1988), 385-389.
- [4]. Kramosil and Michalek. Fuzzy metric and statistical metric spaces, Kybernetika 11 (1975), 326-334.
- [5]. Singh B. and Jain S. Semicompactibility and fixed-point theorems in fuzzy metric space using implicit relation, Internat. J. Math. Math. Sci 16 (2005) 2167-2629.
- [6]. Singh B. and Jain S. A fixed point theorem in Menger spaces through weak compatibility, J. Math. Anal. Appl. 301 (2005), 439-448.
- [7]. Tripathi Piyush Kumar and Agrawal A. K. Fixed Point Theorem in Probabilistic Metric Spaces, International Journal of Scientific Research and Review Vol 7(3), 2019, 3653-3657.