



SPC Using the Weighted Akshaya Distribution

Amin Shaka Aunali^{1*} and D. Venkatesan²

^{1*,2}Department of Statistics, Annamalai University, India

Abstract: The methodology of statistical process control is mainly based on the use of control charts and frequency distributions of process and quality characteristics data. In some manufacturing processes, the distribution are usually skewed. In this case, classical control charts based on normality assumptions can lead to erroneous conclusions regarding the stability and the capability of the process. Such wrong conclusions would cost manufacturing and lost customers to competitors. To avoid such problems, many researchers develop the control charts based on different distributions regarding to the type of process they are working with. In this paper, the control charts have been developed using the weighted Akshaya distribution to check the performances of the production process.

Keywords: Control Charts, Weighted Distributions, Akshaya distribution

I. Introduction

Statistical process control is a methodology to improve the processes widely used by modern production and service organizations. This methodology is mainly based on the use of control charts and frequency distributions of process and quality characteristics data. Common and well established in 1920s control charts include the Shewhart control chart which was developed by Prof. Shewhart Walter. Since then, several researchers developed different types of control charts. In fact, customers expect companies to offer the best products and services, with panels that help them continuously improve their processes, making them more stable and capable of producing high quality, meeting customer specifications and to achieve the excellence of society. Shewhart control charts designed on the assumption that the process being monitored produces a quality characteristic that can be approximated by a symmetrical normal distribution, when only the innate sources of variability are present in the system. The central limit theorem can be used to approximate distributions to the normal distribution provided that the samples being measured and monitored would be large enough. However, in many industrial cases, this cannot be assured and the process output may not be always normally distributed and heavy tailed and skewed. In some manufacturing processes, the distribution are usually skewed. In this case, classical control charts based on normality assumptions can lead to erroneous conclusions regarding the stability and the capability of the process. Such wrong conclusions would cost manufacturing and lost customers to competitors. With the advents in statistical theories and computing facilities, this can be easily solved, by understanding of distributions that provide good model for what type of quality characteristics the industries are faced with. Some researchers, Amin and Venkatesan (2017) proposed the comparison of Bayesian method and classical charts in detection of small shifts in the control charts. And recently in 2019, they discussed the recent developments in control charts techniques. Derya and Canan (2012) developed standard control charts based on Weibull, gamma and lognormal distributions. Sherill and Johnson (2009) demonstrated the ability to use Weibull, Lognormal and exponential distribution to transform non-normal data into process control and process capability calculations. Santiago and Smith (2013) have recently proposed the control charts, called the t-chart, when the time between events follows the exponential distribution. They used the variable transformation proposed by Nelson (1994) to transform exponentially distributed data to an approximate normal data. Aslam *et al.* (2014) proposed a new control charts for the exponential distribution using the transformed variable and the repetitive sampling. The use of the weighted distributions to model control charts that can also be used for process improvement purposes.

The concept of weighted distributions was provided by Fisher (1934) where he studied how the methods of ascertainment can influence the form of the distribution of recorded observations and then Rao (1965) introduced it and formulated it generally in relation to the modeling of statistical data when it was determined that the usual practice in the use of standard distributions was not appropriate. Weighted distributions are utilized to modulate the probabilities of the events as observed and transcribed. The theory of weighted distributions provides a collective access for the problems of model specification and data interpretation. It provides a technique for fitting models to the unknown weight functions when samples can be taken both from the original distribution and the

developed distribution. Weighted distributions take into account the method of ascertainment, by adjusting the probabilities of the actual occurrence of events to arrive at a specification of the probabilities of those events as observed and recorded.

In this paper, the techniques of weighted distributions have been applied to the Akshaya distribution which was proposed by Shanker (2017). The Akshaya distribution is a new one parametric life-time distribution which has a better flexibility in handling lifetime data as compared to exponential distribution. Aafaq and Subramanian (2018) discussed a new generalization of Akshaya distribution with applications in engineering science. Several distributions have been used by many researchers to monitor the process control, and in this paper, the process control chart has been derived using the Weighted Akshaya distribution to monitor the process. The organization of this paper is as follows.

Section 2 provides the description of the distribution. The Performance measures of the weighted Akshaya distribution is provided in section 3. The Performance of Control limits for the weighted Akshaya distribution is provided in section 4. Whereas a numerical example is provided in section 5, and finally section 6 provides the Conclusion.

II. Description of the Distribution

Definition: Let a non-negative continuous random variable X with density function (pdf) $f(x)$. Let $w(x)$ be the non-negative weight function, then the probability density function of the weighted random variable X_w is given by:

$$f_w(x) = \frac{w(x)f(x)}{\mu_w}; \quad x > 0 \quad (1)$$

Where; $\mu_w = E(w(x)) = \int w(x)f(x)dx < \infty$

Definition: The density function (pdf) of Akshaya distribution is given as

$$h(x; \theta) = \frac{\theta^4}{\theta^3 + 3\theta^2 + 6\theta + 6} (1+x)^3 e^{-\theta x}; \quad x > 0, \theta > 0 \quad (2)$$

and the respective cumulative distribution function (cdf) is

$$H(x) = 1 - \left[1 + \frac{\theta^3 x^3 + 3\theta^2 (\theta + 1)x^2 + 3\theta(\theta^2 + 2\theta + 2)x}{\theta^3 + \theta^2 + 6\theta + 6} \right] e^{-\theta x}; \quad x, \theta > 0 \quad (3)$$

To obtain the weighted Akshaya distribution, one can consider the weight function as $w(x) = x^c$.

Definition: The probability density of weighted Akshaya distribution is given as:

$$f_w(x; c, \theta) = \frac{x^c f(x, \theta)}{E(x^c)}; \quad x > 0$$

$$f_w(x; c, \theta) = \frac{x^c \theta^{c+4} (1+x)^3 e^{-\theta x}}{c! [\theta^3 + 3\theta^2(c+1) + 3\theta(c+2)(c+1) + (c+3)(c+2)(c+1)]}$$

Where;

$$E(x^c) = \frac{c! [\theta^3 + 3\theta^2(c+1) + 3\theta(c+2)(c+1) + (c+3)(c+2)(c+1)]}{\theta^c (\theta^3 + 3\theta^2 + 6\theta + 6)}; \quad x, c, \theta > 0$$

The cdf of weighted Akshaya Distribution is then obtained as:

$$F_w(x; c, \theta) = \int_0^x f_w(x; c, \theta) dx$$

$$= \int_0^x \frac{x^c \theta^{c+4}}{c! [\theta^3 + 3\theta^2(c+1) + 3\theta(c+2)(c+1) + (c+3)(c+2)(c+1)]} (1+x)^3 e^{-\theta x} dx$$

Putting $x = t$, then $X = dt$. And as x tends to 0, t also tends to 0. When x tends to x , then t tends to θx . Therefore,

$$F_w(x; c, \theta) = \frac{\theta^3 \gamma(c+1, \theta x) + 3\theta^2 \gamma(c+2, \theta x) + 3\theta \gamma(c+3, \theta x) + \gamma(c+4, \theta x)}{c! [\theta^3 + 3\theta^2(c+1) + 3\theta(c+2)(c+1) + (c+3)(c+2)(c+1)]};$$

Where; $\gamma(s, x) = \int_0^x t^{s-1} e^{-t} dt$ is the lower incomplete gamma function and $x, c, \theta > 0$.

III. Performance measures of the weighted Akshaya distribution

From the above weighted Akshaya probability density function (pdf), one can derive the r^{th} moment $E(X^r)$ of the weighted Akshaya distribution, thus

$$\begin{aligned} \mu'_r &= E(X_w^r) = \int_0^\infty x^r f_w(x; c, \theta) dx \\ &= \int_0^\infty \frac{x^c \theta^{c+4}}{c! [\theta^3 + 3\theta^2(c+1) + 3\theta(c+2)(c+1) + (c+3)(c+2)(c+1)]} (1+x)^3 e^{-\theta x} dx \end{aligned}$$

Therefore,

$$\mu'_r = \frac{(c+r)! [\theta^3 + 3\theta^2(c+r+1) + 3\theta(c+r+2)(c+r+1) + (c+r+3)(c+r+2)(c+r+1)]}{\theta^r c! [\theta^3 + 3\theta^2(c+1) + 3\theta(c+2)(c+1) + (c+3)(c+2)(c+1)]} \quad (4)$$

One can get the mean and second moment from equation (4) by substituting $r=1$ and $r=2$ respectively and are given by:

$$E(X) = \frac{(c+1) [\theta^3 + 3\theta^2(c+2) + 3\theta(c+3)(c+2) + (c+4)(c+3)(c+2)]}{\theta [\theta^3 + 3\theta^2(c+1) + 3\theta(c+2)(c+1) + (c+3)(c+2)(c+1)]} \quad (5)$$

And

$$E(X^2) = \frac{(c+2)(c+1) [\theta^3 + 3\theta^2(c+3) + 3\theta(c+4)(c+3) + (c+5)(c+4)(c+3)]}{\theta^2 [\theta^3 + 3\theta^2(c+1) + 3\theta(c+2)(c+1) + (c+3)(c+2)(c+1)]} \quad (6)$$

From the above equations (5) and (6), the variance is given as:

$$\begin{aligned} V(X) &= E(X^2) - [E(X)]^2 \\ V(X) &= \frac{(c+2)(c+1) [\theta^3 + 3\theta^2(c+3) + 3\theta(c+4)(c+3) + (c+5)(c+4)(c+3)]}{\theta^2 [\theta^3 + 3\theta^2(c+1) + 3\theta(c+2)(c+1) + (c+3)(c+2)(c+1)]} \end{aligned}$$

$$-\frac{(c+1)^2 [\theta^3 + 3\theta^2(c+2) + 3\theta(c+3)(c+2) + (c+4)(c+3)(c+2)]^2}{\theta^2 [\theta^3 + 3\theta^2(c+1) + 3\theta(c+2)(c+1) + (c+3)(c+2)(c+1)]^2} \quad (7)$$

Using mean and variance, one can construct the UCL and LCL to study the process and are given in the following.

IV. Performance of Control Limits for the weighted Akshaya distribution

A typical control chart has several values and control limit sets. Therefore, when the process is under control, almost all points are within the upper control (UCL) and lower (LCL) limits. Therefore, from equations (5) and (7), the control limits are given by:

$$UCL = \frac{(c+1) [\theta^3 + 3\theta^2(c+2) + 3\theta(c+3)(c+2) + (c+4)(c+3)(c+2)]}{\theta [\theta^3 + 3\theta^2(c+1) + 3\theta(c+2)(c+1) + (c+3)(c+2)(c+1)]} + 3 \sqrt{\frac{(c+2)(c+1) [\theta^3 + 3\theta^2(c+3) + 3\theta(c+4)(c+3) + (c+5)(c+4)(c+3)]}{\theta^2 [\theta^3 + 3\theta^2(c+1) + 3\theta(c+2)(c+1) + (c+3)(c+2)(c+1)]} - \frac{(c+1)^2 [\theta^3 + 3\theta^2(c+2) + 3\theta(c+3)(c+2) + (c+4)(c+3)(c+2)]^2}{\theta^2 [\theta^3 + 3\theta^2(c+1) + 3\theta(c+2)(c+1) + (c+3)(c+2)(c+1)]^2}}$$

$$CL = \frac{(c+1) [\theta^3 + 3\theta^2(c+2) + 3\theta(c+3)(c+2) + (c+4)(c+3)(c+2)]}{\theta [\theta^3 + 3\theta^2(c+1) + 3\theta(c+2)(c+1) + (c+3)(c+2)(c+1)]}$$

$$LCL = \frac{(c+1) [\theta^3 + 3\theta^2(c+2) + 3\theta(c+3)(c+2) + (c+4)(c+3)(c+2)]}{\theta [\theta^3 + 3\theta^2(c+1) + 3\theta(c+2)(c+1) + (c+3)(c+2)(c+1)]} - 3 \sqrt{\frac{(c+2)(c+1) [\theta^3 + 3\theta^2(c+3) + 3\theta(c+4)(c+3) + (c+5)(c+4)(c+3)]}{\theta^2 [\theta^3 + 3\theta^2(c+1) + 3\theta(c+2)(c+1) + (c+3)(c+2)(c+1)]} - \frac{(c+1)^2 [\theta^3 + 3\theta^2(c+2) + 3\theta(c+3)(c+2) + (c+4)(c+3)(c+2)]^2}{\theta^2 [\theta^3 + 3\theta^2(c+1) + 3\theta(c+2)(c+1) + (c+3)(c+2)(c+1)]^2}}$$

Where; $\theta, c > 0$

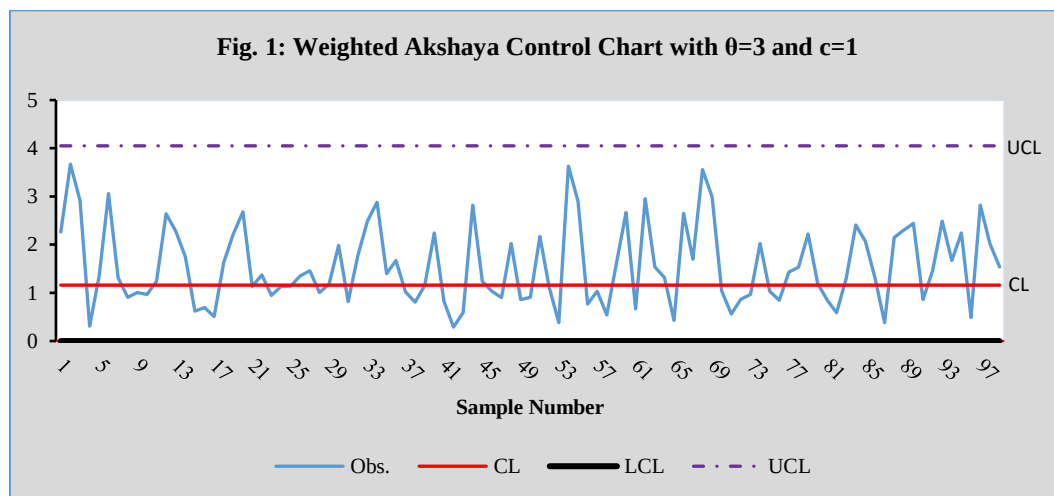
V. Numerical illustration

An example regarding the construction of control limits is considered for illustrating the applications of the proposed method. The control limits of the weighted Akshaya distribution are obtained by using simulated data set for parameters θ and c being random variables the table 1 is constructed.

Table 1: Control limits using weighted Akshaya Distribution

C	θ	CL	LCL	UCL
0.3	5	0.42	-1.13	1.96
	7	0.27	-0.76	1.29
	8	0.23	-0.65	1.10
	9	0.19	-0.56	0.95
	10	0.17	-0.49	0.84
0.5	3	0.93	-1.94	3.80
	5	0.47	-1.21	2.15
	7	0.31	-0.82	1.43
	8	0.26	-0.71	1.22
	9	0.22	-0.61	1.06
0.7	10	0.20	-0.54	0.93
	3	1.02	-1.91	3.95
	5	0.53	-1.28	2.33
	7	0.34	-0.89	1.57
	8	0.29	-0.76	1.34
1	9	0.25	-0.67	1.17
	10	0.22	-0.59	1.03
	3	1.16	-1.74	4.05
	5	0.61	-1.37	2.58
	7	0.40	-0.98	1.77
3	8	0.34	-0.84	1.52
	9	0.29	-0.74	1.32
	10	0.26	-0.66	1.17
	5	1.10	-1.01	3.21
	7	0.75	-1.27	2.76
5	8	0.64	-1.19	2.47
	9	0.56	-1.10	2.22
	10	0.49	-1.01	2.00
	7	1.07	-0.66	2.80
	8	0.92	-1.02	2.87
7	9	0.81	-1.12	2.75
	10	0.72	-1.13	2.57
	9	1.06	-0.45	2.56
	10	0.94	-0.83	2.72

It is observed from the Table 1, that for the fixed value of the parameter c , with θ as random variable, the target value decrease whenever the parameter θ increases and the deviation's limits also decrease, the weighted Akshaya Control Chart is shown in Fig. 1 for $\theta = 3, c = 1$ as:



It is also observed that, the observations of the process control must be $0 < x < \theta$; $k > 0$; $\theta, c > 0$, otherwise the process will be out of control. That is, depends on the manufacturing products, the manufacturing engineers should fix the parameters value based on what type of data they are working with. As it can be seen on Table 1, the increasing of parameter c , the parameter θ should be high in order to get the control limits because whenever the parameter c increases with lesser parameter θ , the variance values become negative also if c tends to zero.

VI. Conclusion

In this article, the process control is made using a weighted Akshaya distribution. A novel algorithm is given for sentencing the process while manufacturing. The Control limits are obtained for the weighted Akshaya distribution with different values of parameters θ and c . Table is constructed to help to the selection of the parameters based on the type of data the manufacturing engineer is faced with. The control chart is drawn by considering the special case $\theta=3$ and $c=1$ where all the observations are showing in control. Hence it is advisable to keep θ greater than the observed process data.

References

- [1] A. J. Duncan (1986), *"Quality Control and Industrial Statistics"*, 5th ed., Irwin, Homewood, IL.
- [2] Aafaq A. Rather and C. Subramanian (2018), *"A New Generalization of Akshaya Distribution with Applications in Engineering Science"*, International Journal of Management, Technology and Engineering, Vol. 9, No. 4, pp. 2175-2184.
- [3] C. R. Rao (1965), *"On discrete distributions arising out of methods of ascertainment, in Classical and Contagious Discrete Distribution"*, G. P. Patil, ed., Pergamon Press and Statistical Publishing Society, Calcutta, pp. 320-332.
- [4] D. C. Montgomery (2012), *"Introduction to Statistical Quality Control"*, 7th ed., Wiley, New York.
- [5] Eduardo Santiago & Joel Smith (2013), *"Control Charts Based on the Exponential Distribution: Adapting Runs Rules for the t Chart"*, Quality Engineering, Vol. 25, No. 2, pp. 85-96.
- [6] K. Derya and H. Canan (2012), *"Control Charts for Skewed Distributions: Weibull, Gamma, and Lognormal"*, Metodoloski zvezki, Vol. 9, No. 2, pp. 95-106.
- [7] L. S. Nelson (1984), *"The Shewhart control chart-tests for special causes"*, Journal of Quality Technology, Vol. 16 pp. 238-239.
- [8] L. S. Nelson (1994), *"A control chart for parts-per-million nonconforming items"*, Journal of Quality Technology, Vol. 26 pp. 239-240.
- [9] M. Aslam, M. Azam, and C. H. Jun (2014), *"A new exponentially weighted moving average sign chart using repetitive sampling"*, Journal of Process Control, Vol. 24, No. 7, pp. 1149-1153.
- [10] R. A. Fisher (1934), *"The effects of methods of ascertainment upon the estimation of frequencies"*, The Annals of Eugenics, Vol. 6, pp. 13-25.
- [11] R. Shanker (2017), *"Akshaya Distribution and Its Application"*, American Journal of Mathematics and Statistics, Vol. 7, No. 2, pp. 51-59.
- [12] R. W. Sherill and L. A. Johnson (2009), *"Calculated Decisions"*, Quality Progress, Vol. 42, No. 1, pp. 30-35.
- [13] S. P. Mukherjee and A. Islam (1983), *"A finite-range distribution of failure times"*, Naval Research Logistics, Vol. 30, No. 3, pp. 487-491.
- [14] Shaka A. Amin and D. Venkatesan (2017), *"Comparison of Bayesian Method and Classical Charts in Detection of Small Shifts in the Control Charts"*, International Journal of Operations Research and Optimization, Vol. 8, No. 1, pp. 23-35.
- [15] Shaka A. Amin and D. Venkatesan (2019), *"Recent Developments in Control Charts Techniques"*, Universal Review Journal, Vol. 8, No. 4, pp. 746-756.