



Some Linear Generating Relations Associated With Gegenbauer Polynomials

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Abstract: In this paper, we obtain some linear generating relations involving classical Gegenbauer polynomials by using series decomposition technique and De Moivre's theorem.

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1 INTRODUCTION AND PRELIMINARIES

Throughout the present work, we use the following standard notations:

$\mathbb{N} := \{1, 2, 3, \dots\}$, $\mathbb{N}_0 := \{0, 1, 2, 3, \dots\} = \mathbb{N} \cup \{0\}$ and $\mathbb{Z}^- := \{-1, -2, -3, \dots\} = \mathbb{Z}_0^- \setminus \{0\}$.

Here, as usual, $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}_+$, $\mathbb{R}_-$ denote the sets of positive and negative real numbers respectively and $\mathbb{C}$ denotes the set of complex numbers.

The Pochhammer symbol (or the shifted factorial) $(\lambda)_\nu$ ($\lambda, \nu \in \mathbb{C}$) is defined, in terms of the familiar Gamma function, by

$$(\lambda)_\nu := \frac{\Gamma(\lambda + \nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu = 0; \lambda \in \mathbb{C} \setminus \{0\}) \\ \lambda(\lambda + 1) \dots (\lambda + \nu - 1) & (\nu = n \in \mathbb{N}; \lambda \in \mathbb{C}) \end{cases} \quad (1.1)$$

it being understood conventionally that $(0)_0 = 1$ and assumed tacitly that the Gamma quotient exists.

The following results will be required in our present investigations:

The Gegenbauer polynomials $C_n^\nu(x)$ [13, p.276 (1) and p.279 (15)] are defined by

$$\sum_{n=0}^{\infty} C_n^\nu(x) t^n = (1 - 2xt + t^2)^{-\nu}, \quad (1.2)$$

where

$$C_n^\nu(x) = \frac{(2\nu)_n}{n!} {}_2F_1 \left[\begin{matrix} -n, 2\nu + n; \\ \nu + \frac{1}{2}; \end{matrix} \frac{1-x}{2} \right]. \quad (1.3)$$

The Kampé de Fériet's double hypergeometric function of higher order in the modified notation of Srivastava and Panda [20, pp. 423(26), 424(27)], is given by

$$F_{j;m;n}^{p;q;k} \left[\begin{matrix} (a_p) : (b_q) : (d_k) \\ (g_j) : (e_m) : (h_n) \end{matrix} ; x, y \right] = \sum_{r,s=0}^{\infty} \frac{\prod_{i=1}^p (a_i)_{r+s} \prod_{i=1}^q (b_i)_r \prod_{i=1}^k (d_i)_s}{\prod_{i=1}^j (g_i)_{r+s} \prod_{i=1}^m (e_i)_r \prod_{i=1}^n (h_i)_s} \frac{x^r y^s}{r! s!}, \quad (1.4)$$

where (a_p) abbreviates the array of p parameters given by $a_1, a_2, \dots, a_p$ with similar interpretations for $(b_q), (d_k)$ et cetera and for convergence of double hypergeometric series (1.4), we have

- (i) $p + q < j + m + 1$, $p + k < j + n + 1$, $|x| < \infty$ and $|y| < \infty$ or
(ii) $p + q = j + m + 1$, $p + k = j + n + 1$, and

$$\begin{cases} |x|^{\frac{1}{p-j}} + |y|^{\frac{1}{p-j}} < 1, & \text{if } p > j \\ \max\{|x|, |y|\} < 1, & \text{if } p \leq j. \end{cases}$$

The Appell's double hypergeometric functions F_1 , F_2 , F_3 and F_4 [19, p.53 (4,5,6,7)] are denoted by $F_{1;0;0}^{1;1;1}$, $F_{0;1;1}^{1;1;1}$, $F_{1;0;0}^{0;2;2}$ and $F_{0;1;1}^{2;0;0}$ respectively.

The notation $\Delta(\ell; \lambda)$ abbreviates the array of ℓ parameters given by

$$\frac{\lambda}{\ell}, \frac{\lambda+1}{\ell}, \dots, \frac{\lambda+\ell-1}{\ell}; \quad \ell = 1, 2, 3, 4, \dots$$

The Appell's function of first kind F_1 [19, p.53(4)] is defined by

$$F_1[a, b, b'; c; x, y] = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n} (b)_m (b')_n x^m y^n}{(c)_{m+n} m! n!} \quad (1.5)$$

where $\max\{|x|, |y|\} < 1$.

The idea of separation of a power series into its even and odd terms [19, p.200 (1), p.214 Q.N.8; see also 18, p.196], exhibited by the elementary identity

$$\sum_{n=0}^{\infty} \Psi(n) = \sum_{n=0}^{\infty} \Psi(2n) + \sum_{n=0}^{\infty} \Psi(2n+1), \quad (1.6)$$

is at least as old as the series themselves.

$$\begin{aligned} \sum_{m,n=0}^{\infty} \Phi(m, n) &= \sum_{m,n=0}^{\infty} \Phi(2m, 2n) + \sum_{m,n=0}^{\infty} \Phi(2m+1, 2n+1) + \\ &+ \sum_{m,n=0}^{\infty} \Phi(2m, 2n+1) + \sum_{m,n=0}^{\infty} \Phi(2m+1, 2n), \end{aligned} \quad (1.7)$$

provided that above multiple series are absolutely convergent.

The Gauss hypergeometric function [13, p.73 (1)] is defined by

$${}_2F_1 \left[\begin{matrix} a, b; \\ d; \end{matrix} z \right] = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(d)_n} \frac{z^n}{n!} \quad (1.8)$$

where $d \neq 0, -1, -2, \dots$ and $|z| < 1$.

Suppose

$$A + iB = \lambda \exp(i\mu) = \lambda \cos \mu + i\lambda \sin \mu, \quad (1.9)$$

then modulus

$$\lambda = \sqrt{(A^2 + B^2)} \quad (1.10)$$

and the amplitude μ is given by any one of the following

$$\mu := \begin{cases} \arctan\left(\frac{B}{A}\right) & (A, B \in \mathbb{R}_+) \\ -\arctan\left(\frac{|B|}{A}\right) & (A \in \mathbb{R}_+; B \in \mathbb{R}_-) \\ \pi - \arctan\left(\frac{B}{|A|}\right) & (A \in \mathbb{R}_-; B \in \mathbb{R}_+) \\ -\pi + \arctan\left(\frac{|B|}{|A|}\right) & (A, B \in \mathbb{R}_-). \end{cases} \quad (1.11)$$

Motivated by the work of Barr [1], Carlson [2], MacRobert [5-6], Manocha [7], Chaudhary et al. [3,4,8], Qureshi and Ahmad [9], Qureshi, Quraishi and Pal [11], Qureshi, Yasmeen and Pathan [12], Qureshi, Kabra and Khan [10], Sharma [14-16] and Srivastava [17,18], we shall obtain some linear generating relations associated with Gegenbauer polynomials of even and odd degree.

II MAIN GENERATING RELATIONS

Any values of parameters and variables leading to the results given in this section which do not make sense, are tacitly excluded (Suppose α , λ , μ , ν and x are real numbers), then

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{\left(\frac{\lambda}{2}\right)_n \left(\frac{\lambda+1}{2}\right)_n}{\left(\frac{\mu}{2}\right)_n \left(\frac{\mu+1}{2}\right)_n} C_{2n}^{\alpha}(x) t^n = \\ & F_{2;1}^{2;2} \left[\begin{matrix} \Delta(2; \lambda) : \Delta(2; \alpha); \Delta(2; \alpha); \\ \Delta(2; \mu) : \end{matrix} \frac{1}{2}; \frac{1}{2}; (x + \sqrt{x^2 - 1})^2 t, (x - \sqrt{x^2 - 1})^2 t \right] + \frac{\lambda(\lambda+1)\alpha^2 t}{\mu(\mu+1)} \times \\ & \times F_{2;1}^{2;2} \left[\begin{matrix} \Delta(2; \lambda+2) : \Delta(2; \alpha+1); \Delta(2; \alpha+1); \\ \Delta(2; \mu+2) : \end{matrix} \frac{3}{2}; \frac{3}{2}; (x + \sqrt{x^2 - 1})^2 t, (x - \sqrt{x^2 - 1})^2 t \right], \end{aligned} \quad (2.1)$$

where $\max\left\{(x + \sqrt{x^2 - 1})^2 t, |(x - \sqrt{x^2 - 1})^2 t|\right\} < 1$

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{\left(\frac{\lambda+1}{2}\right)_n \left(\frac{\lambda+2}{2}\right)_n}{\left(\frac{\mu+1}{2}\right)_n \left(\frac{\mu+2}{2}\right)_n} C_{2n+1}^{\alpha}(x) t^n = (x - \sqrt{x^2 - 1})\alpha \times \\ & \times F_{2;1}^{2;2} \left[\begin{matrix} \Delta(2; \lambda+1) : \Delta(2; \alpha); \Delta(2; \alpha+1); \\ \Delta(2; \mu+1) : \end{matrix} \frac{1}{2}; \frac{3}{2}; (x + \sqrt{x^2 - 1})^2 t, (x - \sqrt{x^2 - 1})^2 t \right] + \\ & + (x + \sqrt{x^2 - 1})\alpha \times \\ & \times F_{2;1}^{2;2} \left[\begin{matrix} \Delta(2; \lambda+1) : \Delta(2; \alpha+1); \Delta(2; \alpha); \\ \Delta(2; \mu+1) : \end{matrix} \frac{3}{2}; \frac{1}{2}; (x + \sqrt{x^2 - 1})^2 t, (x - \sqrt{x^2 - 1})^2 t \right], \end{aligned} \quad (2.2)$$

where $\max\left\{(x + \sqrt{x^2 - 1})^2 t, |(x - \sqrt{x^2 - 1})^2 t|\right\} < 1$

$$\sum_{n=0}^{\infty} C_n^{\nu}(x) \cos(n\theta) = \frac{\cos \nu \theta}{2^{\nu} |x - \cos \theta|^{\nu}}, \quad (2.3)$$

$$\sum_{n=0}^{\infty} C_n^{\nu}(x) \sin(n\theta) = \frac{-\sin \nu \theta}{2^{\nu} |x - \cos \theta|^{\nu}}. \quad (2.4)$$

III. DERIVATIONS

To prove the results (2.1) and (2.2) we consider the following generating relation [19, p.126 (7)]

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{(\mu)_n} C_n^{\alpha}(x) t^n = F_1 \left[\lambda, \alpha, \alpha; \mu; (x + \sqrt{x^2 - 1})t, (x - \sqrt{x^2 - 1})t \right] \quad (3.1)$$

$$= \sum_{m,n=0}^{\infty} \frac{(\lambda)_{m+n} (\alpha)_m (\alpha)_n (x + \sqrt{x^2 - 1})^m (x - \sqrt{x^2 - 1})^n t^m t^n}{(\mu)_{m+n} m! n!}, \quad (3.2)$$

where

$$\max \left\{ |(x + \sqrt{x^2 - 1})t|, |(x - \sqrt{x^2 - 1})t| \right\} < 1,$$

Now using the series identities (1.6) and (1.7) in equations (3.1) and (3.2), we get

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{(\lambda)_{2n}}{(\mu)_{2n}} C_{2n}^{\alpha}(x) t^{2n} + \sum_{n=0}^{\infty} \frac{(\lambda)_{2n+1}}{(\mu)_{2n+1}} C_{2n+1}^{\alpha}(x) t^{2n+1} \\ &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\lambda)_{2m+2n} (\alpha)_{2m} (\alpha)_{2n} (x + \sqrt{x^2 - 1})^{2m} (t^2)^m (x - \sqrt{x^2 - 1})^{2n} (t^{2n})}{(\mu)_{2m+2n} (2m!) (2n!)} + \\ &+ t \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\lambda)_{2m+2n+1} (\alpha)_{2m} (\alpha)_{2n+1} (x + \sqrt{x^2 - 1})^{2m} (t^{2m}) (x - \sqrt{x^2 - 1})^{2n+1} (t^{2n})}{(\mu)_{2m+2n+1} (2m!) (2n+1)!} + \\ &+ t \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\lambda)_{2m+2n+1} (\alpha)_{2m+1} (\alpha)_{2n} (x + \sqrt{x^2 - 1})^{2m+1} (t^{2m}) (x - \sqrt{x^2 - 1})^{2n} (t^{2n})}{(\mu)_{2m+2n+1} (2m+1)! (2n)!} + \\ &+ t^2 \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\lambda)_{2m+2n+2} (\alpha)_{2m+1} (\alpha)_{2n+1} (x + \sqrt{x^2 - 1})^{2m+1} t^{2m} (x - \sqrt{x^2 - 1})^{2n+1} t^{2n}}{(\mu)_{2m+2n+2} (2m+1)! (2n+1)!} \end{aligned} \quad (3.3)$$

Put $t = iT$ or $t^2 = -T^2$ in equation (3.3), equating real and imaginary parts, we get

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{\left(\frac{\lambda}{2}\right)_n \left(\frac{\lambda+1}{2}\right)_n}{\left(\frac{\mu}{2}\right)_n \left(\frac{\mu+1}{2}\right)_n} C_{2n}^{\alpha}(x) (-T^2)^n = \\ & \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\left(\frac{\lambda}{2}\right)_{m+n} \left(\frac{\lambda+1}{2}\right)_{m+n} \left(\frac{\alpha}{2}\right)_m \left(\frac{\alpha+1}{2}\right)_m \left(\frac{\alpha}{2}\right)_n \left(\frac{\alpha+1}{2}\right)_n (x + \sqrt{x^2 - 1})^{2m} (-T^2)^m (x - \sqrt{x^2 - 1})^{2n} (-T^2)^n}{\left(\frac{\mu}{2}\right)_{m+n} \left(\frac{\mu+1}{2}\right)_{m+n} \left(\frac{1}{2}\right)_m \left(\frac{1}{2}\right)_n m! n!} + \\ & + \frac{\lambda(\lambda+1)\alpha^2 (-T^2)}{\mu(\mu+1)} \times \end{aligned}$$

$$\times \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\left(\frac{\lambda+2}{2}\right)_{m+n} \left(\frac{\lambda+3}{2}\right)_{m+n} \left(\frac{\alpha+1}{2}\right)_m \left(\frac{\alpha+2}{2}\right)_m \left(\frac{\alpha+1}{2}\right)_n \left(\frac{\alpha+2}{2}\right)_n (x + \sqrt{x^2 - 1})^{2m} (-T^2)^m (x - \sqrt{x^2 - 1})^{2n} (-T^2)^n}{\left(\frac{\mu+2}{2}\right)_{m+n} \left(\frac{\mu+3}{2}\right)_{m+n} \left(\frac{3}{2}\right)_m \left(\frac{3}{2}\right)_n m! n!} \quad (3.4)$$

and

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{\left(\frac{\lambda+1}{2}\right)_n \left(\frac{\lambda+2}{2}\right)_n}{\left(\frac{\mu+1}{2}\right)_n \left(\frac{\mu+2}{2}\right)_n} C_{2n+1}^{\alpha}(x) (-T^2)^n = (x - \sqrt{x^2 - 1}) \alpha \times \\ & \times \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\left(\frac{\lambda+1}{2}\right)_{m+n} \left(\frac{\lambda+2}{2}\right)_{m+n} \left(\frac{\alpha}{2}\right)_m \left(\frac{\alpha+1}{2}\right)_m \left(\frac{\alpha+1}{2}\right)_n \left(\frac{\alpha+2}{2}\right)_n (x + \sqrt{x^2 - 1})^{2m} (-T^2)^m (x - \sqrt{x^2 - 1})^{2n} (-T^2)^n}{\left(\frac{\mu+1}{2}\right)_{m+n} \left(\frac{\mu+2}{2}\right)_{m+n} \left(\frac{1}{2}\right)_m \left(\frac{3}{2}\right)_n m! n!} + \\ & + (x + \sqrt{x^2 - 1}) \alpha \times \\ & \times \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\left(\frac{\lambda+1}{2}\right)_{m+n} \left(\frac{\lambda+2}{2}\right)_{m+n} \left(\frac{\alpha+1}{2}\right)_m \left(\frac{\alpha+2}{2}\right)_m \left(\frac{\alpha}{2}\right)_n \left(\frac{\alpha+1}{2}\right)_n (x + \sqrt{x^2 - 1})^{2m} (-T^2)^m (x - \sqrt{x^2 - 1})^{2n} (-T^2)^n}{\left(\frac{\mu+1}{2}\right)_{m+n} \left(\frac{\mu+2}{2}\right)_{m+n} \left(\frac{3}{2}\right)_m \left(\frac{1}{2}\right)_n m! n!} \end{aligned} \quad (3.5)$$

Put $T = i\sqrt{t}$ or $T^2 = -t$ in equations (3.4) and (3.5) and apply the definition of Kampé de Fériet's double hypergeometric function (1.4), we get the desired result (2.1) and (2.2).

To prove the results (2.3) and (2.4), we proceed as follows:

Put $t = \cos\theta + i\sin\theta$ in equation (1.2) and apply De Moivre's theorem, we get

$$\sum_{n=0}^{\infty} C_n^{\nu}(x) (\cos n\theta + i\sin n\theta) = [(1 - 2x\cos\theta + \cos 2\theta) + i(\sin 2\theta - 2x\sin\theta)]^{-\nu} \quad (3.6)$$

Suppose $(\sin 2\theta - 2x\sin\theta)$ and $(1 - 2x\cos\theta + \cos 2\theta)$ are positive real numbers

Let $r\cos\phi = (1 - 2x\cos\theta + \cos 2\theta)$ and $r\sin\phi = (\sin 2\theta - 2x\sin\theta)$

Again using De Moivre's theorem in the right hand side of equation (3.6), we get

$$\sum_{n=0}^{\infty} C_n^{\nu}(x) (\cos n\theta + i\sin n\theta) = r^{-\nu} (\cos\nu\phi - i\sin\nu\phi), \quad (3.7)$$

where $r = 2|x - \cos\theta|$ and $\phi = \theta$.

Equating real and imaginary parts in equation (3.7), we get the desired results (2.3) and (2.4).

We conclude our present investigation, by observing that several other linear generating relations can be obtained from known generating relations, in analogous manner.

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