



A survey of results on a generalization of Ramanujan sum

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Abstract. Ramanujan sum is a sum of powers of primitive roots of unity defined by Srinivasa Ramanujan. After this, there have been many generalizations of the Ramanujan sum one of which was given by E. Cohen. In a series of articles, he proved that several interesting properties of the classical Ramanujan sum extends to his generalization as well. Many other authors followed the footsteps of Cohen to give various such generalized results. In this survey article, we list some of the most important properties of the original sum and the generalization and also give some expected results using the generalized sum

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I. Introduction

Certain properties of a sum of powers of primitive roots of unity were discussed in ([17]) by Sreenivasa Ramanujan. He discussed various infinite series expressions in the form of a power series for many well known arithmetical functions. His interest in the sums originated in his desire to "obtain expressions for a variety of well-known arithmetical functions of n in the form of a series $\sum_s a_s c_s(n)$ ". This power series expansion which is analytic in its behaviour and related topics attracted the attention of many in the years there after. In more recent years, Ramanujan sums have appeared in various problems. Being interested in the applications of the Ramanujan sums, many mathematicians later tried to generalize it to find more and more applications. One such generalization was given by E. Cohen. After that, various other generalizations were discussed in many papers including that of C. S. Venkataraman and R. Sivaramakrishnan [20] and J. Chidambaraswamy ([6]). E. Cohen in a series of papers established various properties and identities for the generalized Ramanujan sums. They are mainly the multiplicative property of the sum, Holders type evaluation and orthogonality relations that the sum holds. Cohen proved most of these identities in his generalization. We state and explain all these results, and in addition suggest some directions which may give results in such generalized settings.

2. Notations

Most of the notations, functions, and identities we mention below are standard and can be found in [3] or [12]. We use the following standard notations: N the set of all natural numbers, Z the set of all integers, μ the Mobius function on N , (m, n) the gcd of m and n . Note that for any given $k \in N$, we have

$$\sum_{d|k} \mu(d) = 0$$

For two arithmetical functions f and g , $f * g$ denote their Dirichlet convolution (Dirichlet product). Note that this product is commutative.

Next we define the generalized gcd function:

Definition 1. The generalized GCD function on $N \times N$ denoted by $(j, k)_s$ is defined to be the largest $d^s \in N$ such that $d^s | k$ and $d^s | j$. Therefore, $(j, k)_s = 1$ means that no natural number $d^s > 1$ exists such that $d^s | k$ and $d^s | j$.

When $s = 1$, the generalized GCD function becomes the usual GCD function.

The Jordan totient function of order $s \in N$ can be defined using

$$J_s(n) = n^s \prod_{p|n, p \text{ prime}} \left(1 - \frac{1}{p^s}\right)$$

where $n \in \mathbb{N}$. This can also be defined as the number of j less than n^s such that $(j, n^s)_s = 1$ [7, Section 2]. The function satisfies ([3], Chapter 2, Exer. 17)

$$J_s(n) = \sum_{d|n} d^s \mu\left(\frac{n}{d}\right) = \sum_{d|n} \left(\frac{n}{d}\right)^s \mu(d) \text{ for } n \geq 1.$$

For $k \in \mathbb{N}$, $j \in \mathbb{Z}$, the classical Ramanujan sum is defined as the sum

$$c_k(j) = \sum_{m=1, (m,k)=1} e^{2\pi i j m / k}$$

We will now define Bernoulli Polynomials and Bernoulli numbers as follows: For any $x \in \mathbb{C}$ define the functions $B_n(x)$ by the equation

$$\frac{ze^{xz}}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} z^n \text{ where } |z| < 2\pi$$

If we put $x=0$, we get

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n \text{ where } |z| < 2\pi$$

Note that, by convention, $B_n := B_n(0)$. $B_n(x)$ are called Bernoulli polynomials and $B_n = B_n(0)$ are called Bernoulli numbers. Any other notations and definitions we use will be introduced later as and when required.

3. Results and generalizations

In this exposition, we list a few properties of the Ramanujan sum which yields to natural generalizations in terms of the generalized sum of Cohen. We state the original results and the generalized results in this exposition. Most of the results in this exposition are taken from already appeared papers listed in the references. We suggest two new results also in this paper which might be proved in a sequel to this work. The aim of this survey of results is to provide the reader a bird's eye view on the naturality of the generalization given by Cohen and hence to enable a reader to dig deeply into the works of Cohen. We state the definition of the generalization ([7]) here calling this sum as the generalized Ramanujan sum:

Definition 2. For $s, k \in \mathbb{N}$, $j \in \mathbb{Z}$, the generalized Ramanujan sum is defined as

$$c_k^{(s)}(j) = \sum_{m=1, (j, k^s)_s=1}^k e^{2\pi i j m / k^s} :$$

One of the properties of the original Ramanujan sum (which we here after call simply as the Ramanujan Sum) is the following multiplicative property.

Proposition 3. ([14], Cor. 2.2) For all j, k and k' with $(k, k') = 1$, $c_{kk'}(j) = c_k(j)c_{k'}(j)$

It was proved by Cohen [7, Theorem 1] analogously that

Proposition 4. For all j, k, k' and s with $(k, k') = 1$, $c_{kk'}^{(s)}(j) = c_k^{(s)}(j)c_{k'}^{(s)}(j)$.

Therefore the Ramanujan sum c_k and the general sum $c_k^{(s)}$ both are multiplicative in k . Another notable property of the Ramanujan sum is the following identity [14, Proposition 2.1].

Proposition 5. For all k, j ,

$$c_k(j) = \sum_{d|(k,j)} d \mu\left(\frac{k}{d}\right)$$

This was generalized by Cohen ([7] Section 2) to

Proposition 6. For all k, j, s ,

$$c_k^{(s)}(j) = \sum_{d|(j, k^s)} d^s \mu\left(\frac{k}{d}\right)$$

It has been observed that when the modulus is a prime, or power of a prime, the Ramanujan sum satisfies

Proposition 7. [14, Chapter 2] For all j, p prime, $\alpha > 0$

$$c_{p^\alpha}(j) = \begin{cases} p^\alpha - p^{\alpha-1} & \text{if } p^\alpha / n \\ -p^{\alpha-1} & \text{if } p^\alpha \nmid n \text{ but } p^{\alpha-1} / n \\ 0 & \text{otherwise} \end{cases}$$

It is natural to hope for such an identity in the case of the generalized sum because the above identity follows from the expression for $c_k(j)$ in terms of the divisor μ product summation. Such a relation holds true for the generalized sum also. Cohen verified the following.

Proposition 8. [7, Theorem 3] For all j, s, p prime, $\alpha > 0$

$$c_{p^\alpha}^{(s)}(j) = \begin{cases} p^{s\alpha} - p^{s(\alpha-1)} & \text{if } p^{s\alpha} / j \\ -p^{s(\alpha-1)} & \text{if } p^{s\alpha} \nmid j \text{ but } p^{s(\alpha-1)} / j \\ 0 & \text{otherwise} \end{cases}$$

From the multiplicative property of the generalized Ramanujan sum and the above proposition, as in the case of the Ramanujan sum, it follows that the generalized version also gives integers all the time. That is, just like $c_k(j), c_k^{(s)}(j)$ is also an integer for all s, k, j . This can be deduced from Proposition (6) as well since the Mobius function always give integer values.

The following orthogonality property was proved by Carmichael in [5]:

Proposition 9. For $d_1 / k, d_2 / k$, and for every n ,

$$\sum_{n=a+b \pmod{k}} c_{d_1}(a) c_{d_2}(b) = \begin{cases} k c_d(n) & \text{if } d_1 = d_2 = d \\ 0 & \text{if } d_1 \neq d_2 \end{cases}$$

A generalization of this relation and analogous results for other functions were obtained in [9] and [10]. The Carmichael relation can be easily deduced from the generalization of Cohen. These results are found to be of fundamental importance in the theory of arithmetical functions (mod r) and quadratic congruences to the modulus r . Later, in [11, Section 3], Cohen gave the following generalization to the orthogonality relation using the generalized Ramanujan sum.

Proposition 10. For $d_1 / k, d_2 / k$, and for every n, s

$$\sum_{n=a+b \pmod{k^s}} c_{d_1}^{(s)}(a) c_{d_2}^{(s)}(b) = \begin{cases} k^s c_d^{(s)}(n) & \text{if } d_1 = d_2 = d \\ 0 & \text{if } d_1 \neq d_2 \end{cases}$$

Cohen proved several similar identities (similar to the orthogonality properties) later in the above mentioned paper, but since we would like to just make a comparison between the Ramanujan sum identities and the similar identities for the general version, we confine only to the above one.

Another interesting identity involving the Ramanujan sum is the Holder's evaluation [13] of the Ramanujan sum. It can be stated as

$$c_k(j) = \frac{\phi(k) \mu(k/(k, j))}{\phi(k/(k, j))} \text{ where } k \in N, j \in Z$$

For the generalized sum, Cohen showed that ([11], section 2)

Proposition 11. (Generalized Holder's identity)

$$(1) \quad c_k^{(s)}(j) = \frac{J_s(k) \mu(k/(j, k^s)_s)}{J_s(k/(j, k^s)_s)} \text{ where } k \in N, j, s \in Z$$

For $r \in \mathbb{N}$. Alkan established an identity for the weighted average

$$\frac{1}{k^{r+1}} \sum_{j=1}^k j^r c_k(j)$$

He used this identity to establish several important results in [1] and [2]. Denoting the above weighted average by $S_r(k)$, Alkan proved that

Proposition 12.

$$(2) \quad S_r(k) = \frac{\phi(k)}{2k} + \frac{1}{r+1} \sum_{m=0}^{\lfloor r/2 \rfloor} \binom{r+1}{2m} B_{2m} \prod_{p|k} \left(1 - \frac{1}{p^{2m}}\right)$$

He used this identity to prove exact formulas for certain mean square averages of special values of L-functions. Alkan described these mean square averages in [1] and proved an asymptotic formula for $\sum_{k \leq x} S_r(k)$ in [2] based on the identity given above. How did he proceed to establish the identity is, that, he used Holder's evaluation of the Ramanujan sums [13] and applied the formula

$$(3) \quad \sum_{j=1, \gcd(j,n)=1}^n j^r = \frac{n^{r+1}}{r+1} \sum_{m=0}^{\lfloor r/2 \rfloor} \binom{r+1}{2m} \frac{B_{2m}}{n^{2m}} \prod_{p|n} (1 - p^{2m-1}) \quad (n, r \in \mathbb{N}, n > 1)$$

which appeared in [18]) and then considered the cases r even and r odd separately.

A simpler proof this identity (2) appeared in a paper [19] of Toth. In addition, in the same paper, Toth derived identities for some other weighted averages of the Ramanujan sums with weights concerning logarithms, values of arithmetic functions for gcd's, the gamma function, the Bernoulli numbers, binomial coefficients, and the exponential function.

This identity involving Ramanujan sum was generalized by the second author K V Namboothiri in [15] to

Proposition 13. For $s, k, r \in \mathbb{N}$, define the weighted average $S_r^{(s)}(k)$ by

$$(4) \quad S_r^{(s)}(k) = \frac{1}{k^{s(r+1)}} \sum_{j=1}^{k^s} j^r c_k^{(s)}(j)$$

Then,

$$(5) \quad S_r^{(s)}(k) = \frac{J_s(k)}{2k} + \frac{1}{r+1} \sum_{m=0}^{\lfloor r/2 \rfloor} \binom{r+1}{2m} B_{2m} \frac{J_s(k)}{k^{2ms}}$$

Most of the other weighted averages involving Ramanujan sum given by Toth was generalized by Namboothiri to give new identities generalized Ramanujan sums. Namboothiri in [16] gave a generalization of a restricted congruence problem which involved generalized Ramanujan sum. The results given by Namboothiri there was proved by Bibak *et al* in [4]. All these results imply the natural correspondence between various results involving the original Ramanujan sum and Cohen's generalization of it. It appears that many other results appearing in the literature using properties of Ramanujan sums yield natural generalization to results involving generalized Ramanujan sums.

In addition to the above mentioned basic identities on Ramanujan sums, Cohen was able to derive series expressions for various arithmetic functions in terms of the generalizations in [7], [8], [11], [9] and [10]. In a forthcoming sequel to this exposition, we plan to explore series expansions of various arithmetical functions in terms of the above discussed generalized sums.

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