



Intuitionistic Fuzzy Bi-ideals of BCK-Algebras

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Abstract: We consider the intuitionistic fuzzification of the concept of sub-algebras and Bi-ideals in BCK-algebras and explore some of their properties. We suggest the perception of equivalence relations on the clan of all intuitionistic fuzzy Bi-ideals of a BCK-algebra and explore some related properties.

Keywords: (Intuitionistic)fuzzysubalgebra, (Intuitionistic)fuzzy ideal,(Intuitionistic)fuzzy Bi-ideal, homomorphism.

I. Introduction

Behind the initiation of the concept of fuzzy set by Zadeh [6] several researches were directed on the generalizations of the thought of fuzzy sets. The intention of “Intuitionistic fuzzy set” was first publicized by Atanassov [5], as a generalization of the knowledge of fuzzy set. The author (together with Hong, Kim, Meng, Roh and Song) considered the fuzzification of ideals and sub-algebras in BCK-algebras ([1, 2,3,4]). In this paper, using the Atanassov’s and Young Bae Jun intention, we enact the intuitionistic fuzzification of the concept of sub-algebras and Bi-ideals in BCK-algebras, and examine some of their properties. We suggest the perception of equivalence relations on the clan of all intuitionistic fuzzy Bi-ideals of a BCK-algebra and explore some related properties.

II. Preliminaries

Definition2.1:[1]By a BCK-algebra we mean a non-empty set X with a binary operation $*$ and a constant 0 satisfying the following condition:

- (a) $((x*y)*(x*z))*(z*y) = 0$
- (b) $(x*(x*y))*y = 0$
- (c) $x*x = 0$
- (d) $0*x = 0$
- (e) $x*y = 0$ and $y*x = 0$ imply that $x = y$ for all $x, y, z \in X$.

Definition2.2:[6] A partial ordering “ $\leq$ ” on X can be defined by $x \leq y$ iff $x*y = 0$.

Definition2.3:[1] A non-empty subset S of a BCK-algebra X is called a sub -algebra of X if $x*y \in S$ whenever $x, y \in S$.

Definition2.4:[1] A non-empty subset I of a BCK-algebra X is called an ideal of X if

- (a) $0 \in I$
- (b) $x*y \in I$ and $y \in I$ imply that $x \in I$ for all $x, y \in X$.

Definition2.5:[6] A fuzzy set μ in a non-empty set X we mean a function $\mu : X \rightarrow [0,1]$.

Definition2.6:[6] The complement of μ , denoted by $\bar{\mu}$ is the fuzzy set in X given by $\bar{\mu}(x) = 1 - \mu(x)$ for all $x \in X$.

Definition2.7:[1] A fuzzy set μ in a BCK-algebra X is called a fuzzy sub-algebra of X if $\mu(x) \geq \min \{ \mu(x), \mu(y) \}$ for all $x, y \in X$.

Definition2.8:[1] A fuzzy set μ in a BCK-algebra X is called a fuzzy ideal of X if

- (a) $\mu(0) \geq \mu(x)$ for all $x \in X$
- (b) $\mu(x) \geq \min \{ \mu(x*y), \mu(y) \}$ for all $x, y \in X$.

III. Intuitionistic Fuzzy Bi-ideal

Definition3.1:[5] A non-empty set X is an object having the form $A = \{ (x, \lambda_A(x), \mu_A(x)) / x \in X \}$ where the functions $\lambda : X \rightarrow [0,1]$ and $\mu : X \rightarrow [0,1]$ denote the degree of membership and the degree of non-membership of each element $x \in X$ to the set A and $0 \leq \lambda_A(x) + \mu_A(x) \leq 1$ for all $x \in X$.

An intuitionistic fuzzy set $A = \{ (x, \lambda_A(x), \mu_A(x)) / x \in X \}$ in X can be identified to an ordered pair (λ_A, μ_A) in $I^X \times I^X$. For the sake of simplicity, we use the symbol $A = (\lambda_A, \mu_A)$ for the IFS, $A = \{ (x, \lambda_A(x), \mu_A(x)) / x \in X \}$.

Definition 3.2: A fuzzy subset μ in a BCK-algebra X is called fuzzy Bi-ideal if

- (a) $\mu_A(0) \geq \mu_A(x)$
- (b) $\mu_A(x * y) \geq \min \{ \mu_A(x * y * z), \mu_A(z) \}$ for all $x, y, z \in X$.

Definition 3.3:[5] An Intuitionistic fuzzy set $A = (\lambda_A, \mu_A)$ in X is called an intuitionistic fuzzy sub-algebra of X if it satisfies:

- (a) $\lambda_A(x * y) \geq \min \{ \lambda_A(x), \lambda_A(y) \}$
- (b) $\mu_A(x * y) \leq \max \{ \mu_A(x), \mu_A(y) \}$ for all $x, y \in X$.

Proposition 3.4: Every intuitionistic fuzzy sub-algebra $A = (\lambda_A, \mu_A)$ of X satisfies the inequalities

- (a) $\lambda_A(0) \geq \lambda_A(x)$ and
- (b) $\mu_A(0) \leq \mu_A(x)$ for all $x \in X$.

Proof: Straightforward.

Definition 3.5:[1] An Intuitionistic fuzzy set $A = (\lambda_A, \mu_A)$ in X is called an intuitionistic fuzzy ideal of X if it satisfies the following inequalities:

- (a) $\lambda_A(0) \geq \lambda_A(x)$ and $\mu_A(0) \leq \mu_A(x)$
- (b) $\lambda_A(x) \geq \min \{ \lambda_A(x * y), \lambda_A(y) \}$
- (c) $\mu_A(x) \leq \max \{ \mu_A(x * y), \mu_A(y) \}$ for all $x, y \in X$.

Definition 3.6: An intuitionistic fuzzy set $A = (\lambda_A, \mu_A)$ in X is called an intuitionistic fuzzy Bi-ideal of X if it satisfies:

- (a) $\lambda_A(0) \geq \lambda_A(x)$ and $\mu_A(0) \leq \mu_A(x)$
- (b) $\lambda_A(x * y) \geq \min \{ \lambda_A(x * y * z), \lambda_A(z) \}$
- (c) $\mu_A(x * y) \leq \max \{ \mu_A(x * y * z), \mu_A(z) \}$ for all $x, y, z \in X$.

Example 3.7: Consider a BCK-algebra $X = \{0, x, y, z\}$ with the following cayley table.

*	0	x	y	z
0	0	0	0	0
x	x	0	0	x
y	y	x	0	y
z	z	z	z	0

Define an IFS, $A = (\lambda_A, \mu_A)$ in X as follows :

$$\lambda_A(0) = \lambda_A(x) = 0.6, \lambda_A(y) = 0.2, \lambda_A(z) = 0.4$$

$$\mu_A(0) = \mu_A(x) = 0.5, \mu_A(y) = 0.3, \mu_A(z) = 0.2$$

Then by routine calculation $A = (\lambda_A, \mu_A)$ is an intuitionistic fuzzy Bi-ideal of X .

Lemma 3.8: Let an IFS $A = (\lambda_A, \mu_A)$ in X be an intuitionistic fuzzy Bi-ideal of X . If the inequality $x * y \leq z$ holds in X , then

- (a) $\lambda_A(x * y) \geq \min \{ \lambda_A(x), \lambda_A(z) \}$
- (b) $\mu_A(x * y) \leq \max \{ \mu_A(x), \mu_A(z) \}$

Proof: Let $x, y, z \in X$ be such that $x * y \leq z$

$$\text{Now (a) } \lambda_A(x * y) \geq \min \{ \lambda_A(x * y * z), \lambda_A(z) \}$$

$$\geq \min \{ \min \{ \lambda_A(x), \lambda_A(z) \}, \lambda_A(z) \}$$

$$= \min \{ \lambda_A(x), \lambda_A(z) \}$$

$$\text{Therefore, } \lambda_A(x * y) \geq \min \{ \lambda_A(x), \lambda_A(z) \}$$

$$\text{(b) } \mu_A(x * y) \leq \max \{ \mu_A(x * y * z), \mu_A(z) \}$$

$$\leq \max \{ \max \{ \mu_A(x), \mu_A(z) \}, \mu_A(z) \}$$

$$= \max \{ \mu_A(x), \mu_A(z) \}$$

$$\text{Therefore, } \mu_A(x * y) \leq \max \{ \mu_A(x), \mu_A(z) \}$$

$$\text{Hence (a) } \lambda_A(x * y) \geq \min \{ \lambda_A(x), \lambda_A(z) \}$$

$$\text{(b) } \mu_A(x * y) \leq \max \{ \mu_A(x), \mu_A(z) \} \text{ for all } x, y, z \in X.$$

Lemma 3.9: Let $A = (\lambda_A, \mu_A)$ be an intuitionistic fuzzy Bi-ideal of X . If $x * y \leq z$ in X , then $\lambda_A(x * y) \geq \lambda_A(z)$, $\mu_A(x * y) \leq \mu_A(z)$ that is λ_A is order-reserving and μ_A is order-preserving.

Proof: Let $x, y, z \in X$ be such that $x*y \leq z$. Then $x*y*z = 0$ and so

$$\begin{aligned} (a) \lambda_A(x*y) &\geq \min \{ \lambda_A(x*y*z), \lambda_A(z) \} \\ &\geq \min \{ \min \{ \lambda_A(x), \lambda_A(z) \}, \lambda_A(z) \} \\ &= \min \{ \lambda_A(x), \lambda_A(z) \} \\ &\geq \min \{ \lambda_A(0), \lambda_A(z) \} \\ &= \lambda_A(z) \end{aligned}$$

Therefore, $\lambda_A(x*y) \geq \lambda_A(z)$

$$\begin{aligned} (b) \mu_A(x*y) &\leq \max \{ \mu_A(x*y*z), \mu_A(z) \} \\ &\leq \max \{ \max \{ \mu_A(x), \mu_A(z) \}, \mu_A(z) \} \\ &= \max \{ \mu_A(x), \mu_A(z) \} \\ &\leq \max \{ \mu_A(0), \mu_A(z) \} \\ &= \mu_A(z) \end{aligned}$$

Therefore, $\mu_A(x*y) \leq \mu_A(z)$

Hence λ_A is order-reserving and μ_A is order-preserving.

Theorem 3.10: If $A = (\lambda_A, \mu_A)$ be an intuitionistic fuzzy Bi-ideal of X , then for any $x*y, a_1, a_2, \dots, a_n \in X$, $(\dots((x*y*a_1)*a_2)*\dots)*a_n = 0$ implies $\lambda_A(x) \geq \min \{ \lambda_A(a_1), \lambda_A(a_2), \dots, \lambda_A(a_n) \}$ and $\mu_A(x) \leq \max \{ \mu_A(a_1), \mu_A(a_2), \dots, \mu_A(a_n) \}$.

Proof: Using induction on n and Lemma 3.8 and 3.9, the proof is straightforward.

Theorem 3.11: Every intuitionistic fuzzy Bi-ideal of X is an intuitionistic fuzzy sub-algebra of X .

Proof: Let $A = (\lambda_A, \mu_A)$ be an intuitionistic fuzzy bi-ideal of X .

Since $x*y \leq x$ for all $x, y \in X$, it follows from lemma 3.9 that, $\lambda_A(x*y) \geq \lambda_A(x)$, $\mu_A(x*y) \leq \mu_A(x)$.

Now, (a) $\lambda_A(x*y) \geq \lambda_A(x)$

$$\geq \min \{ \lambda_A(x*y), \lambda_A(y) \}$$

$$\geq \min \{ \lambda_A(x), \lambda_A(y) \}$$

Therefore, $\lambda_A(x*y) \geq \min \{ \lambda_A(x), \lambda_A(y) \}$

$$(b) \mu_A(x*y) \leq \mu_A(x)$$

$$\leq \max \{ \mu_A(x*y), \mu_A(y) \}$$

$$\leq \max \{ \mu_A(x), \mu_A(y) \}$$

Therefore, $\mu_A(x*y) \leq \max \{ \mu_A(x), \mu_A(y) \}$

Thus $A = (\lambda_A, \mu_A)$ is an intuitionistic fuzzy sub-algebra of X .

Theorem 3.12: Let $A = (\lambda_A, \mu_A)$ be an intuitionistic fuzzy sub-algebra of X such that $\lambda_A(x) \geq \min \{ \lambda_A(y), \lambda_A(z) \}$, $\mu_A(x) \leq \max \{ \mu_A(y), \mu_A(z) \}$ for all $x, y, z \in X$ satisfying the inequality $x*y \leq z$. Then $A = (\lambda_A, \mu_A)$ is an intuitionistic fuzzy Bi-ideal of X .

Proof: Let $A = (\lambda_A, \mu_A)$ be an intuitionistic fuzzy sub-algebra of X .

Recall that $\lambda_A(0) \geq \min \lambda_A(x*y)$ and $\mu_A(0) \leq \max \mu(x*y)$ for all X .

Since $(x*x)*(x*z) \leq z$, it follows from the hypothesis that,

$$(a) \lambda_A(x*y) \geq \min \{ \lambda_A(x*y*z), \lambda_A(z) \}$$

$$(b) \mu_A(x*y) \leq \max \{ \mu_A(x*y*z), \mu_A(z) \}$$

Hence $A = (\lambda_A, \mu_A)$ is an intuitionistic fuzzy bi-ideal of X .

Lemma 3.13: An IFS $A = (\lambda_A, \mu_A)$ is an intuitionistic fuzzy Bi-ideal of X iff the fuzzy sets λ_A and $\overline{\mu_A}$ are fuzzy Bi-ideals of X .

Proof: Let $A = (\lambda_A, \mu_A)$ be an intuitionistic fuzzy bi-ideal of X .

For every $x, y, z \in X$, we have,

$$(a) \overline{\mu_A}(0) = 1 - \mu_A(0)$$

$$\geq 1 - \mu_A(x*y)$$

$$= \overline{\mu_A}(x*y)$$

Therefore, $\overline{\mu_A}(0) \geq \overline{\mu_A}(x*y)$

$$\begin{aligned} \text{(b)} \quad \overline{\mu_A}(x*y) &= 1 - \mu_A(x*y) \\ &\geq 1 - \max \{ \mu_A(x*y*z), \mu_A(z) \} \\ &= \min \{ 1 - \mu_A(x*y*z), 1 - \mu_A(z) \} \\ \text{Therefore, } \overline{\mu_A}(x*y) &= \min \{ \overline{\mu_A}(x*y*z), \overline{\mu_A}(z) \} \end{aligned}$$

Hence $\overline{\mu_A}$ is a fuzzy bi-ideal of X.

Conversely assume that λ_A and $\overline{\mu_A}$ are fuzzy bi-ideals of X.

For every $x, y, z \in X$, we have,

$$\begin{aligned} \lambda_A(0) &\geq \lambda_A(x*y), \quad 1 - \mu_A(0) = \overline{\mu_A}(0) \geq \overline{\mu_A}(x*y) = 1 - \mu_A(x*y) \\ \text{That is, } \mu_A(0) &\leq \mu(x*y); \lambda_A(x*y) \geq \min \{ \lambda_A(x*y*z), \lambda_A(z) \} \text{ and} \end{aligned}$$

$$\begin{aligned} 1 - \mu_A(x*y) &= \overline{\mu_A}(x*y) \\ &\geq \min \{ \overline{\mu_A}(x*y*z), \overline{\mu_A}(z) \} \\ &= \min \{ 1 - \mu_A(x*y*z), 1 - \mu_A(z) \} \\ &= 1 - \max \{ \mu_A(x*y*z), \mu_A(z) \} \\ \mu_A(x*y) &\leq \max \{ \mu_A(x*y*z), \mu_A(z) \} \end{aligned}$$

Hence $A = (\lambda_A, \mu_A)$ is an intuitionistic fuzzy Bi-ideal of X.

Theorem 3.14: Let $A = (\lambda_A, \mu_A)$ is an intuitionistic fuzzy set in X. Then $A = (\lambda_A, \mu_A)$ is an intuitionistic fuzzy Bi-ideal of X iff $A = (\lambda_A, \overline{\lambda_A})$ and $A = (\mu_A, \overline{\mu_A})$ are intuitionistic fuzzy Bi-ideal of X.

Proof: If $A = (\lambda_A, \mu_A)$ is an intuitionistic fuzzy bi-ideal of X, then $\lambda_A = \overline{\overline{\lambda_A}}$ and μ_A are fuzzy Bi-ideals of X from lemma 3.12.

Hence $A = (\lambda_A, \overline{\lambda_A})$ and $A = (\mu_A, \overline{\mu_A})$ are intuitionistic fuzzy Bi-ideals of X.

Conversely, if $A = (\lambda_A, \overline{\lambda_A})$ and $A = (\mu_A, \overline{\mu_A})$ are intuitionistic fuzzy Bi-ideals of X, then the fuzzy sets λ_A and $\overline{\mu_A}$ are fuzzy Bi-ideals of X.

Hence $A = (\lambda_A, \mu_A)$ is an intuitionistic fuzzy Bi-ideal of X.

Definition 3.15:[1] For any $t \in [0,1]$ and a fuzzy set μ in a non-empty set X, the set $U(\mu;t) = \{x \in X / \mu(x) \geq t\}$ is called an upper t-level cut of μ and the set $L(\mu;t) = \{x \in X / \mu(x) \leq t\}$ is called a lower t-level cut of μ .

Definition 3.16:[1] A mapping $f: X \rightarrow Y$ of BCK-algebras is called a homomorphism if $f(x*y) = f(x)*f(y)$ for all $x, y \in X$

Note that if $f: X \rightarrow Y$ is a homomorphism of BCK-algebras then $f(0) = 0$.

Let $f: X \rightarrow Y$ be a homomorphism of BCK-algebras. For any IFS $A = (\lambda_A, \mu_A)$ in Y, we define a new IFS $A^f = (\lambda_A^f, \mu_A^f)$ in X by,

$$\lambda_A^f(x) = \lambda_A(f(x)), \quad \mu_A^f(x) = \mu_A(f(x)) \text{ for all } x \in X.$$

Theorem 3.17: Let $f: X \rightarrow Y$ be a homomorphism of BCK-algebra. If an IFS $A = (\lambda_A, \mu_A)$ in Y is an intuitionistic fuzzy Bi-ideal of Y, then an IFS $A^f = (\lambda_A^f, \mu_A^f)$ in X is an intuitionistic fuzzy Bi-ideal of X.

Proof: We first have that

$$\begin{aligned} \text{(a)} \quad \lambda_A^f(x*y) &= \lambda_A(f(x*y)) \leq \lambda_A(0) \\ &= \lambda_A(f(0)) \end{aligned}$$

$$\text{Therefore, } \lambda_A^f(x*y) = \lambda_A^f(0)$$

$$\begin{aligned} \text{(b)} \quad \mu_A^f(x*y) &= \mu_A(f(x*y)) \geq \mu_A(0) \\ &= \mu_A(f(0)) \end{aligned}$$

$$\text{Therefore, } \mu_A^f(x*y) = \mu_A^f(0) \text{ for all } x*y \in X.$$

Let $x, y, z \in X$. Then

$$\begin{aligned} \text{(a)} \quad \min \{ \lambda_A^f(x*y*z), \lambda_A^f(z) \} &= \min \{ \lambda_A(f(x*y*z)), \lambda_A(f(z)) \} \\ &= \min \{ \lambda_A(f(x*y)*z), \lambda_A(f(z)) \} \\ &\leq \min \{ \lambda_A(f(x)*f(y)*f(z)), \lambda_A(f(z)) \} \\ &= \min \{ \lambda_A(f(x*y), \lambda_A(f(z))), \lambda_A(f(z)) \} \\ &\leq \min \{ \lambda_A(f(x*y)) \} \end{aligned}$$

$$= \lambda_A^f(x*y)$$

Therefore, $\min \{ \lambda_A^f(x*y*z), \lambda_A^f(z) \} \leq \lambda_A^f(x*y)$

$$\begin{aligned} (b) \max \{ \mu_A^f(x*y*z), \mu_A^f(x) \} &= \max \{ \mu_A(f(x*y*z)), \mu_A(f(x)) \} \\ &= \max \{ \mu_A(f(x*y)*z), \mu_A(f(x)) \} \\ &\geq \max \{ \mu_A(f(x)*f(y))*f(z), \mu_A(f(x)) \} \\ &= \max \{ \mu_A(f(x*y)), \mu_A(f(z)), \mu_A(f(x)) \} \\ &= \mu_A(f(x*y)) \\ &\geq \mu_A^f(x*y) \end{aligned}$$

Therefore, $\max \{ \mu_A^f(x*y*z), \mu_A^f(x) \} \geq \mu_A^f(x*y)$

Hence $A^f = (\lambda_A^f, \mu_A^f)$ in X is an intuitionistic fuzzy Bi-ideal of X .

Theorem 3.18: Let $f: X \rightarrow Y$ be an epimorphism of BCK-algebras and let $A = (\lambda_A, \mu_A)$ be an IFS in Y . If $A^f = (\lambda_A^f, \mu_A^f)$ is an intuitionistic fuzzy Bi-ideal of X , then $A = (\lambda_A, \mu_A)$ is an intuitionistic fuzzy Bi-ideal of Y .

Proof: For any $x, y \in Y$, there exists $a \in X$ such that $f(a) = x*y$. Then

$$\begin{aligned} (a) \lambda_A(x*y) &= \lambda_A(f(a)) = \lambda_A^f(a) \leq \lambda_A^f(0) \\ &= \lambda_A^f(f(0)) \\ &= \lambda_A(0) \end{aligned}$$

Therefore, $\lambda_A(x*y) = \lambda_A(0)$

$$\begin{aligned} (b) \mu_A(x*y) &= \mu_A(f(a)) = \mu_A^f(a) \geq \mu_A^f(0) \\ &= \mu_A^f(f(0)) \\ &= \mu_A(0) \end{aligned}$$

Therefore, $\mu_A(x*y) = \mu_A(0)$

Let $x, y, z \in Y$, then $f(a) = x, f(b) = y$ and $f(c) = z$ for some $a, b, c \in X$. It follows that

$$\begin{aligned} (a) \lambda_A(x*y) &= \lambda_A(f(a)*f(b)) = \lambda_A^f(a*b) \\ &\geq \min \{ \lambda_A^f(a*b*c), \lambda_A^f(c) \} \\ &= \min \{ \lambda_A(f(a*b*c)), \lambda_A(f(c)) \} \\ &= \min \{ \lambda_A(f(a)*f(b)*f(c)), \lambda_A(f(c)) \} \end{aligned}$$

Therefore, $\lambda_A(x*y) \geq \min \{ \lambda_A(x*y*z), \lambda_A(z) \}$

$$\begin{aligned} (b) \mu_A(x*y) &= \mu_A(f(a)*f(b)) = \mu_A^f(a*b) \\ &\leq \max \{ \mu_A^f(a*b*c), \mu_A^f(c) \} \\ &= \max \{ \mu_A(f(a*b*c)), \mu_A(f(c)) \} \\ &= \max \{ \mu_A(f(a)*f(b)*f(c)), \mu_A(f(c)) \} \end{aligned}$$

Therefore, $\mu_A(x*y) \leq \max \{ \mu_A(x*y*z), \mu_A(z) \}$

Hence $A = (\lambda_A, \mu_A)$ is an intuitionistic fuzzy Bi-ideal of Y .

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