



Evaluation of Integral Time Square Error using Optimal Cost Function to stabilize the Non Linear Time Varying System

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Abstract: The foundation of optimization based feedback stabilization for a non-linear system has been developed. The time-varying system parameters are adjusted to fix the response of the system at an optimum level of performance.

The effective form of performance index is the Integral Square Error. Integral of the Square Error (ISE) and Integral Time Square Error (ITSE) are the performance indices and also these are the cost function based differential equation. A system is considered as an Optimum control system, when the system parameters are adjusted so that the indices are reached its optimum level, generally a minimum value. The aim of the thesis is a specific unified form which requires a performance index.

Keywords: Hamilton-Jacobi-Bellman Method, Partial Differential Equations, ISE, ITSE, Affine System,

I. INTRODUCTION

The use of Optimum Regulator reflects the state of art in modern predictive control technology. Dynamic programming provides a computational numeric method to apply the principle of Optimality to a sequence of decision which defines an Optimum control policy. The Principles were mostly developed by Pontryagin in the 1950s and his co-workers [1]. However, before the invention of Optimal regulator in the year 1960 by R.E. Kalman[27], Linear Quadratic Gaussian Analysis had played over a period of fifty years an optimum role in the design of controllers as well as in electro-control mechanical system. The Performance Index is selected to carry forward the system response towards the desired behavior of the Dynamical System. However, solving the associated Hamilton-Jacobi-Bellman (HJB) equation demands a very large computational cost in terms of time and storage requirement. The numerical complexity is overcome by using an Approximate Dynamic Programming (ADP) formulation proposed by Werbos[12].

II. OPTIMUM CONTROL

Around 40 years ago, the optimal control was begun in Soviet Union [1]. The 'Pontryagin Maximum Principle' is nothing but minor edition of 'Calculus of Variation'. L.Marks describes this explanation in International Congress of Mathematicians in the year 1958. Primarily, the principle was applied in Military purpose. The Calculus of Variation has standardized form to minimize 'J', Performance Index (PI)

$$J_{\min} = \int_a^b \Gamma(x(t), \dot{x}(t), t) dt \quad \text{subject to } x(a) = \bar{q} \quad \text{and } x(b) = \hat{q} \quad \dots \dots \dots (1)$$

Eqⁿ(1) have the following Optimize form minimize 'J*', Optimum Performance Index (PI)

$$J_{\min}^* = \int_a^b \Gamma(x(t), u(t), t) dt \quad \text{initialize with } x(a) = \bar{q} \quad \text{and } x(b) = \hat{q} \quad \text{for } a \leq t \leq b \quad \dots \dots \dots (2)$$

In particular, Optimum Control theory shows its reverse characteristic with comparison to the Calculus of Variation, where the controlled variable, $\Gamma \equiv 1$ i.e., completely trivial, and so in Optimal Control, interesting action occurs via Dynamic Programming. So for considering $\Gamma = 1$ among a set of curves where it is desired to minimize time, it set $t \rightarrow x(t)$ that satisfies end point constraints as in Eqⁿ(2) and the solution of

$\dot{q}(t) = \Gamma(q(t), u(t), t)$ for some control $t \rightarrow u(t)$, is called minimum time problem. Therefore It clearly shows the difference between Optimal Control and the Calculus of Variations.

III. HJB FORMULATION FOR NON LINEAR SYSTEM

Consider a continuous-time non linear system as, $\dot{x} = F(x, u, t)$ and $x(t_0) = x_0$ (3)
 $y = h(x(t))$

where $x(t) \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ are the state vector and control input. y , is the system output. 'F' and 'h' are the system matrices. x_0 , is the operating point of the system.

The Eqⁿ(3) is unique for any finite initial condition ' x_0 ' and control u .

The stability of Non linear system can be highly sensitive to initial condition.

For Stabilize system there exists a continuous feedback law $u(x(t))$ such that the closed loop system is asymptotically stable.

The cost function for the system (Eqⁿ(3.1)) can be defined as :

$$J(x(t), u(t), t) = C(x(T), T) + \int_t^T \Gamma(x(\tau), u(\tau), \tau) d\tau \quad \dots\dots\dots (4)$$

where 'C' is constant parameter of terminating point of state of the system and 'Γ' is instantaneous cost

function. Based on the expression of Hamiltonian, $\Gamma(x(\tau), u(\tau), \tau) + \frac{\partial J^*}{\partial x} F(x(t), u(t), t)$, the HJB equation

$$\text{is written as, } \frac{\partial J^*}{\partial t} + \min_u (\Gamma(x(t), u(t), t) + \frac{\partial J^*}{\partial x} (F(x, u, t))) \quad [7]. \quad \dots\dots\dots (5)$$

IV. CONTRIBUTION OF THE THESIS WORK : EVALUATION OF INTEGRAL TIME SQUARE ERROR USING OPTIMAL COST FUNCTION

The Optimal Control problem is to find an admissible control u^* such that the cost in Eqⁿ(4) associated with the system Eqⁿ(3) is minimized.

$$\text{Now, the Optimal cost function is } J^*(x(t), t) = \min_u [J(x(t), u(t), t)] \quad \dots\dots\dots (6)$$

With the trajectory of reverse characteristic of *Principle of Optimal Control*, the Optimum Performance Index with disturbance is expressed on the interval $[(t+\Delta t), T]$ as :

$$J_\Psi = \int_{t+\Delta t}^T (x(t)' Q x(t) + [u(t) + \Psi(t)]' R [u(t) + \Psi(t)]) dt \quad \dots\dots\dots (7)$$

where ' Ψ ' is disturbance of the system.

The instantaneous cost function with disturbance excitation is,-

$$\Gamma_\Psi = \{x(t)' Q x(t) + [y(t) + \Psi(t)]' R [u(t)]\} \text{ and consider, } e(t) = y(t) - \Gamma_\Psi \quad \dots\dots\dots (8)$$

So $J^*(e(\Delta t), \Delta t, \Delta u)$ is Integral Time Square Error performance parameter.

So minimization of this error ' $e(t)$ ' is evaluated by the following process,-

$$\text{Optimal cost function, } J^* = \min \int_0^\infty [e^2(t)] dt \quad \dots\dots\dots (9)$$

For non linear and time varying system, the performance index is modified as,

$$J = \int_t^T [e^2(t) + \gamma u^2(t)] dt \quad \dots\dots\dots (10)$$

When the constraint of the system is under controlled, the system matrix ‘ γ ’ is properly optimized.

In discrete time system, the minimization of error vector is optimized by developing the performance index as,

$$J^*(x(n), n) = \frac{1}{2} e^T(N) P e(N) + \frac{1}{2} \sum_{n=0}^{N-1} e^T(n) M e(n) + u^T(n) N u(n)$$

$$J^*(x(n), n) = \frac{1}{2} e^T(N) P e(N) + \frac{1}{2} \sum_{n=0}^{n+\Delta n} e^T(n) M e(n) + u^T(n) N u(n) + \frac{1}{2} \sum_{n=n+\Delta n}^{N-1} e^T(n) M e(n) + u^T(n) N u(n)$$

$$J^*(x(n), n) = \frac{1}{2} e^T(N) P e(N) + \frac{1}{2} \sum_{n=0}^{n+\Delta n} e^T(n) M e(n) + u^T(n) N u(n) + J^*(x(n + \Delta n), n + \Delta n) \quad \dots\dots\dots (11)$$

Expanding $J^*(x(n + \Delta n), n + \Delta n)$ in *Taylor's* series about $(x(n), n)$ and neglecting the Higher Order Term

$$J^*(x(n + \Delta n), n + \Delta n) = \frac{1}{2} \sum_{n=0}^{n+\Delta n} e^T(n) M e(n) + u^T(n) N u(n) + J^*(x(n + \Delta n), n + \Delta n) + \Delta J^*(n + \Delta n - n) + \Delta J^*(x(n + \Delta n) - x(n)) + H.O.T$$

$$0 = \frac{1}{2} \sum_{n=0}^{n+\Delta n} e^T(n) M e(n) + u^T(n) N u(n) + \Delta J^*(\Delta n) + \Delta J^*(\dot{x} \Delta n) + H.O.T$$

(H.O.T – Higher Order Terms) \dots\dots\dots (12)

For small Δn ,

$$0 = e^T(\Delta n) M e(\Delta n) + u^T(\Delta n) N u(\Delta n) + 2\Delta J^*(\Delta n) + 2\Delta J^*(\dot{x} \Delta n) \quad \dots\dots\dots (13)$$

In Continuous Time System, the above equation is written in the following way,

$$0 = [e^T(\Delta t) M e(\Delta t) + u^T(\Delta t) N u(\Delta t)] + 2 \frac{\partial J^*}{\partial t}(x(t), t) + 2 \frac{\partial J^*}{\partial x}(F(x, u, t)) \quad \dots\dots\dots (14)$$

For, small Δt the optimal point of *Integral Time Square Error* is determined by,

$$J^*(e(\Delta t), \Delta t, \Delta u) \Big|_{\min u} = \left[2t \frac{\partial J^*}{\partial t}(x(t), t) + 2t \frac{\partial J^*}{\partial x}(F(x, u, t)) \right] \quad \dots \quad \dots\dots\dots (15)$$

V. NON LINEAR SYSTEM : HAMILTONIAN FORM

Minimum value of the Performance Indices are determined by from Eqⁿ(4),

$$J_{\min} = \min \left\{ C(x(T), T) + \int_t^T \Gamma(x(\tau), u(\tau), \tau) d\tau \right\} \dots \dots \dots (16)$$

Now, for considering the Hamiltonian Form we have,

$$H^*(x, \dot{J}^*(x), u) = \Gamma(x, u) + \frac{\partial J^{*T}}{\partial x} [\dot{x}] \dots \dots \dots (17)$$

Again, for non linear system consider,

$$\dot{x} = f(x) + Y(x)u \dots \dots \dots (18)$$

VI.

where $f(\cdot)$, $Y(\cdot)$ are sufficiently smooth functions on a neighborhood of the origin $x = 0$

Again considering a quadratic utility function in discrete time system which is given by ,

$$\Gamma(x(n), u(n)) = \frac{1}{2} (x(n)^T M x(n) + u(n)^T N u(n)) \dots \dots \dots (19)$$

VI. RELATION BETWEEN OPTIMUM POINT OF ITSE AND HAMILTONIAN PERFORMANCE INDEX

For an affine system, with optimal policy, the performance indices are expressed as,

$$\frac{\partial J(n)}{\partial x(n)} = N u(n) + Y^T(x(n), \beta(n+1)) = 0 \dots \dots \dots (20)$$

Let, $\beta = \frac{\partial J}{\partial x}$ then,

$$\frac{\partial J(n)}{\partial u(n)} = N u(n) + Y^T(x(n), \beta(n+1)) = 0 \dots \dots \dots (21)$$

From Eqⁿ(20), the plan input $u(n)$ is represented as,

$$u(n) = -N^{-1} Y^T \beta(n+1) \dots \dots \dots (22)$$

Differentiating again Eqⁿ(20) we have,

$$\frac{\partial^2 J(n)}{\partial u^2(n)} = N > 0 \dots \dots \dots (23)$$

$$\text{For Optimum control affine system, the } \beta = \frac{\partial J}{\partial x} \text{ is written as, } \beta^* = \frac{\partial J^*}{\partial x} \dots \dots \dots (24)$$

$$\text{The Eq}^n(11) \text{ is then written as, } u^* = -N^{-1} Y^T \beta^* \dots \dots \dots (25)$$

Which gives the Eqⁿ (17) is a new form, that is

$$H(x^*, \beta^*, u^*) = \frac{1}{2} x^{*T} M x^* + \frac{1}{2} \beta^{*T} Y N^{-1} Y^T \beta + \beta^{*T} [f - Y N^{-1} Y^T \beta^*] \\ \Rightarrow H^* = \frac{1}{2} x^{*T} M x^* - \frac{1}{2} \beta^{*T} Y N^{-1} Y^T \beta^* + \beta^{*T} f \dots \dots \dots (26)$$

Now, Optimum control of performance indices is followed by the HJB equation in following way

$$\frac{\partial J^*}{\partial t} + H(x, \beta^*, u, t) \Big|_{\min} = 0 \dots \dots \dots (27)$$

Now, Comparing the Eqⁿ (4.12) with Eqⁿ (3.10) a relation is created between the optimum point of Integral Time Square Error (ITSE) and Hamiltonian Performance Index which is explained below :

$$J^*(e(\Delta t), \Delta t, \Delta u) \Big|_{\min u} = \left[2t \frac{\partial J^*}{\partial t}(x(t), t) + 2t \left| H(x, \beta^*, u, t) \right| \right] \dots\dots\dots (28)$$

$$\text{So it will be said that, } -\frac{\partial J^*}{\partial t}(\Gamma(x, u, t)) = \left| H(x, \beta^*, u, t) \right| \dots\dots\dots (29)$$

where ,

$$\beta^* = \frac{\partial J^*}{\partial x} \text{ is known as Costate Vector}$$

VII. CONCLUSION

In Optimal Control, Optimum Regulator involves optimization of the performance of control system by minimizing or maximizing the characteristics of system parameters. The Performance Index (PI) provides a gateway between the amount of energy used to achieve desired performance and the specified desired performance. Using Linear Quadratic Optimum Regulator, the Time In Varying or Time Varying system was regulated. But for Non Linear system, It is very difficult to designed a well defined regulator that optimally unaltered the characteristics of response in presence of Time Varying System Parameters.

Here, using Hamiltonian, Jacobi and Bellman (HJB) method, I have tried to evaluate the Integral Time Square Error (ITSE) in Non Linear System by Optimum Regulator. The most common performance indices are Integral Square error (ISE) used in practice. When this ISE is time dependent parameter then it becomes ITSE. I also try to make the relation in between the ITSE and Optimum Hamiltonian Performance Index to get the Optimum Value Function satisfied by HJB method.

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References

- [1] Hector J. Sussmann, Jan C. Willems, " 300 Years of Optimal Control : From The Brachystochrone to the Maximum Principle ", History Session of the 35th Conference on Decision and Conference on Decision and Control, pp. 32-44, in Kobe. Japan on December. 11, 1996 .
- [2] A.K. Nanda Kumaran, " Calculus of Variation to Optimal Control ", Lecture notes at Winter School on Stochastic Analysis and Control of Fluid Flow during the period of December. 03-20, 2012, pp. 1-23, at Department of Mathematics, IISER, Trivandrum.
- [3] J. Balaram, G.N. Saridis, " Design of Sub Optimal Regulators For Nonlinear Systems ", Proceedings of 24th IEEE Conference on Decision and Control Ft. Lauderdale, Fl, pp. 273-278, December. 1985.
- [4] J.A. Nickerson, " A New Linear Quadratic Optimal Controller for the 34- Meter High Efficiency Antenna Position Looop ", TDA Progress Report - NASA Technical Reports Server. (NTRS): The Telecommunications and Data Acquisition Report 42-90, pp. 136-141, April-June. 1987
- [5] Alessio Merola, Carlo Cosentino Domenico Colacino, Francesco Amato, "Optimal Control of Uncertain Nonlinear Quadratic Systems with Constrained Inputs", arXiv: 1701.03008 v1 [cs.SY], pp.1-12, 11 January 2017.
- [6] R. Oldenberger and T. Nakada, "Signal Stabilisation of self Oscillating Systems" In Proc. Int. Federal Autom. Control, 1960, Vol.1 . pp – 233-238.
- [7] Guoshan Zhang¹, Lei Liu¹ and Wanquan Liu "LINEAR QUADRATIC OPTIMAL CONTROL BASED ON DYNAMIC COMPENSATION", *International Journal of Innovative Computing, Information and Control* Volume 8, Number 5(B), May 2012 ISSN 1349-4198 pp. 3743-3754.
- [8] J.V. Breakwell J.L. Speyer and A.E. Bryson, " Optimization and Control of Non linear system using the second variation", SIAM J. Control. Vol.1.No.2 (1963) pp-193-223.
- [9] J.S. Tyler and F.B. Tuteur "The Use Of A Quadratic Performance Index To Design Multivariable Control System", IEEE. Trans. Auto Control Vol. AC-II No.1 (January 1966) pp-84-92.
- [10] Brian D.O. Anderson and John.B. Moore, "Optimal Control", Prentice Hall of India Private Limited 2004. [11] A.E. Bryson and Y.C. Ho. "Applied Optimal Control". Taylor and Francis. 1975.
- [12] P.J. Werbos. "Approximate Dynamic Programming for real time control and neural modeling. In D.A. Sofge (Eds.), Handbook of intelligent control, Multiscience Press, 1992
- [13] S. Ferrari and R.F. Stengel. "Model based adaptive critic designs", In Jennie Si, A.G. Barto, W.B. Powell, and D. Wunsch II (Eds.), Handbook of learning and Approximate Dynamic Programming, IEEE Press, 2004.
- [14] D.V. Prokhorov and D.C. Wunsch II. "Adaptive critic designs.", IEEE Transactions on Neural Networks, 8(5): 997-1007, September 1997.
- [15] Chuan-Kai Lin, "Adaptive critic autopilot design of bank-to-turn missiles using fuzzy basis function networks.", IEEE Trans. On SMC B, 35(2) : pp-197-207, 2005.