



Stability and Bifurcation in a Fractional Order Prey – Predator Model with Omnivore and its Discretization

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Abstract: In this present work, we discuss the dynamical behavior of prey - predator model in the presence of an omnivore. A discretization process is applied to obtain a discrete version of fractional order system and the impact of the omnivore on the ecosystem is discussed. Existence and Local stability of the equilibrium states of the fractional order system are established. Using growth rate of prey as the bifurcation parameter, it is shown that the system undergoes a period doubling (flip) and Hopf bifurcation around axial and positive equilibrium states. It has been found that the dynamical behavior of the model is very sensitive to the parameter values and initial conditions. The trajectories and phase plane diagrams are plotted for biologically meaningful sets of parameter values. Bifurcation of the model is also discussed for selected range of growth parameter. Numerical simulations of the model are performed to demonstrate the analytical results and the rich dynamic nature of the discretized system is exhibited. The Lyapunov exponents are numerically computed to characterize the asymptotic stability of the system dynamic response and estimate the amount of chaos in the system.

Keywords: Fractional Order, Prey, Predator – Omnivore, Equilibrium States, Discretization, Period Doubling Bifurcation, Hopf Bifurcation, Chaos.

I. Introduction

Mathematical models are of importance in examining the complex dynamics of interacting populations. Lotka and Volterra introduced the first mathematical model which described the interaction of populations [4, 11]. We consider three species model which includes prey, predator and omnivore in an ecosystem. An omnivore is a predator feeding on more than one trophic level [1, 3] and it was commonly observed in some three or more species food chain models. The omnivore considered in this work is introduced as a scavenger top-predator, which does not only consume the carcasses of the predator but also predaes the original prey. In recent years, there are several mathematical models involving Omnivores and Scavenger [6, 7, 8, 10]. Fractional order differential equations have also been successfully applied in population dynamics [5].

II. Fractional Order Prey - Predator Model with an Omnivore

In this discussion, we consider the following three dimensional fractional order predator - prey - omnivore system:

$$\begin{aligned} {}^c D_t^\alpha x(t) &= rx(t)(1-x(t)) - ax(t)y(t) - x(t)z(t) \\ {}^c D_t^\alpha y(t) &= \tau x(t)y(t) - cy(t) \\ {}^c D_t^\alpha z(t) &= x(t)z(t) - bz(t) + \mu y(t)z(t) - \eta z^2(t) \end{aligned} \quad (1)$$

with the initial conditions $x(t), y(t), z(t)|_{t=0} = (x_0, y_0, z_0)$ where ${}^c D_t^\alpha$ is the Caputo fractional order derivative which satisfies $0 < \alpha \leq 1$. Especially, when $\alpha = 1$, the system (1) is a classical integer order system. All the constant coefficients $\alpha, r, a, b, c, \tau, \mu$ and η are positive real numbers. In the model $x(t) > 0, y(t) > 0$ and $z(t) > 0$ represent the densities of prey, predator and omnivore species at time t .

III. Existence and Uniqueness

In this section, we establish the existence and uniqueness of solutions for the system (1). Here, the fractional order prey – predator model with omnivore (1) can be rewritten in the form

$${}^c D_t^\alpha X(t) = AX(t) + x(t)BX(t) + y(t)CX(t) + z(t)DX(t), \quad X(0) = X_0 \quad (2)$$

where $0 < \alpha \leq 1$, $t \in (0, T]$ and $X(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$, $X_0 = \begin{pmatrix} x(0) \\ y(0) \\ z(0) \end{pmatrix}$, $A = \begin{pmatrix} r & 0 & 0 \\ 0 & -c & 0 \\ 0 & 0 & -b \end{pmatrix}$, $B = \begin{pmatrix} -r & -a & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$,

$$C = \begin{pmatrix} 0 & 0 & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & \mu & -\eta \end{pmatrix}.$$

Definition 3.1. For $X(t) = (x(t), y(t), z(t))^T$, let $C_1^*[0, T]$ be the set of continuous column vectors $X(t)$ on the interval $[0, T]$. The norm of $X(t) \in C_1^*[0, T]$ is given by $\|X(t)\| = \max |x(t)|$.

Theorem 3.1. The fractional order prey – predator with omnivore system (2) has a unique solution if $X(t) \in C_1^*[0, T]$.

Proof. Let $F(X(t)) = AX(t) + x(t)BX(t) + y(t)CX(t) + z(t)DX(t)$, then $X(t) \in C_1^*[0, T]$ implies $F(X(t)) \in C_1^*[0, T]$. In addition, we take $X_1(t), X_2(t) \in C_1^*[0, T]$ and $X_1(t) \neq X_2(t)$, then the following inequality holds:

$$\begin{aligned} \|F(X_1(t)) - F(X_2(t))\| &= \|A(X_1(t) - X_2(t)) + x_1(t)BX_1(t) - x_2(t)BX_2(t) + y_1(t)CX_1(t) \\ &\quad - y_2(t)CX_2(t) + z_1(t)DX_1(t) - z_2(t)DX_2(t)\| \\ &\leq \|A(X_1(t) - X_2(t))\| + \|Bx_1(t)(X_1(t) - X_2(t))\| + \|Bx_2(t)(x_1(t) - x_2(t))\| \\ &\quad + \|Cy_1(t)(X_1(t) - X_2(t))\| + \|CX_2(t)(y_1(t) - y_2(t))\| + \|Dz_1(t)(X_1(t) - X_2(t))\| \\ &\quad + \|DX_2(t)(z_1(t) - z_2(t))\| \\ &\leq [\|A\| + \|B\|(|x_1(t)| + \|X_2(t)\|) + \|C\|(|y_1(t)| + \|X_2(t)\|) \\ &\quad + \|D\|(|z_1(t)| + \|X_2(t)\|)] \|X_1(t) - X_2(t)\| \\ \|F(X_1(t)) - F(X_2(t))\| &\leq L \|X_1(t) - X_2(t)\|. \end{aligned}$$

where $L = \|A\| + [\|B\| + \|C\| + \|D\|](M_1 + M_2) > 0$ and M_1 & M_2 are positive and satisfy $\|X_1(t)\| \leq M_1$ and $\|X_2(t)\| \leq M_2$ as result of $X_1(t), X_2(t) \in C_1^*[0, T]$. Based on Theorem 2.1 and 2.2 in [1], system (2) has a unique solution.

IV. Equilibrium States and Local Stability Analysis

In this section, we analyze the equilibrium states and apply a discretization process of a fractional-order System (1) outlined in [2, 7, 8]. We obtain the discrete fractional order omnivore system as follows:

$$\begin{aligned} x_{t+1} &= x_t + \frac{s^\alpha}{\alpha\Gamma(\alpha)} (rx_t(1-x_t) - ax_t y_t - x_t z_t) \\ y_{t+1} &= y_t + \frac{s^\alpha}{\alpha\Gamma(\alpha)} (\tau x_t y_t - cy_t) \\ z_{t+1} &= z_t + \frac{s^\alpha}{\alpha\Gamma(\alpha)} (x_t z_t - bz_t + \mu y_t z_t - \eta z_t^2) \end{aligned} \quad (3)$$

Now, we discuss the local dynamical behaviors of the discretized fractional-order system (3) which is determined by the parameters $\alpha, s, r, a, b, c, \tau, \mu$ and η . The Jacobian matrix of the state variable is given by

$$J(x, y, z) = \begin{bmatrix} 1 + \frac{s^\alpha}{\alpha\Gamma(\alpha)} (r[1-2x] - ay - z) & -\frac{s^\alpha}{\alpha\Gamma(\alpha)} ax & -\frac{s^\alpha}{\alpha\Gamma(\alpha)} x \\ \frac{s^\alpha}{\alpha\Gamma(\alpha)} \tau y & 1 + \frac{s^\alpha}{\alpha\Gamma(\alpha)} (\tau x - c) & 0 \\ \frac{s^\alpha}{\alpha\Gamma(\alpha)} z & \frac{s^\alpha}{\alpha\Gamma(\alpha)} \mu z & 1 + \frac{s^\alpha}{\alpha\Gamma(\alpha)} (x - b + \mu y - 2\eta z) \end{bmatrix} \quad (4)$$

4.1. Extinction Equilibrium State: In system of equation (3), if $x=0$ and $z=0$, we have the extinction equilibrium state $E_0=(0,0,0)$.

Theorem 4.1. Since $r > 0$, then $|\lambda_1| > 1$. The equilibrium state E_0 is source if $s > \max\left\{\sqrt[\alpha]{\frac{2\alpha\Gamma(\alpha)}{b}}, \sqrt[\alpha]{\frac{2\alpha\Gamma(\alpha)}{c}}\right\}$, E_0 is saddle if $0 < s < \min\left\{\sqrt[\alpha]{\frac{2\alpha\Gamma(\alpha)}{b}}, \sqrt[\alpha]{\frac{2\alpha\Gamma(\alpha)}{c}}\right\}$ and E_0 is non-hyperbolic if, either $s = \sqrt[\alpha]{\frac{2\alpha\Gamma(\alpha)}{b}}$ or $s = \sqrt[\alpha]{\frac{2\alpha\Gamma(\alpha)}{c}}$.

4.2. Exclusion Equilibrium State: In order to find the exclusion equilibrium state, we take $x \neq 0, y=0$ and $z=0$ in system of equations (3). Hence we obtain $E_1=(1,0,0)$, always exist.

Theorem 4.2. The equilibrium state E_1 is sink if $0 < s < \min\left\{\sqrt[\alpha]{\frac{2\alpha\Gamma(\alpha)}{r}}, \sqrt[\alpha]{\frac{2\alpha\Gamma(\alpha)}{c-\tau}}, \sqrt[\alpha]{\frac{2\alpha\Gamma(\alpha)}{b-1}}\right\}$, E_1 is a source if $s > \max\left\{\sqrt[\alpha]{\frac{2\alpha\Gamma(\alpha)}{r}}, \sqrt[\alpha]{\frac{2\alpha\Gamma(\alpha)}{c-\tau}}, \sqrt[\alpha]{\frac{2\alpha\Gamma(\alpha)}{b-1}}\right\}$, E_1 is saddle if $s > \sqrt[\alpha]{\frac{2\alpha\Gamma(\alpha)}{r}}$ and $0 < s < \min\left\{\sqrt[\alpha]{\frac{2\alpha\Gamma(\alpha)}{c-\tau}}, \sqrt[\alpha]{\frac{2\alpha\Gamma(\alpha)}{b-1}}\right\}$ and E_1 is non-hyperbolic if, either $s = \sqrt[\alpha]{\frac{2\alpha\Gamma(\alpha)}{r}}$ or $s = \sqrt[\alpha]{\frac{2\alpha\Gamma(\alpha)}{c-\tau}}$ or $s = \sqrt[\alpha]{\frac{2\alpha\Gamma(\alpha)}{b-1}}$.

4.3. The First Boundary Equilibrium State: In the equation (3), if $x \neq 0, y \neq 0$ and $z=0$ then we have the boundary equilibrium state in xy -plane is $E_2=\left(\frac{c}{\tau}, \frac{r(\tau-c)}{a\tau}, 0\right)$, exist when $c < \tau$.

Theorem 4.3. The first boundary equilibrium state E_2 is locally asymptotically stable if

$$r < \frac{a}{\mu(c-\tau)}\left(\frac{2\alpha\Gamma(\alpha)}{s^\alpha} + c - b\tau\right) \text{ and } \frac{2\left(\frac{s^\alpha}{\alpha\Gamma(\alpha)}\right)rc + \left(\frac{s^\alpha}{\alpha\Gamma(\alpha)}\right)^2 rc^2}{4 + \left(\frac{s^\alpha}{\alpha\Gamma(\alpha)}\right)^2 rc} < \tau < \frac{\alpha\Gamma(\alpha)}{s^\alpha} + c.$$

The state E_2 becomes saddle if

$$\text{if } r > \frac{a}{\mu(c-\tau)}\left(\frac{2\alpha\Gamma(\alpha)}{s^\alpha} + c - b\tau\right) \text{ and } \frac{2\left(\frac{s^\alpha}{\alpha\Gamma(\alpha)}\right)rc + \left(\frac{s^\alpha}{\alpha\Gamma(\alpha)}\right)^2 rc^2}{4 + \left(\frac{s^\alpha}{\alpha\Gamma(\alpha)}\right)^2 rc} < \tau < \frac{\alpha\Gamma(\alpha)}{s^\alpha} + c.$$

4.4. The Second Boundary Equilibrium State: In the equation (3), if $x \neq 0, y=0$ and $z \neq 0$ then we have the second boundary equilibrium state in xz -plane is $E_3=(\hat{x}, 0, \hat{z})=\left(\frac{b+r\eta}{1+r\eta}, 0, \frac{r(1-b)}{1+r\eta}\right)$, exist when $b < 1$.

Theorem 4.4. The second boundary equilibrium state E_3 is locally asymptotically stable if $c < \tau\hat{x} + \frac{2\alpha\Gamma(\alpha)}{s^\alpha}$ and

$$\sqrt[\alpha]{\frac{\alpha\Gamma(\alpha)\left[(r\hat{x}+\eta\hat{z}) \pm \sqrt{(r\hat{x}-\eta\hat{z})^2 - 4\hat{x}\hat{z}}\right]}{\hat{x}\hat{z}(1+r\eta)}} < s < \sqrt[\alpha]{\frac{\alpha\Gamma(\alpha)(r\hat{x}+\eta\hat{z})}{\hat{x}\hat{z}(1+r\eta)}}.$$

The state E_3 becomes saddle if

$$c > \tau\hat{x} + \frac{2\alpha\Gamma(\alpha)}{s^\alpha} \text{ and } \sqrt[\alpha]{\frac{\alpha\Gamma(\alpha)\left[(r\hat{x}+\eta\hat{z}) \pm \sqrt{(r\hat{x}-\eta\hat{z})^2 - 4\hat{x}\hat{z}}\right]}{\hat{x}\hat{z}(1+r\eta)}} < s < \sqrt[\alpha]{\frac{\alpha\Gamma(\alpha)(r\hat{x}+\eta\hat{z})}{\hat{x}\hat{z}(1+r\eta)}}.$$

Example 4.1: Considering the following suitable parameter values $\alpha=0.11, s=0.96, r=0.191, a=0.599, b=0.197, c=0.26, \mu=0.291, \tau=0.93, \eta=0.137$ with the initial values $x_0=0.4, y_0=0.3$ and $z_0=0.2$. The second boundary state $E_3=(0.2175, 0, 0.1495)$ and the eigenvalues are $\lambda_{1,2}=0.9674 \pm i0.1891$ and $\lambda_3=0.9393$ so that $|\lambda_{1,2}|=0.9857 < 1$ and $|\lambda_3| < 1$. Hence the system (3) is stable, see Fig. 1.

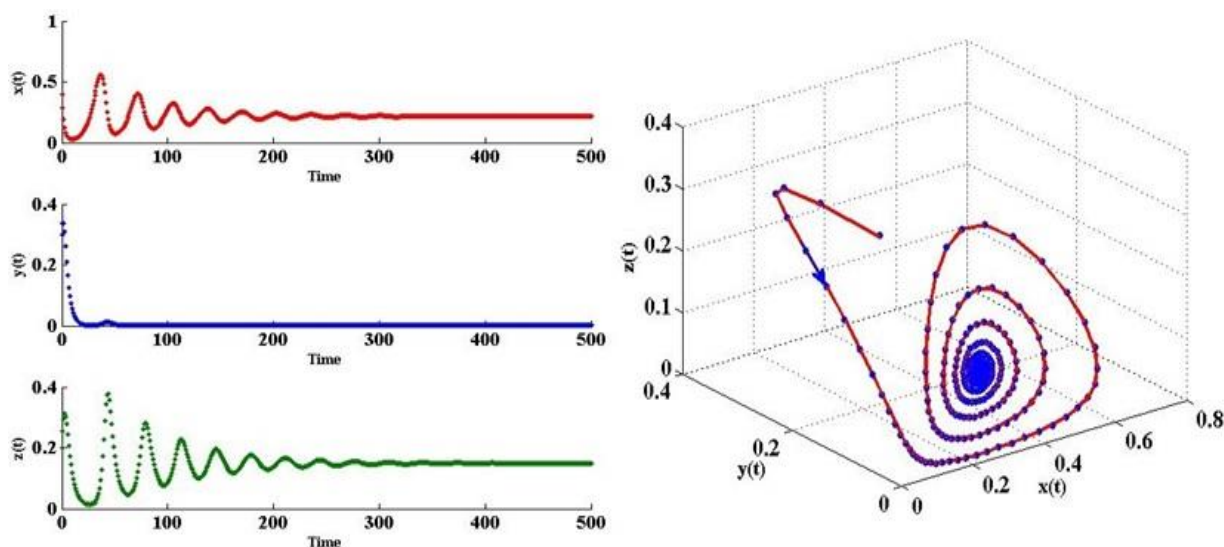


Figure 1: Time series and Phase portrait are stable at E_3 .

4.5. Coexistence Equilibrium State: In order to find the coexistence equilibrium state, we take $x \neq 0, y \neq 0$ and $z \neq 0$ in system of equations (3). Hence we obtain $E_4 = (x^*, y^*, z^*)$ with $x^* = \frac{c}{\tau}$, $y^* = \frac{\tau(b+r\eta) - c(1+r\eta)}{\tau(\mu+a\eta)}$, $z^* = \frac{a(c-b\tau) - r\mu(c-\tau)}{\tau(\mu+a\eta)}$ exist when $c(1+r\eta) < \tau(b+r\eta)$ and $r\mu(c-\tau) < a(c-b\tau)$.

Theorem 4.5. The coexistence equilibrium state E_4 of system (3) is locally asymptotically stable if the following conditions are satisfied: (i) $|\delta_1 + \delta_3| < 1 + \delta_2$, (ii) $|\delta_1 - 3\delta_3| < 3 - \delta_2$ and (iii) $\delta_3^2 + \delta_2 - \delta_1\delta_3 < 1$.

Proof: The Jacobian matrix for system (3) at $E_4 = (x^*, y^*, z^*)$ has the form

$$J(E_4) = \begin{bmatrix} 1 - \frac{s^\alpha}{\alpha\Gamma(\alpha)}(\xi_1) & -\frac{s^\alpha}{\alpha\Gamma(\alpha)}ax^* & -\frac{s^\alpha}{\alpha\Gamma(\alpha)}x^* \\ \frac{s^\alpha}{\alpha\Gamma(\alpha)}\tau y^* & 1 - \frac{s^\alpha}{\alpha\Gamma(\alpha)}(\xi_2) & 0 \\ \frac{s^\alpha}{\alpha\Gamma(\alpha)}z^* & \frac{s^\alpha}{\alpha\Gamma(\alpha)}\mu z^* & 1 - \frac{s^\alpha}{\alpha\Gamma(\alpha)}(\xi_3) \end{bmatrix}$$

The characteristic equation of $J(E_4)$ is $\lambda^3 + \delta_1\lambda^2 + \delta_2\lambda + \delta_3 = 0$ (5)

where $\delta_1 = \frac{s^\alpha}{\alpha\Gamma(\alpha)}(\xi_1 + \xi_2 + \xi_3) - 3$, $\delta_2 = 3 - 2\frac{s^\alpha}{\alpha\Gamma(\alpha)}(\xi_1 + \xi_2 + \xi_3) + \left(\frac{s^\alpha}{\alpha\Gamma(\alpha)}\right)^2(\xi_1\xi_2 + \xi_2\xi_3 + \xi_3\xi_1 + a\tau x^*y^* + x^*z^*)$ and $\delta_3 = \frac{s^\alpha}{\alpha\Gamma(\alpha)}(\xi_1 + \xi_2 + \xi_3) - \left(\frac{s^\alpha}{\alpha\Gamma(\alpha)}\right)^2(\xi_1\xi_2 + \xi_2\xi_3 + \xi_3\xi_1 + a\tau x^*y^* + x^*z^*) + \left(\frac{s^\alpha}{\alpha\Gamma(\alpha)}\right)^3(\xi_1\xi_2\xi_3 + \xi_2x^*z^* + \xi_3a\tau x^*y^* + \tau\mu x^*y^*z^*) - 1$ such that $\xi_1 = r(2x^* - 1) + ay^* + z^*$, $\xi_2 = c - \tau x^*$ and $\xi_3 = b - x^* - \mu y^* + 2\eta z^*$. Now, applying Jury's condition [9], the positive equilibrium state E_4 of system (3) is locally asymptotically stable if the following conditions are satisfied: $|\delta_1 + \delta_3| < 1 + \delta_2$, $|\delta_1 - 3\delta_3| < 3 - \delta_2$ and $\delta_3^2 + \delta_2 - \delta_1\delta_3 < 1$.

Example 4.2: If $r < a$ for the following set of parameter values $\alpha = 0.91, s = 0.96, r = 0.599, a = 0.61, b = 0.397, c = 0.16, \mu = 0.651, \tau = 0.673, \eta = 0.17$ with the initial values are $x_0 = 0.4, y_0 = 0.3$ and $z_0 = 0.2$. We obtain the coexistence equilibrium state $E_4 = (0.2377, 0.3139, 0.2651)$ and the eigenvalues are $\lambda_{1,2} = 0.9612 \pm i0.2995$ and $\lambda_3 = 0.8904$ so that $|\lambda_{1,2}| = 1.0068 > 1$ and $|\lambda_3| < 1$. Hence the system (3) is unstable, see Fig. 2.

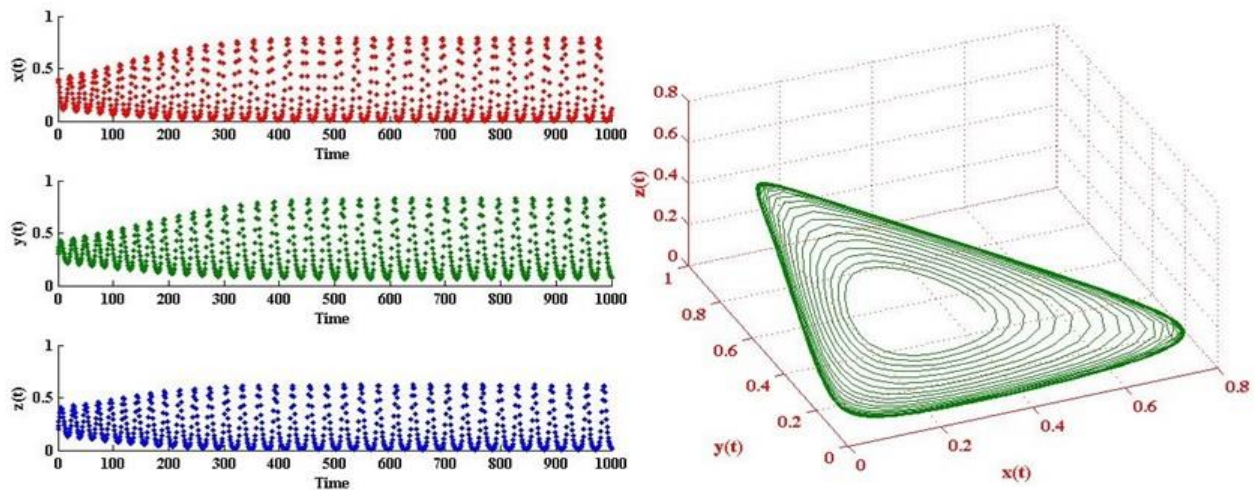


Figure 2: Time series and Phase portrait are unstable at E_4 .

Whereas with $r = 0.991$ and keeping the other parameter values are same. If $r > a$, then the positive equilibrium state $E_4 = (0.2377, 0.3812, 0.5229)$ and the eigenvalues are $\lambda_{1,2} = 0.9149 \pm i0.3852$ and $\lambda_3 = 0.8462$ so that $|\lambda_{1,2}| = 0.9927 < 1$ and $|\lambda_3| < 1$. We observe the system (3) is stable, see Fig. 3.

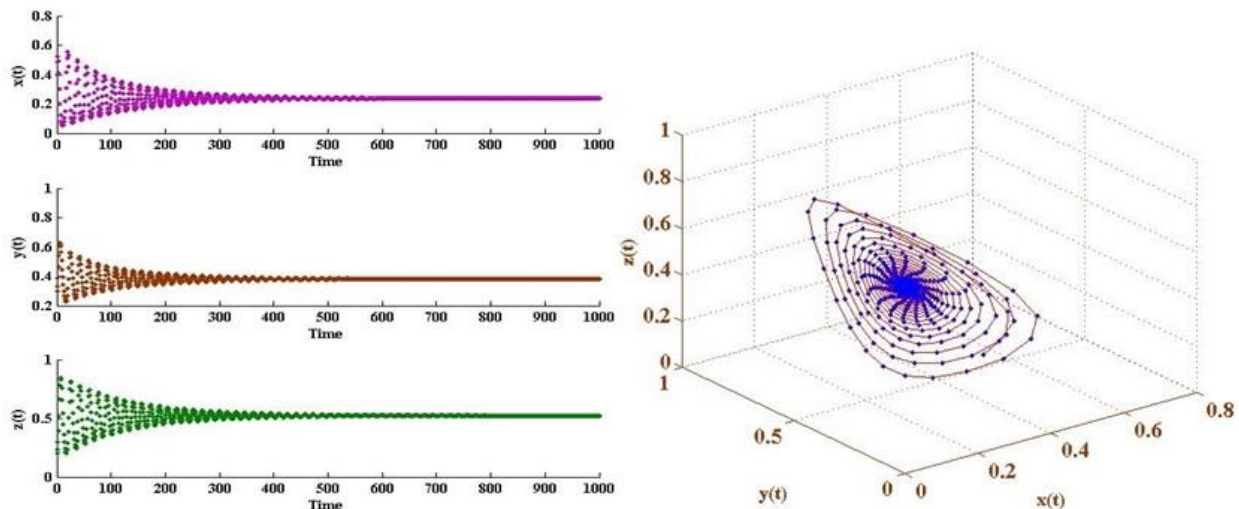


Figure 3: Time series and Phase portrait are stable at E_4 .

V. Bifurcation Analysis

In this section, we investigate the bifurcation parametric conditions for the existence of period doubling (flip) bifurcation at the axial equilibrium state and a discrete Hopf bifurcation at the coexistence equilibrium state, maximum Lyapunov exponents to support the analytical analysis and the complex dynamics of fractional order discrete prey - predator – omnivore system (3) with the help of numerical simulations.

5.1. Period Doubling Bifurcation: We can see easily that for axial equilibrium state E_1 if r varies in the small neighborhood of FB_{E_1} , then the flip bifurcation will appear in the system (3) where

$$FB_{E_1} = \left\{ (\alpha, s, r, a, b, c, \mu, \tau, \eta) : r = r_0 = \frac{2\alpha\Gamma(\alpha)}{s^\alpha}, c \neq \frac{2\alpha\Gamma(\alpha)}{s^\alpha} + \tau, b \neq \frac{2\alpha\Gamma(\alpha)}{s^\alpha} + 1, r > 1, \alpha, s, a, b, c, \mu, \tau, \eta > 0 \right\}.$$

Example 5.1. First, we take $\alpha = 0.181, s = 0.577, a = 0.789, b = 0.997, c = 0.816, \mu = 0.41, \tau = 0.827, \eta = 0.39$ and $r \in [1.7, 3.1]$ with initial values $x_0 = 0.4, y_0 = 0.3$ and $z_0 = 0.1$, then the system (3) undergoes period doubling (flip) bifurcation as it emerges from the axial equilibrium state $E_1 = (1, 0, 0)$ as the intrinsic growth bifurcation parameter r varies in the small neighborhood of $r_0 = 2.0402$ and the system dynamics converges to a periods- 2 orbit. The corresponding bifurcation diagram is shown in Fig. 4(a). The characteristic polynomial evaluated at this state is given by $\lambda^3 - 1.0137\lambda^2 - \lambda + 1.0138 = 0$ (6)

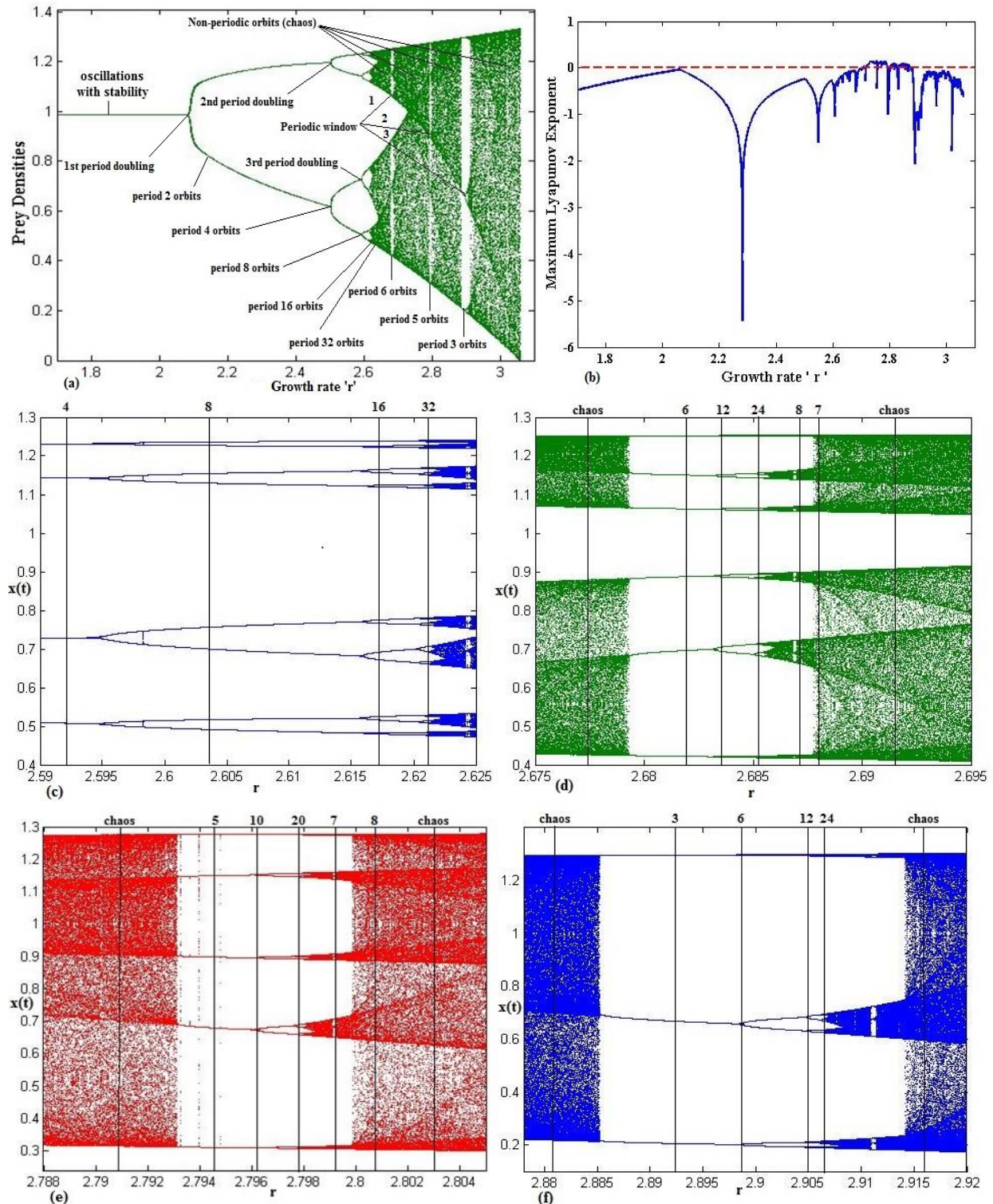


Figure 4: (a) Flip bifurcation diagram for system (3) around E_1 in $(r-x)$ plane, (b) Maximum Lyapunov exponents corresponding to 4(a), (c) – (f) Local amplifications of Figure 4 (a) showing periodic windows.

Furthermore, the roots of (6) are $\lambda_1 = -1$, $\lambda_2 = 1.0108$ and $\lambda_3 = 1.0029$ so that $\lambda_1 = -1$ and $\lambda_{2,3} \neq \pm 1$. Thus we have $(r = 2.0402, b \neq 3.0402, c \neq 2.8672) \in FB_{E_1}$. From Fig. 4, we observe that the axial equilibrium state E_1 of map (3) is stable for $r < 2.0402$ and loses its stability through a period doubling bifurcation for $r = 2.0402$ and for $r > 2.0402$, we observe period doubling cascade in orbits of periods – 2, 4, 8, 16, 32 and non-periodic oscillations appear, that is usually referred to as chaos, see Fig. 4(c). The maximum Lyapunov exponent (LEs) related to Fig. 4(a) is computed and plotted in Fig. 4(b) which confirms the existence of the chaotic area and a periodic orbits in the parametric space. It is observed that some LE values are positive and some are negative so

there exists stable equilibrium states or stable period windows in the chaotic region. In general the positive Lyapunov exponent is considered to be one of the characteristics implying the existence of chaos. Fig. 4(d) - 4(f) are local amplifications of the bifurcation diagram which presents three periodic windows that occurs in the chaotic region. In these periodic windows, periods - 6, 5 and 3 orbits appear. Furthermore, in Fig. 4(d) periodic windows occur in orbits of periods - 6, 12 and 24, in the second periodic window cascade in orbits of periods - 5, 10, and 20 appear, see Fig. 4(e) and in Fig. 4(f) a third periodic window in orbits of periods - 3, 6, 12 and 24 appear.

5.2. Hopf Bifurcation: In order to discuss a discrete Hopf bifurcation for the system (3) at the coexistence equilibrium state E_4 , we choose r as bifurcation parameter. From (5) it is easy to see that $F(\lambda) = 0$ must have a complex conjugate root with modulus one. Clearly equation (5) will have two pure imaginary roots and one real root. Let $\delta_1\delta_2 = \delta_3$, for some values of r , say $r = r^*$, the equation becomes $(\lambda^2 + \delta_2)(\lambda + \delta_1) = 0$ which has three roots $\lambda_{1,2} = \pm i\sqrt{\delta_2}$ and $\lambda_3 = -\delta_1$. Hence the system (3) undergoes a discrete Hopf bifurcation at the coexistence state E_4 if r varies in the small neighborhood of HB_{E_4} , where

$$HB_{E_4} = \{(\alpha, s, r, a, b, c, \mu, \tau, \eta) : |\delta_2| = 1, \delta_1 \neq \pm 1, r > 1, \alpha, s, a, b, c, \mu, \tau, \eta > 0\}$$

Example 5.2. Let $\alpha = 0.84, s = 0.96, a = 0.63, b = 0.47, c = 0.16, \mu = 0.71, \tau = 0.62, \eta = 0.29$ and $r \in [0, 1]$ with the initial values are $x_0 = 0.2, y_0 = 0.4$ and $z_0 = 0.5$, then the system (3) undergoes a discrete Hopf bifurcation emerges from the coexistence equilibrium state $E_4 = (0.2581, 0.3918, 0.2284)$ at bifurcation parameter $r = 0.6405$. The corresponding bifurcation diagram is shown in Fig. 5. The characteristic polynomial evaluated at E_4 is given by $\lambda^3 - (2.7627)\lambda^2 + (2.6403)\lambda - 0.8638 = 0$ (7)

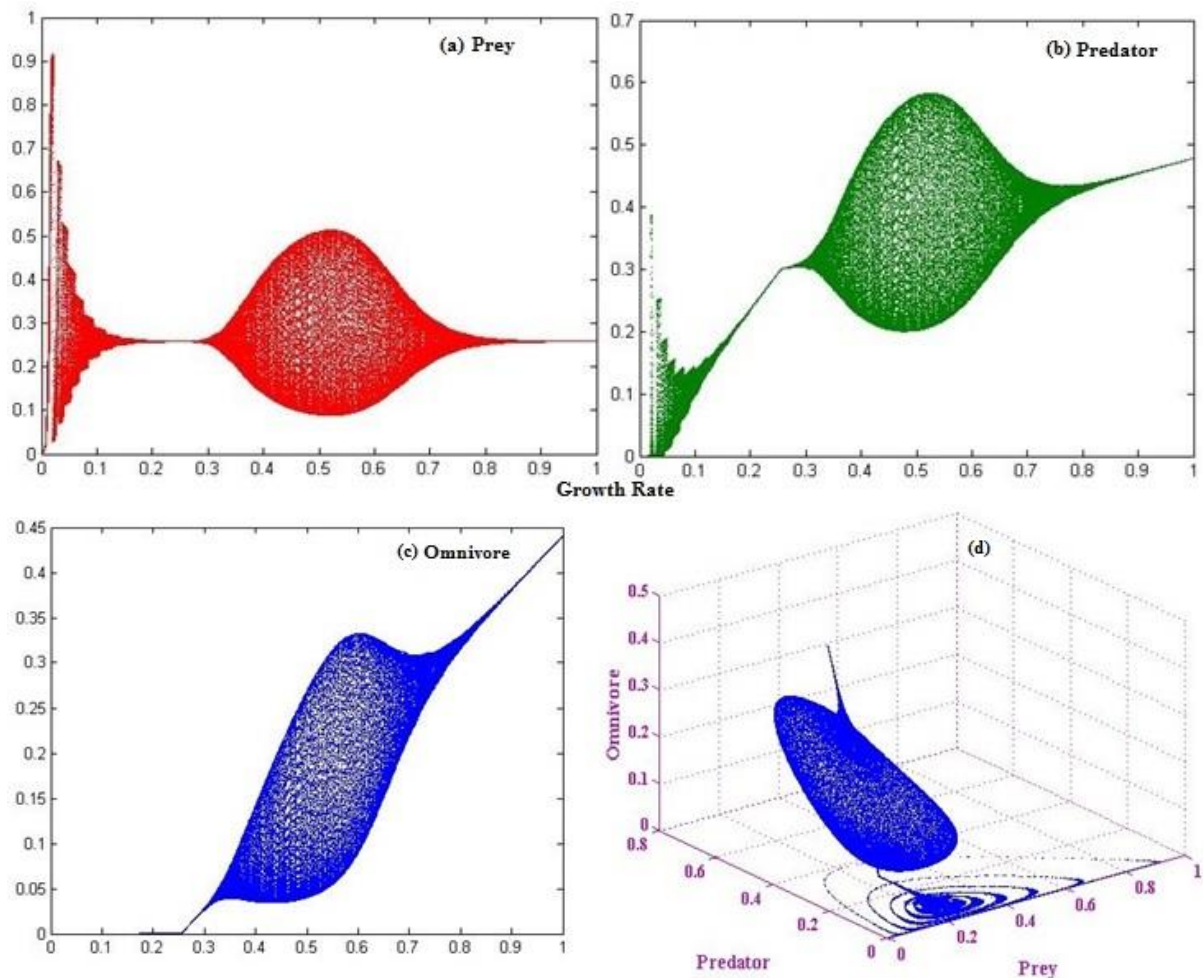


Figure 5: (a - c) Hopf bifurcation diagrams of system (3) in $(r-x)$, $(r-y)$ and $(r-z)$ planes, (d) Hopf bifurcation diagram in $(x-y-z)$ space. Furthermore, the roots of (7) are $\lambda_{1,2} = 0.9494 \pm i0.3140$ and $\lambda_3 = 0.8639$ with $\lambda_{1,2} = \delta_2 = 1$ and $\lambda_3 = \delta_1 \neq \pm 1$. From Fig. 5, we observe that coexistence equilibrium state E_4 of map (3) is stable for $r < 0.6405$ and loses its stability through a discrete Hopf bifurcation for $r = 0.6405$ and an invariant circle appears for $r > 0.6405$.

VI. Sensitive Dependence on Initial Values

The sensitivity to initial conditions is a characteristic of chaos. Let us consider the following suitable parameter values $\alpha = 0.181, s = 0.577, a = 0.789, b = 0.997, c = 0.816, \mu = 0.41, \tau = 0.827, \eta = 0.39$ and $r = 2.712$. In order to demonstrate the sensitivity to initial values of system (3), we compute 2 orbits for prey populations with the initial conditions (x_0, y_0, z_0) and $(x_0 + 0.0001, y_0, z_0)$ respectively. From Fig. 6(a-b), it is clearly observe that, at the beginning, the time plots are indistinguishable but after a number of iterations, the difference between them builds up rapidly. The x -coordinates of initial conditions differ by 0.0001 and the other coordinates remain the same, see Fig. 6(b).

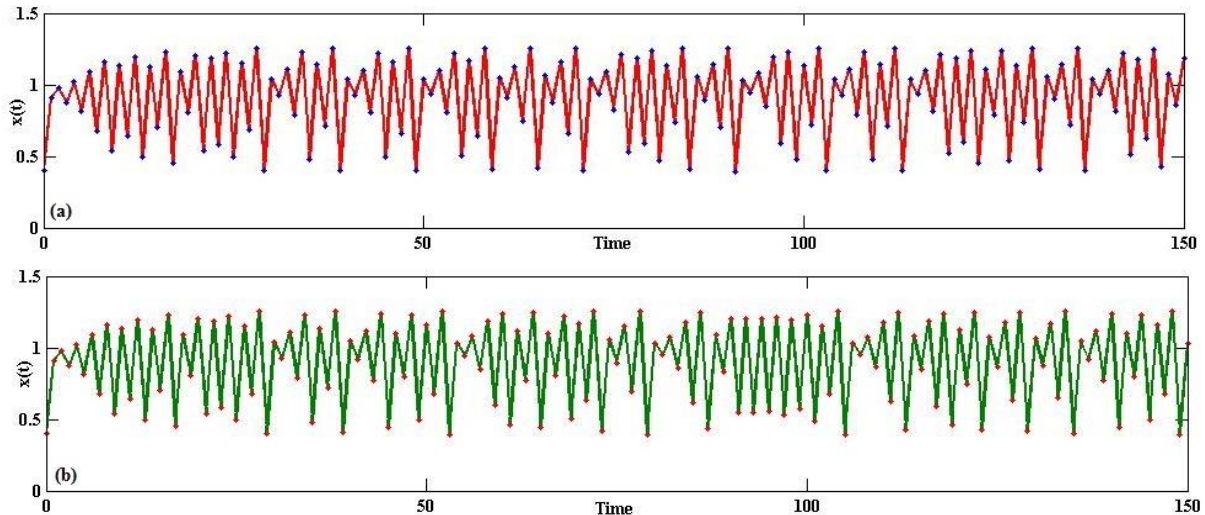


Figure 6: Time series x_t corresponding to the initial conditions (a) $(0.4, 0.3, 0.1)$ and (b) $(0.4001, 0.3, 0.1)$ of system (3).

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