



CHARACTERIZATION OF DOM STRONG DOMINATION NUMBER AND CONNECTIVITY OF A GRAPH

¹M.HajMeeral and ²S.Gayathri Devi

¹Department of Mathematics, The Quaide Milleth College for Men, Medavakkam,
Chennai - 600 100, Tamil Nadu, India.

²Department of Mathematics, Mohamed Sathak College of Arts and Science, Shollinganallur,
Chennai - 600 119, TamilNadu, India.

Abstract: A subset S of V is called a dom strong dominating set if for every vertex $v \in V-S$, there exists $u_1, u_2 \in S$ such that $u_1v, u_2v \in E(G)$ and $d(u_1) \geq d(v)$. The minimum cardinality of a dom strong dominating set is called the dom strong domination number and is denoted by $\gamma_{ds}(G)$. The connectivity $\kappa(G)$ of a graph G is the minimum number of vertices whose removal results in a disconnected or trivial graph. In this paper the sum of the dom strong domination number and connectivity of graph less than or equal to $2n-4$ is obtained and characterized the corresponding extremal graphs.

Keywords: Dom strong dominating set, Dom strong domination number and Connectivity.

I. Introduction

Let $G=(V,E)$ be a simple undirected graph. A graph G consists of a pair $(V(G), E(G))$ where $V(G)$ is non-empty finite set whose elements are called vertices and $E(G)$ is a set of unordered pairs of distinct elements of $V(G)$. The connectivity $\kappa(G)$ of a graph G is the minimum number of vertices whose removal results in a disconnected or trivial graph. Any set M of independent edges of a graph G is called a matching of G . If $uv \in M$, u and v are matched under M . The vertices u and v are M -saturated. A Matching M is called perfect matching if every vertex of G is M -saturated. M is called a maximum matching if there is no matching M' in G such that $|M'| > |M|$. The degree of a vertex u in G is the number of edges incident with u and is denoted by $\deg(u)$ or $d(u)$. The minimum and maximum degree of a graph is denoted by $\delta(G)$ and $\Delta(G)$. For terminology, we refer to Harary [4].

A set $S \subseteq V$ is a dominating set of G if every vertex of G is dominated by some vertex in S . The minimum cardinality of a dominating set is called the domination number of G and is denoted by $\gamma(G)$. Double domination introduced by Haynes [3] serves as a model for the type of fault tolerance where each computer has access to at least two fileservers and each of the fileservers has direct access to at least one backup fileserver. Sampathkumar and Pushpalatha [7] have introduced the concept of strong weak domination in graphs. A combination of the concepts of double domination and strong weak domination is the concept of domination strong domination where in for every vertex outside the dominating set, there are two vertices inside the dominating set, one of which dominates the outside vertex and the other strongly dominates the outside vertex. The concept of dom strong domination number of a graph is defined in [8]. The dom strong dominating set is defined as a subset S of V is called a dom strong dominating set, then for every vertex $v \in V-S$, there exists $u_1, u_2 \in S$ such that $u_1v, u_2v \in E(G)$ and $d(u_1) \geq d(v)$. The minimum cardinality of a dom strong dominating set is called the dom strong domination number and is denoted by $\gamma_{ds}(G)$. In this paper the sum of the dom strong domination number and connectivity of graph less than or equal to $2n - 4$ is obtained and characterized the corresponding extremal graphs.

II. Results

Theorem 2.1: [6] For any connected graph G , $\kappa(G) \leq \delta(G)$.

Theorem 2.2: [5] For any connected graph G , $2 \leq \gamma_{ds}(G) \leq n$.

Theorem 2.3: [5] For any connected graph G , $\gamma_{ds}(G) = n$ if and only if G is a star.

Notation 2.4: $C_n(P_k)$ is the graph by attaching the end vertices of P_k path graph to any one vertex of the cycle graph. $K_n(P_k)$ is the graph by attaching the end vertices of P_k path graph to any one vertex of the complete graph.

Theorem 2.5: For any connected graph G , $\gamma_{ds}(G) + \kappa(G) \leq 2n-1$ and the equality holds if and only if G is a complete graph of order 2.

Proof: If G is connected, $\gamma_{ds}(G) + \kappa(G) \leq n + \delta \leq n + n - 1 = 2n - 1$. Let $\gamma_{ds}(G) + \kappa(G) = 2n - 1$. Then $\gamma_{ds}(G) = n$ implies G is a star and since $\kappa(G) = n-1$, G is complete. Hence $G \cong K_2$. Conversely, if G is a complete graph of order 2 then $\gamma_{ds}(G) + \kappa(G) = 2n-1$.

Theorem 2.6: For any connected graph G , $\gamma_{ds}(G) + \kappa(G) = 2n-2$ if and only if $G \cong K_3$ or $K_{1,2}$.

Proof: If $G \cong K_3$ or $K_{1,2}$, it is obvious that $\gamma_{ds}(G) + \kappa(G) = 2n - 2$. Suppose $\gamma_{ds}(G) + \kappa(G) = 2n - 2$. Then the possible cases are (i) $\gamma_{ds}(G) = n$ and $\kappa(G) = n - 2$ (ii) $\gamma_{ds}(G) = n - 1$ and $\kappa(G) = n - 1$. For case (i), since $\gamma_{ds}(G) = n$ which implies G is a star, for which $\kappa(G) = 1$. Therefore $G \cong K_{1,2}$. For case (ii), since $\kappa(G) = n - 1$, G is a complete graph. Then $\gamma_{ds}(G) = 2$, which results to $G \cong K_3$.

Theorem 2.7: For any connected graph G , $\gamma_{ds}(G) + \kappa(G) = 2n - 3$ if and only if $G \cong K_{1,3}$ or K_4 .

Proof: If $G \cong K_{1,3}$ or K_4 then $\gamma_{ds}(G) + \kappa(G) = 2n - 3$. Conversely, Suppose $\gamma_{ds}(G) + \kappa(G) = 2n - 3$, then the possible cases are (i) $\gamma_{ds}(G) = n$ and $\kappa(G) = n - 3$ (ii) $\gamma_{ds}(G) = n - 1$ and $\kappa(G) = n - 2$ (iii) $\gamma_{ds}(G) = n - 2$ and $\kappa(G) = n - 1$.

Case (i): $\gamma_{ds}(G) = n$ and $\kappa(G) = n - 3$.

Since $\gamma_{ds}(G) = n$ which implies G is a star for which $\kappa(G) = 1$, Therefore $G \cong K_{1,3}$.

Case (ii): $\gamma_{ds}(G) = n-1$ and $\kappa(G) = n - 2$.

Since G is connected and $\kappa(G) = n - 2$, $\delta(G) \geq n - 2$. If $\delta(G) = n - 1$, then G is complete, is a contradiction.

If $\delta(G) = n-2$, then $G \cong K_n - Q$ where Q is a matching in K_n . Hence $\gamma_{ds}(G) \leq 3$. Suppose $\gamma_{ds}(G) = 2$ or 3 then $G \cong K_3 - Q$ or $K_4 - Q$ respectively, which is a contradiction.

Case (iii): $\gamma_{ds}(G) = n - 2$ and $\kappa(G) = n - 1$.

Since $\kappa(G) = n-1$, G is a complete on K_n . But for complete graphs, $\gamma_{ds}(G) = 2$. Then $G \cong K_4$.

Theorem 2.8: For any connected graph G , $\gamma_{ds}(G) + \kappa(G) = 2n - 4$ if and only if $G \cong K_{1,4}$ or K_5 or P_4 or C_4 or $K_3(P_2)$ or $K_4 - e$ or $K_5 - e$ or $K_5 - Q$ where Q is a maximum matching in K_5 .

Proof: If $G \cong K_{1,4}$ or K_5 or P_4 or C_4 or $K_3(P_2)$ or $K_4 - e$ or $K_5 - e$ or $K_5 - Q$ where Q is a maximum matching in K_5 , then $\gamma_{ds}(G) + \kappa(G) = 2n - 4$. Conversely, suppose $\gamma_{ds}(G) + \kappa(G) = 2n - 4$, then the possible cases are (i) $\gamma_{ds}(G) = n$ and $\kappa(G) = n - 4$ (ii) $\gamma_{ds}(G) = n - 1$ and $\kappa(G) = n - 3$ (iii) $\gamma_{ds}(G) = n - 2$ and $\kappa(G) = n - 2$ (iv) $\gamma_{ds}(G) = n - 3$ and $\kappa(G) = n - 1$.

Case (i): $\gamma_{ds}(G) = n$ and $\kappa(G) = n - 4$.

Since $\gamma_{ds}(G) = n$ which implies G is a star for which $\kappa(G) = 1$. Therefore $G \cong K_{1,4}$.

Case (ii): $\gamma_{ds}(G) = n - 1$ and $\kappa(G) = n - 3$.

Since $\kappa(G) = n - 3$, $\delta(G) \geq n - 3$. If $\delta(G) = n - 1$, then G is complete. Hence $\gamma_{ds}(G) = 2$ implies $G \cong K_3$ is a contradiction. If $\delta(G) = n - 2$, then $G \cong K_n - Q$ where Q is a matching in K_n . Therefore $\gamma_{ds} \leq 3$. Suppose $\gamma_{ds}(G) = 2$ or 3 then $G \cong K_3 - Q$ or $K_4 - Q$ respectively, which is a contradiction. If $\delta(G) = n - 3$. Let $X = \{v_1, v_2, v_3, \dots, v_{n-3}\}$ be the minimum cut vertex of G and $V - X = \{u, v, w\}$.

Sub case 1: $\langle V - X \rangle = \overline{K_3}$

Let us consider $\langle X \rangle$ is connected. If $|X| = 1$ then G is isomorphic to $K_{1,3}$ which is a contradiction. If $|X| = 2$ then G is isomorphic to graph obtained from $K_{2,3}$ by joining the vertices of degree 3 by an edge which is a contradiction. If $|X| \geq 3$ and let $\deg(v_2) \geq \deg(v_i)$, $i=1$ or 3 then $\gamma_{ds} = 4$ implies $n = 4$ is a contradiction.

Sub case 2: $\langle V - X \rangle = K_2 \cup K_1$

Let $uv \in K_2$ and $w \in K_1$. Suppose $\langle X \rangle$ is connected. Since $\delta(G) = n - 3$, w is adjacent to all vertices of X . If $|X| = 1$ and if $\deg(u) = \deg(v) = n - 2$ then $G \cong K_3(P_2)$. If $\deg(u) = n - 2$ and $\deg(v) < n - 2$ then $G \cong P_4$. Suppose $\deg(u) = \deg(v) = n - 3$. If v_1 is a vertex in X such that u and v is not adjacent to v_1 . Then graph does not exist. Hence v_1 is adjacent to at least one vertex of $uv \in K_2$ and adjacent to $w \in K_1$. If $|X| = 2$ then $\gamma_{ds}(G) + \kappa(G) \neq 2n - 4$. If $|X| \geq 3$ then as above sub case, is a contradiction. Suppose $\langle X \rangle$ is disconnected. If $\langle X \rangle = \overline{K_2}$, Let $v_1, v_2 \in X$ is adjacent to u, v and w in $\langle V - X \rangle$ then graph does not exist. Let $|X| \geq 3$ and $\deg(v_2) = \deg(v_i)$, $i=1, 3$. We obtain graphs with $\gamma_{ds} = 3$, which is a contradiction.

Case (iii): $\gamma_{ds}(G) = n - 2$ and $\kappa(G) = n - 2$.

If $\kappa(G) = n - 2$ then $\delta(G) \geq n - 2$. If $\delta(G) = n - 1$ then $G \cong K_n$ is a contradiction. If $\delta(G) = n - 2$ then $G \cong K_n - Q$ where Q is a matching in K_n . Therefore $\gamma_{ds} \leq 3$. Suppose $\gamma_{ds} = 2$ then $n=4$. Hence $G \cong K_4 - Q$

or C_4 where Q is a matching in K_4 with $|Q| = 2$. If $|Q| = 1$ then $G \cong K_4 - e$. Suppose $\gamma_{ds} = 3$ then $n=5$. Hence $G \cong K_5 - Q$ where Q is a matching in K_5 with $|Q| = 2$. If $|Q| = 1$ then $G \cong K_5 - e$.

Case (iv): $\gamma_{ds}(G) = n - 3$ and $\kappa(G) = n - 1$.

Since $\kappa(G) = n - 1$, G is complete on K_n . For complete graphs, $\gamma_{ds}(G) = 2$. Hence $G \cong K_5$.

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