



Fourier series Expansions Involving G-Function of Two Variables

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Abstract: The subject of Fourier series for generalized hypergeometric functions occupies a prominent place in the literature of special functions and boundary value problems. Certain double Fourier series of generalized hypergeometric functions play an important role in the development of the theories of special functions and two-dimensional boundary value problems. The object of the present paper is to establish two Fourier series expansion formulae involving G-Function of two variables.

Key Words: G-Function of two variables, Fourier series expansion, Generalized Hypergeometric Functions, Double Fourier Series, G-function of one variable.

I. INTRODUCTION:

Recently, Beg [2], Ayant Frédéric [1], Dubey [3] and others have evaluated certain number of Fourier series expansion involving generalized hypergeometric functions. Following them in this paper we shall try to obtain some new Fourier Series expansion involving G-function of two variables, which play an important role in the development of the theories of special functions and two-dimensional boundary value problems.

The G-function of two variables was defined by Shrivastava and Joshi [7, p. 471] in terms of Mellin-Barnes type integrals as follows:

$$G_{p_1, q_1; p_2, q_2; p_3, q_3}^{0, n_1; m_2, n_2; m_3, n_3} [x, y] = \frac{(a_j; 1)_{1, p_1} (c_j, 1)_{1, p_2} (e_j, 1)_{1, p_3}}{(b_j; 1, 1)_{1, q_1} (d_j, 1)_{1, q_2} (f_j, 1)_{1, q_3}} \int_{L_1} \int_{L_2} \phi_1(\xi, \eta) \theta_2(\xi) \theta_3(\eta) x^\xi y^\eta d\xi d\eta \quad (1)$$

where

$$\phi_1(\xi, \eta) = \frac{\prod_{j=1}^{n_1} \Gamma(1 - a_j + \xi + \eta)}{\prod_{j=n_1+1}^{p_1} \Gamma(a_j - \xi - \eta) \prod_{j=1}^{q_1} \Gamma(1 - b_j + \xi + \eta)},$$

$$\theta_2(\xi) = \frac{\prod_{j=1}^{m_2} \Gamma(d_j - \xi) \prod_{j=1}^{n_2} \Gamma(1 - c_j + \xi)}{\prod_{j=m_2+1}^{q_2} \Gamma(1 - d_j + \xi) \prod_{j=n_2+1}^{p_2} \Gamma(c_j - \xi)}$$

$$\theta_3(\eta) = \frac{\prod_{j=1}^{m_3} \Gamma(f_j - \eta) \prod_{j=1}^{n_3} \Gamma(1 - e_j + \eta)}{\prod_{j=m_3+1}^{q_3} \Gamma(1 - f_j + \eta) \prod_{j=n_3+1}^{p_3} \Gamma(e_j - \eta)}$$

x and y are not equal to zero, and an empty product is interpreted as unity p_i, q_i, n_i and m_j are non negative integers such that $p_i \geq n_i \geq 0, q_i \geq m_j \geq 0, (i = 1, 2, 3; j = 2, 3)$.

The contour L_1 is in the ξ -plane and runs from $-\infty$ to $+\infty$, with loops, if necessary, to ensure that the poles of $\Gamma(d_j - \xi)$ ($j = 1, \dots, m_2$) lie to the right, and the poles of $\Gamma(1 - c_j + \xi)$ ($j = 1, \dots, n_2$), $\Gamma(1 - a_j + \xi + \eta)$ ($j = 1, \dots, n_1$) to the left of the contour.

The contour L_2 is in the η -plane and runs from $-\infty$ to $+\infty$, with loops, if necessary, to ensure that the poles of $\Gamma(f_j - \eta)$ ($j = 1, \dots, m_3$) lie to the right, and the poles of $\Gamma(1 - e_j + \eta)$ ($j = 1, \dots, n_3$), $\Gamma(1 - a_j + \xi + \eta)$ ($j = 1, \dots, n_1$) to the left of the contour.

and the double integral converges if

$$2(n_1 + m_2 + n_2) > (p_1 + q_1 + p_2 + q_2)$$

$$2(n_1 + m_3 + n_3) > (p_1 + q_1 + p_3 + q_3)$$

$$\text{and } |\arg x| < \frac{1}{2} U, |\arg y| < \frac{1}{2} V$$

$$\text{where } U = [n_1 + m_2 + n_2 - \frac{1}{2}(p_1 + q_1 + p_2 + q_2)]$$

$$V = [n_1 + m_3 + n_3 - \frac{1}{2}(p_1 + q_1 + p_3 + q_3)]$$

These assumptions for the G-function of two variables will be adhered to throughout this research work. The following formulae are required in the proof:

From MacRobert [5, 6]:

$$\frac{\sqrt{\pi}\Gamma(2-s)}{2\Gamma(\frac{3}{2}-s)}(\sin\theta)^{1-2s} = \sum_{r=0}^{\infty} \frac{(s)_r}{(2-s)_r} \sin(2r+1)\theta, \quad (2)$$

where $0 < \theta \leq \pi$, $\operatorname{Re} s \leq \frac{1}{2}$.

$$\frac{\sqrt{\pi}\Gamma(1-s)}{\Gamma(\frac{1}{2}-s)}\left(\sin\frac{\theta}{2}\right)^{-2s} = 1 + 2\sum_{r=0}^{\infty} \frac{(s)_r}{(1-s)_r} \cos r\theta, \quad (3)$$

where $0 < \theta \leq \pi$.

II. MAIN RESULT:

In this section, we shall establish following Fourier series:

$$\begin{aligned} & \sum_{r=0}^{\infty} G_{p_1, q_1; p_2, q_2; p_3+1, q_3+2}^{0, n_1; m_2, n_2; m_3+1, n_3+1} \left[\begin{matrix} (a_j; 1, 1)_{1, p_1}; (c_j, 1)_{1, p_2}; (1-r, 1), (e_j, 1)_{1, p_3}; (2+r, 1) \\ (b_j; 1, 1)_{1, q_1}; (d_j, 1)_{1, q_2}; (\frac{3}{2}, 1), (f_j, 1)_{1, q_3}; (1, 1) \end{matrix} \right] \sin(2r+1)\theta \\ &= \frac{\sqrt{\pi}}{2} \sin\theta G_{p_1, q_1; p_2, q_2; p_3, q_3}^{0, n_1; m_2, n_2; m_3, n_3} \left[\begin{matrix} (a_j; 1, 1)_{1, p_1}; (c_j, 1)_{1, p_2}; (e_j, 1)_{1, p_3} \\ (b_j; 1, 1)_{1, q_1}; (d_j, 1)_{1, q_2}; (f_j, 1)_{1, q_3} \end{matrix} \right] \end{aligned} \quad (4)$$

where $0 \leq \theta \leq \pi$, $|\arg z_2| < (m_3 + n_3 - \frac{p_3}{2} - \frac{q_3}{2})\pi$.

$$\begin{aligned} & G_{p_1, q_1; p_2, q_2; p_3+1, q_3+1}^{0, n_1; m_2, n_2; m_3+1, n_3} \left[\begin{matrix} (a_j; 1, 1)_{1, p_1}; (c_j, 1)_{1, p_2}; (e_j, 1)_{1, p_3}; (1, 1) \\ (b_j; 1, 1)_{1, q_1}; (d_j, 1)_{1, q_2}; (\frac{1}{2}, 1), (f_j, 1)_{1, q_3} \end{matrix} \right] \\ &+ 2\sum_{r=0}^{\infty} G_{p_1, q_1; p_2, q_2; p_3+2, q_3+2}^{0, n_1; m_2, n_2; m_3+1, n_3+1} \left[\begin{matrix} (a_j; 1, 1)_{1, p_1}; (c_j, 1)_{1, p_2}; (1-r, 1), (e_j, 1)_{1, p_3}; (1+r, 1) \\ (b_j; 1, 1)_{1, q_1}; (d_j, 1)_{1, q_2}; (\frac{3}{2}, 1), (f_j, 1)_{1, q_3}; (1, 1) \end{matrix} \right] \cos r\theta \\ &= \sqrt{\pi} G_{p_1, q_1; p_2, q_2; p_3, q_3}^{0, n_1; m_2, n_2; m_3, n_3} \left[\begin{matrix} (a_j; 1, 1)_{1, p_1}; (c_j, 1)_{1, p_2}; (e_j, 1)_{1, p_3} \\ (b_j; 1, 1)_{1, q_1}; (d_j, 1)_{1, q_2}; (f_j, 1)_{1, q_3} \end{matrix} \right] \end{aligned} \quad (5)$$

where $0 < \theta \leq \pi$, $|\arg z_2| < (m_3 + n_3 - \frac{p_3}{2} - \frac{q_3}{2})\pi$.

Proof of (4):

To prove (4), expressing the G-function on the left-hand side as Mellin-Barnes type integral (1), we have

$$\sum_{r=0}^{\infty} \frac{(-1)^r}{4\pi^2} \int_{L_1} \int_{L_2} \phi_1(\xi, \eta) \theta_2(\xi) \theta_3(\eta) \frac{\Gamma(\frac{3}{2}-\eta)\Gamma(r+\eta)}{\Gamma(\eta)\Gamma(2+r-\eta)} (z_1)^\xi (z_2)^\eta \sin(2r+1)\theta d\xi d\eta$$

On changing the order of integration and summation which is easily seen to be justified, the above expression becomes

$$\begin{aligned} & \frac{(-1)}{4\pi^2} \int_{L_1} \int_{L_2} \phi_1(\xi, \eta) \theta_2(\xi) \theta_3(\eta) \frac{\Gamma(\frac{3}{2}-\eta)}{\Gamma(2-\eta)} \times \\ & \times \left[\sum_{r=0}^{\infty} \frac{(\eta)_r}{(2-\eta)_r} \sin(2r+1)\theta \right] (z_1)^\xi (z_2)^\eta d\xi d\eta. \end{aligned}$$

and on using the relation (2), it takes the form

$$\frac{\sqrt{\pi}}{2} \sin\theta \cdot \frac{(-1)}{4\pi^2} \int_{L_1} \int_{L_2} \phi_1(\xi, \eta) \theta_2(\xi) \theta_3(\eta) (z_1)^\xi (z_2/\sin^2\theta)^\eta d\xi d\eta.$$

which is just the expression on the right side of (4). (4) is the Fourier sine series for the G-function of two variables.

The Fourier cosine series (5) is proved in an analogous by using (3).

III. SPECIAL CASES:

On specializing the parameters in (4) and (5), we get following Fourier series in terms of G-function of one variable, which is a result given by Kesarwani [4, p. 1 (1.1) and (1.2)]:

$$\sum_{r=0}^{\infty} G_{p+2, q+2}^{m+1, n+1} \left[\begin{matrix} (1-r, 1), (a_j, 1)_{1, p}; (2+r, 1) \\ (\frac{3}{2}, 1), (b_j, 1)_{1, q}; (1, 1) \end{matrix} \right] \sin(2r+1)\theta$$

$$= \frac{\sqrt{\pi}}{2} \sin \theta G_{p,q}^{m,n} \left[\frac{z}{\sin^2 \theta} \middle| \begin{matrix} (a_j, 1)_{1,p} \\ (b_j, 1)_{1,q} \end{matrix} \right] \quad (6)$$

where $0 \leq \theta \leq \pi$, $|\arg z| < (m + n - \frac{p}{2} - \frac{q}{2})\pi$.

$$\begin{aligned} & G_{p+1,q+1}^{m+1,n} \left[z \middle| \begin{matrix} (a_j, 1)_{1,p}, (1, 1) \\ (\frac{1}{2}, 1), (b_j, 1)_{1,q} \end{matrix} \right] + 2 \sum_{r=0}^{\infty} G_{p+2,q+2}^{m+1,n+1} \left[z \middle| \begin{matrix} (1-r, 1), (a_j, 1)_{1,p}, (1+r, 1) \\ (\frac{1}{2}, 1), (b_j, 1)_{1,q}, (1, 1) \end{matrix} \right] \cos r\theta \\ &= \sqrt{\pi} G_{p,q}^{m,n} \left[\frac{z}{\sin^2(\frac{\theta}{2})} \middle| \begin{matrix} (a_j, 1)_{1,p} \\ (b_j, 1)_{1,q} \end{matrix} \right] \end{aligned} \quad (7)$$

where $0 \leq \theta \leq \pi$, $|\arg z| < (m + n - \frac{p}{2} - \frac{q}{2})\pi$.

IV. CONCLUSION:

On specializing the parameters, G-function of two variables may be reduced to several other higher transcendental functions. Therefore the result (4) and (5) are of general nature and may reduce to be in different forms, which will be useful in the literature on applied Mathematics and other branches.

REFERENCES

- [1]. AyantFrédéric and Kumar Dinesh: A unified study of Fourier series involving the Aleph-function and the, Kampé de Fériet's function, International Journal of Mathematics Trends and Technology (ISSN 2231-5373), Vol. 35, No. 1, July 2016, pp. 40-48.
- [2]. Beg, Mehphooj and Shrivastava, S. S.: Fourier Series Involving H-Function of Two Variables, International Research Journal of Engineering and Technology (IRJET) (p-ISSN: 2395-0072), Volume: 04 Issue: 05 | May -2017, pp. 876-878.
- [3]. Dubey, Shanti Swarup and Shrivastava, S. S.: Fourier Series Involving I-Function of One Variable, International Research Journal of Mathematics, Engineering and IT (ISSN: (2349-0322)), Vol. 4, Issue 4, April 2017, pp. 1-3.
- [4]. Kesarwani, Roop Narain: Fourier Series for Meijer's G-Functions, Compositio Mathematica, tome 17 (1965-1966), p. 149-151. [1](#)
- [5]. Macrobert, T. M.: Fourier series for E-function, Math. Zeitschlie, 75, 79-82 (1961).
- [6]. Macrobert, T. M.: Infinite series for E-function, Math. Zeitschlie, 71, 143-154 (1959).
- [7]. Shrivastava, H. M. & Joshi, C. M.: Integration of Certain Products Associated with a Generalized Meijer Function, Proc. Cambridge Philos. Soc. 65, 471-477.