



Cototal Domination Number of the Jump Graph of a Graph

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Abstract: Let $D \subseteq V[J(G)]$ be a dominating set of the jump graph $J(G)$ of a graph $G(p, q)$. If $[V[J(G)] - D]$ contains no isolated vertex, then D is called the cototal dominating set of the jump graph $J(G)$. The minimum cardinality of a cototal dominating set, is the cototal domination number of the jump graph, denoted by $\gamma_{ct}[J(G)]$. In this paper bounds been established, exact value of some standard graphs are obtained and the graph theoretic relations of the cototal domination number with other graph parameters of $J(G)$ is discussed.

Keywords: Domination number of the jump graph, Cototal Dominating set of the jump graph, Cototal Domination number of the jump graph, Well Domination in jump graphs.

1. INTRODUCTION

Let $G(p, q)$ be a finite, simple, undirected, connected graph with $p = |V|$ vertices and $q = |E|$ edges. For all other graph terminology and notations refer [13]. Graph whose vertices are the edges of G is known as the line graph $L(G)$ of graph G . Any two vertices in $L(G)$ are adjacent if and only if their corresponding edges are adjacent in G .

The complement of a line graph $L(G)$ is called as the **jump graph of G** , denoted as $J(G)$. Thus, the jump graph $J(G)$ of a graph G is the graph defined on the edge set E of G , where two vertices are adjacent if and only if their corresponding edges are not adjacent in G . The isolated vertices of the graph G (if any) plays no part in both line graph $L(G)$ and jump graph $J(G)$, as both the line and jump graphs are defined on the edges set of G .

A non-empty subset $D \subseteq V(G)$ is a dominating set of G , if every vertex in $V - D$ is adjacent to some vertex in D . The domination number, $\gamma(G)$ of G is the minimum cardinality of the minimal dominating set of G . Let D be the dominating set of the graph G . If $V - D$ contains a dominating set D' of G , then D' is called an inverse dominating set of G with respect to D . The minimum cardinality of the inverse dominating set is the inverse domination number, $\gamma^{-1}(G)$.

A non-empty subset $D \subseteq V[J(G)]$ is called the dominating set of $J(G)$, if each vertex not in D is adjacent to a vertex in D . The domination number, $\gamma[J(G)]$ is the minimum cardinality of a dominating set in $J(G)$. Let $D \subseteq V[J(G)]$ be the minimum dominating set of $J(G)$. If, $[V[J(G)] - D]$ contain a dominating set D' then D' is called an inverse dominating set of the jump graph $J(G)$ with respect to the set D of $J(G)$. The inverse domination number, $\gamma^{-1}[J(G)]$ of the jump graph, is the minimum cardinality of the minimal inverse dominating set of $J(G)$ [5,6].

In our study, we have discussed the parameter, cototal domination number of a jump graph and the well domination number for some standard graphs been obtained. All the simple connected graph G considered in our study have $q = |E| \geq 4$.

II. COTOTAL DOMINATION NUMBER OF A JUMP GRAPH.

Definition: 2.1. Let $D \subseteq V[J(G)]$ be the dominating set of the jump graph $J(G)$ of a graph G . If the induced subgraph $[V[J(G)] - D]$ contains no isolated vertex, then D is termed as the cototal dominating set of the jump graph $J(G)$. The cototal domination number, $\gamma_{ct}[J(G)]$ - is the minimum cardinality taken over all the minimal cototal dominating sets of $J(G)$.

Definition: 2.2. A cototal dominating set of $J(G)$ with maximum cardinality is said to be the upper cototal domination number of $J(G)$, denoted by $\Gamma_{ct}[J(G)]$.

Theorem: 2.3.

Exact values for some Standard graphs

(i). For any Path P_p with $p \geq 5$, $\gamma_{ct}[J(G)] = 2$

- (ii). For a cycle C_p with $p \geq 5$, $\gamma_{ct}[J(G)] = \begin{cases} 3, & p = 5 \\ 2, & p \geq 6 \end{cases}$
- (iii). For the complete graph K_p , $\gamma_{ct}[J(G)] = \begin{cases} 4, & p = 5 \\ 3, & p \geq 6 \end{cases}$
- (iv). For a complete bipartite graph K_{p_1, p_2} ,
- $$\gamma_{ct}[J(G)] = \begin{cases} 2, & p_1 = 2, p_2 \geq 3 \\ 3, & p_1, p_2 \geq 3 \end{cases}$$
- (v). For a wheel graph $W_p = C_{p-1} + K_1$, $\gamma_{ct}[J(G)] = \begin{cases} 4, & p = 5 \\ 3, & p = 6 \\ 2, & p \geq 7 \end{cases}$
- (vi). For a Corona graph of two graphs, $G(p, q)$,
- a). $G = C_p \circ K_1$. $\gamma_{ct}[J(G)] = \begin{cases} 3, & p = 3 \\ 2, & p \geq 4 \end{cases}$
- b). $G = P_p \circ K_1$. $\gamma_{ct}[J(G)] = \begin{cases} 3, & p = 3 \\ 2, & p \geq 4 \end{cases}$
- (vii). For Petersen graph, $G = (10, 15)$,
- $$\gamma_{ct}[J(G)] = 2.$$
- (viii). If G is a Fan graph, $G = P_{p-1} + K_1$, $\gamma_{ct}[J(G)] = \begin{cases} 3, & p = 5 \\ 2, & p \geq 6 \end{cases}$
- (ix). For a Friendship graph $G = F_p$,
- $$\gamma_{ct}[J(G)] = \begin{cases} 4, & p = 5 \\ 2, & p \geq 6 \end{cases}$$

Remark: 2.4.

When the graph G contains atleast an edge e , such that $\deg(e) = q-1$, where q is the number of edges of G , then the jump graph $J(G)$ of G will have an isolated vertex. (i.e) $J(G)$ will be a disconnected graph.

Remark: 2.5.

If there exists an edge e in G such that $\deg(e) = q-2$, then the corresponding jump graph $J(G)$ of G will have a pendent vertex.

Theorem: 2.6: [7] For any connected graph $G(p, q)$, $\gamma_{ct}[J(G)] \geq 2$.

Proof: Let G be a simple connected graph with q edges $e_1, e_2, \dots, e_q$. Then the jump graphs of G is a graph, having the vertex set, $|V[J(G)]| = |E(G)| = q$. Thus, if D is a dominating set of $J(G)$, then D will contain atleast two vertices. Hence if $\gamma_{ct}[J(G)]$ exists then it has to be greater than or equal to 2.

Remark: 2.7: Cototal domination number of the jump graph does not exist for all graphs.

Theorem: 2.8: Let G be a Star graph or a Bistar, then G does not have the cototal domination number.

Proof: When G is either a Star graph or a Bistar graph, then there is an edge (edges) with $\deg(e) = q-1$. Therefore, $J(G)$ will have an isolated vertex (vertices).

Theorem: 2.9: Every pendent vertex of $J(G)$ is a member of the cototal dominating set.

Proof: Assume the contradiction. Let D be the cototal dominating set of $J(G)$. If D does not contain the pendent vertices of $J(G)$, if it exists. Then those vertices adjacent to the pendent vertices will be in D . This implies, $\langle V[J(G)] - D \rangle$ will have isolated vertices. Hence every pendent vertex of $J(G)$ must be in cototal dominating set of G .

Theorem: 2.10. For a connected graph $G(p, q)$,

$\gamma_{ct}[J(G)] \geq |m_i|$, where m_i is the set of all pendent vertices of $J(G)$

Proof: The cototal dominating set will contain every pendent vertex of $J(G)$, if it exists (Theorem.2.9). Hence the cototal dominating set will have atleast $|m_i|$ members.

Theorem: 2.11. Let $G(p, q)$ be isomorphic to any graph of P_5 or $P_3 \circ K_1$ or $C_3 \circ K_1$ or the spider graph of $K_{1,2}$, then,

$\gamma_{ct}[J(G)] + |m_i| = q$, where m_i is the set of all pendent vertices of $J(G)$ and q is the size of G .

Proof: For the graphs P_5 , $P_3 \circ K_1$, the spider graph of $K_{1,2}$ or $C_3 \circ K_1$, the set of all pendent vertices along with its adjacent vertex or the pendent vertex in the corresponding jump graphs forms the $\gamma_{ct}[J(G)]$ - set.

Theorem: 2.12. For a Spider graph of $G = K_{1, p-1}$, $p \geq 3$, $\gamma_{ct}[J(G)] = 2$.

Proof: Let $v_1, v_2, \dots, v_{p-1}$ be the pendent vertices of the spider graph of $K_{1, p-1}$. If e_{i1}, e_{j1} are the edges adjacent to the pendent vertex v_i and its support v_j , in G , then the vertices v_1', v_2' of $J(G)$, corresponding to the edges e_{i1}, e_{j1} of G , forms the minimum dominating set D of the $J(G)$. Also the induced subgraph $\langle V[J(G)] - D \rangle$ contains no isolated vertices. Thus the set D forms a minimum cototal dominating set of $J(G)$.

Theorem: 2.13[7]. Let T be a tree and $\text{diam}(T)$ is not less than 4, then

$$\gamma_{ct}[J(T)] = \begin{cases} 3, & \text{if } \deg(e) = q - 2 \\ 2, & \text{OTHERWISE} \end{cases}$$

Proof: Assume, T to be a tree with $\text{diam}(T)$ greater than three, then there must exist at least two vertices v_i and v_j in T , such that the distance between them is maximum. Let e_i, e_j be the edges adjacent to the vertices v_i and v_j in T . In the jump graph $J(T)$ of T , the vertices v_i', v_j' corresponding to the edges e_i and e_j , forms a minimum dominating set D . Clearly, the $< V[J(T)] - D >$ of the jump graph will have no isolated vertex.

Thus, D is the $\gamma_{ct}[J(T)]$ -set of $J(T)$. Moreover, if T contains an edge e_k , such that its degree is equal to $q - 2$, then the corresponding vertex v_k' of the edge e_k will be a pendent vertex of $J(G)$. By theorem 7, this pendent vertex must be a member of the $\gamma_{ct}[J(G)]$ -set of $J(G)$, along with the vertices v_i', v_j' . Hence the theorem.

Theorem: 2.14. [6] CHARACTERIZATION THEOREM.

A cototal dominating set D of the jump graph $J(G)$ is minimal if and only if for each $v_i' \in D$, one of the following conditions holds;

- (i). there exists a vertex $v_j' \in V[J(G)] - D$, such that, $N(v_j') \cap D = \{v_i'\}$;
- (ii). v_i' is an isolated vertex in the $< D >$;
- (iii). v_i' is an isolated vertex in the $< (V[J(G)] - D) \cup \{v_i'\} >$.

Proof: Let D be the cototal dominating set of $J(G)$, satisfying the three conditions. If D is not a minimal cototal dominating set of $J(G)$, then for any vertex $v_i' \in D$, $D - v_i'$ will be the minimal cototal dominating set of $J(G)$. That is, there is a vertex in $D - v_i'$, which is adjacent to v_i' . Hence v_i' is not an isolated vertex of $J(G)$, a contradiction to (ii). Also, if $D - v_i'$ is the minimal cototal dominating of $J(G)$, then v_i' will be in the set of $(V[J(G)] - D)$, contradicting (iii) and (i). Thus D is the minimal cototal dominating set of $J(G)$. Conversely, let D be the minimal cototal dominating set of $J(G)$. If any $v_i' \in D$ does not hold the three conditions of the theorem, then $D' = D - v_i'$, will be minimal dominating set of $J(G)$ and the $< V[J(G)] - D' >$ will have no isolated vertex. Thus D' will be the cototal dominating set of the jump graph, contradicting the assumption that D is the minimal cototal dominating set of $J(G)$.

RELATION WITH OTHER GRAPH PARAMETERS

Observation: 2.15. For any connected graph $G(p, q)$,

$$\gamma[J(G)] \leq \gamma_{ct}[J(G)].$$

For the graphs $P_n, n \geq 5, C_n, K_n, n \geq 6$, the bound is sharp.

Proof: Every cototal dominating set of $J(G)$ is a dominating set of $J(G)$.

Theorem: 2.16. For a graph $G(p, q)$, $\gamma[J(G)] + \gamma_{ct}[J(G)] \leq q$. Bound is attained for P_5, C_5 .

Proof: The vertex set of the jump graph $J(G)$ is the edge set of G . Let the set of all vertices with maximum degree in $J(G)$ be denoted by D_1 . If a subset $D_2 \subseteq D_1$ dominates all other vertices in $J(G)$. Then D_2 will be the dominating set of $J(G)$. Also if the $< V[J(G)] - D_2 >$ contains no isolated vertex, then D_2 will be the minimum cototal dominating set. Else there will be another subset D_3 , satisfying the conditions. Since, $D_2 \cup D_3 \subseteq V[J(G)]$, we conclude $\gamma[J(G)] + \gamma_{ct}[J(G)] \leq q$.

When the graph G is isomorphic to P_5 or C_5 the bound holds.

Theorem: 2.17. Let G be a connected graph, then

$$\gamma_c[J(G)] \leq \gamma_{ct}[J(G)].$$

Proof: Let $D \subseteq V[J(G)]$ be $\gamma_c[J(G)]$ -set of the jump graph $J(G)$ of a connected graph G . If the induced graph of $< V[J(G)] - D >$, contains no isolated vertex, then D will be the cototal dominating set of the jump graph $J(G)$.

Observation: 2.18.[13] Let G be a connected graph, then

$$\gamma[J(G)] \leq \gamma_c[J(G)] \leq \gamma_{ct}[J(G)] \leq \gamma_{ns}[J(G)].$$

Theorem: 2.19. For the complete graph K_5 , $\gamma_m[J(G)] = \gamma_{ct}[J(G)]$, where $\gamma_m[J(G)]$ -denotes the maximum domination number of $J(G)$.

Proof: Let $G = K_5$. Then we can find vertices v_1, v_2, v_3 in K_5 such that $< v_1, v_2, v_3 >$ forms a complete graph K_3 . Let e_i, e_j, e_k be the edges of K_3 . If the corresponding vertices of the edges e_i, e_j, e_k of G are v_i', v_j', v_k' in $J(G)$, then these vertices form a minimum dominating set D of $J(G)$ and on removing these vertices the jump graph $J(G)$ will contain an isolated vertex. Choose an edge e_l in G , such that, e_l is non-adjacent to each of e_i, e_j, e_k . Then the vertex v_l' in $J(G)$ corresponding to the edge e_l , along with v_i', v_j', v_k' forms the minimal cototal dominating set $D = \{v_i', v_j', v_k', v_l'\}$. In K_5 we can find exactly one such non-adjacent vertex for any K_3 also the complement $V[J(G)] - D$ is not a dominating set of the jump graph. Thus D is the $\gamma_m[J(G)]$ -set of $J(G)$.

III. WELL DOMINATION IN COTOTAL DOMINATING SET OF THE JUMP GRAPH.

Definition: 3.1.[13]

The jump graph $J(G)$ of a connected graph G is said to be well dominated if , $\gamma_{ct}[J(G)] = \Gamma_{ct}[J(G)]$.

Theorem:3.2

The jump graph of Path graph P_p , $p \geq 5$ is well dominated.

Proof:

$\gamma_{ct}[J(G)] = 2$, for $G = P_p$, $p \geq 5$. Also for the jump graph of a path graph, every pair of corresponding vertices of the edge set (e_i, e_j) forms a minimal cototal dominating set when they don't have a common edge between them. Thus every cototal dominating set of $J(G)$ is the minimum cototal dominating set.

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