



Nonexpansive Mappings in the Framework of Ordered S-Metric Spaces and Their Fixed Points

Bhawna Parkhey^{1,*}, Manoj Ughade², R. D. Daheriya³

^{1,3}Department of Mathematics, Government J.H. Post Graduate College, Betul,
Madhya Pradesh, India-460001

²Department of Mathematics, Institute for Excellence in Higher Education, Bhopal,
Madhya Pradesh, India- 462016

*Corresponding author

Abstract: We introduce nonexpansive and generalized nonexpansive condition for a map in an ordered S-metric space and derive fixed point theorems for a non-decreasing mapping. To illustrate our results, we give throughout the paper an example.

Keywords: Partially ordered set; nonexpansive mapping; non-decreasing map; fixed point; S-metric space.

I. Introduction and Preliminaries

The generalization of metric spaces in various structure has attracted attention of scientists due to the development and generalization of fixed point theory in metric spaces. Particularly, Bakhtin [5] generalized the concept of metric space by introducing b-metric space, many researchers studied fixed point results for various mappings satisfying certain conditions in b-metric space. Mustafa and Sims [13] introduced the concept of G-metric spaces to overcome fundamental flaws in Dhage's theory of generalized metric spaces. Sedghi et al. [25] familiarized the notion of a D^* -metric space. Every G-metric space is a D^* -metric space. The converse, however, is false in general; a D^* -metric space is not necessarily a G-metric space. Sedghi et al. [26] introduced a new generalized metric space called an S-metric space.

Definition 1.1 (see [26]) Let X be a non-empty set. An S-metric on X is a function $S: X \times X \times X \rightarrow [0, +\infty)$ that satisfies the following conditions, for each $x, y, z, a \in X$,

(S1). $S(x, y, z) \geq 0$,

(S2). $S(x, y, z) = 0$ if and only if $x = y = z$,

(S3). $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$.

Then S is called an S-metric on X and (X, S) is called an S-metric space.

The following is the intuitive geometric example for S-metric spaces.

Example 1.2 (see [26]) Let $X = \mathbb{R}^2$ and d be the ordinary metric on X . Put $S(x, y, z) = d(x, y) + d(x, z) + d(y, z)$ for all $x, y, z \in \mathbb{R}^2$, that is, S is the perimeter of the triangle given by $S(x, y, z)$. Then S is an S-metric on X .

Lemma 1.3 (see [26]) Let (X, S) be an S-metric space. Then $S(x, x, y) = S(y, y, x)$ for all $x, y \in X$.

Lemma 1.4 (see [10]) Let (X, S) be an S-metric space. Then $S(x, x, z) \leq 2S(x, x, y) + S(y, y, z)$ and $S(x, x, z) \leq 2S(x, x, y) + S(z, z, y)$ for all $x, y, z \in X$.

Definition 1.5 (see [26]) Let X be an S-metric space.

(1). A sequence $\{x_n\}$ converges to x if and only if $\lim_{n \rightarrow \infty} S(x_n, x_n, x) = 0$. That is, for each $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $S(x_n, x_n, x) < \epsilon$ and we denote this by $\lim_{n \rightarrow \infty} x_n = x$.

(2). A sequence $\{x_n\}$ is called a Cauchy if $\lim_{n, m \rightarrow \infty} S(x_n, x_n, x_m) = 0$. That is, for each $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for all $n, m \geq n_0$, $S(x_n, x_n, x_m) < \epsilon$.

(3). X is called complete if every Cauchy sequence in X is a convergent.

From (see [26], Examples in page 260), we have the following.

Example 1.6

(a) Let $\mathbb{R}$ be the real line. Then $S(x, y, z) = |x - z| + |y - z|$ for all $x, y, z \in \mathbb{R}$, is an S-metric on $\mathbb{R}$. This S-metric is called the usual S-metric on $\mathbb{R}$. Furthermore, the usual S-metric space $\mathbb{R}$ is complete

(b) Let Y be a non-empty set of $\mathbb{R}$. Then $S(x, y, z) = |x - z| + |y - z|$ for all $x, y, z \in Y$, is an S-metric on Y . If Y is a closed subset of the usual metric space $\mathbb{R}$, then the S-metric space Y is complete.

Lemma 1.7 (see [26]) Let (X, S) be an S-metric space. If the sequence $\{x_n\}$ in X converges to x , then x is unique.

Lemma 1.8 (see [26]) Let (X, S) be an S-metric space. If $x_n \rightarrow x$ and $y_n \rightarrow y$. Then $S(x_n, x_n, y_n) \rightarrow S(x, x, y)$.

Definition 1.9 (see [26]) Let (X, S) be an S-metric space. A mapping $f: X \rightarrow X$ is called Lipschitzian if there exists a number $k \geq 0$ such that $S(fx, fx, fy) \leq kS(x, x, y), \forall x, y \in X$. The mapping f is called contractive if $k < 1$.

In 2012, Sedghi et al. [26] asserted that an S-metric is a generalization of a G-metric, that is, every G-metric is an S-metric, see [26, Remarks 1.3] and [26, Remarks 2.2]. The Example 2.1 and Example 2.2 of Dung et al. [9] shows that this assertion is not correct. Moreover, the class of all S-metrics and the class of all G-metrics are distinct. Jeli and Samet [12] showed that a G-metric is not a real generalization of a metric. Further, they proved that the fixed point theorems proved in G-metric spaces can be obtained by usual metric arguments. The similar approach may be found in [11].

The importance of nonexpansive mappings was outlined, e.g., in 1980 by Bruck [7]. A nonexpansive mapping of a complete metric space need not have a fixed point (consider a translation operator $fx = x + c$ in a Banach space). A fixed point of a nonexpansive mapping need not be unique (consider $f = I$). To ensure the existence and/or uniqueness of fixed points we must assume additional conditions on f and/or the underlying space. Contraction mappings, isometries and orthogonal projection are all nonexpansive mappings. The study of non-expansive mappings has been one of the main features in recent developments of fixed point theory-see for instance [6, 8]. Browder et al. [6] proved that every non-expansive mapping f from a convex bounded closed subset C of a Hilbert space X into C has a fixed point. There are also several interesting unsolved problems. On the other hand, fixed point theory has developed rapidly in metric spaces endowed with a partial ordering. The first result in this direction was given by Ran and Reurings [23] who presented its applications to matrix equations. Subsequently, Nieto and Rodríguez-López [19] extended this result for non-decreasing mappings and applied it to obtain a unique solution for a first order ordinary differential equation with periodic boundary conditions. Thereafter, several authors obtained many fixed point theorems in ordered metric spaces. For more details see [1-2, 4, 11, 115-18, 20-22, 24, 28] and the references cited therein.

We will prove in this paper some fixed point theorems for generalized nonexpansive mappings in complete S-metric spaces endowed with a partial order. We furnish suitable examples to demonstrate the validity of the hypotheses of our results. Presented theorems are extensions of the results of Ćirić [8] is S-metric space.

II. Main Results

First, we introduce some definitions in ordered S-metric spaces.

Definition 2.1 Let (X, S) be an S-metric space. A mapping $f: X \rightarrow X$ is called nonexpansive if there exists a number $k = 1$ such that $S(fx, fx, fy) \leq kS(x, x, y)$ for all $x, y \in X$.

Definition 2.2 Let $(X, S, \leq)$ be an ordered S-metric space. A mapping $f: X \rightarrow X$ is said to be generalized nonexpansive (of Ćirić type [8]) if

$$S(fx, fx, fy) \leq a \max \{S(x, x, y), S(x, x, fx), S(y, y, fy), \frac{1}{3}[S(x, x, fy) + S(y, y, fx)]\} \\ + b \max \{S(x, x, fx), S(y, y, fy)\} + c[S(x, x, fy) + S(y, y, fx)] \quad (2.1)$$

holds for all comparable $x, y \in X$ where $a \geq 0, b > 0, c > 0$ satisfy $a + b + 3c = 1$.

Definition 2.3

1. If $(X, \leq)$ is a partially ordered set then $x, y \in X$ are called comparable if $x \leq y$ or $y \leq x$ holds.
2. A subset K of X is said to be totally ordered if every two elements of K are comparable.
3. If $f: X \rightarrow X$ is such that, for $x, y \in X$, $x \leq y$ implies $fx \leq fy$, then the mapping f is said to be nondecreasing.

Definition 2.4 Let X be a nonempty set. Then $(X, S, \leq)$ is called an ordered S-metric space if

- (a) (X, S) is a metric space,
- (b) $(X, S, \leq)$ is a partially ordered set.

Definition 2.5 The space $(X, S, \leq)$ is called regular if the following hypothesis holds: if $\{x_n\}$ is a non-decreasing sequence in X with respect to $\leq$ such that $x_n \rightarrow x \in X$ as $n \rightarrow \infty$, then $x_n \leq x$.

Now, we state and prove our first result.

Theorem 2.6 Let $(X, S, \leq)$ be a partially ordered complete S-metric space. Suppose that $f: X \rightarrow X$ is a nondecreasing generalized nonexpansive mapping. Also suppose that there exists $x_0 \in X$ with $x_0 \leq fx_0$. If f has closed graph (in particular, if it is continuous), then f has a fixed point. Moreover, the set of fixed points of f is totally ordered if and only if it is a singleton.

Proof If $fx_0 = x_0$, then the proof is completed. Suppose $fx_0 \neq x_0$. Now since $x_0 \leq fx_0$ and T is non-decreasing

$$x_0 \leq fx_0 \leq f^2x_0 \leq \dots \leq f^nx_0 \leq f^{n+1}x_0 \leq \dots \quad (2.2)$$

Put $x_0 = f^n x_0$ and so $x_{n+1} = f x_n$. If there exists $n_0 \in \{1, 2, \dots\}$ such that right-hand side of (2.1) is 0 for $x = x_{n_0-1}$ and $y = x_{n_0}$, then it is clear that $x_{n_0-1} = x_{n_0} = f x_{n_0-1}$ and so we have finished. Now we claim that $S(x_{n+1}, x_{n+1}, x_n) \leq S(x_n, x_n, x_{n-1})$ for all $n \geq 1$. Suppose this is not true, that is, there exists $n_0 \geq 1$ such that $S(x_{n_0+1}, x_{n_0+1}, x_{n_0}) > S(x_{n_0}, x_{n_0}, x_{n_0-1})$. Now since $x_{n_0-1} \leq x_{n_0}$, we can use the inequality (2.1) for these elements, and we obtain

$$\begin{aligned} S(x_{n_0}, x_{n_0}, x_{n_0+1}) &= S(f x_{n_0-1}, f x_{n_0-1}, f x_{n_0}) \\ &\leq a \max\{S(x_{n_0-1}, x_{n_0-1}, x_{n_0}), S(x_{n_0-1}, x_{n_0-1}, f x_{n_0-1}), S(x_{n_0}, x_{n_0}, f x_{n_0}), \\ &\quad \frac{1}{3}[S(x_{n_0-1}, x_{n_0-1}, f x_{n_0}) + S(x_{n_0}, x_{n_0}, f x_{n_0-1})]\} \\ &\quad + b \max\{S(x_{n_0-1}, x_{n_0-1}, f x_{n_0-1}), S(x_{n_0}, x_{n_0}, f x_{n_0})\} \\ &\quad + c[S(x_{n_0-1}, x_{n_0-1}, f x_{n_0}) + S(x_{n_0}, x_{n_0}, f x_{n_0-1})] \\ &\leq a \max\{S(x_{n_0-1}, x_{n_0-1}, x_{n_0}), S(x_{n_0-1}, x_{n_0-1}, x_{n_0}), S(x_{n_0}, x_{n_0}, x_{n_0+1}), \\ &\quad \frac{1}{3}[S(x_{n_0-1}, x_{n_0-1}, x_{n_0+1}) + S(x_{n_0}, x_{n_0}, x_{n_0})]\} \\ &\quad + b \max\{S(x_{n_0-1}, x_{n_0-1}, x_{n_0}), S(x_{n_0}, x_{n_0}, x_{n_0+1})\} \\ &\quad + c[S(x_{n_0-1}, x_{n_0-1}, x_{n_0+1}) + S(x_{n_0}, x_{n_0}, x_{n_0})] \end{aligned}$$

From Lemma 1.3, (S2) and (S3), we get

$$\begin{aligned} S(x_{n_0+1}, x_{n_0+1}, x_{n_0}) &\leq a \max\{S(x_{n_0}, x_{n_0}, x_{n_0-1}), S(x_{n_0}, x_{n_0}, x_{n_0-1}), S(x_{n_0+1}, x_{n_0+1}, x_{n_0}), \\ &\quad \frac{1}{3}S(x_{n_0-1}, x_{n_0-1}, x_{n_0+1})\} \\ &\quad + b \max\{S(x_{n_0}, x_{n_0}, x_{n_0-1}), S(x_{n_0+1}, x_{n_0+1}, x_{n_0})\} + cS(x_{n_0-1}, x_{n_0-1}, x_{n_0+1}) \\ &\leq a \max\{S(x_{n_0}, x_{n_0}, x_{n_0-1}), S(x_{n_0}, x_{n_0}, x_{n_0-1}), S(x_{n_0+1}, x_{n_0+1}, x_{n_0}), \\ &\quad \frac{1}{3}[S(x_{n_0-1}, x_{n_0-1}, x_{n_0}) + S(x_{n_0-1}, x_{n_0-1}, x_{n_0}) + S(x_{n_0+1}, x_{n_0+1}, x_{n_0})]\} \\ &\quad + b \max\{S(x_{n_0}, x_{n_0}, x_{n_0-1}), S(x_{n_0+1}, x_{n_0+1}, x_{n_0})\} \\ &\quad + c[S(x_{n_0-1}, x_{n_0-1}, x_{n_0}) + S(x_{n_0-1}, x_{n_0-1}, x_{n_0}) + S(x_{n_0+1}, x_{n_0+1}, x_{n_0})] \\ &\leq a \max\{S(x_{n_0}, x_{n_0}, x_{n_0-1}), S(x_{n_0}, x_{n_0}, x_{n_0-1}), S(x_{n_0+1}, x_{n_0+1}, x_{n_0}), \\ &\quad \frac{1}{3}[2S(x_{n_0}, x_{n_0}, x_{n_0-1}) + S(x_{n_0+1}, x_{n_0+1}, x_{n_0})]\} \\ &\quad + b \max\{S(x_{n_0}, x_{n_0}, x_{n_0-1}), S(x_{n_0+1}, x_{n_0+1}, x_{n_0})\} \\ &\quad + c[2S(x_{n_0}, x_{n_0}, x_{n_0-1}) + S(x_{n_0+1}, x_{n_0+1}, x_{n_0})] \\ &= (a + b) \max\{S(x_{n_0}, x_{n_0}, x_{n_0-1}), S(x_{n_0+1}, x_{n_0+1}, x_{n_0})\} \\ &\quad + c[2S(x_{n_0}, x_{n_0}, x_{n_0-1}) + S(x_{n_0+1}, x_{n_0+1}, x_{n_0})] \end{aligned}$$

Since $c > 0$ this implies that

$S(x_{n_0+1}, x_{n_0+1}, x_{n_0}) < (a + b)S(x_{n_0+1}, x_{n_0+1}, x_{n_0}) + 3cS(x_{n_0+1}, x_{n_0+1}, x_{n_0}) = S(x_{n_0+1}, x_{n_0+1}, x_{n_0})$ a contradiction. Thus $S(f x_n, f x_n, x_n) \leq S(f x_{n-1}, f x_{n-1}, x_{n-1})$. From Lemma 1.3, we have $S(x_n, x_n, f x_n) \leq S(x_{n-1}, x_{n-1}, f x_{n-1})$. Hence

$$S(x_n, x_n, f x_n) \leq S(x_0, x_0, f x_0) \quad (2.3)$$

Using (2.1) and (2.3), we have

$$\begin{aligned} S(x_1, x_1, f x_2) &= S(f x_0, f x_0, f x_2) \\ &\leq a \max\{S(x_0, x_0, x_2), S(x_0, x_0, f x_0), S(x_2, x_2, f x_2), \frac{1}{3}[S(x_0, x_0, f x_2) + S(x_2, x_2, f x_0)]\} \\ &\quad + b \max\{S(x_0, x_0, f x_0), S(x_2, x_2, f x_2)\} + c[S(x_0, x_0, f x_2) + S(x_2, x_2, f x_0)] \\ &\leq a \max\{S(x_0, x_0, f x_1), S(x_0, x_0, f x_0), S(x_2, x_2, f x_2), \frac{1}{3}[S(x_0, x_0, f x_2) + S(x_2, x_2, f x_0)]\} \\ &\quad + b \max\{S(x_0, x_0, f x_0), S(x_2, x_2, f x_2)\} + c[S(x_0, x_0, f x_2) + S(x_2, x_2, f x_0)] \quad (2.4) \end{aligned}$$

From (2.3) and (S3), we get

$$\begin{aligned} \frac{1}{3}[S(x_0, x_0, f x_2) + S(x_2, x_2, f x_0)] &\leq \frac{1}{3}[S(x_0, x_0, f x_0) + S(x_0, x_0, f x_0) + S(f x_2, f x_2, f x_0) + S(x_2, x_2, f x_0)] \\ &= \frac{1}{3}[S(x_0, x_0, f x_0) + S(x_0, x_0, f x_0) + S(f x_0, f x_0, f x_2) + S(f x_0, f x_0, x_2)] \\ &= \frac{1}{3}[S(x_0, x_0, f x_0) + S(x_0, x_0, f x_0) + S(x_1, x_1, f x_2) + S(x_1, x_1, f x_1)] \\ &\leq \frac{1}{3}[S(x_0, x_0, f x_0) + S(x_0, x_0, f x_0) + S(x_1, x_1, f x_1)] \\ &\quad + \frac{1}{3}[S(x_1, x_1, f x_1) + S(x_1, x_1, f x_1) + S(f x_2, f x_2, f x_1)] \\ &= \frac{1}{3}[S(x_0, x_0, f x_0) + S(x_0, x_0, f x_0) + S(x_1, x_1, f x_1)] \\ &\quad + \frac{1}{3}[S(x_1, x_1, f x_1) + S(x_1, x_1, f x_1) + S(f x_1, f x_1, f x_2)] \\ &= \frac{1}{3}[S(x_0, x_0, f x_0) + S(x_0, x_0, f x_0) + S(x_1, x_1, f x_1)] \end{aligned}$$

$$\begin{aligned} & + \frac{1}{3}[S(x_1, x_1, f x_1) + S(x_1, x_1, f x_1) + S(x_2, x_2, f x_2)] \\ & \leq 2S(x_0, x_0, f x_0) \end{aligned} \quad (2.5)$$

Substituting (2.5) in (2.4), we have

$$\begin{aligned} S(x_1, x_1, f x_2) & \leq 2aS(x_0, x_0, f x_0) + bS(x_0, x_0, f x_0) + 6cS(x_0, x_0, f x_0) \\ & = (2a + b + 6c)S(x_0, x_0, f x_0) \end{aligned}$$

From $a + b + 3c = 1$, we get

$$S(x_1, x_1, f x_2) \leq (2 - b)S(x_0, x_0, f x_0) \quad (2.6)$$

From (2.1), (2.3), (2.5) and (2.6), we have

$$\begin{aligned} S(x_2, x_2, f x_2) & = S(f x_1, f x_1, f x_2) \\ & \leq a \max \left\{ S(x_1, x_1, x_2), S(x_1, x_1, f x_1), S(x_2, x_2, f x_2), \frac{1}{3}[S(x_1, x_1, f x_2) + S(x_2, x_2, f x_1)] \right\} \\ & \quad + b \max \{S(x_1, x_1, f x_1), S(x_2, x_2, f x_2)\} + c[S(x_1, x_1, f x_2) + S(x_2, x_2, f x_1)] \\ & = \max \left\{ S(x_1, x_1, f x_1), S(x_1, x_1, f x_1), S(x_2, x_2, f x_2), \frac{1}{3}[S(x_1, x_1, f x_2) + S(x_2, x_2, f x_1)] \right\} \\ & \quad + b \max \{S(x_1, x_1, f x_1), S(x_2, x_2, f x_2)\} + c[S(x_1, x_1, f x_2) + S(x_2, x_2, f x_1)] \\ & = a \max \left\{ S(x_1, x_1, f x_1), S(x_1, x_1, f x_1), S(x_2, x_2, f x_2), \frac{1}{3}S(x_1, x_1, f x_2) \right\} \\ & \quad + b \max \{S(x_1, x_1, f x_1), S(x_2, x_2, f x_2)\} + cS(x_1, x_1, f x_2) \\ & \leq aS(x_0, x_0, f x_0) + bS(x_0, x_0, f x_0) + (2 - b)cS(x_0, x_0, f x_0) \\ & \leq (a + b + 2c - bc)S(x_0, x_0, f x_0) = (1 - c(1 - b))S(x_0, x_0, f x_0) \end{aligned}$$

Hence $S(x_2, x_2, f x_2) \leq (1 - c(1 - b))S(x_0, x_0, f x_0)$. Proceeding in this manner we obtain

$$S(x_2, x_2, f x_2) \leq (1 - c(1 - b))^{\left[\frac{n}{2}\right]} S(x_0, x_0, f x_0) \quad (2.7)$$

for all $n = 1, 2, \dots$, where $\left[\frac{n}{2}\right]$ denotes the greatest integer not exceeding $\frac{n}{2}$. Since $1 - c(1 - b) < 1$, from (2.7), we conclude that $\{f^n x_0\}$ is a Cauchy sequence. From the completeness of X , there exists $p \in X$ such that $\lim_{n \rightarrow \infty} f^n x_0 = p$. Passing to the limit as $n \rightarrow \infty$ in the relation $f f^n x_0 = f^{n+1} x_0$, and using that f has closed graph, we get that $f p = p$, i.e., p is a fixed point of f . Now, suppose that the set of fixed points of f is totally ordered. We claim that there is a unique fixed point of f . Assume that $f u = u$ and $f v = v$. By supposition, we can replace x by u and y by v in (2.1), and we obtain

$$\begin{aligned} S(u, u, v) & = S(f u, f u, f v) \leq a \max \left\{ S(u, u, v), S(u, u, f u), S(v, v, f v), \frac{1}{3}[S(u, u, f v) + S(v, v, f u)] \right\} \\ & \quad + b \max \{S(u, u, f u), S(v, v, f v)\} + c[S(u, u, f v) + S(v, v, f u)] \\ & = (a + 2c)S(u, u, v) = (1 - (b + c))S(u, u, v) \end{aligned} \quad (2.8)$$

Since $b + c > 0$, this implies that $u = v$. The converse is trivial.

We are also able to prove the existence of a fixed point of a mapping without using that it has closed graph. More precisely, we have the following theorem.

Theorem 2.7 Let $(X, S, \leq)$ and $f: X \rightarrow X$ satisfy all the assumptions of Theorem 2.6, except that condition of f having closed graph is substituted by the condition that $(X, S, \leq)$ is regular. Then the same conclusions as in Theorem 2.6 hold.

Proof Following the proof of Theorem 2.6, we have that $\{x_n\}$ is a Cauchy sequence in (X, S) which is complete. Then, there exists $p \in X$ such that $x_n \rightarrow p$. Now suppose that $S(p, p, f p) > 0$. From regularity of X , by (2.2), we have $x_n \leq p$ for all $n \in \mathbb{N}$. Hence, we can apply the considered nonexpansive condition. Then, setting $x = x_n$ and $y = p$ in (2.1), we obtain:

$$\begin{aligned} S(x_{n+1}, x_{n+1}, f p) & = S(f x_n, f x_n, f p) \\ & \leq a \max \left\{ S(x_n, x_n, p), S(x_n, x_n, f x_n), S(p, p, f p), \frac{1}{3}[S(x_n, x_n, f p) + S(p, p, f x_n)] \right\} \\ & \quad + b \max \{S(x_n, x_n, f x_n), S(p, p, f p)\} + c[S(x_n, x_n, f p) + S(p, p, f x_n)] \\ & = a \max \left\{ S(x_n, x_n, p), S(x_n, x_n, x_{n+1}), S(p, p, f p), \frac{1}{3}[S(x_n, x_n, f p) + S(p, p, x_{n+1})] \right\} \\ & \quad + b \max \{S(x_n, x_n, x_{n+1}), S(p, p, f p)\} + c[S(x_n, x_n, f p) + S(p, p, x_{n+1})] \end{aligned} \quad (2.9)$$

Passing to the limit as $n \rightarrow \infty$ and using that $x_n \rightarrow p$, we obtain $S(p, p, f p) \leq (a + b + c)S(p, p, f p) < S(p, p, f p)$ a contradiction. Therefore $S(p, p, f p) = 0$ and consequently $p = f p$, that is, f has a fixed point. The uniqueness of fixed point of f follows in the same way as in Theorem 2.6.

Corollary 2.8 Let $(X, S, \leq)$ be a partially ordered complete S-metric space. Suppose that $f: X \rightarrow X$ is a mapping satisfying

$$S(f x, f x, f y) \leq b \max \{S(x, x, f x), S(y, y, f y)\} + c[S(x, x, f y) + S(y, y, f x)] \quad (2.10)$$

for all comparable $x, y \in X$, where $b > 0, c > 0$ satisfy $b + 3c = 1$. If f has closed graph or X is regular, then f has a fixed point. Moreover, the set of fixed points of f is totally ordered if and only if it is a singleton.

Now, we present an example showing how our results can be used.

Example 2.9 Let $X = [0, +\infty[$ equipped with the S-metric $S(x, y, z) = |x - z| + |y - z|$ for all $x, y \in X$ and the order defined by $x \leq y \Leftrightarrow (x = y) \vee (x, y \in [0, \frac{1}{2}] \wedge x \geq y)$. Consider the following self-mapping on X : $fx = \frac{1}{2}x, 0 \leq x \leq \frac{1}{2}$ and $fx = 2x, x > \frac{1}{2}$. Then, it is easy to show that all the conditions of Theorem 2.7 are fulfilled, the non-expansive condition (2.1) with $a = \frac{1}{3}, b = \frac{1}{6}$ and $c = \frac{1}{6}$. Indeed, let $x, y \in X$ be comparable. If $x = y$, then (2.1) holds trivially. Let $x \neq y$ and, e.g., $x \leq y$. Then $x, y \in [0, \frac{1}{2}]$ and $x > y$. Condition (2.1) takes the form

$$\begin{aligned} |x - y| &= 2 \left| \frac{1}{2}x - \frac{1}{2}y \right| = |fx - fy| + |fx - fy| = S(fx, fx, fy) \\ &\leq \frac{1}{3} \max \left\{ 2|x - y|, 2 \left| x - \frac{1}{2}x \right|, 2 \left| y - \frac{1}{2}y \right|, \frac{1}{3} \left[2 \left| x - \frac{1}{2}y \right| + 2 \left| y - \frac{1}{2}x \right| \right] \right\} \\ &\quad + \frac{1}{6} \max \left\{ 2 \left| x - \frac{1}{2}x \right|, 2 \left| y - \frac{1}{2}y \right| \right\} + \frac{1}{6} \left[2 \left| x - \frac{1}{2}y \right| + 2 \left| y - \frac{1}{2}x \right| \right] \\ &\leq \frac{2}{3} \max \left\{ (x - y), x, y, \frac{1}{3} \left[\left(x - \frac{1}{2}y \right) + \left| y - \frac{1}{2}x \right| \right] \right\} + \frac{1}{6} \max \{x, y\} + \frac{1}{3} \left[\left(x - \frac{1}{2}y \right) + \left| y - \frac{1}{2}x \right| \right] \end{aligned}$$

Using substitution $y = tx, t \geq 0$, the last inequality reduces to

$$1 - t \leq \frac{2}{3} \max \left\{ 1 - t, 1, t, \frac{1}{3} \left[\left(1 - \frac{1}{2}t \right) + \left| t - \frac{1}{2} \right| \right] \right\} + \frac{1}{6} \max \{1, t\} + \frac{1}{3} \left[\left(1 - \frac{1}{2}t \right) + \left| t - \frac{1}{2} \right| \right]$$

and can be checked by discussion on possible values for $t \geq 0$. Here $a + b + 3c = 1$. Hence, all the conditions of Theorem 2.7 are satisfied and T has a fixed point (which is 0). Note that T does not satisfy the non-expansive condition for arbitrary $x, y \in X$.

III. Acknowledgments

The authors have benefited a lot from the referee's report. So, they would like to express their gratitude for his/her constructive suggestions which improved the presentation and readability.

IV. References

- [1]. Abbas M., Nazir T., Radenovic S.: Common fixed point of four maps in partially ordered metric spaces. *Appl. Math. Lett.* 24, 1520–1526 (2011).
- [2]. Agarwal RP, El-Gebeily MA, O'Regan D: Generalized contractions in partially ordered metric spaces. *Appl. Anal.* 87, 1–8 (2008).
- [3]. An TV, Dung NV, Hang VTL: A new approach to fixed point theorems on G-metric spaces, *Topology Appl.* 160(2013), pp. 1486-1493.
- [4]. Aydi H, Nashine HK, Samet B, Yazidi H: Coincidence and common fixed point results in partially ordered cone metric spaces and applications to integral equations. *Nonlinear Anal.* 74(17), 6814–6825 (2011).
- [5]. Bakhtin IA: The contraction principle in quasi metric spaces, *Funct. Anal., Gos. Ped. Inst. Unianowsk*, 30 (1989) 26–37.
- [6]. Browder FE, Petryshyn WV: Construction of fixed points of nonlinear mappings in Hilbert spaces. *J. Math. Anal. Appl.* 20, 197–228 (1967).
- [7]. Bruck RE: Asymptotic behaviour of non-expansive mappings. In: Sine, R.C. (eds.) *Contemporary Mathematics*, vol 18. Fixed Points and Nonexpansive Mappings. AMS, Providence (1980).
- [8]. Ćirić LjB: On some nonexpansive mappings and fixed points. *Indian J. Pure Appl. Math.* 24(3), 145–149 (1993).
- [9]. Dung NV, Hieu NV, Radojevic S: Fixed point theorems for g-monotone maps on partially ordered S-metric spaces, *Filomat*. 2014;28(9):1885-1898, DOI 10.2298/FIL1409885D, (2014).
- [10]. Dung NV: On coupled common fixed points for mixed weakly monotone maps in partially ordered S-metric spaces, *Fixed Point Theory Appl.*, 48 (2013), pp. 1-17.
- [11]. Harjani J, Sadarangani K.: Fixed point theorems for weakly contractive mappings in partially ordered sets. *Nonlinear Anal.* 71, 3403–3410 (2009).
- [12]. Jleli M, Samet B: Remarks on G-metric spaces and fixed point theorems. *Fixed Point Theory Appl.*, 201(2012).
- [13]. Mustafa Z, Sims B: A new approach to generalized metric spaces. *J. Nonlinear Convex Anal.* 2006;7(2):289-297.
- [14]. Mustafa Z: A New Structure for Generalized Metric Spaces with Applications to Fixed Point Theory, Ph. D. Thesis, The University of Newcastle, Callaghan, Australia, 2005.
- [15]. Nashine HK, Altun I: A common fixed point theorem on ordered metric spaces. *Bull. Iranian Math. Soc.* 38, 925–934 (2012).
- [16]. Nashine HK, Altun I: Fixed point theorems for generalized weakly contractive condition in ordered metric spaces. *Fixed Point Theory Appl.* (2011). Article ID 132367.
- [17]. Nashine HK, Samet B, Vetro C: Monotone generalized nonlinear contractions and fixed point theorems in ordered metric spaces. *Math. Comput. Modelling* 54, 712–720 (2011).
- [18]. Nashine HK, Samet B: Fixed point results for mappings satisfying (ψ, ϕ) -weakly contractive condition in partially ordered metric spaces. *Nonlinear Anal.* 74, 2201–2209 (2011).
- [19]. Nieto JJ, López RR.: Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations. *Order* 22, 223–239 (2005).
- [20]. O'Regan D, Petrusel A: Fixed point theorems for generalized contractions in ordered metric spaces. *J. Math. Anal. Appl.* 341, 1241–1252 (2008).
- [21]. Petrusel A, Rus IA: Fixed point theorems in ordered L-spaces. *Proc. Am. Math. Soc.* 134, 411–418 (2006).
- [22]. Radenovic S, Kadelburg Z: Generalized weak contractions in partially ordered metric spaces. *Comput. Math. Appl.* 60(6), 1776–1783 (2010).
- [23]. Ran ACM, Reurings MCB: A fixed point theorem in partially ordered sets and some applications to matrix equations. *Proc. Am. Math. Soc.* 132, 1435–1443 (2004).
- [24]. Saadati R, Vaezpour SM: Monotone generalized weak contractions in partially ordered metric spaces. *Fixed Point Theory* 11, 375–382 (2010).
- [25]. Sedghi S, Rao KPR, Shobe N: Common fixed point theorems for six weakly compatible mappings in D^* -metric spaces, *Internat. J. Math. Math. Sci.*, 6(2007), pp. 225-237.
- [26]. Sedghi S, Shobe N, Aliouche A: A generalization of fixed point theorem in S-metric spaces, *Mat. Vesnik*. 2012;64:258-266.
- [27]. Sedghi S, Shobe N, Zhou H: A common fixed point theorem in D^* -metric spaces. *Fixed Point Theory Appl.* 2007;13. Article ID 27906.
- [28]. Shatanawi W, Samet B: On (ψ, ϕ) -weakly contractive condition in partially ordered metric spaces. *Comput. Math. Appl.* 62, 3204–3214 (2011).