



Adjacent vertex distinguishing total coloring of tensor product of graphs

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Abstract: In this paper, we prove the adjacent vertex distinguishing total coloring conjecture for the graphs namely, tensor product and 2-tensor product of the graphs Path by Path and Path by Cycle in detail. Also, we prove that P_n , C_n and K_n admits adjacent vertex distinguishing total coloring and their color classes are discussed.

Keywords: simple graph, adjacent vertex distinguishing total coloring, adjacent vertex distinguishing total chromatic number, tensor product, 2-tensor product.

I. Introduction

A graph G consists of a set of vertices $V(G)$ and a set of edges $E(G)$. All graphs considered in this paper are simple and finite. For every vertex $u, v \in V(G)$, the edge connecting two vertices is denoted by $uv \in E(G)$ and their distance denoted by $d_G(u, v)$ is the length of the shortest path between u and v . Let $\Delta(G)$ denote the maximum degree of a graph G . If the edge set is empty then G is null graph. For basic terminology and concepts of graph theory, we refer [2], [3]. A graph is connected if any two of its vertices are the endpoints of one of its paths. Otherwise, it is disconnected. For disconnected graphs, it is often useful to consider the connected subgraphs that are maximal with respect to inclusion. They are uniquely determined and called connected components or simply components. If every component of a disconnected graph consists of a single vertex, we say the graph is totally disconnected. The totally disconnected graph on n vertices is denoted by D_n . For graphs G_1 and G_2 , we let $G_1 \cup G_2$ denotes their union, that is, $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$. Throughout the paper, we denote the path graph, cycle graph, complete graph with n vertices by P_n , C_n and K_n respectively.

Definition 1.1

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two connected graphs. [5] The tensor product of G_1 and G_2 denoted by $G = G_1 \otimes G_2$ is the graph with vertex set $V(G) = V(G_1) \times V(G_2)$ and the edge set and the edge set $E(G) = \{(u, v) \text{ adjacent to } (u', v') \mid d_{G_1}(u, u') = 1 \text{ and } d_{G_2}(v, v') = 1\}$.

Definition 1.2

[1] Let 2- tensor product $G = G_1 \otimes_2 G_2$ of G_1 and G_2 has the vertex set $V(G) = V(G_1) \otimes_2 V(G_2)$ and the edge set $E(G_1 \otimes_2 G_2) = \{(u, v) \text{ adjacent to } (u', v') \mid d_{G_1}(u, u') = 2 \text{ and } d_{G_2}(v, v') = 2\}$.

Definition 1.3. [8] A total k -coloring of G is a mapping $f: V(G) \cup E(G) \rightarrow \{1, 2, \dots, k\}$, $k \in \mathbb{Z}^+$ such that any two adjacent or incident elements in $V(G) \cup E(G)$ have different colors. A proper total k -coloring of G is adjacent vertex distinguishing, or a total- k -avd-coloring, if $C_f(u) \neq C_f(v)$ whenever $uv \in E(G)$, where $C_f(v)$ is the color set (or color class) of the vertex v (with respect to f), we denote $C_f(v)$ as $C(v)$.

$$C(v) = \{f(v)\} \cup \{f(vw) \mid vw \in E(G)\} \text{ and } \bar{C}(v) = \{1, 2, \dots, k\} \setminus C(v)$$

The minimum number of colors required to give an adjacent vertex distinguishing total coloring (abbreviated as AVDTTC) to G is denoted by $\chi_{avt}(G)$. The well-known AV DTC conjecture, made by Zhang et al [8] says that every simple graph G has $\chi_{avt}(G) \leq \Delta(G) + 3$. More recently, Wang [6] and Chen independently confirmed this conjecture for graph G with $\Delta(G) \leq 3$. Xiang'en Chen [7] proved $\chi_{avt}(G) \leq 6$ for graphs with maximum degree

$\Delta(G) = 3$. If G is a bipartite graph, then $\chi_{\text{avt}}(G) \leq \Delta(G) + 2$. Also, J. Hulan [4] has given a short proof for an upper bound on the adjacent vertex distinguishing total chromatic number of graphs of maximum degree 3.

II. AVDTTC of P_n , C_n and K_n

ZHANG Zhongfu [8] presented the concept of adjacent vertex distinguishing total coloring of some special graphs. In this section, we propose a procedure for AV DTC of P_n , C_n and K_n and also, we discuss their color sets. Let $v_1, v_2, \dots, v_n$ be the vertices of the graph G . The vertices and edges are colored from the color set $\{1, 2, \dots, k\}$ using the function ' f '. Here $f(v_i)$ denotes the color of the vertex v_i and $f(v_i v_j)$ denotes the color of the edge $v_i v_j$.

Theorem 2.1. The graph P_n admits AV DTC and $\chi_{\text{avt}}(P_n) = \begin{cases} 3, & n = 2, 3 \\ 4, & n \geq 4 \end{cases}$

Proof. Let $v_1, v_2, \dots, v_n$ be the vertices of the path P_n . Define $f: V(P_n) \cup E(P_n) \rightarrow \{1, 2, \dots, k\}$ as follows.

It can be easily proved for $n = 2$.

For $n = 3$, $f(v_i) = i$ for $i = 1, 2, 3$ and $f(v_i v_{i+1}) = \begin{cases} 3, & i = 1 \\ 1, & i = 2 \end{cases}$

The color classes are $C(v_1) = C(v_3) = \{3, 1\}$ and $C(v_2) = \{1, 3, 2\}$

Now, we prove for $n \geq 4$. For $1 \leq i \leq n$, $f(v_i) = \begin{cases} 1, & \text{if } i \equiv 1 \pmod{2} \\ 2, & \text{if } i \equiv 0 \pmod{2} \end{cases}$

For $1 \leq i \leq n-1$, $f(v_i v_{i+1}) = \begin{cases} 3, & \text{if } i \equiv 1 \pmod{2} \\ 4, & \text{if } i \equiv 0 \pmod{2} \end{cases}$

Moreover, for $i \equiv 0 \pmod{2}$, $C(v_i) = \{3, 4, 2\}$ and $C(v_{i+1}) = \{4, 3, 1\}$

Both $C(v_1)$ and $C(v_n)$ are two-element set. $C(v_1) = \{3, 1\}$ and $C(v_n) = \begin{cases} \{3, 2\}, & \text{if } n \equiv 0 \pmod{2} \\ \{4, 1\}, & \text{if } n \equiv 1 \pmod{2} \end{cases}$

Clearly, the color set of any two adjacent vertices are different.

$$\therefore \chi_{\text{avt}}(P_n) = \begin{cases} 3, & n = 2, 3 \\ 4, & n \geq 4 \end{cases}$$

Theorem 2.2. The graph C_n admits AVDTTC and $\chi_{\text{avt}}(C_n) = 4$, for $n \geq 4$.

Proof: Let $v_1, v_2, \dots, v_n$ be the vertices of the cycle C_n with n vertices and n edges.

Define $f: V(C_n) \cup E(C_n) \rightarrow \{1, 2, \dots, k\}$ as follows.

Case-1. If $n \equiv 1 \pmod{2}$.

For $1 \leq i \leq n-1$, $f(v_i) = \begin{cases} 1, & \text{for } i \equiv 1 \pmod{2} \\ 2, & \text{for } i \equiv 0 \pmod{2} \end{cases}$.

For $1 \leq i \leq n-2$, $f(v_i v_{i+1}) = \begin{cases} 3, & \text{for } i \equiv 1 \pmod{2} \\ 4, & \text{for } i \equiv 0 \pmod{2} \end{cases}$ and $f(v_n) = 4$, $f(v_n v_1) = 2$, $f(v_{n-1} v_n) = 1$.

The color classes are $C(v_1) = C(v_{n-1}) = \{2, 3, 1\}$, $C(v_n) = \{1, 2, 4\}$.

For $2 \leq i \leq n-2$, $C(v_i) = \begin{cases} \{4, 3, 1\}, & \text{for } i \equiv 1 \pmod{2} \\ \{3, 4, 2\}, & \text{for } i \equiv 0 \pmod{2} \end{cases}$

Case-2. If $n \equiv 0 \pmod{2}$.

For $1 \leq i \leq n$, $f(v_i) = \begin{cases} 1, & \text{for } i \equiv 1 \pmod{2} \\ 2, & \text{for } i \equiv 0 \pmod{2} \end{cases}$

For $1 \leq i \leq n-1$, $f(v_i v_{i+1}) = \begin{cases} 3, & \text{for } i \equiv 1 \pmod{2} \\ 4, & \text{for } i \equiv 0 \pmod{2} \end{cases}$ and $f(v_n v_1) = 4$.

The color classes are $C(v_i) = \begin{cases} \{4, 3, 1\}, & \text{if } i \equiv 1 \pmod{2} \\ \{3, 4, 2\}, & \text{if } i \equiv 0 \pmod{2} \end{cases}$.

Therefore $\chi_{\text{avt}}(C_n) = 4$, for $n \geq 4$.

Theorem 2.3. The graph K_n admits AVDTTC and $\chi_{\text{avt}}(K_n) = \begin{cases} n+1, & \text{if } n \equiv 0 \pmod{2} \\ n+2, & \text{if } n \equiv 1 \pmod{2} \end{cases}$.

Proof: Let $v_1, v_2, \dots, v_n$ be the vertices of K_n with n vertices and $\frac{n(n-1)}{2}$ edges.

$$V(K_n) = \{v_1, v_2, \dots, v_n\}, \quad n \geq 3 \quad \text{and} \quad E(K_n) = \{v_i v_j \mid 1 \leq i \leq n \text{ and } i \neq j\}.$$

Case-1. If $n \equiv 0 \pmod{2}$

$$f(v_i) = i, \quad \text{for } 1 \leq i \leq n$$

$$f(v_i v_j) = i+j, \quad \text{for } i+j = n+1$$

For $i+j \neq n+1$,

$$f(v_i v_j) = \begin{cases} \frac{i+j}{2}, & \text{for } i+j \equiv 0 \pmod{2} \\ \left\lceil \frac{n+2}{2} \right\rceil (i+j) \pmod{n+1}, & \text{for } i+j \equiv 1 \pmod{2} \end{cases}.$$

The color classes are for $i = 1, 2, \dots, \frac{n}{2}$, $\bar{C}(v_{2i}) = \{i\}$, $\bar{C}(v_{2i-1}) = \left\{\frac{n}{2} + i\right\}$.

Case-2. If $n \equiv 1 \pmod{2}$

$$f(v_i) = i, \quad \text{for } 1 \leq i \leq n$$

$$f(v_i v_j) = i+j, \quad \text{for } i+j = n+2$$

For $i+j \neq n+2$,

$$f(v_i v_j) = \begin{cases} \frac{i+j}{2}, & \text{for } i+j \equiv 0 \pmod{2} \\ \left\lceil \frac{n+2}{2} \right\rceil (i+j) \pmod{n+2}, & \text{for } i+j \equiv 1 \pmod{2} \end{cases}.$$

The color classes are

$$\bar{C}(v_{2i}) = \left\{i, \left\lceil \frac{n}{2} \right\rceil + i\right\}, \quad \text{for } 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor; \quad \bar{C}(v_{2i-1}) = \left\{\left\lceil \frac{n}{2} \right\rceil + i, i-1\right\}, \quad \text{for } 2 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor; \quad \bar{C}(v_1) = \left\{\left\lceil \frac{n}{2} \right\rceil + 1, n+2\right\}.$$

Clearly, the color set of any two adjacent vertices are different.

$$\therefore \chi_{\text{avt}}(K_n) = \begin{cases} n+1, & \text{if } n \equiv 0 \pmod{2} \\ n+2, & \text{if } n \equiv 1 \pmod{2} \end{cases}.$$

III. AV DTC of $G_1 \otimes G_2$

Next we present the AV DTC of tensor product of two graphs G_1 and G_2 . Let $\{v_1, v_2, \dots, v_m\}$ be the vertices of G_1 and $\{w_1, w_2, \dots, w_n\}$ be the vertices of G_2 . By the definition of tensor product, we obtain a graph with vertex set $\{u_{i,j} \text{ for } i = 1, 2, \dots, m \text{ and } j = 1, 2, \dots, n\}$.

Now we start by investigating a product of two path P_m and P_n , $m, n \in \mathbb{N}$

Theorem 3.1. The tensor product P_m and P_n admits AV DTC and

$$\chi_{\text{avt}}(P_m \otimes P_n) = \begin{cases} 3, & \text{for } m = 2 \text{ and } n = 2, 3 \\ 4, & \text{for } m = 2 \text{ and } n \geq 4 \\ 5, & \text{for } m = 3 \text{ and } n \geq 3 \\ 6, & \text{for } m \geq 4 \text{ and } n \geq 4 \end{cases}$$

Proof. Let the vertex set of P_m and P_n are $\{v_1, v_2, \dots, v_m\}$ and $\{w_1, w_2, \dots, w_n\}$ respectively. The vertices and edges are colored by defining $f: V(P_m \otimes P_n) \cup E(P_m \otimes P_n) \rightarrow \{1, 2, \dots, k\}$. The vertex and edge set of $P_m \otimes P_n$ is given by

$$V(P_m \otimes P_n) = \{u_{i,j} \mid i = 1, 2, \dots, m, j = 1, 2, \dots, n\}$$

$$E(P_m \otimes P_n) = \left\{ \bigcup_{i=1}^{m-1} (u_{i,j} u_{i+1,j}) \cup \left(\bigcup_{i=1}^{m-1} (u_{i+1,j} u_{i,j+1}) \right), j = 1, 2, \dots, (n-1) \right\},$$

Clearly, $P_m \otimes P_n$ has mn vertices and $2(m-1)(n-1)$ edges. First, we prove for $m = 2$, the graph $G = P_2 \otimes P_n$ has two components each of them is path of length $n-1$.

For $m = 2$ and $n = 2$. $f(u_{1,j}) = 1; f(u_{2,j}) = 2; f(u_{2,1} u_{1,2}) = f(u_{1,1} u_{2,2}) = 3, \text{ for } j = 1, 2.$

The color classes of the graph $P_2 \otimes P_2$ is $C(u_{1,1}) = C(u_{1,2}) = \{3, 1\}$ and $C(u_{2,1}) = C(u_{2,2}) = \{3, 2\}$.

For $n = 3$, $f(u_{2,j}) = f(u_{1,j}) = j, \text{ for } j = 1, 2, 3,$

$$f(u_{2,1} u_{1,2}) = 2; f(u_{1,1} u_{2,2}) = 3, f(u_{2,2} u_{1,3}) = f(u_{1,2} u_{2,3}) = 1.$$

The color classes of the graph $P_2 \otimes P_3$ for $i = 1, 2$, $C(u_{i,1}) = C(u_{i,3}) = \{1, 3\}$ and $C(u_{i,2}) = \{1, 2, 3\}$

Next, for $n \geq 4$, $f(u_{i,j}) = 1; \text{ for } i = 1, 2 \text{ and } 1 \leq j \leq n.$

For $i = 1$ and $j = 1, 2, 3, \dots, (n-1)$, $f(u_{i,j} u_{i+1,j+1}) = i+2; f(u_{i+1,j} u_{i,j+1}) = i+3.$

The color classes are $C(u_{1,1}) = \{3, 1\}, C(u_{1,m}) = \{4, 1\}, C(u_{2,1}) = \{4, 2\}, C(u_{2,m}) = \{3, 2\},$

$$C(u_{1,j}) = \{3, 4, 1\}, C(u_{2,j}) = \{3, 4, 2\}, \text{ for } j = 2, 3, \dots, (n-1)$$

$$\text{Then } \chi_{\text{avt}}(P_2 \otimes P_n) = \begin{cases} 3, & n = 2, 3 \\ 4, & n \geq 4 \end{cases}$$

Next, we discuss for $m = 3$ and $n \geq 3$.

Consider the graph $G = P_3 \otimes P_n$ with $3n$ vertices and $4(n-1)$ edges.

$$f(u_{i,j}) = i; \text{ for } i = 1, 2, 3 \text{ and } j = 1, 2, 3, \dots, n$$

For $i = 1, 2$ and $1 \leq j \leq (n-1)$,

$$f(u_{i,j} u_{i+1,j+1}) = i + 2 \quad \text{and} \quad f(u_{i+1,j} u_{i,j+1}) = \begin{cases} 5, & \text{if } (i + 5j - 1) \equiv 0 \pmod{5} \\ (i + 5j - 1) \pmod{5}, & \text{otherwise} \end{cases}$$

The color classes of $P_3 \otimes P_n$ are $C(u_{1,1}) = \{3, 1\}$, $C(u_{1,n}) = \{5, 1\}$

$$C(u_{2,1}) = \{4, 5, 2\}, \quad C(u_{2,n}) = \{1, 3, 2\}; \quad C(u_{3,1}) = \{1, 3\}, \quad C(u_{3,n}) = \{4, 3\}$$

For $2 \leq j \leq (n-1)$, $C(u_{1,j}) = \{3, 5, 1\}$, $C(u_{2,j}) = \{1, 3, 4, 5, 2\}$; $C(u_{3,j}) = \{1, 4, 3\}$.

$$\chi_{\text{avt}}(P_3 \otimes P_n) = 5, \text{ for } n \geq 3.$$

Now, we consider the general graph $P_m \otimes P_n$ for $m \geq 4$ and $n \geq 4$.

$$\text{For } 1 \leq i \leq m \text{ and } 1 \leq j \leq n; \quad f(u_{i,j}) = \begin{cases} 1, & \text{if } i \equiv 1 \pmod{2} \\ 2, & \text{if } i \equiv 0 \pmod{2} \end{cases}$$

For $1 \leq i \leq m-1$ and $1 \leq j \leq n-1$,

$$f(u_{i,j} u_{i+1,j+1}) = \begin{cases} 3, & \text{if } i \equiv 1 \pmod{2} \\ 4, & \text{if } i \equiv 0 \pmod{2} \end{cases} \quad \text{and} \quad f(u_{i+1,j} u_{i,j+1}) = \begin{cases} 5, & \text{if } i \equiv 1 \pmod{2} \\ 6, & \text{if } i \equiv 0 \pmod{2} \end{cases}$$

The color classes are $C(u_{1,1}) = \{3, 1\}$, $C(u_{1,n}) = \{5, 1\}$, $C(u_{1,j}) = \{3, 5, 1\}$ for $j = 2, 3, \dots, n-1$.

For $2 \leq i \leq m-1$ and $2 \leq j \leq n-1$,

$$C(u_{i,j}) = \begin{cases} \{3, 4, 5, 6, 2\}, & \text{for } i \equiv 0 \pmod{2} \\ \{3, 4, 5, 6, 1\}, & \text{for } i \equiv 1 \pmod{2} \end{cases} \quad \text{and} \quad C(u_{i,1}) = \begin{cases} \{5, 4, 2\}, & \text{for } i \equiv 0 \pmod{2} \\ \{3, 6, 1\}, & \text{for } i \equiv 1 \pmod{2} \end{cases}$$

$$C(u_{i,n}) = \begin{cases} \{3, 6, 2\}, & \text{if } i \equiv 0 \pmod{2} \\ \{4, 5, 1\}, & \text{if } i \equiv 1 \pmod{2} \end{cases} \quad \text{and} \quad C(u_{m,j}) = \begin{cases} \{3, 5, 2\}, & \text{if } m \text{ is even} \\ \{4, 6, 1\}, & \text{if } m \text{ is odd} \end{cases}$$

$$C(u_{m,1}) = \begin{cases} \{6, 1\}, & \text{if } m \text{ is odd} \\ \{5, 2\}, & \text{if } m \text{ is even} \end{cases} \quad \text{and} \quad C(u_{m,n}) = \begin{cases} \{3, 2\}, & \text{m and n even} \\ \{4, 1\}, & \text{m and n odd} \end{cases}$$

Therefore the colour classes of any two adjacent vertices are different.

$$\chi_{\text{avt}}(P_m \otimes P_n) = \begin{cases} 3, & \text{for } m = 2 \text{ and } n = 2, 3 \\ 4, & \text{for } m = 2 \text{ and } n \geq 4 \\ 5, & \text{for } m = 3 \text{ and } n \geq 3 \\ 6, & \text{for } m \geq 4 \text{ and } n \geq 4 \end{cases}$$

Hence the theorem. ■

Theorem 3.2. The tensor product $P_m \otimes C_n$ admits AV DTC and

$$\chi_{\text{avt}}(P_m \otimes C_n) = \begin{cases} 4, & \text{for } m = 2 \text{ and } n \geq 3 \\ 5, & \text{for } m = 3 \text{ and } n \geq 3 \\ 6, & \text{for } m \geq 4 \text{ and } n \geq 4 \end{cases}$$

Proof. Let the vertex sets of P_m and C_n are $\{v_1, v_2, \dots, v_m\}$ and $\{w_1, w_2, \dots, w_n\}$ respectively. The vertex and edge set of the graph $P_m \otimes C_n$ is given by $V(P_m \otimes C_n) = \{u_{i,j} \mid i = 1, 2, \dots, m, j = 1, 2, \dots, n\}$

$$E(P_m \otimes C_n) = \left\{ \begin{aligned} & \bigcup_{i=1}^{m-1} (u_{i,j} u_{i+1,j+1}) \cup \left(\bigcup_{i=1}^{m-1} (u_{i+1,j} u_{i,j+1}) \right), j = 1, 2, \dots, (n-1) \\ & \bigcup_{i=1}^{m-1} (u_{i,j} u_{i+1,j-n+1}) \cup \left(\bigcup_{i=1}^{m-1} (u_{i+1,j} u_{i,j-n+1}) \right), j = n \end{aligned} \right\},$$

Clearly, $(P_m \otimes C_n)$ has mn vertices and $2(m-1)(n)$ edges. First, we prove for $m = 2$ and $n \geq 3$. The graph

$G = (P_2 \otimes C_n)$ has $2n$ vertices and $2n$ edges. $\therefore G = P_2 \otimes C_n = \begin{cases} C_{2n} & \text{if } n \equiv 1 \pmod{2} \\ 2C_n & \text{if } n \equiv 0 \pmod{2} \end{cases}$. The vertices and

edges are colored by defining $f: V(P_m \otimes C_n) \cup E(P_m \otimes C_n) \rightarrow \{1, 2, \dots, k\}$.

$$f(u_{i,j}) = i; \text{ for } i = 1, 2 \text{ and } j = 1, 2, 3, \dots, n.$$

For $i = 1$ and $1 \leq j \leq (n-1)$, $f(u_{i,j} u_{i+1,j+1}) = i + 2$; $f(u_{i+1,j} u_{i,j+1}) = i + 3$.

For $i = 1, j = n$, $f(u_{i+1,j-n+1} u_{i,j}) = i + 2$; $f(u_{i,j-n+1} u_{i+1,j}) = i + 3$.

The color classes are $C(u_{1,j}) = \{3, 4, 1\}$, $C(u_{2,j}) = \{3, 4, 2\}$, for $j = 1, 2, \dots, n$.

Then $\chi_{\text{avt}}(P_2 \otimes C_n) = 4, n \geq 3$.

Next, we discuss the case $m = 3$ and $n \geq 3$. $\therefore$ The graph $P_3 \otimes C_n$ has $3n$ vertices and $4n$ edges.

$$f(u_{i,j}) = i; \text{ for } i = 1, 2, 3 \text{ and } j = 1, 2, 3, \dots, n$$

For $i = 1, 2$ and $1 \leq j \leq (n-1)$,

$$f(u_{i,j} u_{i+1,j+1}) = i + 2; \text{ and } f(u_{i+1,j} u_{i,j+1}) = \begin{cases} 5, & \text{if } (i + 5j - 1) \equiv 0 \pmod{5} \\ (i + 5j - 1) \pmod{5}, & \text{otherwise} \end{cases}$$

$$\text{For } i = 1, 2 \text{ and } j = n, f(u_{i,j} u_{i+1,j-n+1}) = i + 2 \text{ and } f(u_{i+1,j} u_{i,j-n+1}) = \begin{cases} 5, & \text{if } i = 1 \\ 1, & \text{if } i = 2 \end{cases}$$

For $1 \leq j \leq n$, the color classes are $C(u_{1,j}) = \{3, 5, 1\}$, $C(u_{2,j}) = \{1, 3, 4, 5, 2\}$; $C(u_{3,j}) = \{1, 4, 3\}$.

$$\chi_{\text{avt}}(P_3 \otimes C_n) = 5; \quad n \geq 3.$$

Next, we prove for the general graph $P_m \otimes C_n$ for $m \geq 4$ and $n \geq 4$.

$$|V(P_m \otimes C_n)| = mn \text{ and } |E(P_m \otimes C_n)| = 2(m-1)n.$$

For $1 \leq i \leq m$ and $1 \leq j \leq n$,

$$f(u_{i,j}) = \begin{cases} 1, & \text{for } i \equiv 1 \pmod{2} \\ 2, & \text{for } i \equiv 0 \pmod{2} \end{cases}$$

For $1 \leq i \leq (m-1)$ and $1 \leq j \leq (n-1)$,

$$f(u_{i,j} u_{i+1,j+1}) = \begin{cases} 3, & \text{if } i \equiv 1 \pmod{2} \\ 4, & \text{if } i \equiv 0 \pmod{2} \end{cases} \text{ and } f(u_{i+1,j} u_{i,j+1}) = \begin{cases} 5, & \text{if } i \equiv 1 \pmod{2} \\ 6, & \text{if } i \equiv 0 \pmod{2} \end{cases}$$

For $1 \leq i \leq (m-1)$ and $j = n$,

$$f(u_{i+1,j} u_{i,j-n+1}) = \begin{cases} 5, & \text{if } i \equiv 1 \pmod{2} \\ 6, & \text{if } i \equiv 0 \pmod{2} \end{cases} \text{ and } f(u_{i+1,j-n+1} u_{i,j}) = \begin{cases} 3, & \text{if } i \equiv 1 \pmod{2} \\ 4, & \text{if } i \equiv 0 \pmod{2} \end{cases}$$

The color classes are $C(u_{1,j}) = \{3, 5, 1\}$ for $j = 1, 2, \dots, n$.

For $2 \leq i \leq (m-1)$ and $1 \leq j \leq n$,

$$C(u_{i,j}) = \begin{cases} \{3, 4, 5, 6, 2\}, & \text{if } i \equiv 0 \pmod{2} \\ \{3, 4, 5, 6, 1\}, & \text{if } i \equiv 1 \pmod{2} \end{cases} \text{ and } C(u_{m,j}) = \begin{cases} \{4, 6, 1\}, & \text{if } m \text{ is odd} \\ \{3, 5, 2\}, & \text{if } m \text{ is even} \end{cases}$$

Here, the color classes of any two adjacent vertices are different.

Thus

$$\chi_{\text{avt}}(P_m \otimes C_n) = \begin{cases} 4, & \text{for } m = 2 \text{ and } n \geq 3 \\ 5, & \text{for } m = 3 \text{ and } n \geq 3 \\ 6, & \text{for } m \geq 4 \text{ and } n \geq 4 \end{cases}. \text{ Hence the theorem.}$$

IV. AVDTTC of $G_1 \otimes_2 G_2$

In this section, we discuss a detailed procedure for 2 - tensor product of some graphs and their AV DTC.

The 2-tensor product of two graphs $G_1(V_1; E_1)$ and $G_2(V_2; E_2)$ is denoted by $G_1 \otimes_2 G_2$. Let $G_1 = P_m$ and $G_2 = P_n$

be the two path graphs with m and n vertices respectively. The graph $P_m \otimes_2 P_n$ is a null graph (isolated vertices) if $\Delta(G_1) < 2$ or $\Delta(G_2) < 2$. Consider the vertex set $V(P_m) = \{v_1, v_2, \dots, v_m\}$ and $V(P_n) = \{w_1, w_2, \dots, w_n\}$ respectively. Since $P_2 \otimes_2 P_n$ is a null graph, we consider $m, n \geq 3$.

Theorem 4.1. The graph $P_m \otimes_2 P_n$ admits AVDTTC and

$$\chi_{\text{avt}}(P_m \otimes_2 P_n) = \begin{cases} 3, & \text{for } m = 3, 4 \text{ and } n = 3, 4, 5, 6 \\ 4, & \text{for } m = 3, 4 \text{ and } n \geq 7 \\ 5, & \text{for } m = 5, 6 \text{ and } n \geq 5 \\ 6, & \text{for } m \geq 7 \text{ and } n \geq 7 \end{cases}.$$

Proof: Let the vertex sets of P_m and P_n are $\{v_1, v_2, \dots, v_m\}$ and $\{w_1, w_2, \dots, w_n\}$ respectively. The vertex and edge set of $P_m \otimes_2 P_n$ is given by $V(P_m \otimes_2 P_n) = \{u_{i,j} \mid i = 1, 2, 3, \dots, m \text{ and } j = 1, 2, 3, \dots, n\}$

$$E(P_m \otimes_2 P_n) = \left\{ \left(\bigcup_{i=1}^{m-2} u_{i,j} u_{i+2,j+2} \right) \cup \left(\bigcup_{i=1}^{m-2} u_{i+2,j} u_{i,j+2} \right), j = 1, 2, 3, \dots, (n-2) \right\}$$

Then graph $P_m \otimes_2 P_n$ has mn vertices and $2(n-2)(m-2)$ edges. First, we prove for $m=3, 4$ and $n=3, 4$.

If $n \equiv 1 \pmod{2}$, the graph $P_3 \otimes_2 P_n$ has two path graphs each of length $\frac{n+1}{2}$ and $\left\lfloor \frac{n}{2} \right\rfloor$ with D_n vertices. These

D_n vertices are $(u_{2,j})$, $j=1,2,3,\dots,n$. If $n \equiv 0 \pmod{2}$, the graph $P_3 \otimes_2 P_n$ has four path graphs each of length $\frac{n}{2}$ with D_n vertices. The graph $P_4 \otimes_2 P_n$ has four path each of length $\frac{n+1}{2}$ and $\frac{n-1}{2}$, for $n \equiv 1 \pmod{2}$ and it has

eight path graphs each of length $\frac{n}{2}$, for $n \equiv 0 \pmod{2}$. The vertices and edges are colored by defining

$f: V(P_m \otimes_2 P_n) \cup E(P_m \otimes_2 P_n) \rightarrow \{1,2,\dots,k\}$. We discuss the case for $m=3,4$, $n=3,4$ and $m \geq n$.

For $1 \leq j \leq n$,
$$f(u_{i,j}) = \begin{cases} 1, & \text{for } i=1,2 \\ 2, & \text{otherwise} \end{cases}$$

For $1 \leq i \leq m-2$ and $1 \leq j \leq n-2$, $f(u_{i,j}u_{i+2,j+2}) = f(u_{i+2,j}u_{i,j+2}) = 3$.

Next, For the case $n=5,6$. We have for $1 \leq i \leq m$ and $1 \leq j \leq n$,
$$f(u_{i,j}) = \begin{cases} 1, & \text{for } j=1,2 \\ 2, & \text{for } j=3,4 \\ 3, & \text{for } j=5,6 \end{cases}.$$

For $1 \leq i \leq (m-2)$ and $1 \leq j \leq (n-2)$, $f(u_{i,j}u_{i+2,j+2}) = f(u_{i+2,j}u_{i,j+2}) = \begin{cases} 3, & \text{for } j=1,2 \\ 1, & \text{otherwise} \end{cases}.$

Now, for $n \geq 7$, we have for $1 \leq i \leq m$ and $1 \leq j \leq n$, $f(u_{i,j}) = \begin{cases} 1, & \text{for } i=1,2 \\ 2, & \text{for } i=3,4 \end{cases}$ and

For $1 \leq i \leq (m-2)$ and $1 \leq j \leq (n-2)$, $f(u_{i,j}u_{i+2,j+2}) = 3$, and $f(u_{i+2,j}u_{i,j+2}) = 4$.

Next, we discuss the case for $m=5,6$ and $n \geq 5$. For $1 \leq i \leq m$ and $1 \leq j \leq n$, $f(u_{i,j}) = \begin{cases} 1, & \text{for } i=1,2 \\ 2, & \text{for } i=3,4 \\ 3, & \text{otherwise} \end{cases}$ and

For $1 \leq i \leq (m-2)$ and $1 \leq j \leq (n-2)$, we have

$$f(u_{i,j}u_{i+2,j+2}) = \begin{cases} 3, & \text{for } i=1,2 \\ 4, & \text{otherwise} \end{cases} \text{ and } f(u_{i+2,j}u_{i,j+2}) = \begin{cases} 5, & \text{for } i=1,2 \\ 1, & \text{otherwise} \end{cases}.$$

Now, we discuss the color classes of $P_3 \otimes_2 P_n$, for $n \geq 3$.

The color classes of $P_3 \otimes_2 P_3$, for $j=1,3$, $C(u_{1,j})=\{1,3\}$, $C(u_{3,j})=\{2,3\}$, $C(u_{1,2})=\{1\}$, $C(u_{3,2})=\{2\}$ and $C(u_{2,j})=\{1\}$, for $j=1,2,3$.

The color classes of the graph $P_3 \otimes_2 P_4$, for $j=1,2,3,4$, we have
$$C(u_{i,j}) = \begin{cases} \{3,1\}, & \text{for } i=1 \\ \{1\}, & \text{for } i=2 \\ \{2,3\}, & \text{for } i=3 \end{cases}$$

The color classes of the graph $P_3 \otimes_2 P_5$, for $i=1,3$, we have

$$C(u_{i,j}) = \begin{cases} \{3,1\}, & \text{for } j=1,2,5 \\ \{1,2,3\}, & \text{for } j=3 \\ \{2,3\}, & \text{for } j=4 \end{cases} \text{ and } C(u_{2,j}) = \begin{cases} \{1\}, & \text{for } j=1,2 \\ \{2\}, & \text{for } j=3,4 \\ \{3\}, & \text{for } j=5 \end{cases}$$

The color classes of the graph $P_3 \otimes_2 P_6$, for $i=1,3$, we have

$$C(u_{i,j}) = \begin{cases} \{1,3\}, & \text{for } j=1,2,5,6 \\ \{1,2,3\}, & \text{for } j=3,4 \end{cases} \text{ and } C(u_{2,j}) = \begin{cases} \{1\}, & \text{for } j=1,2 \\ \{2\}, & \text{for } j=3,4 \\ \{3\}, & \text{for } j=5,6 \end{cases}$$

The color classes of the graph $P_3 \otimes_2 P_n$, $n \geq 7$ are $C(u_{2,j})=\{1\}$,

$$C(u_{1,j}) = \begin{cases} \{1,3\}, & \text{for } j=1,2 \\ \{1,3,4\}, & \text{for } j=3,4,\dots,n-2 \\ \{1,4\}, & \text{for } j=n-1,n \end{cases} \text{ and } C(u_{3,j}) = \begin{cases} \{2,4\}, & \text{for } j=1,2 \\ \{2,3,4\}, & \text{for } j=3,4,\dots,n-2 \\ \{2,3\}, & \text{for } j=n-1,n \end{cases}.$$

Next, for $m=4$ and $n=4$. For $1 \leq j \leq n$, the color classes of the graph $P_4 \otimes_2 P_4$ are

$$C(u_{i,j}) = \begin{cases} \{3,1\}, & \text{for } i = 1,2 \\ \{2,3\}, & \text{for } i = 3,4 \end{cases}.$$

The color classes of the graph $P_4 \otimes_2 P_5$, for $i=1,2,3,4$,

$$C(u_{i,j}) = \begin{cases} \{3,1\}, & \text{for } j = 1,2,5 \\ \{1,2,3\}, & \text{for } j = 3 \\ \{2,3\}, & \text{for } j = 4 \end{cases}.$$

The color classes of the graph $P_4 \otimes_2 P_6$ for $i=1,2,3,4$, $C(u_{i,j}) = \begin{cases} \{3,1\}, & \text{for } j = 1,2,5,6 \\ \{1,2,3\}, & \text{for } j = 3,4 \end{cases}.$

Now, we discuss the color classes of $P_4 \otimes_2 P_n$, for $n \geq 7$. For $i=1,2$, $C(u_{i,j}) = \begin{cases} \{3,1\}, & \text{for } j = 1,2 \\ \{1,3,4\}, & \text{for } j = 3,4,\dots,n-2 \\ \{1,4\}, & \text{for } j = n-1, n \end{cases}$

and for $i=3,4$, $C(u_{i,j}) = \begin{cases} \{2,4\}, & \text{for } j = 1,2 \\ \{2,3,4\}, & \text{for } j = 3,4,\dots,n-2 \\ \{2,3\}, & \text{for } j = n-1, n \end{cases}.$

The color classes of the graph $P_m \otimes_2 P_n$, for $m=5$ and $n \geq 5$, we have

$$C(u_{1,j}) = \begin{cases} \{1,3\}, & \text{for } j = 1,2 \\ \{1,3,5\}, & \text{for } j = 3,4,\dots,n-2 \\ \{3,4\}, & \text{for } j = n-1, n \end{cases}, \quad C(u_{3,j}) = \begin{cases} \{2,4,5\}, & \text{for } j = 1,2 \\ \{1,2,3,4,5\}, & \text{for } j = 3,4,\dots,n-2 \\ \{1,2,3\}, & \text{for } j = n-1, n \end{cases}$$

$$C(u_{5,j}) = \begin{cases} \{1,3\}, & \text{for } j = 1,2 \\ \{1,3,4\}, & \text{for } j = 3,4,\dots,n-2 \\ \{3,4\}, & \text{for } j = n-1, n \end{cases}, \quad C(u_{2,j}) = \begin{cases} \{3,1\}, & \text{for } j = 1,2 \\ \{1,3,4\}, & \text{for } j = 3,4,\dots,n-2 \\ \{1,4\}, & \text{for } j = n-1, n \end{cases}$$

$$C(u_{4,j}) = \begin{cases} \{2,4\}, & \text{for } j = 1,2 \\ \{2,3,4\}, & \text{for } j = 3,4,\dots,n-2 \\ \{2,3\}, & \text{for } j = n-1, n \end{cases}.$$

The color classes of $P_m \otimes_2 P_n$, for $m = 6$ and $n \geq 5$, we have

For $i=1,2$, $C(u_{i,j}) = \begin{cases} \{1,3\}, & \text{for } j = 1,2 \\ \{1,3,5\}, & \text{for } j = 3,4,\dots,n-2 \\ \{1,5\}, & \text{for } j = n-1, n \end{cases};$ For $i=3,4$, $C(u_{i,j}) = \begin{cases} \{2,4,5\}, & \text{for } j = 1,2 \\ \{1,2,3,4,5\}, & \text{for } j = 3,4,\dots,n-2 \\ \{1,2,3\}, & \text{for } j = n-1, n \end{cases}$

For $i=5,6$, $C(u_{i,j}) = \begin{cases} \{1,3\}, & \text{for } j = 1,2 \\ \{1,3,4\}, & \text{for } j = 3,4,\dots,n-2 \\ \{3,4\}, & \text{for } j = n-1, n \end{cases}.$

Now, for the general graph $P_m \otimes_2 P_n$, $m \geq 7$, $n \geq 7$

For $1 \leq i \leq m$ and $1 \leq j \leq n$, $f(u_{i,j}) = \begin{cases} 1, & \text{for } i \equiv 1,2 \pmod{4} \\ 2, & \text{for } i \equiv 0,3 \pmod{4} \end{cases}.$

For $1 \leq i \leq (m-2)$ and $1 \leq j \leq (n-2)$,

$$f(u_{i,j}u_{i+2,j+2}) = \begin{cases} 3, & \text{for } i \equiv 1,2 \pmod{4} \\ 4, & \text{for } i \equiv 0,3 \pmod{4} \end{cases} \text{ and } f(u_{i+2,j}u_{i,j+2}) = \begin{cases} 5, & \text{for } i \equiv 1,2 \pmod{4} \\ 6, & \text{for } i \equiv 0,3 \pmod{4} \end{cases}$$

Now, we discuss the color classes for $i=1,2$ of the graph $P_m \otimes_2 P_n$, for $m \geq 7$, $n \geq 7$.

$$C(u_{i,j}) = \begin{cases} \{1,3\}, & \text{for } j = 1,2 \\ \{1,3,5\}, & \text{for } j = 3,4,\dots,n-2 \\ \{1,5\}, & \text{for } j = n-1, n \end{cases}.$$

$$\text{For } 3 \leq i \leq m-2, \quad C(u_{i,j}) = \begin{cases} \left\{ \begin{array}{l} \{1,3,6\}, \text{ for } j=1,2 \\ \{1,3,4,5,6\}, \text{ for } 3 \leq j \leq n-2 \\ \{1,4,5\}, \text{ for } j=n-1, n \end{array} \right\} & \text{if } i \equiv 1,2 \pmod{4} \\ \left\{ \begin{array}{l} \{2,4,5\}, \text{ for } j=1,2 \\ \{2,3,4,5,6\}, \text{ for } 3 \leq j \leq n-2 \\ \{2,3,6\}, \text{ for } j=n-1, n \end{array} \right\} & \text{if } i \equiv 0,3 \pmod{4} \end{cases}$$

$$\text{If } m \equiv 2 \pmod{4}, \text{ then } C(u_{m-1,j}) = C(u_{m,j}) = \begin{cases} \{1,6\}, \text{ for } j=1,2 \\ \{1,4,6\}, \text{ for } 3 \leq j \leq n-2 \\ \{1,4\}, \text{ for } j=n-1, n \end{cases}$$

$$\text{If } m \equiv 0 \pmod{4}, \text{ then } C(u_{m-1,j}) = C(u_{m,j}) = \begin{cases} \{2,5\}, \text{ for } j=1,2 \\ \{2,3,5\}, \text{ for } 3 \leq j \leq n-2 \\ \{2,3\}, \text{ for } j=n-1, n \end{cases}$$

If $m \equiv 1 \pmod{4}$, then

$$C(u_{m,j}) = \begin{cases} \{1,6\}, \text{ for } j=1,2 \\ \{1,4,6\}, \text{ for } 3 \leq j \leq n-2 \\ \{1,4\}, \text{ for } j=n-1, n \end{cases} \text{ and } C(u_{m-1,j}) = \begin{cases} \{2,5\}, \text{ for } j=1,2 \\ \{2,3,5\}, \text{ for } 3 \leq j \leq n-2 \\ \{2,3\}, \text{ for } j=n-1, n \end{cases}$$

If $m \equiv 3 \pmod{4}$, then

$$C(u_{m-1,j}) = \begin{cases} \{1,6\}, \text{ for } j=1,2 \\ \{1,4,6\}, \text{ for } 3 \leq j \leq n-2 \\ \{1,4\}, \text{ for } j=n-1, n \end{cases} \text{ and } C(u_{m,j}) = \begin{cases} \{2,5\}, \text{ for } j=1,2 \\ \{2,3,5\}, \text{ for } 3 \leq j \leq n-2 \\ \{2,3\}, \text{ for } j=n-1, n \end{cases}$$

Clearly, the color classes of any two adjacent vertices are different. Continuing the process, we have

$$\chi_{\text{avt}}(P_m \otimes_2 C_n) = \begin{cases} 3, & \text{for } m=3,4 \text{ and } n=3,4,5,6 \\ 4, & \text{for } m=3,4 \text{ and } n \geq 7 \\ 5, & \text{for } m=5,6 \text{ and } n \geq 5 \\ 6, & \text{for } m \geq 7 \text{ and } n \geq 7 \end{cases}. \text{ Hence the theorem.}$$

Theorem 4.2. The graph $P_m \otimes_2 C_n$ admits AVDTTC and $\chi_{\text{avt}}(P_m \otimes_2 C_n) = \begin{cases} 3 & \text{for } m=3,4 \text{ and } n=4 \\ 4 & \text{for } m=3,4 \text{ and } n > 4 \\ 5 & \text{for } m=5,6 \text{ and } n \geq 5 \\ 6 & \text{for } m \geq 7 \text{ and } n \geq 7 \end{cases}$

Proof: The graph $P_m \otimes_2 C_n$ has the vertex $V(P_m \otimes_2 C_n) = \{u_{i,j} \mid i=1,2,3,\dots,m \text{ and } j=1,2,3,\dots,n\}$

$$\text{and the edge set } E(P_m \otimes_2 C_n) = \left\{ \begin{array}{l} \left(\bigcup_{i=1}^{m-2} u_{i,j} u_{i+2,j+2} \right) \cup \left(\bigcup_{i=1}^{m-2} u_{i+2,j} u_{i,j+2} \right) \text{ for } j=1,2,\dots,(n-2) \\ \cup \left(\bigcup_{i=1}^{m-2} u_{i,j-n+2} u_{i+2,j} \right) \cup \left(\bigcup_{i=1}^{m-2} u_{i+2,j-n+2} u_{i,j} \right) \text{ for } j=n-1, n \end{array} \right\}$$

$\therefore P_m \otimes_2 C_n$ has mn vertices and $2n(m-2)$ edges. The vertices and edges are colored by defining $f: V(P_m \otimes_2 C_n) \cup E(P_m \otimes_2 C_n) \rightarrow \{1,2,\dots,k\}$. First, for $m=3,4$ and $n \geq 4$.

$$\text{For } 1 \leq i \leq m \text{ and } 1 \leq j \leq n, \quad f(u_{i,j}) = \begin{cases} 1, & \text{for } i=1,2 \\ 2, & \text{otherwise} \end{cases}.$$

$$\text{For } n=4, \text{ for } 1 \leq i \leq (m-2) \text{ and } 1 \leq j \leq (n-2), \quad f(u_{i,j} u_{i+2,j+2}) = f(u_{i+2,j} u_{i,j+2}) = 3.$$

Next, for $m=3,4$ and $n > 4$, The graph $P_m \otimes_2 C_n$ has D_n vertices with four cycles each of length $\frac{n}{2}$, when $n \equiv 0 \pmod{4}$ or a cycle of length $2n$, when $n \equiv 1,3 \pmod{4}$ or two cycles each of length n , when $n \equiv 2 \pmod{4}$. The graph $P_m \otimes_2 C_n$ has $4n$ vertices and $4n$ edges and it has all bipartite components with two cycles each of length $2n$ for $n \equiv 1,3 \pmod{4}$, eight cycles each of length $\frac{n}{2}$ for $n \equiv 0 \pmod{4}$ and four cycles each of length n for $n \equiv 2 \pmod{4}$.

$$\text{For } 1 \leq i \leq (m-2) \text{ and } 1 \leq j \leq (n-2), \quad f(u_{i,j} u_{i+2,j+2}) = 3; f(u_{i+2,j} u_{i,j+2}) = 4 \text{ and}$$

$$\text{For } j = (n-1), n, \quad f(u_{i+2, j-n+2} u_{i, j}) = 3; f(u_{i, j-n+2} u_{i+2, j}) = 4$$

$$\text{The color set for } 1 \leq j \leq n \text{ of the graph } P_3 \otimes_2 C_n \text{ for } n > 4, \quad C(u_{i, j}) = \begin{cases} \{1, 3, 4\}, & \text{for } i = 1 \\ \{2, 3, 4\}, & \text{for } i = 3 \\ \{1\}, & \text{for } i = 2 \end{cases}.$$

$$\text{The color set for } 1 \leq j \leq n \text{ of the graph } P_4 \otimes_2 C_n, \text{ for } n > 4, \quad C(u_{i, j}) = \begin{cases} \{1, 3, 4\}, & \text{for } i = 1, 2 \\ \{2, 3, 4\}, & \text{for } i = 3, 4 \end{cases}.$$

$$\text{Now, we discuss the case for } m=5, 6 \text{ and } n \geq 5. \text{ For } 1 \leq i \leq m \text{ and } 1 \leq j \leq n, \quad f(u_{i, j}) = \begin{cases} 1, & \text{for } i = 1, 2 \\ 2, & \text{for } i = 3, 4 \\ 3, & \text{otherwise} \end{cases}$$

$$\text{For } 1 \leq i \leq (m-2) \text{ and } 1 \leq j \leq (n-2), \quad f(u_{i, j} u_{i+2, j+2}) = \begin{cases} 3, & \text{for } i = 1, 2 \\ 4, & \text{otherwise} \end{cases} \text{ and } f(u_{i+2, j} u_{i, j+2}) = \begin{cases} 5, & \text{for } i = 1, 2 \\ 1, & \text{otherwise} \end{cases}.$$

$$\text{For } j=(n-1), n, \text{ we have } f(u_{i, j-n+2} u_{i+2, j}) = \begin{cases} 5, & \text{for } i = 1, 2 \\ 1, & \text{otherwise} \end{cases} \text{ and } f(u_{i+2, j-n+2} u_{i, j}) = \begin{cases} 3, & \text{for } i = 1, 2 \\ 4, & \text{otherwise} \end{cases}$$

$$\text{The color classes of } P_5 \otimes_2 C_n, \text{ for } n \geq 5, \text{ we have } 1 \leq j \leq n, \quad C(u_{i, j}) = \begin{cases} \{1, 3, 5\}, & \text{for } i = 1 \\ \{1, 4, 3\}, & \text{for } i = 2, 5 \\ \{1, 2, 3, 4, 5\}, & \text{for } i = 3 \\ \{2, 3, 4\}, & \text{for } i = 4 \end{cases}$$

$$\text{The color classes of } P_6 \otimes_2 C_n, \text{ for } n \geq 5, \text{ we have for } 1 \leq j \leq n, \quad C(u_{i, j}) = \begin{cases} \{1, 3, 5\}, & \text{for } i = 1, 2 \\ \{1, 2, 3, 4, 5\}, & \text{for } i = 3, 4 \\ \{1, 4, 3\}, & \text{for } i = 5, 6 \end{cases}$$

Now, we prove the general graph for $m \geq 7$ and $n \geq 7$.

$$\text{For } 1 \leq i \leq m \text{ and } 1 \leq j \leq n, \quad f(u_{i, j}) = \begin{cases} 1, & \text{for } i \equiv 1, 2 \pmod{4} \\ 2, & \text{for } i \equiv 0, 3 \pmod{4} \end{cases}.$$

$$\text{For } 1 \leq i \leq (m-2) \text{ and } 1 \leq j \leq (n-2), \quad f(u_{i, j} u_{i+2, j+2}) = \begin{cases} 5, & \text{for } i \equiv 1, 2 \pmod{4} \\ 6, & \text{for } i \equiv 0, 3 \pmod{4} \end{cases} \text{ and } f(u_{i+2, j} u_{i, j+2}) = \begin{cases} 5, & \text{for } i \equiv 1, 2 \pmod{4} \\ 6, & \text{for } i \equiv 0, 3 \pmod{4} \end{cases}$$

$$\text{For } j=(n-1), n, \quad f(u_{i, j-n+2} u_{i+2, j}) = \begin{cases} 5, & \text{for } i \equiv 1, 2 \pmod{4} \\ 6, & \text{for } i \equiv 0, 3 \pmod{4} \end{cases} \text{ and } f(u_{i+2, j-n+2} u_{i, j}) = \begin{cases} 3, & \text{for } i \equiv 1, 2 \pmod{4} \\ 4, & \text{for } i \equiv 0, 3 \pmod{4} \end{cases}.$$

The color classes are $C(u_{i, j}) = \{1, 3, 5\}$, for $i = 1, 2$ and $j = 1, 2, \dots, n$.

$$\text{For } 3 \leq i \leq m-2, \quad C(u_{i, j}) = \begin{cases} \{1, 3, 4, 5, 6\}, & \text{if } i \equiv 1, 2 \pmod{4} \\ \{2, 3, 4, 5, 6\}, & \text{if } i \equiv 0, 3 \pmod{4} \end{cases}.$$

$$\text{If } m \equiv 1 \pmod{4} \text{ and } 1 \leq j \leq n, \quad C(u_{i, j}) = \begin{cases} \{2, 3, 5\}, & \text{if } i = m-1 \\ \{1, 4, 6\}, & \text{if } i = m \end{cases}$$

$$\text{If } m \equiv 2 \pmod{4} \text{ and } 1 \leq j \leq n, \quad C(u_{m-1, j}) = C(u_{m, j}) = \{1, 4, 6\}.$$

$$\text{If } m \equiv 0 \pmod{4} \text{ and } 1 \leq j \leq n, \quad C(u_{m-1, j}) = C(u_{m, j}) = \{2, 3, 5\}.$$

$$\text{If } m \equiv 3 \pmod{4} \text{ and } 1 \leq j \leq n, \quad C(u_{i, j}) = \begin{cases} \{2, 3, 5\}, & \text{if } i = m \\ \{1, 4, 6\}, & \text{if } i = m-1 \end{cases}.$$

The color classes are different for any two adjacent vertices.

$$\chi_{\text{avt}}(P_m \otimes_2 C_n) = \begin{cases} 3 & \text{for } m = 3, 4 \text{ and } n = 4 \\ 4 & \text{for } m = 3, 4 \text{ and } n > 4 \\ 5 & \text{for } m = 5, 6 \text{ and } n \geq 5 \\ 6 & \text{for } m \geq 7 \text{ and } n \geq 7 \end{cases}. \text{ Hence the theorem.}$$

Conclusion: We found the adjacent vertex distinguishing total chromatic number of tensor product and 2-tensor product of path by path and path by cycle. Also, we are working in the direction of finding the adjacent vertex distinguishing total chromatic number of 2-tensor product of path P_3 with some special graphs like wheel, star, sunlet, path graph and complete bipartite graph.

V. References

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