



Almost Semi-Invariant Submanifolds of a δ -Lorentzian Trans-Sasakian Manifold

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Abstract : The semi-invariant submanifold of an almost contact metric manifold was defined and studied by Bejancu and Papaghuic. In 1984 Bejancu and Papaghuic^[1] defined almost semi-invariant submanifold of sasakian manifold. Tripathi and Singh^[4] in 1996 studied almost semi-invariant submanifold of an ε -framed metric manifold. U.C. De and K. De^[9] introduced the Lorentzian trans-sasakian manifold and obtained some very important properties. Later on Kishore and Gupta^[6], A. Haseeb & others^[2], Bhati and U.C. De^[7] introduced and studied about δ -Lorentzian Trans-Sasakian manifold and find some interesting result. The objective of the present paper is to define and study the almost semi-invariant submanifold of a δ -Lorentzian Trans-Sasakian manifold. The integrability conditions of the distributions have also been studied.

Keywords: δ -Lorentzian Trans-Sasakian manifold, Almost Semi-invariant submanifold, Integrability, Distributions.

I. Introduction

Let $\tilde{M}$ be a $(2n+1)$ dimensional manifold it is said to be δ -almost contact metric manifold if it admits a $(1, 1)$ tensor field J , a structure tensor field U , a 1-form u and a Riemannian metric g such that $\tilde{M}$ satisfy the following conditions :-

$$J^2 = I + u \otimes U, \quad u(U) = -1, \quad J(U) = 0, \quad u \circ J = 0 \quad (1.1)$$

$$g(JX, JY) = g(X, Y) + \delta u(X)u(Y) \quad (1.2)$$

$$u(X) = \delta g(X, U) \quad g(U, U) = -\delta \quad (1.3)$$

$$g(JX, Y) = g(X, JY) \quad (1.4)$$

for all $X, Y \in \Gamma T\tilde{M}$ where δ is such that $\delta^2 = 1$

The above structure (J, U, u, g, δ) on $\tilde{M}$ is called the δ -Lorentzian Structure on $\tilde{M}$.

A δ -Lorentzian manifold with structure (J, U, u, g, δ) is said to be δ -Lorentzian trans-sasakian manifold $\tilde{M}$ of type (α, β) if it satisfies the condition

$$(\tilde{\nabla}_X J)Y = \alpha \{g(X, Y)U - \delta u(Y)X\} + \beta \{g(JX, Y)U - \delta u(Y)JX\} \quad (1.5)$$

for any vector fields X and Y on $\tilde{M}$ where $\tilde{\nabla}$ is the operator of covariant differentiation with respect to g . Therefore we have

$$\tilde{\nabla}_X U = \delta(-\alpha JX - \beta(X + u(X)U)) \quad (1.6)$$

$$\text{and } (\tilde{\nabla}_X u)Y = \alpha g(JX, Y) + \beta \{g(X, Y) + \delta u(X)u(Y)\} \quad (1.7)$$

Let M be an m -dimensional submanifold of $\tilde{M}$ with the vector field U of $\tilde{M}$ be tangent to M and $\{U\}$ be the 1-dimensional distribution spanned by U on M and $\{U^\perp\}$ be the complementary orthogonal distribution to $\{U\}$ in ΓTM .

For any $X \in \Gamma TM$, we have $g(JX, U) = g(X, JU) = 0$

$$\text{Then Put } JX = BX + CX \quad (1.8)$$

Where $BX \in \Gamma\{U^\perp\}$ i.e. $BX \in \Gamma TM$ and $CX \in \Gamma TM^\perp$

Therefore B is an endomorphism of the tangent bundle TM and C is a normal Bundle valued 1-form on M

$$B: TM \rightarrow TM$$

$$C: TM \rightarrow TM^\perp$$

For each $x \in M$, we define the following subspaces

$$D_x = \{X_x \in \{U^\perp\}; C(X_x) = 0\} \quad (1.9)$$

$$D_x^\perp = \{X_x \in \{U^\perp\}; B(X_x) = 0\} \quad (1.10) \quad \therefore$$

$$D = \text{Ker } C$$

$$D^\perp = \text{Ker } B$$

here D_x and $D_x^\perp$ are two orthogonal subspaces of the tangent space $T_x M$.

Therefore we have

$$g(X_x, Y_x) = g(JX_x, JY_x) = g(BX_x, CY_x) = 0$$

$$\text{for any } X_x \in D_x \text{ and } Y_x \in D_x^\perp$$

II. Almost semi-invariant submanifold of a δ -Lorentzian Trans-Sasakian manifold

Definition: A submanifold M of a δ -Lorentzian trans-sasakian manifold $\tilde{M}$ is called an almost semi-invariant submanifold if its tangent bundle admits the orthogonal distributions

$$TM = D \oplus D^\perp \oplus \tilde{D} \oplus \{U\}$$

Where

(i) The distribution D is maximal invariant distribution of J on M that is ,

$$J(D_x) = D_x \text{ for each } x \in M$$

$$\text{i.e. } D: x \rightarrow D_x \subset T_x M$$

(ii) The Distribution $D^\perp$ is maximal anti-invariant distribution of J on M , that is,

$$J(D_x^\perp) \subset T_x M^\perp \text{ for each } x \in M$$

$$\text{where } T_x M^\perp \text{ is the normal space to M at } x. D^\perp: x \rightarrow D_x^\perp \subset T_x M$$

(iii) $\tilde{D}$ is neither invariant nor an anti invariant distribution on M

$$\text{i.e. } BX_x \neq 0 \text{ and } CX_x \neq 0 \text{ for any } x \in M \text{ and } X_x \in D_x$$

$\tilde{D}$ is the complementary orthogonal distribution to $D \oplus D^\perp \oplus \{U\}$ in TM.

(iv) $\{U\}$ is the 1-dimensional distribution spanned by the structure vector fields U.

The particular cases of almost semi-invariant submanifold are-

a) Semi-invariant submanifold when $\dim \tilde{D} = 0$

b) Invariant submanifold when $\dim \tilde{D} = 0, \dim D^\perp = 0$

c) Anti-invariant submanifold when $\dim D = 0, \dim \tilde{D} = 0$

Let M be an almost semi-invariant submanifold of the δ -Lorentzian trans-sasakian manifold $\tilde{M}$. We denote by g both Riemannian metric on $\tilde{M}$ and M.

The Projection morphism of TM to the distribution D, $D^\perp$ and $\tilde{D}$ are denoted respectively by P, Q, and L then, for each $X \in \Gamma TM$, we have

$$X = PX + QX + LX + u(X)U \quad (2.1)$$

$$\text{where } u(X) = \delta g(X, U)$$

Now we take $X \in \Gamma \tilde{D}$ and note that $BX_x \neq 0$ and $CX_x \neq 0$

Thus C defines a vector subbundle

$$C\tilde{D}: x \rightarrow C\tilde{D}_x \subset TM^\perp, \text{ of } TM^\perp$$

$$\text{Also we have } B\tilde{D}_x = \tilde{D}_x \text{ and } \tilde{D}_x \cap C\tilde{D}_x = \{0\} \quad (2.2)$$

for each $x \in M$

Hence B is an automorphism of $\tilde{D}$.

For any $N \in \Gamma TM^\perp$, we put

$$JN = RN + FN \quad (2.3)$$

Where RN is the tangential part of JN and FN is the normal part of JN

$$\begin{aligned} \text{i.e. } R : TM^\perp &\rightarrow TM \\ F : TM^\perp &\rightarrow TM^\perp \end{aligned}$$

$$\text{We also have } g(JD^\perp, C\tilde{D}) = 0 \quad (2.4)$$

$$\text{So that, } g(JX, CY) = g(JX, JY) = g(X, Y) = 0 \quad (2.5)$$

for all $X \in \Gamma D^\perp$ & $Y \in \Gamma \tilde{D}$

Therefore vector subbundle $JD^\perp$ is orthogonal to $C\tilde{D}$.

Next, we denote by v the orthogonal complementary vector bundle to $JD^\perp \oplus C\tilde{D}$ in $TM^\perp$. Therefore we have the orthogonal decomposition of $TM^\perp$ as

$$TM^\perp = JD^\perp \oplus C\tilde{D} \oplus v \quad (2.6)$$

Theorem(2.1): The morphism R and F satisfy

$$R(TM^\perp) = D^\perp \oplus \tilde{D} \quad (2.7)$$

$$R(JD^\perp) = D^\perp \quad (2.8)$$

$$R(C\tilde{D}) = \tilde{D} \quad (2.9)$$

$$F(C\tilde{D}) = C\tilde{D} \quad (2.10)$$

Proof: Let $N \in \Gamma TM^\perp$ then

$$\begin{aligned} g(RN, X) &= g(JN - FN, X) \\ &= g(JN, X) - g(FN, X) \\ &= g(JN, X) \\ &= g(N, JX) \\ &= 0 \quad \text{where } X \in \Gamma D \end{aligned}$$

$$\therefore g(RN, X) = 0 \quad \forall X \in \Gamma D$$

$$\begin{aligned} g(RN, U) &= g(JN - FN, U) \\ &= g(JN, U) - g(FN, U) \\ &= g(JN, U) \\ &= g(N, JU) \\ &= 0 \end{aligned}$$

$$\therefore RN \in \Gamma D^\perp \oplus \tilde{D}$$

Hence (2.7) exist

Next, for each $X \in \Gamma D^\perp$ so that $JX \in JD^\perp$

Therefore using (1.1) and (2.3) we have

$$\begin{aligned} X &= J^2X = J(JX) \\ &= R(JX) + F(JX) \\ &= R(JX) \end{aligned}$$

Therefore $R(JD^\perp) = D^\perp$

Hence (2.8) exist.

By direct computation we have

$$Rv = \{0\} \text{ and } R(C\tilde{D}) \subseteq \tilde{D} \quad (2.11)$$

Now, By using (2.8) and first equality from (2.11), we obtain

$$R(TM^\perp) = D^\perp \oplus R(C\tilde{D}) \quad (2.12)$$

from (2.12) and (2.7) taking account of the second relation from (2.11) we get (2.9).

Now, (2.10) follows by using (2.9) and taking account that $\tilde{D}$ is not anti-invariant by J .

Theorem(2.2): Let M be an almost semi-invariant submanifold of the δ -Lorentzian trans-sasakian manifold $\tilde{M}$. Then the endomorphism $B : TM \rightarrow TM$ satisfies $B^3 - B = 0$ if and only if M is a semi-invariant submanifold. (2.13)

Proof: Firstly notice that $(B^3 - B)X = 0$ for any $X \in \Gamma D \oplus D^\perp \oplus \{U\}$

Now, since B is an automorphism of $\tilde{D}$

$$B\tilde{D}_x = \tilde{D}_x$$

We see that

$$\begin{aligned} (B^3 - B)\tilde{D} &= \{0\} \text{ if and only if} \\ (B^2 - I)\tilde{D} &= \{0\} \end{aligned} \quad (2.14)$$

Applying J to (1.8) and using (1.8), (2.3) and (1.1)

$$\begin{aligned} \text{We get } X &= J^2 X \quad \text{for } X \in \Gamma D \\ &= J(JX) \\ &= J(BX + CX) \\ &= JBX + JCX \\ &= B(BX) + C(BX) + R(CX) + F(CX) \\ &= B^2X + C(BX) + R(CX) + F(CX) \end{aligned}$$

equating tangential part in the above equation

$$\begin{aligned} X &= B^2X + R(CX) \\ (B^2 - I)X &= -RCX \end{aligned}$$

$$\therefore (B^2 - I)X = \{0\} \quad \forall X \in \Gamma \tilde{D}$$

If and only if $R(CX) = \{0\} \quad \forall X \in \Gamma \tilde{D}$

$$\Rightarrow R(\tilde{C}\tilde{D}) = \{0\}$$

Finally from (2.9) $\tilde{D} = \{0\}$

Hence the submanifold is semi-invariant submainfold.

Theorem (2.3): Let M be an almost semi-invariant submanifold of the δ -Lorentzian Trans-sasakian manifold $\tilde{M}$. Then M is a semi-invariant submanifold if and only if we have $F^3 - F = 0$ (2.15)

Proof: We see that equation (2.15) exist for $N \in \Gamma(JD^\perp \oplus v)$

if we take $N \in \Gamma(JD^\perp)$, that is, $N_1 \in \Gamma D^\perp$

$$\begin{aligned} JN_1 &= N \in \Gamma JD^\perp \\ J^2N_1 &= JN = RN + FN \quad (\text{using equation (2.3)}) \\ N_1 &= RN + FN \quad (\text{using equation (1.1)}) \end{aligned}$$

equating tangential and normal part

$$N_1 = RN, \quad FN = 0$$

Therefore for $N \in \Gamma(JD^\perp)$ we have $FN = 0$

if $N \in \Gamma v$ that is $N \notin \Gamma(JD^\perp)$ and $N \notin \Gamma(\tilde{C}\tilde{D})$

we have $JN = FN$ (using equation (2.3))

By equation (2.10) F is an automorphism on $\tilde{C}\tilde{D}$. Hence $(F^3 - F)(\tilde{C}\tilde{D}) = \{0\}$ if and only if we have

$$(F^2 - I)(\tilde{C}\tilde{D}) = \{0\}$$

By equation (1.8) and (2.3) for $X \in \Gamma \tilde{D}$

$$JCX = RCX + FCX$$

$$J^2CX = J(RCX) + J(FCX)$$

$$CX = B(RCX) + C(RCX) + R(FCX) + F^2CX$$

Therefore equation $(F^2 - I)(\tilde{C}\tilde{D}) = \{0\}$ is equivalent to $CR(\tilde{C}\tilde{D}) = \{0\}$ which implies that $R(\tilde{C}\tilde{D}) = \{0\}$

By (2.9) $\tilde{D} = \{0\}$

Hence submanifold is a semi-invariant submanifold

Theorem(2.4) : Let M be an almost semi-invariant submanifold of the δ -Lorentzian Trans-sasakian manifold $\tilde{M}$. Then we have

$$(B^2 + RC)X = X + u(X)U \quad ; \quad (CB + FC)X = 0 \quad (2.16)$$

$$(BR + RF)N = 0 \quad ; \quad (F^2 - I + CR)N = 0 \quad (2.17)$$

$$(F^3 - F + CRF)N = 0 \quad (2.18)$$

$$(B^3 - B + RCB)X = 0 \quad (2.19)$$

for any $X \in \Gamma TM$ and $N \in \Gamma TM^\perp$

Proof: Using (1.1), (1.8) and (2.3) for $X \in \Gamma TM$

$$\begin{aligned} X + u(X)U &= J^2 X \\ &= J(JX) \\ &= J(BX + CX) \\ &= JBX + JCX \\ &= B(BX) + C(BX) + R(CX) + F(CX) \\ &= B^2 X + C(BX) + R(CX) + F(CX) \end{aligned}$$

Equating tangential and normal part

$$\begin{aligned} (B^2 + RC)X &= X + u(X)U \\ (CB + FC)X &= 0 \end{aligned}$$

Hence (2.16) exist.

Now, for $N \in \Gamma TM^\perp$ using (1.1), (1.8) & (2.3)

$$\begin{aligned} N + u(N)U &= J^2 N \\ \therefore N \in \Gamma TM^\perp &\Rightarrow u(N) = 0 \\ N &= J^2 N \\ &= J(JN) \\ N &= J(RN + FN) \\ &= JRN + JFN \\ &= B(RN) + C(RN) + R(FN) + F(FN) \\ &= B(RN) + C(RN) + R(FN) + F^2 N \end{aligned}$$

Equating tangential and Normal part

$$\begin{aligned} (BR + RF)N &= 0 \\ (CR + F^2)N &= N \\ \Rightarrow (CR + F^2 - I)N &= 0 \\ \Rightarrow (F^2 - I + CR)N &= 0 \end{aligned}$$

Hence (2.17) exist.

For (2.18) use (2.17) and we can prove it easily

For (2.19) use (2.16) and we can also prove it easily.

The operator of covariant differentiation with respect to the induced connections on $\tilde{M}$ (resp. M) will be denoted by $\tilde{\nabla}$ (resp. ∇). The linear connection induced by $\tilde{\nabla}$ on the normal bundle $TM^\perp$ is denoted by $\nabla^\perp$. Then Gauss and Weingarten equation are

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y) \quad (2.20)$$

$$\tilde{\nabla}_X N = -A_N X + \nabla_X^\perp N \quad (2.21)$$

for all $X, Y \in \Gamma TM$ and $N \in \Gamma TM^\perp$, where h is the second fundamental form of M and A_N is the fundamental tensor of Weingarten with respect to the normal section N . So that $-A_N X$ and $\nabla_X^\perp N$ are the tangential part and normal part of $\tilde{\nabla}_X N$ respectively.

These tensor fields are related by

$$g(h(X, Y), N) = g(A_N X, Y) \quad (2.22)$$

for all $X, Y \in \Gamma TM$ and $N \in \Gamma TM^\perp$

Lemma (3.1): Let M be an almost semi-invariant submanifold of the δ -Lorentzian Trans-sasakian manifold $\tilde{M}$. Then we have

$$A_{JX}Y = -A_{JY}X \quad \text{for all } X, Y \in \Gamma D^\perp \quad (2.23)$$

Proof: From equation (1.1), (1.5), (2.20) and (2.22) for $X, Y \in \Gamma D^\perp$ & $Z \in \Gamma TM$

$$\begin{aligned} g(A_{JX}Y, Z) &= g(h(Y, Z), JX) \\ &= g(\tilde{\nabla}_Z Y - \nabla_Z Y, JX) \\ &= g(\tilde{\nabla}_Z Y, JX) - g(\nabla_Z Y, JX) \\ &= g(\tilde{\nabla}_Z Y, JX) \quad \because X, Y \in \Gamma D^\perp \\ &= g(J\tilde{\nabla}_Z Y, X) \\ &= g(\tilde{\nabla}_Z JY - \alpha\{g(Z, Y)U - \delta u(Y)Z\} - \beta\{g(JZ, Y)U - \delta u(Y)JZ\}, X) \\ &= g(\tilde{\nabla}_Z JY, X) - \alpha g(Z, Y)g(U, X) + \alpha \delta u(Y)g(Z, X) \\ &\quad - \beta g(JZ, Y)g(U, X) + \beta \delta u(Y)g(JZ, X) \\ &= g(\tilde{\nabla}_Z JY, X) \\ &= -g(JY, \tilde{\nabla}_Z X) \\ &= -g(\tilde{\nabla}_Z X, JY) \\ &= -g(h(X, Z), JY) \\ &= -g(A_{JY}X, Z) \quad \text{for } X, Y \in \Gamma D^\perp, Z \in \Gamma TM \quad \therefore A_{JX}Y = -A_{JY}X \end{aligned}$$

Lemma(3.2): Let M be an almost semi-invariant submanifold of the δ -Lorentzian Trans-Sasakian manifold $\tilde{M}$. Then we have

$$\nabla_X U = -\delta\alpha JX - \delta\beta X, \quad h(X, U) = 0 \quad \text{for any } X \in \Gamma D \quad (2.24)$$

$$\nabla_X U = -\delta\beta X, \quad h(X, U) = -\delta\alpha JX \quad \text{for any } X \in \Gamma D^\perp \quad (2.25)$$

$$\nabla_X U = -\delta\alpha BX - \delta\beta X, \quad h(X, U) = -\delta\alpha CX \quad \text{for any } X \in \Gamma \tilde{D} \quad (2.26)$$

$$\nabla_U U = 0, \quad h(U, U) = 0 \quad (2.27)$$

Proof : By using (1.8), (2.1), (2.20) in (1.6) we obtain for $X \in \Gamma TM$

$$\begin{aligned} \tilde{\nabla}_X U &= \delta(-\alpha JX - \beta(X + u(X)U)) \\ \nabla_X U + h(X, U) &= -\delta\alpha JX - \delta\beta(X + u(X)U) \\ &= -\delta\alpha J(PX + QX + LX + u(X)U) - \delta\beta X - \delta\beta u(X)U \\ &= -\delta\alpha(JPX + JQX + JLX) - \delta\beta X - \delta\beta u(X)U \\ &= -\delta\alpha(JPX + JQX + BLX + CLX) - \delta\beta X - \delta\beta u(X)U \\ &= -\delta\alpha JPX - \delta\alpha JQX - \delta\alpha BLX - \delta\alpha CLX - \delta\beta X - \delta\beta u(X)U \\ &= -\delta\alpha JPX - \delta\alpha JQX - \delta\alpha BLX - \delta\alpha CLX - \\ &\quad - \delta\beta(PX + QX + LX + u(X)U) - \delta\beta u(X)U \\ &= -\delta\alpha JPX - \delta\alpha JQX - \delta\alpha BLX - \delta\alpha CLX - \\ &\quad - \delta\beta PX - \delta\beta QX - \delta\beta LX - 2\delta\beta u(X)U \end{aligned} \quad (2.28)$$

Equation (2.24) - (2.27) are follows from (2.28) Now we denote

$$u'(X, Y) = \nabla_X JPY + \nabla_X BLY - A_{JQY}X - A_{CLY}X \quad (2.29)$$

For any $X, Y \in \Gamma TM$.

Lemma (3.3): Let M be an almost semi-invariant submanifold of the δ -Lorentzian Trans-Sasakian manifold $\tilde{M}$. Then we have

$$P(u'(X, Y)) = JP(\nabla_X Y) - \delta\alpha u(Y)PX - \delta\beta u(Y)JPX \quad (2.30)$$

$$Q(u'(X, Y)) = QR(h(X, Y)) - \delta\alpha u(Y)QX \quad (2.31)$$

$$L(u'(X, Y)) = BL(\nabla_X Y) + LR(h(X, Y)) - \delta\alpha u(Y)LX - \delta\beta u(Y)BLX \quad (2.32)$$

$$u(u'(X, Y)) = \alpha(g(X, Y) - \delta\alpha u(Y)u(X)) + \beta g(JX, Y)U \quad (2.33)$$

$$h(X, JPY) + h(X, BLY) + \nabla_X^\perp JQY + \nabla_X^\perp CLY = JQ\nabla_X Y + CL\nabla_X Y + F(h(X, Y) - \delta\beta u(Y)JQX - \delta\beta u(Y)CLX) \quad (2.34)$$

Proof :- By using the decomposition (1.8), (2.1), (2.3) and (2.20), (2.21), (1.2), (1.4) in equation (1.5), we obtain

$$\begin{aligned} & \nabla_X JPY + \nabla_X BLY - A_{JQY}X - A_{CLY}X + h(X, JPY) + h(X, BLY) + \nabla_X^\perp JQY + \nabla_X^\perp CLY \\ &= JP(\nabla_X Y) + JQ(\nabla_X Y) + BL(\nabla_X Y) + CL(\nabla_X Y) + R(h(X, Y)) + Fh(X, Y) \\ &+ \alpha\{g(X, Y)U - \delta u(Y)PX - \delta u(Y)QX - \delta u(Y)LX - \delta u(Y)u(X)U\} \\ &+ \beta\{g(JX, Y)U - \delta u(Y)JPX - \delta u(Y)JQX - \delta u(Y)BLX - \delta u(Y)CLX\} \end{aligned}$$

Then equation (2.30) – (2.34) are obtained by taking the components on each of the vector bundle D , $D^\perp$, $\tilde{D}$, $\{U\}$ and $TM^\perp$ respectively.

Lemma (3.4): Let M be an almost semi-invariant submanifold of the δ -Lorentzian Trans-Sasakian manifold $\tilde{M}$. Then we have

$$\nabla_U U' \in \Gamma D \quad \text{for any } U' \in \Gamma D \quad (2.35)$$

$$\nabla_U V' \in \Gamma D^\perp \quad \text{for any } V' \in \Gamma D^\perp \quad (2.36)$$

$$\nabla_U W' \in \Gamma \tilde{D} \quad \text{for any } W' \in \Gamma \tilde{D} \quad (2.37)$$

Proof: From (2.29) put $X=U$, $Y=U' \in \Gamma D$

$$u'(U, U') = \nabla_U JPU' + \nabla_U BLU' - A_{JQU'}U - A_{CLU'}U$$

$$\Rightarrow u'(U, U') = \nabla_U JPU'$$

therefore from equation (2.31), by putting $X=U$, $Y=U' \in \Gamma D$ we get

$$Q(u'(U, U')) = QR(h(U, U')) - \delta\alpha u(U')QU$$

$$Q(\nabla_U JPU') = QR(h(U, U'))$$

$$= 0 \quad (\text{using equation 2.24})$$

$$\Rightarrow Q(\nabla_U JU') = 0$$

from (2.34) and using (2.24) and above equation

$$h(U, JPU') + h(U, BLU') + \nabla_U^\perp JQU' + \nabla_U^\perp CLU' = JQ\nabla_U U' + CL\nabla_U U' + Fh(U, U') - \delta\beta u(U')JQU - \delta\beta u(U')CLU$$

$$\Rightarrow CL(\nabla_U U') = 0$$

from equation (2.27)

$$g(\nabla_U U', U) = -g(U', \nabla_U U) = 0$$

Since C is an isomorphism of $\tilde{D}$ to $C\tilde{D}$ and J is an isomorphism of D . Therefore from the above three results we have

$$\nabla_U U' \in \Gamma D \quad \text{for any } U' \in \Gamma D$$

Similarly we can show other two results (2.36) and (2.37).

From Lemmas (3.2) and (3.4) we obtain

Lemma(3.5): Let M be an almost semi-invariant submanifold of the δ -Lorentzian Trans-Sasakian manifold $\tilde{M}$. Therefore

$$[X, U] \in \Gamma D \quad \text{for every } X \in \Gamma D \quad (2.38)$$

$$[Y, U] \in \Gamma D^\perp \quad \text{for every } Y \in \Gamma D^\perp \quad (2.39)$$

$$[Z, U] \in \Gamma \tilde{D} \quad \text{for every } Z \in \Gamma \tilde{D} \quad (2.40)$$

III. Integrability of some distributions on an almost semi-invariant submanifold of δ -Lorentzian Trans-Sasakian Manifold

Theorem (3.1): Let M be an almost semi-invariant submanifold of the δ -Lorentzian Trans-Sasakian manifold $\tilde{M}$ then the distribution D is integrable if and only if $h(X, JY) = h(Y, JX)$ for any $X, Y \in \Gamma D$ (3.1)

Proof: Using equation (2.24)

$$\begin{aligned} g([X, Y], U) &= g(\nabla_X Y - \nabla_Y X, U) \\ &= g(\nabla_X Y, U) - g(\nabla_Y X, U) \\ &= -g(Y, \nabla_X U) + g(X, \nabla_Y U) \\ &= -g(Y, -\delta\alpha JX - \delta\beta X) + g(X, -\delta\alpha JY - \delta\beta Y) \\ &= \delta\alpha g(Y, JX) + \delta\beta g(Y, X) - \delta\alpha g(X, JY) - \delta\beta g(X, Y) \\ &= \delta\alpha g(Y, JX) + \delta\beta g(X, Y) - \delta\alpha g(JX, Y) - \delta\beta g(X, Y) \\ &= \delta\alpha g(Y, JX) - \delta\alpha g(Y, JX) \\ &= 0 \quad \text{for any } X, Y \in \Gamma D \end{aligned}$$

Now from (2.34)

$$h(X, JY) = JQ(\nabla_X Y) + CL(\nabla_X Y) + Fh(X, Y) \quad \text{for any } X, Y \in \Gamma D$$

Similarly

$$h(Y, JX) = JQ(\nabla_Y X) + CL(\nabla_Y X) + Fh(Y, X) \quad \text{for any } X, Y \in \Gamma D$$

Hence

$$\begin{aligned} h(X, JY) - h(Y, JX) &= JQ(\nabla_X Y - \nabla_Y X) + CL(\nabla_X Y - \nabla_Y X) + F\{h(X, Y) - h(Y, X)\} \\ &= JQ[X, Y] + CL([X, Y]) \quad \text{for any } X, Y \in \Gamma D \end{aligned} \quad (3.2)$$

hence by (3.1) D is integrable.

Corollary (3.1): The distribution $D \oplus \{U\}$ is integrable if and only if $h(X, JY) = h(Y, JX)$ is satisfied.

Proof: In view of theorem (3.1) and equation (2.38), we can prove the above result.

Theorem (3.2): The Distribution $D^\perp$ is not integrable.

Proof: By using (2.25) for any $X, Y \in \Gamma D^\perp$

$$\begin{aligned} g([X, Y], U) &= g(\nabla_X Y - \nabla_Y X, U) \\ &= g(\nabla_X Y, U) - g(\nabla_Y X, U) \\ &= -g(Y, \nabla_X U) + g(X, \nabla_Y U) \\ &= -g(Y, -\delta\beta X) + g(X, -\delta\beta Y) \\ &= \delta\beta g(Y, X) - \delta\beta g(X, Y) \\ &= \delta\beta g(X, Y) - \delta\beta g(X, Y) \\ &= 0 \end{aligned}$$

Now from (2.29) for all $X, Y \in \Gamma D^\perp$

$$u'(X, Y) = -A_{JX}Y$$

Apply J to (2.30) and (1.1)

$$\begin{aligned} JP(-A_{JY}X) &= J^2 P(\nabla_X Y) \\ &= P(\nabla_X Y) + u(P(\nabla_X Y))U \\ &= P(\nabla_X Y) \end{aligned}$$

$$P(\nabla_X Y) = -JP(A_{JY}X)$$

$$\text{Similarly } P(\nabla_Y X) = -JP(A_{JX}Y)$$

$$P(\nabla_X Y - \nabla_Y X) = -JP(A_{JY} X - A_{JX} Y)$$

$$P[X, Y] = -JP(A_{JY} X - A_{JX} Y)$$

for any $X, Y \in \Gamma D^\perp$

But from Lemma (3.1)

$$P[X, Y] = 2JP(A_{JX} Y)$$

Therefore $D^\perp$ is not integrable.

Corollary (3.2): The distribution $D^\perp \oplus \{U\}$ is not integrable.

Proof: In view of theorem (3.2) and equation (2.39) we can prove the above result.

Theorem (3.3): Let M be an almost semi-invariant submanifold of the δ -Lorentzian Trans-Sasakian manifold $\tilde{M}$ with $\dim \tilde{D} \neq 0$. Then we have $\dim \tilde{D} \geq 2$ but the distribution $\tilde{D}$ is integrable.

Proof: Let there exist $X \in \Gamma TM$ such that $X \in \Gamma \tilde{D}$ then $g(X, BX) = g(X, JX) \neq 0$ which together with (2.2) implies $\dim \tilde{D} \geq 2$

Now take $X, Y \in \Gamma \tilde{D}$ and by using (2.26)

$$\begin{aligned} g([X, Y], U) &= g(\nabla_X Y - \nabla_Y X, U) \\ &= g(\nabla_X Y, U) - g(\nabla_Y X, U) \\ &= -g(Y, \nabla_X U) + g(X, \nabla_Y U) \\ &= -g(Y, -\delta\alpha BX - \delta\beta X) + g(X, -\delta\alpha BY - \delta\beta Y) \\ &= \delta\alpha g(Y, BX) + \delta\beta g(Y, X) - \delta\alpha g(X, BY) - \delta\beta g(X, Y) \\ &= \delta\alpha g(Y, BX) - \delta\alpha g(BX, Y) \\ &= 0 \end{aligned}$$

Therefore the distribution $\tilde{D}$ is integrable.

Corollary(3.3): The distribution $\tilde{D} \oplus \{U\}$ is integrable.

Proof: By theorem (3.3) and equation (2.40) we can prove the above result.

Theorem(3.4): Let M be an almost semi-invariant submanifold of the δ -Lorentzian Trans-Sasakian manifold $\tilde{M}$ then the distribution $D^\perp \oplus \tilde{D} \oplus \{U\}$ is integrable if and only if

$$\nabla_X BLY - \nabla_X BLX + A_{CX} Y - A_{CY} X \in \Gamma(D^\perp \oplus \tilde{D} \oplus \{U\})$$

for any $X, Y \in \Gamma(D^\perp \oplus \tilde{D} \oplus \{U\})$

Proof: Take $X \in \Gamma(D^\perp \oplus \tilde{D} \oplus \{U\})$ in Equation (2.29), that is,

$$u'(X, Y) = \nabla_X BLY - A_{JQY} X - A_{CLY} X$$

Now applying J to (2.30) we get

$$JP(\nabla_X BLY - A_{JQY} X - A_{CLY} X) = P(\nabla_X Y)$$

for any $X, Y \in \Gamma(D^\perp \oplus \tilde{D} \oplus \{U\})$

Similarly

$$JP(\nabla_Y BLX - A_{JQX} Y - A_{CLX} Y) = P(\nabla_Y X)$$

Therefore

$$P(\nabla_X Y - \nabla_Y X) = JP(\nabla_X BLY - \nabla_Y BLX + A_{JQX} Y + A_{CLX} Y - A_{JQY} X - A_{CLY} X)$$

Since $Y = QY + LY + u(Y)U$

So applying J , we get $CY = JQY + CLY$

$$\begin{aligned} P([X, Y]) &= JP(\nabla_X BLY - \nabla_Y BLX + A_{JQX} Y + A_{CLX} Y - A_{JQY} X - A_{CLY} X) \\ &= JP(\nabla_X BLY - \nabla_Y BLX + A_{CX} Y - A_{CY} X) \end{aligned}$$

For any $X, Y \in \Gamma(D^\perp \oplus \tilde{D} \oplus \{U\})$

Hence proof follows from the above equation.

Theorem(3.5): Let M be an almost semi-invariant submanifold of the δ -Lorentzian Trans-Sasakian manifold $\tilde{M}$ then the distribution $D \oplus D^\perp \oplus \{U\}$ is integrable if and only if

$$\nabla_X JPY - \nabla_Y JPX + A_{JQX} Y - A_{JQY} X \in \Gamma(D \oplus D^\perp \oplus \{U\})$$

For any $X, Y \in \Gamma(D \oplus D^\perp \oplus \{U\})$

Proof: from equation (2.32) we get for $X, Y \in \Gamma(D \oplus D^\perp \oplus \{U\})$

$$L\{\nabla_X JPY - A_{JQY} X\} = BL(\nabla_X Y) + LRh(X, Y)$$

$$BL(\nabla_X Y) = L\{\nabla_X JPY - A_{JQY} X - Rh(X, Y)\}$$

Similarly

$$BL(\nabla_Y X) = L\{\nabla_Y JPX - A_{JQX} Y - Rh(Y, X)\}$$

$$BL(\nabla_X Y - \nabla_Y X) = L\{\nabla_X JPY - \nabla_Y JPX + A_{JQX} Y - A_{JQY} X + Rh(Y, X) - Rh(X, Y)\}$$

$$BL([X, Y]) = L\{\nabla_X JPY - \nabla_Y JPX + A_{JQX} Y - A_{JQY} X\}$$

For any $X, Y \in \Gamma(D \oplus D^\perp \oplus \{U\})$

from the above equation theorem is proved as B is an automorphism of $\tilde{D}$

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