



A Survey on Fuzzy Labeling Graphs

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Abstract: After introducing and developing fuzzy set theory, a lot of studies have been done in this field and then a result appeared as a fuzzy graph (Combination of Graph theory and Fuzzy set theory). This is now known as Fuzzy graph theory. In this Paper, we have studied the survey of selected recent results in Fuzzy Labeling Graphs (FLGs). A graph is said to be FLG if it has fuzzy labeling Fuzzy subgraph, Fuzzy end vertices, Fuzzy cut vertices, Fuzzy labeling tree, Blocks in FLG, Distances in FLG, Center in FLG, m-polar FLG, some other types of fuzzy labeling like magic, Bi-magic, Anti-magic, graceful labeling of FLGs have been reported.

Key Words: Fuzzy Labeling Graph, Blocks in FLG, Distances in FLG, Center in FLG, m-polar FLG.

I. Introduction

It is quite well known that graphs are simply models of relations. A graph is a convenient way of representing information involving relationship between objects. The objects are represented by vertices and relations by edges. When there is vagueness in the description of the objects or in its relationships or in both, it is natural that we need to design a 'Fuzzy Graph Model'. We know that a graph is a symmetric binary relation on a nonempty set V . Similarly, a fuzzy graph is a symmetric binary fuzzy relation on a fuzzy subset. The concept of fuzzy sets and fuzzy relations was introduced by Zadeh. L.A. in 1965 [37] and further studied in [38]. It was Rosenfeld [22] who considered fuzzy relations on fuzzy sets and developed the theory of fuzzy graphs in 1975. The concept of fuzzy trees, blocks, bridges and cut nodes in fuzzy graph has been studied in [22].

The Concept of Fuzzy Labeling was introduced by A.Nagoor gani and D.Rajalaxmi. They discussed the Properties of FLG in [13]. The concepts of Fuzzy labeling and Fuzzy magic labeling graph were introduced in [11]. They proved that every fuzzy magic graph is a fuzzy labeling graph but the converse is not true. They discussed about some properties related to fuzzy bridge and fuzzy cut node.

The concept of metric in FLG were introduced [16]. Some results related with μ -length, eccentricity, diameter and radius of fuzzy labeling graph have been derived. A necessary Condition for a graph G to have more than one center is given. It deals with μ -distance related to FLG also. Rashmanlou.H and Borzooei.R.A. [20] have extended the definitions of Metric and Distances in FLG with some applications. They discussed four distances which are a metric in FLGs. Also it is proved that every connected fuzzy labeling graph is ss-self-centered as well as self-centered.

Nagoor gani.A et al., [15] extended the FL concept to fuzzy labeling tree. Different properties of FL trees were conversed and also bipartite FLG was defined. They had given the algorithm for finding maximum spanning, strong arcs and fuzzy bridges of any FLG. Samanta.S et al., [18] have extended the above concept and they defined some new connectivity concepts in FLGs. A new type of labeling graphs called θ -fuzzy labeling graph was defined in the same.

Akram.M and Arooj Adeel [1] have introduced the concept of m-polar FLGs. They introduced several types of distances in m-polar FLG. Also they have extended this concept into bipartite m-polar FLGs [2]. Interval valued Fuzzy magic labeling was discoursed in [10]. Kishore Kumar.P.K. et al., [9] have deliberated on Magic labeling on interval valued intuitionistic fuzzy graphs. Ameenal Bibi.K and Devi.M [3,4,5,6,7,8] have extended the concepts of Fuzzy labeling to Fuzzy Bi-magic labeling, Fuzzy Anti-magic labeling, Interval valued Fuzzy Bi-magic labeling and Bi-Magic labeling on interval valued intuitionistic fuzzy graphs. Hesitancy fuzzy magic labeling was studied in [21]. Nagarani and Vimala [29,30] have introduced Energy of FLGs. Yahya.M [36] have discussed about Matching in FLG. Jebasty Shajila and Vimala [31,32,33] have introduced Fuzzy graceful labeling. Basic Concepts of Fuzzy Sets were found in [11,39] and for concepts in fuzzy graph theory, we refer [22,28].

II Fuzzy Labeling graph for Subgraph and Union:

Definition 2.1.1: [14] A bijection ω is a function from the set of all vertices and edges of G^* to $[0,1]$ which assigns each vertices $\sigma^\omega(u), \sigma^\omega(v)$ and each $\mu^\omega(u, v)$ a membership value such that $\mu^\omega(u, v) < \sigma^\omega(u) \wedge \sigma^\omega(v)$ for all

$u, v \in V$ is called Fuzzy labeling. A graph is said to be fuzzy labeling graph if it has a fuzzy labeling and it is denoted by G^ω .

Definition 2.1.2: [14] A Cycle graph G^* is said to be a fuzzy labeling cycle if it has fuzzy labeling.

Definition 2.1.3: [14] The fuzzy labeling graph $H^\omega = (\tau^\omega, \rho^\omega)$ is called a fuzzy labeling subgraph of $G^\omega = (\sigma^\omega, \mu^\omega)$ if $\tau^\omega(u) \leq \sigma^\omega(u)$ for all $u \in V$ and $\rho^\omega(u, v) \leq \mu^\omega(u, v)$ for all $u, v \in V$.

Proposition 2.1.4: [14] If $(\tau^\omega, \rho^\omega)$ is fuzzy labeling subgraph of $(\sigma^\omega, \mu^\omega)$ then $\rho^{\omega\omega}(u, v) \leq \mu^{\omega\omega}(u, v)$ for all $u, v \in V$.

Proposition 2.1.5: [14] Union of any two fuzzy labeling graphs G_1^ω and G_2^ω is also a fuzzy labeling graph, if the membership value of the edges between G_1^ω and G_2^ω are distinct.

2.2 Properties of Fuzzy Labeling Cycle:

Proposition 2.2.1: [14] If G^* is a cycle then the fuzzy labeling cycle G^ω has exactly only one weakest arc.

Corollary: No cycle is a fuzzy cycle in fuzzy labeling graph.

Proposition 2.2.2: [14] Let G^ω be a fuzzy labeling cycle such that G^* is a cycle then it has $(n-1)$ bridges.

Proposition 2.2.3: [14] Let $G = (\sigma^\omega, \mu^\omega)$ be a fuzzy graph such that G^* is a cycle. Then a vertex is a fuzzy cut vertex of G iff it is a common vertex of two fuzzy bridges.

Proposition 2.2.4: [14] If G^* is a cycle with fuzzy labeling, then it has $(n-2)$ cut vertices.

Proposition 2.2.5: [14] If G^* is a cycle with fuzzy labeling, then the graph has exactly two end vertices.

Proposition 2.2.6: [14] If G^* is a cycle with fuzzy labeling then the vertex of G^ω is either a cut vertex or end vertices.

Proposition 2.2.7: [14] If G^ω is a fuzzy labeling cycle graph, then every bridges is strong and vice-versa.

2.3 Fuzzy Labeling with Bridge and Strong edge:

Proposition 2.3.1: [14] The following statements are equivalent:

- i) (x, y) is a fuzzy bridge.
- ii) $\mu^{\omega\omega}(x, y) < \mu(x, y)$
- iii) (x, y) is not a weakest arc of any cycle.

Proposition 2.3.2: [14] If G^ω is a fuzzy labeling graph, then G^ω has atleast one fuzzy bridge. The converse is not true.

Proposition 2.3.3: [14] If G^ω is a connected fuzzy labeling graph, then there exists a strong path between every pair of vertices.

Proposition 2.3.4: [14] Every FLG has atleast one weakest arc.

Proposition 2.3.5: [14] For any FLG G^ω , $\delta(G^\omega)$ is a fuzzy end vertex of G^ω such that the no. of arcs incident on $\delta(G^\omega)$ is atleast two.

Proposition 2.3.6: [14] Every FLG has atleast one end vertices.

Proposition 2.3.7: [14] Every FLG has atleast one cut vertex.

III Fuzzy Labeling Trees:

Azriel Rosenfeld in 1975 [22] developed the structure of fuzzy graphs and obtained analogs of several graph theoretical concepts like bridges and trees.

Further Sunitha [24] established some characterization of fuzzy trees using fuzzy bridges and fuzzy cut nodes. Nagoor gani and Rajalaxmi introduced the concept of fuzzy labeling tree (FLT).

Definition 3.1: [15] A Connected fuzzy graph $G = (\sigma, \mu)$ is a fuzzy tree if it has a fuzzy spanning subgraph $F = (\sigma, \gamma)$ which is a tree, where for all arcs (u, v) not in F , $\mu(u, v) < \gamma^{\omega\omega}(u, v)$.

Proposition 3.2: [15] If G is a FLT, then the arcs of F are fuzzy bridges of G .

Proposition 3.3: [15] Every FLG is a FLT. This is not true for general fuzzy graph.

Proposition 3.4: [15] If $G = (\sigma, \mu)$ is a FLT, then its spanning subgraph $F = (\sigma, \gamma)$ is also a FLG.

Remark 3.5: [15] All the Properties of FLG hold good for FLT. As in the fuzzy graph, here also internal vertices of F are cut vertices since the arcs are fuzzy bridges. Here also the fuzzy spanning subgraph F is unique and which is the maximum spanning tree and as μ is bijective, one cannot conclude that the lower weighted arc will not be there in F .

For example, consider fig (5) in [15] in which arc (x, y) is in F but (x, v) is not in F .

Proposition 3.6: [15] If G is a FLT and F is its spanning subgraph, then $(G-F)^*$ is tree.

Remark 3.7: The above is not true if G^* is complete.

Proposition 3.8: [15] Let G be a FLT and F be its spanning subgraph such that G^* is complete. Then $d_G(u) \neq d_F(u)$ for all $u, v \in V$.

Remark 3.9: [15] The above proposition is not true for general fuzzy tree and other fuzzy Labeling trees.

Proposition 3.10: [15] Let G be a FLT such that G^* is a cycle then it has $(n-1)$ bridges.

Proposition 3.11: [15] If G is a FLT such that G^* is a cycle then its spanning subgraph F has $(n-1)$ fuzzy bridges. The above is true for all fuzzy labeling trees.

Proposition 3.12: [15] Let G be a FLT such that G^* is complete. Then every fuzzy bridge of G is strong and the converse is also true.

Proposition 3.13: [15] Let $G = (\sigma, \mu)$ be a FLT and $F = (\sigma, \gamma)$ be the fuzzy spanning subgraph of G , for all (x, y) not in F then $\gamma^\infty(u, v) \neq \text{height of } G$.

Proposition 3.14: [15] Every bridge is strong, but a strong arc need not be a bridge.

Proposition 3.15: [15] If G is a FLT, such that G^* is complete with $|V| \geq 5$, then the spanning subgraph F has at least two disjoint strong paths.

Proposition 3.16: [15] If G is a FLT then there exists a unique strong path between any two vertices of G .

Definition 3.17: [15] A FLG $G = (\sigma, \mu)$ is bipartite if the vertex set V can be partitioned into two non-empty sets V_1 and V_2 such that V_1 and V_2 are fuzzy independent sets.

Proposition 3.18: [15] If G is a connected FLG then there exists a strong path between any pair of vertices.

Proposition 3.19: [15] Every FLT is fuzzy bipartite graph.

Proposition 3.20: [15] If G is a FLT such that G^* is $K_{1,n}^*$, then G is a Complete bipartite graph.

Corollary 3.21: [15] Every FLG is not a complete bipartite graph. Also $K_{2,n}^*$ is not a Complete bipartite graph.

Proposition 3.22: [15] Let $G = (\sigma, \mu)$ be a fuzzy graph such that G^* is a cycle. Then a vertex is a fuzzy cut vertex of G iff it is a common vertex of two fuzzy bridges.

Proposition 3.23: [15] If G is a FLG with $n \geq 4$, it has at least one vertex as cut vertex in each independent set.

An algorithm for finding the spanning subgraph F of a FLT G , such that G^* is complete is given in [15]. Note that, the resulting graph is the spanning subgraph F of a fuzzy labeling graph whose arcs are fuzzy bridges.

Partial Tree 3.24:

A Connected FLG $G = (\sigma, \mu)$ is called partial tree if G has a spanning subgraph $F = (\sigma, \mu')$ which is a tree, where for all arcs (x, y) of G which are not in F , we have $CONN_G(x, y) > \mu(x, y)$.

When the graph G is not Connected and the condition is satisfied by all components of G , then G is called partial forest.

Theorem 3.25: [18] A Connected FLG $G = (\sigma, \mu)$ is called partial tree iff any cycle C of G , there exists an arc $e = (x, y)$ such that $\mu_e < CONN_{G-e}(x, y)$, where $G-e$ is the subgraph of G obtained by deleting the arc e from G .

Theorem 3.26: [18] If G is a Partial tree and is not a tree, then there exists at least one arc (u, v) for which $\mu(u, v) < CONN_G(u, v)$.

Theorem 3.27: [18] If G is a Partial tree and F , the spanning tree in the definition, then the arcs of F are the Partial bridges of G .

IV Blocks in Fuzzy Labeling Graphs:

The concept of block was introduced by A. Rosenfeld [22] and an excellent study on this can be found in [24, 25, 26]. Rajab Ali Borzooei et al., studied Partial blocks in "A Study on fuzzy labeling graphs".

Definition 4.1: [22] A fuzzy graph is said to be a block (also called non-separable) if it is connected and has no fuzzy cut vertices. Note that in a graph, a block cannot have bridge. But in fuzzy graphs, a block may have fuzzy bridges.

Definition 4.2: [18] Let G be a FLG. A vertex w is called a Partial cut vertex (P-cut vertex) of G if there exists a pair of vertices u, v in G such that $u \neq v \neq w$ and $CONN_{G-w}(u, v) < CONN_G(u, v)$. A Connected FLG having no P-cut vertices is called a Partial block (P-block).

Theorem 4.3: [18] A Connected FLG G is a partial block iff any two vertices $u, v \in V(G)$ such that (u, v) is not α -strong and joined by two internally disjoint strongest paths.

Theorem 4.4: [18] If G is P-block then the following conditions hold and are equivalent:

- Every two vertices of G lie on a common strong Cycle.
- Each vertex and a strong arc of G lie on a common strong Cycle.
- Any two strong arcs of G lie on a common strong Cycle.
- For two given vertices and a strong arc in G , there exists a strong path joining the vertices containing the arc.
- For every three distinct vertices of G , there exists strong paths joining any two of them containing the third.
- For every three vertices of G , there exists strong paths joining any two of them which does not contain the third.

V. Distances in Fuzzy labeling graphs:

In this chapter, we discuss four distances which are a metric in FLG namely, ω -distance d_ω , Strong geodesic distance d_{sg} , Strongest strong distance d_{ss} and δ -distance. They are all different metrics in FLGs.

Theorem 5.1.1: [16] If G is a FLG, such that $\mu(x, y) \in G$ is a bridge of G , then $\delta(x, y) = \frac{1}{\mu(x, y)}$.

Theorem 5.1.2: [16] If G is a FLG, such that G^* is a cycle, then $\delta(x, y) = \frac{1}{\mu(x, y)}$ for all $(x, y) \in V \times V$.

Observation 5.1.3: [16]The Maximum μ -length of a fuzzy labeling graph G is unique.

Proposition 5.1.4: [16]If the maximum μ -length of a graph G exists between the vertices u and v then there exists a strongest path between the vertices u and v .

Proposition 5.1.5: [16]Let G be a FLG such that G^* is a cycle. If $\{e(x), e(y)\} = \text{diameter of } G$ then there exists a weakest arc of G between the vertices of x and y .

Theorem 5.1.6: [16]Let G be a FLG, if $\mu(x, y) \in V \times V$ such that $\mu(x, y) < \mu_i$'s then $\max\{e(x), e(y)\} = \text{diam of } G$.

Remark 5.1.7: The converse is not true.

5.2 Center in Fuzzy Labeling Graphs:

Theorem 5.2.1: [16]If $x \in G$ be the center of G , then $r(G) = e(x_i)$ for all $\mu(x, y_i) > 0$. This is not necessary for a vertex to be the center.

Theorem 5.2.2: [16]The FLG G has exactly only one center, if G^* is complete. This is not true if G^* is a cycle.

Theorem 5.2.3: [16]If $x \in G$ is the center of G , then it is a cut vertex of G and the converse is not true.

Proposition 5.2.4: [16]Removal of a center reduces the strength of connectedness between the vertices.

Proposition 5.2.5: [16]If $x \in G$ is the center of a Fuzzy labeling graph, then it is not super strong of vertex of G .

5.3 Diametrical & Eccentric vertices in Fuzzy Labeling Graphs:

Proposition 5.3.1: [16]If each vertex of a Fuzzy labeling graph G is eccentric, then it has two centers.

Proposition 5.3.2: [16]If G is a FLG with exactly one center $x \in G$, then x is not an eccentric vertex of G .

Proposition 5.3.3: [16]If G is a FLG then the diametrical vertices of G are eccentric vertices of G .

Proposition 5.3.4: [16]If G is a FLG such that G^* is a tree, then the diametrical vertices are the only eccentric vertices in G .

Proposition 5.3.5: [16]If G is a FLG such that G^* is a path then the end vertices of G are the diametrical vertices of G .

Definition 5.3.6: [20]Let G be a FLG. The ω -distance between two distinct vertices u and v in G , denoted by $d_\omega(u, v)$ is defined as the smallest ω -length of any $u - v$ path, where ω -length of a path $P = u_0, u_1, \dots, u_n$ is denoted as $L_\omega(P) = \sum_{i=1}^n \frac{1}{\mu(u_{i-1}, u_i)}$. Also $d_\omega(u, v) = 0$ for every vertex u in G . If u and v are not connected by a path, then $d_\omega(u, v) = \infty$.

Theorem 5.3.7: [20]Let G be a FLG with vertex set V then d_ω is a metric on V .

Definition 5.3.8: [20]The strong Geodesic distance or sg -distance between two vertices u and v in a FLG G denoted by $d_{sg}(u, v)$, is defined as the length of the shortest $u - v$ strong path. If u and v are not connected by a path, then $d_{sg}(u, v) = \infty$.

Theorem 5.3.9: [20]Let G be a FLG with vertex set V then $d_{sg}(u, v)$ is a metric on V .

Definition 5.3.10: [20] Let G be a FLG. The strongest strong distance between two vertices u and v in G , denoted by $d_{ss}(u, v)$, is defined as $d_{ss}(u, v) = \frac{1}{CONN_G(u, v)}$ and $d_{ss}(u, v) = 0$ for all $u \in V$. If G is disconnected and two vertices (say) u and v are not connected by a path, then $CONN_G(u, v) = 0$ and $d_{ss}(u, v) = \infty$.

Theorem 5.3.11: [20]Let G be a FLG with vertex set V then $d_{ss}(u, v)$ is a metric on V .

Definition 5.3.12: [20]The δ -distance between two vertices u and v in connected FLG G , denoted by $\delta(u, v)$, is defined as $\delta(u, v) = 1 + \Delta_\omega - CONN_G(u, v)$ where Δ_ω is the maximum member degree of all arcs and $\delta(u, v) = 0$ for every $u \in V$.

Theorem 5.3.13: [20] δ -distance in a connected FLG G with vertex set V is a metric on V .

Definition 5.3.14: [20]A connected FLG G is self-centered with respect to the metric d if each vertex is a central vertex with respect to d .

Theorem 5.3.15: [20]Let G be a connected FLG and d be any one of the metric d_w, d_{sg}, d_{ss} or δ , then G is self-centered with respect to d if $CONN_G(u, v) = \mu(u, v)$ for all $u, v \in V$ and

$$(i) \quad r d_w(G) = \frac{1}{\mu_0}, \text{ where } \mu_0 \text{ is the minimum membership degree of all arcs in } G.$$

$$(ii) \quad r_{sg}(G) = 1.$$

$$(iii) \quad r_{ss}(G) = \frac{1}{\mu_0}, \text{ where } \mu_0 \text{ is the least among all the membership degree of all arcs in } G$$

$$(iv) \quad r\delta(G) = 1 + \Delta_\omega - \mu_0, \text{ where } \mu_0 \text{ is the least among all the membership degree of all arcs in } G.$$

Theorem 5.3.16: [20] Every Connected FLG G is ss self-centered as well as self-centered.

Finally [20] give some applications of distance in the FLG.

VI. Distances in m-polar Fuzzy Labeling Graphs:

M.Akram and Arooj adeel introduced the concept of m-polar Fuzzy sets in labeling graphs and introduce the concept of m-polar FLG and properties of m-polar FL Cycle and m-polar FL with bridge and cut vertex has discussed. They introduced several types of distances in m-polar FLG.

Definition 6.1.1: [1] An m -polar fuzzy set (or a $[0,1]^m$ -set) on V is a mapping $C: V \rightarrow [0,1]^m$. The set of all m -polar fuzzy sets on V is denoted by $m(V)$.

Akram and Neha [35] defined m -polar fuzzy graphs. Chen.J et al., [34] defined m -polar fuzzy sets.

Definition 6.1.2: [1] An arc (x, y) is called as m -polar fuzzy bridge of G if its removal reduces the strength of connectedness between some other pair of vertices in G .

Definition 6.1.3: [1] A vertex is an m -polar fuzzy cut vertex of G if its removal reduces the strength of connectedness between some other pair of vertices in G .

Definition 6.1.4: [1] A vertex x is an m -polar fuzzy end vertex of G if it has exactly one strong neighbour of G .

Definition 6.1.5: [1] An arc (x, y) of an m -polar fuzzy graph is called strong arc if its weight is as great as the strength of connectedness of its m -polar fuzzy end vertices.

Definition 6.1.6: [1] An m -polar fuzzy path $x - y$ is said to be strongest m -polar fuzzy path if its strength equal to its connectedness.

Definition 6.1.7: [1] An m -polar fuzzy weakest arc is an arc having least degree of membership.

Definition 6.1.8: [1] A graph $G_p^\omega = (C_p^\omega, D_p^\omega)$ is said to be an m -polar FLG if $C_p^\omega: V \rightarrow [0,1]^m$ and $D_p^\omega: V \times V \rightarrow [0,1]^m$ are bijective such that the membership values of vertices and edges are distinct and $p_i o D_p^\omega(x, y) < p_i o C_p^\omega(x) \wedge p_i o C_p^\omega(y)$ for all $(x, y) \in V, i = 1, 2, \dots, m$.

Definition 6.1.9: [1] An m -polar FLG $H_p^\omega = (E_p^\omega, F_p^\omega)$ is called an m -polar FL Subgraph of $G_p^\omega = (C_p^\omega, D_p^\omega)$ if $p_i o E_p^\omega(x) \leq p_i o C_p^\omega(x)$ for all $x \in V$ and $p_i o F_p^\omega(x, y) \leq p_i o D_p^\omega(x, y)$ for all $(x, y) \in V$.

Proposition 6.1.10: [1] If H_p^ω is an m -polar FL Subgraph of G_p^ω then $p_i o F_p^\omega(x, y) \leq p_i o D_p^\omega(x, y)$ for all $(x, y) \in V$.

Definition 6.1.11: [1] A Cycle is said to be an m -polar FL Cycle if it has an m -polar FL.

Definition 6.1.12: [1] A Star in m -polar FG can be defined as having two m -polar fuzzy vertex sets x and y with $|X| = 1$ and $|Y| > 1$, such that $p_i o D(xy_j) > 0$ and $p_i o D(y_j y_{j+1}) = 0, 1 \leq j \leq n$. It is denoted by $S_p(1, n)$.

Theorem 6.1.13: [1] Let G^* be a cycle and $e_1, e_2, \dots, e_n$ are edges of an m -polar FL cycle. If the degree of membership of these edges are in increasing order. (ie), $u_1, u_2, \dots, u_m < v_1, v_2, \dots, v_m < \dots < w_1, w_2, \dots, w_m$. Then G_p^ω has exactly one m -polar fuzzy weakest arc.

Theorem 6.1.14: [1] If G^* is a cycle with m -polar FL and $u_i < v_i < \dots < w_i$ for all $i=1, 2, \dots, m$ are degree of membership of edges then G_p^ω has $(n-1)$ m -polar fuzzy bridges.

Proposition 6.1.15: [1] If G_p^ω is an m -polar FL cycle then a vertex is called an m -polar fuzzy cut vertex iff it is a common vertex of two m -polar fuzzy bridges.

Theorem 6.1.16: [1] If G^* is an m -polar FL cycle and $u_i < v_i < \dots < w_i$ for all $i=1, 2, \dots, m$. Then G_p^ω has $(n-2)$ m -polar fuzzy cut vertices.

Theorem 6.1.17: [1] If G^* is an m -polar FL cycle and $u_i < v_i < \dots < w_i$ for all $i=1, 2, \dots, m$. Then G_p^ω has exactly two m -polar fuzzy end vertices.

Theorem 6.1.18: [1] If G^* is an m -polar FL cycle then the vertices of C are either m -polar fuzzy cut vertices or m -polar fuzzy end vertices.

Theorem 6.1.19: [1] If the degree of membership of edges of an m -polar FL cycle G_p^ω is an increasing order such that $u_i < v_i < \dots < w_i$ for all $i=1, 2, \dots, m$, then every m -polar fuzzy bridge is strong.

Theorem 6.1.20: [1] If the degree of membership of edges of an m -polar FL cycle G_p^ω is an increasing order (ie), $u_1, u_2, \dots, u_m < v_1, v_2, \dots, v_m < \dots < w_1, w_2, \dots, w_m$. Then for every G_p^ω , there exists atleast one m -polar fuzzy weakest arc.

Theorem 6.1.21: [1] If the degree of membership of edges of an m -polar FL cycle G_p^ω is an increasing order (ie), $u_1, u_2, \dots, u_m < v_1, v_2, \dots, v_m < \dots < w_1, w_2, \dots, w_m$. Then for every G_p^ω , there exists atleast one m -polar fuzzy bridge

Theorem 6.1.22: [1] For an m -polar FLG G_p^ω . In which degree of membership of edges are of form $(u_i < v_i < \dots < w_i)$ for all $i=1, 2, \dots, m$. Then there exists atleast one m -polar fuzzy cut vertex.

Theorem 6.1.23: [1] For an m -polar FLG G_p^ω . In which degree of membership of edges are of form $(u_i < v_i < \dots < w_i)$ for all $i=1, 2, \dots, m$. Then there exists atleast one m -polar fuzzy end vertices.

6.2 Distances in m -polar Fuzzy Labeling Graphs:

Definition 6.2.1: [1] Let G_p^ω be a connected m -polar FLG. The ω -distance in m -polar FLG is the shortest ω -length of any $x - y$ m -polar fuzzy path P in G_p^ω , where $(x, y) \in V$.

$$d_p^\omega(xy) = (d_{p_1}^\omega(xy), d_{p_2}^\omega(xy), \dots, d_{p_m}^\omega(xy)),$$

$$d_{p_i}^\omega(xy) = \inf\{l_{w_i}(p): P \text{ is a } x - y \text{ path}\}.$$

Where ω -length of an m -polar fuzzy path P is defined as $l_{w_i}(p) = \sum_{j=1}^{n-1} \frac{1}{p_i o D(x_j x_{j+1})}, i=1, 2, \dots, m$. Also $d_p^\omega(xx) = 0$ for each vertex $x \in G_p^\omega$. If x and y are not connected by an m -polar fuzzy path, then $d_p^\omega(xy) = \infty$.

Theorem 6.2.2: [1] Let G_p^ω be an m -polar FLG with vertex set V then d_p^ω is a metric on V .

Definition 6.2.3: [1] Let G_p^ω be connected m-polar FLG, the strongest strong distance between two vertices x and y in G_p^ω is defined as $d_p^{ss} = \frac{1}{CONN_{G_p^\omega}(xy)}$ where $CONN_{G_p^\omega}(xy) = (p_1 o D(xy))^\infty$,

$$= (p_2 o D(xy))^\infty, \dots, \\ = (p_m o D(xy))^\infty.$$

Also $d_p^{ss}(xx)=0$ for each vertex $x \in G_p^\omega$. If x and y are not connected by an m-polar fuzzy path, then $d_p^{ss}(xy) = \infty$.

Theorem 6.2.4: [1] Let G_p^ω be an m-polar FLG with vertex set V . Then strongest strong distance d_p^{ss} in m-polar FLG is metric on V .

Definition 6.2.5: [1] Let G_p^ω be connected m-polar FLG, the strong geodesic distance between two vertices x and y in G_p^ω is defined as the length of the shortest x - y m-polar fuzzy strong path. If x and y are not connected by an m-polar fuzzy path, then $d_p^{sg}(xy) = \infty$.

Theorem 6.2.6: [1] The strong geodesic distance d_p^{sg} in connected m-polar FLG G_p^ω with vertex set V is a metric on V .

Definition 6.2.7: [1] Let G_p^ω be connected m-polar FLG, the distance d_p^δ between two vertices x and y in G_p^ω is defined as $d_p^\delta(x, y) = 1 + \Delta_p^\omega - CONN_{G_p^\omega}(x, y)$ where $\Delta_p^\omega = \max\{p_i o D(xy), \forall x, y \in V\}$. Also $d_p^\delta(x, x) = 0$ for each vertex $x \in G_p^\omega$. If x and y are not connected by an m-polar fuzzy path, then $d_p^\delta(x, y) = \infty$.

Theorem 6.2.8: [1] δ -distance d_p^δ is connected m-polar FLG with vertex set V is a metric on V .

Definition 6.2.9: [1] A Connected m-polar FLG G_p^ω is self-centered with respect to any metric d_p , if each vertex of G_p^ω is a central vertex. That is $r(G_p^\omega) = e(v)$ for all $v \in V$.

Theorem 6.2.10: [1] Let G_p^ω be connected m-polar FLG and d be any one of the metric $d_p^\omega, d_p^{sg}, d_p^{ss}, d_p^\delta$ then G_p^ω is self-centered with respect to d if $CONN_{G_p^\omega} = p_i o D_{p_o}^\omega(xy)$ for all $x, y \in V$, $i=1, 2, \dots, m$ and

- (i) $rd_p^\omega(G_p^\omega) = \frac{1}{p_i o D_{p_o}^\omega}$, where $p_i o D_{p_o}^\omega$ is the minimum membership degree of all edges in G_p^ω .
- (ii) $rd_p^{sg}(G_p^\omega) = 1$.
- (iii) $rd_p^{ss}(G_p^\omega) = \frac{1}{p_i o D_{p_o}^\omega}$, where $p_i o D_{p_o}^\omega$ is the minimum membership degree of all edges in G_p^ω .
- (iv) $rd_p^\delta(G_p^\omega) = 1 + \Delta_p^\omega - p_i o D_{p_o}^\omega$, where $p_i o D_{p_o}^\omega$ is the minimum membership degree of all edges in G_p^ω .

Theorem 6.2.11: [1] Every Connected m-polar FLG G_p^ω is ss-self-centered as well as δ -self-centered.

VII. Bipartite m-polar Fuzzy Labeling Tree:

Akram.M and Arooj adeel introduced the concept of m-polar Fuzzy labeling tree G_p^ω generated by m-polar spanning subgraph S_p^ω and investigate some of its properties. They presented the concepts of bipartite m-polar FLGs.

Definition 7.1: [2] A bipartite m-polar FLG $G_p^\omega = (C_p^\omega, D_p^\omega)$ is defined as, if set of vertices X can be distributed into two non-empty m-polar fuzzy independent sets X_1 and X_2 , where as two vertices of an m-polar fuzzy graph are called m-polar fuzzy independent. If there does not exist any strong arc between them.

Proposition 7.2: [2] If any pair of vertices, there will be a strong m-polar fuzzy path if G_p^ω is connected m-polar FLG.

Proposition 7.3: [2] Every m-polar FL tree is a bipartite m-polar FG.

Proposition 7.4: [2] If G^* is $K_{1,n}^*$ and G_p^ω is an m-polar FL tree, then G_p^ω is a complete bipartite m-polar fuzzy graph.

Remark 7.5 : [2] Every m-polar FLG is not a complete bipartite m-polar fuzzy graph.

Algorithm for finding m-polar fuzzy spanning subgraph S_p^ω of an m-polar FL tree G_p^ω , when degree of membership of edges are in increasing order such that $e_1 < e_2 < \dots < e_n$ and $e_i = (r_1, r_2, \dots, r_m)$, where G^* is complete.

VIII. Other types of Fuzzy Labeling Graphs:

Mishra S.N. and Anita pal [10] introduced the concept of magic labeling of interval-valued fuzzy graph. It is well-known about the concept of magic square, its beauty and applications, from which the concept of magic labeling is inspired. Discussing the significant perception over magic labeling of interval-valued fuzzy graph and obtained some of its properties and also incur some structures and bring it into the operation of interval-valued fuzzy magic labeling graph. They have established neighborhood intervals and obtained some bounds over the size and shape of the interval-valued fuzzy graph and confine the membership values of the nodes and edges.

Kishore Kumar.P.K. et al., [9] introduced the concept of interval valued intuitionistic fuzzy graph and define magic labeling of interval valued intuitionistic fuzzy graph.

Ameenal Bibi.K and Devi.M [5] discussed the concept of fuzzy Bi-Magic labeling in Graphs. They defined Fuzzy Bi-Magic labeling for cycle and star graph. Further, the Properties of such labeling on these graphs have been investigated. A fuzzy graph $G: (\sigma, \mu)$ is known as fuzzy Bi-Magic graph if there exists two bijective functions $\sigma: V \rightarrow [0,1]$ and $\mu: V \times V \rightarrow [0,1]$ such that $\mu(u, v) < \sigma(u) \wedge \sigma(v)$ with the property that the sum of the labels on the vertices and the labels of their incident edges is one of the constants either k_1 or k_2 , independent of the choice of vertex and also investigated that fuzzy Cycle graphs and fuzzy Star graphs are fuzzy Bi-Magic graphs. Further, some properties related to fuzzy bridge and fuzzy cut node have been discussed. The concept of Bi-Magic labeling of Interval valued fuzzy graph was discussed in [4]. They attained neighborhood intervals and defined the membership values of the vertices and edges. The concept of interval valued intuitionistic fuzzy graphs was discussed and defined Bi-Magic labeling of interval valued intuitionistic fuzzy graphs in [6]. They have also investigated the existence of some bounds over the size and shape of the interval-valued intuitionistic fuzzy graphs based on δ -neighborhood and confined the membership values of the vertices and edges of them. Jebasty Shajila and Vimala [31,32,33] have introduced Fuzzy graceful labeling. A graph which admits a fuzzy graceful labeling is called a fuzzy graceful graph. Some of the authors have contributed their exertion in some more concepts of Fuzzy Labeling graphs like Hesitancy fuzzy magic labeling, Matching in FLG, Energy in FLG and etc.

IX Conclusion:

In this Paper, we have made an extensive survey of selected recent results in Fuzzy Labeling Graphs (FLGs). Fuzzy graphs play an important role in many fields including Decision making, Computer networking, Artificial intelligence and Operations Research. we also discussed results related to Fuzzy Labeling Graphs like Blocks, Distances, Center, m-polar, Magic, Bi-magic and some more types.

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