



RELATION BETWEEN COLOUR LAPLACIAN ENERGY AND COLOUR SIGNLESS LAPLACIAN ENERGY OF A COMPLETE GRAPH

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Abstract: Let G be a simple, finite, connected and undirected graph of order n and size m . Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be the eigenvalues of the colour adjacency matrix of G , and let $\mu_1, \mu_2, \dots, \mu_n$ be the eigenvalues of the colour Laplacian matrix. Then, the colour signless Laplacian energy of G is defined as $LE_C^+(G) = \sum_{i=1}^n \left| \mu_i - \frac{2m}{n} \right|$, where $\frac{2m}{n}$ is the average degree of the graph G . In this paper, we found the relation between colour Laplacian energy and colour signless Laplacian energy of complete graph and also attained their bounds.

Keywords: complete graph, colour signless Laplacian matrix, colour signless Laplacian energy of a graph.

I. Introduction

Let G be a simple, finite, connected and undirected graph with vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$ and edges $E(G) = \{e_1, e_2, \dots, e_m\}$. Let $A(G)$ and $D(G)$ be the adjacency matrix and the diagonal matrix with the vertex degrees of the graph G on the diagonal respectively. The matrices $L(G) = D(G) - A(G)$ and $L^+(G) = D(G) + A(G)$ are the signless Laplacian matrices of the graph G . For more results on the spectral properties of signless Laplacian matrix, one may refer to [1],[2],[3],[4],[5],[6],[7]. Let $\{\mu_1, \mu_2, \dots, \mu_n\}$ be the signless eigen values of the graph G .

The signless Laplacian energy [8] of the graph G is defined as $LE^+(G) = \sum_{i=1}^n \left| \mu_i - \frac{2m}{n} \right|$.

A colouring of a graph G is an assignment of colours to its vertices such that no two adjacent vertices share the same colour. The minimum number of colours needed for the colouring of a graph G is called the chromatic number of G and is denoted by $\chi(G)$. Let G be a coloured graph. The entries of the colour adjacency matrix $A_C(G)$ are as follows: If $C(v_i)$ is the colour of vertex v_i , then

$$a_{ij} = \begin{cases} 1, & \text{if } v_i \text{ and } v_j \text{ are adjacent with } C(v_i) \neq C(v_j) \\ -1, & \text{if } v_i \text{ and } v_j \text{ are non adjacent with } C(v_i) = C(v_j) \\ 0, & \text{otherwise} \end{cases}$$

The eigen values $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$ of $A_C(G)$ are called the colour eigen values of G . The colour energy of a graph denoted by $E_C(G)$ is defined as the sum of the absolute values of the colour signless eigen values of G , i.e., $E_C(G) =$

$$\sum_{i=1}^n |\lambda_i|. \text{ In 2013, Ch. Adiga, E. Sampathkumar, M.A. Sriraj and A.S. Shrikanth [17],[18] have studied the energy}$$

of the coloured graph G .

II Preliminaries

A. Proper colouring of a graph

A proper colouring of a graph is an assignment of colours to the vertices of the graph so that no two adjacent vertices have the same colour.

B. Complete graph

A complete graph is a graph in which each pair of graph vertices is connected by an edge. The assignment of colours to the vertices of the complete graph are different to each other. A complete graph K_n of n vertices requires n colours. So the chromatic number of the complete graph K_n is n .

C. Laplacian matrix

Let G be a graph with order n and size m . The Laplacian matrix of the graph G is denoted by $L = (L_{ij})$ is a square matrix defined by

$$L_{ij} = \begin{cases} -1, & \text{if } v_i \text{ and } v_j \text{ are adjacent} \\ 0, & \text{if } v_i \text{ and } v_j \text{ are non adjacent} \\ d_i, & \text{if } v_i = v_j \end{cases}$$

Where d_i is the degree of the vertex v_i .

D. Laplacian energy

Let G be a connected graph of order n with Laplacian eigenvalues

$\mu_1 \geq \mu_2 \geq \dots \geq \mu_{n-1} \geq \mu_n = 0$. The Laplacian is symmetric positive semidefinite. The Laplacian energy of the graph

$$G \text{ is defined as } LE(G) = \sum_{i=1}^n \left| \mu_i - \frac{2m}{n} \right|.$$

E. Signless Laplacian matrix

For a given graph, the matrix $Q = D + A$ is called the signless Laplacian matrix, where A is the adjacency matrix and D is the diagonal matrix of vertex degrees d_i .

F. Colour Laplacian energy

The colour Laplacian matrix of G is defined as $L_C(G) = D(G) - A_C(G)$. The eigen values $\{\mu_1, \mu_2, \dots, \mu_n\}$ of $L_C(G)$ are called colour Laplacian eigen values of the graph G .

$$\text{The colour Laplacian energy of } G \text{ denoted by } LE_C(G) = \sum_{i=1}^n \left| \mu_i - \frac{2m}{n} \right|.$$

G. Colour signless Laplacian energy

Let G be a coloured graph on n vertices and m edges. We defined the colour signless Laplacian matrix of G as $L_C^+(G) = D(G) + A_C(G)$. The eigen values $\{\mu_1^+, \mu_2^+, \dots, \mu_n^+\}$ of $L_C^+(G)$ are called colour signless Laplacian eigen values of the graph G .

Let G be a coloured graph of order n and size m . Then the colour signless Laplacian energy of G , denoted by

$$LE_C^+(G) = \sum_{i=1}^n \left| \mu_i - \frac{2m}{n} \right|.$$

H. Spectrum of G

The set of graph eigen values of the adjacency matrix is called the spectrum of the graph.

I. Colour spectrum of G

The sum of the absolute values of the distinct colour eigen values of $A_C(G)$ are If $\mu_1 > \mu_2 > \dots \mu_{n-1} > \mu_r$, $r \leq n$ with their multiplicities are $m_1, m_2, \dots, m_r$ is defined as color spectrum of G is written by

$$\text{Spec}_C(G) = \left\{ \begin{matrix} \mu_1, & \mu_2, & \dots & \mu_r \\ m_1, & m_2, & \dots & m_r \end{matrix} \right\}.$$

J. Colour Laplacian spectrum of G

If $\mu_1 > \mu_2 > \dots \mu_{n-1} > \mu_r$, $r \leq n$ are the distinct colour Laplacian eigen values of coloured graph with their multiplicities are $m_1, m_2, \dots, m_r$ is defined as colour Laplacian spectrum of a graph G is written as

$$L\text{Spec}_C(G) = \left\{ \begin{matrix} \mu_1, & \mu_2, & \dots & \mu_r \\ m_1, & m_2, & \dots & m_r \end{matrix} \right\} \text{ color spectrum of } G.$$

III Results

A. Lemma: If $\{\lambda_1, \lambda_2, \dots\}$ is the colour spectrum of a k -regular graph G , then $\{k - \lambda_1, k - \lambda_2, \dots, k - \lambda_n\}$ is the colour Laplacian spectrum of G .

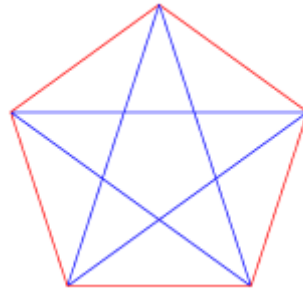
B. Lemma: If the graph G is regular, then $LE_C(G) = E_C(G)$.

C. Lemma: For $n \geq 2$, the colour Laplacian Energy of complete graph is $2(n-1)$.

D. Lemma: If $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$ is the colour spectrum of a k -regular graph G , then $\{k + \lambda_1, k + \lambda_2, \dots, k + \lambda_n\}$ is the colour signless Laplacian spectrum of G .

IV Relation between colourlaplacian energy and coloursignlesslaplacian energy of a complete graph.

Each vertex colour of a complete graph are different because they are adjacent with each other, so the chromatic number of complete graph is n . Consider complete graph K_5 which is given below:



Fig(i) $G=K_5$

The colourlaplacian matrix of k_5 is as follows:

$$L_C(k_5) = \begin{pmatrix} 4 & -1 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 & -1 \\ -1 & -1 & 4 & -1 & -1 \\ -1 & -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & -1 & 4 \end{pmatrix}$$

The characteristic equation is $\mu(\mu - 5)^5 = 0$.

The eigen values of k_5 are 0, 5, 5, 5, 5, 5.

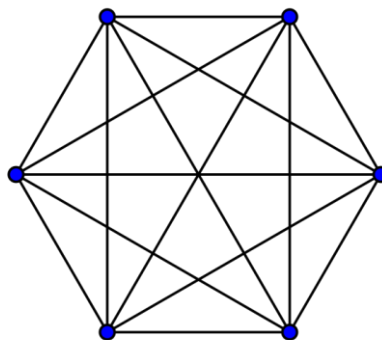
The coloursignlesslaplacian matrix for complete graph K_5 is

$$L_C^+(k_5) = \begin{pmatrix} 4 & 1 & 1 & 1 & 1 \\ 1 & 4 & 1 & 1 & 1 \\ 1 & 1 & 4 & 1 & 1 \\ 1 & 1 & 1 & 4 & 1 \\ 1 & 1 & 1 & 1 & 4 \end{pmatrix}$$

The Characteristic equation is $(\mu-3)^4(\mu-8)=0$

The eigen values are $\mu^+ = 3$ (4 times), $\mu^+ = 8$

consider the following graph K_6 :



Fig(ii) $G=k_6$

The colourlaplacian matrix of k_6 as follows :

$$L_C(k_6) = \begin{pmatrix} 5 & -1 & -1 & -1 & -1 & -1 \\ -1 & 5 & -1 & -1 & -1 & -1 \\ -1 & -1 & 5 & -1 & -1 & -1 \\ -1 & -1 & -1 & 5 & -1 & -1 \\ -1 & -1 & -1 & -1 & 5 & -1 \\ -1 & -1 & -1 & -1 & -1 & 5 \end{pmatrix}$$

The characteristic equation is $\mu(\mu - 6)^5 = 0$

The eigen values are 0, 6, 6, 6, 6, 6

The coloursignlesslaplacian matrix of k_6 is as follows :

$$L_C^+(k_6) = \begin{pmatrix} 5 & 1 & 1 & 1 & 1 & 1 \\ 1 & 5 & 1 & 1 & 1 & 1 \\ 1 & 1 & 5 & 1 & 1 & 1 \\ 1 & 1 & 1 & 5 & 1 & 1 \\ 1 & 1 & 1 & 1 & 5 & 1 \\ 1 & 1 & 1 & 1 & 1 & 5 \end{pmatrix}$$

The Characteristic equation is $(\mu - 4)^5(\mu - 10) = 0$

The eigen values are $\mu = 4$ (5 times), $\mu = 10$

Continuing the above process, the colourlaplacian matrix of complete graph k_n is

$$L_C(k_n) = \begin{pmatrix} n-1 & -1 & -1 & \dots & -1 \\ -1 & n-1 & -1 & \dots & -1 \\ & -1 & -1 & n-1 & \dots & -1 \\ & \dots & \dots & \dots & \dots & \dots \\ & -1 & -1 & -1 & \dots & n-1 \end{pmatrix}$$

The characteristic equation is $\mu(\mu - n)^{n-1} = 0$.

The eigen values are 0, n ($n-1$) times

The signlesscolourlaplacian matrix of complete graph k_n is as follows :

$$L_C^+(k_n) = \begin{pmatrix} n-1 & 1 & 1 & \dots & 1 \\ 1 & n-1 & 1 & \dots & 1 \\ & 1 & 1 & n-1 & \dots & 1 \\ & \dots & \dots & \dots & \dots & \dots \\ & 1 & 1 & 1 & \dots & n-1 \end{pmatrix}$$

The characteristic equation is $(\mu - (n-1))^{n-1}(\mu - 2(n-1)) = 0$

The eigen values are $[n-1](n-1)$ times, $2(n-1)$

Hence the average degree of complete graph is $n-1$.

The Colour Laplacian Energy of complete graph is

$$L_C(k_n) = |0 - (n-1)| + |n - (n-1)|(n-1)$$

$$= n-1 + n-1$$

$$= 2n - 2$$

$$= 2(n-1).$$

$$L_C^+(k_n) = |(n-2) - (n-1)|(n-1) + |2n-2 - (n-1)|$$

$$= n-1 + n-1$$

$$= 2n - 2$$

$$= 2(n - 1).$$

Thus, the colour Laplacian Energy and signless color Laplacian Energy of the complete graph K_n are the same and it is equal to $2(n-1)$.

V. Bounds for colour energy of graphs:

In this section, we presented some new bounds for the colour energy of a graph in terms of Zagreb index $Z_g(G)$, laplacian energy $LE(G)$ and the signlesslaplacian energy $LE^+(G)$.

Bounds for colour energy in terms of zagreb index:

For a graph G of order n and size m having d_i as the degree of the i^{th} vertex

We know that $\sum_{i=1}^n d_i = 2m \longrightarrow (1)$

then Zagreb index is

$$Z_g(G) = \sum_{i=1}^n d_i^2 \longrightarrow (2)$$

Bounds for colour energy in terms of laplacian energy and signlesslaplacian energy:

In this section, we presented some bounds for colour energy in terms of laplacian energy and the signlesslaplacian energy. The following results proved by Abreu et al. (see [1]).

$$LE^+(G) \leq E(G) + \sqrt{nZ_g(G) - 4m^2} \longrightarrow (3)$$

$$|LE^+(G) - LE(G)| \leq 2E(G) \longrightarrow (4)$$

5.1 Theorem

Let G be a graph with n order and size m . Let $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_n$ be the absolute colour laplacian eigen values of G . If σ_1 is repeated $(n-1)$ times then

$$\sigma_1 \leq \frac{1}{n-1} \left(\sqrt{2 \binom{n-1}{2} + \sum_{i=1}^n (d_i(G))^2 2(n-1)} - \sum_{i=2}^{2(n-1)} \sigma_i \right)_{n \neq 1}$$

Proof:

Here we compare the absolute colour laplacian eigen values of G with the absolute eigen values of the graph $H = (\bigcup_{n-1} (n-1))_{(n, \frac{n-1}{2})}$.

Select n and $\frac{n-1}{2}$ such that $n = (n-1)[n + (\frac{n-1}{2})]$. The number of vertices of H is n and the number of edges is $(n-1)(n)(\frac{n-1}{2})$. Its absolute values of eigen values spectrum is

$$\begin{pmatrix} \sqrt{\frac{n(n-1)}{2}} & 0 \\ 2(n-1) & (n - (n-1)) \end{pmatrix}$$

By Cauchy's Schwarz inequality,

$$\sigma_1 \sqrt{\frac{n(n-1)}{2}} + \dots + \sigma_{n-1} \sqrt{\frac{n(n-1)}{2}} + \sigma_n \sqrt{\frac{n(n-1)}{2}} + \dots + \sigma_{2(n-1)} \sqrt{\frac{n(n-1)}{2}} + \sigma_{2n-1}(0) \dots + \sigma_n(0) \leq \sqrt{2m + \sum_{i=1}^n (d_i(G))^2 2(n-1)(n)(\frac{n-1}{2})}$$

But $\sigma_1 = \sigma_2 = \dots = \sigma_{n-1}$

$$\therefore \sigma_1(n-1) \sqrt{\frac{n(n-1)}{2}} + \sqrt{\frac{n(n-1)}{2}} \sum_{i=n}^{2(n-1)} \sigma_i \leq \sqrt{2m + \sum_{i=1}^n (d_i(G))^2 2(n-1)(n)(\frac{n-1}{2})}$$

$$\sigma_1(n-1) + \sum_{i=n}^{2(n-1)} \sigma_i \leq \sqrt{2m + \sum_{i=1}^n (d_i(G))^2 2(n-1)}$$

$$\sigma_1 \leq \frac{1}{n-1} \sqrt{2m + \sum_{i=1}^n (d_i(G))^2 2(n-1)} - \sum_{i=n-1}^{2(n-1)} \sigma_i$$

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