



The Z_k -Magic Labeling On Inverse Graphs from Finite Cyclic Group

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Abstract: Let any nontrivial abelian group A under addition a graph G is said to be A -magic if there exists a labeling $f: E(G) \rightarrow A - \{0\}$ such that the vertex labeling $f^+(v)$ defined as $f^+(v) = \sum_{uv \in E(G)} f(uv)$ taken over all edges uv incident at v is a constant. The corresponding constant x is called an A -magic value. An A -magic graph G is said to be Z_k -magic graph if the group A is Z_k , the group of integers modulo k . In this paper, we proved that the inverse graph constructed from finite cyclic group is admits Z_k -magic labeling.

Keywords: A -magic labeling, Z_k -magic labeling, integer-magic spectrum, inverse graph

I. Introduction

Let G be a finite simple connected graph whose vertex set $V(G)$ and edge set $E(G)$ with $|V(G)| = p$ and $|E(G)| = q$. The open neighbourhood $N(v)$ of a vertex $v \in V(G)$ is the set of all vertices which are adjacent to v . A labeling is assignment of integers to the vertices or edges, or both subject to the certain condition. For more about graph labeling, refer the survey [1]. For graph theoretic terminology, we refer [6]. The concept of an A -magic graph was introduced by Sedlacek [4] as follows: A graph with real-valued edge labeling such that distinct edges have distinct non-negative labels and the sum of the labels of the edges incident to a particular vertex is same for all the vertices.

Definition: 1. 1. [3] For any nontrivial abelian group A under addition a graph G is said to be A -magic if there exists a labeling $f: E(G) \rightarrow A - \{0\}$ such that the vertex labeling $f^+(v)$ defined as $f^+(v) = \sum_{uv \in E(G)} f(uv)$ taken over all edges uv incident at v is a constant. The corresponding constant x is called an A -magic value. An A -magic graph G is said to be Z_k -magic graph if the group A is Z_k the group of integers modulo k . These Z_k -magic graphs are referred to as k -magic graphs.

In [5], the integer-magic spectrum of a graph G is the set $IM(G) = \{k \mid G \text{ is } Z_k\text{-magic and } k \geq 1\}$. Note that the integer-magic spectrum of a graph is different from the set of achievable magic values.

Definition: 1. 2. [2] Let $(\Gamma, *)$ be a finite group and $S = \{u \in \Gamma \mid u \neq u^{-1}\}$. We define the inverse graph $GS(\Gamma)$ associated with Γ as the graph whose set of vertices coincides with Γ such that two distinct vertices u and v are adjacent if and only if either $uv \in S$ or $vu \in S$.

Theorem: 1. 1. [2] For any finite abelian group Γ with at least three elements and a non-empty subset S of non-self-invertible elements, then the graph $GS(\Gamma)$ is connected.

Theorem: 1. 2. [2] Let $(Z_n, +)$ be the finite group of integers modulo n under addition.

Case(i): If n is odd. In this case, Z_n has the identity 0 as the only self-invertible element.

$$\deg(v) = \begin{cases} n-1 & \text{if } v = 0 \\ n-2 & \text{if } v \neq 0 \end{cases}$$

Case(ii): If n is even,

$$\deg(v) = \begin{cases} n-2 & \text{if } v \in \{0, \frac{n}{2}\} \text{ or } v^2 \in \{0, \frac{n}{2}\} \\ n-3 & \text{if otherwise.} \end{cases}$$

II. Main results

We prove the following theorems for the inverse graphs constructed from finite cyclic group admit Z_k -magic labeling.

Theorem: 2. 1. Let $n \equiv 1 \pmod{4}$ and $k (\geq 3)$, then the inverse graph $GS(Z_n)$ admits Z_k -magic labeling with magic constant $x = (n-1)a$.

Proof. Since $n \equiv 1 \pmod{4}$, then the only self-invertible element is 0. Let $G = GS(Z_n)$ be the inverse graph associated with finite cyclic group. Let $V(G) = \{v_0, v_1, \dots, v_{n-1}\}$ and $E(G) = \{v_i v_{i+j} : 0 \leq i \leq n-1, 1 \leq j \leq \lfloor n/2 \rfloor \text{ and } j \neq n-2i\}$. Note that, $v_i v_{n-i} \notin E(G)$ for all i ($0 \leq i \leq n-1$). Define the function $f: E(G) \rightarrow Z_k \setminus \{0\}$ by

$$f(v_i v_{i+1}) = \begin{cases} a & \text{if } i = 0, 2, 4, \dots, n-1 \\ 2a & \text{if } i = 1, 3, 5, \dots, n-2 \end{cases}$$

$$f(v_i v_{i+j}) = a \quad \text{if } 0 \leq i \leq n-1, 2 \leq j \leq \lfloor n/2 \rfloor$$

Then the induced vertex set labeling $f^+(v) = \sum_{uv \in E(G)} f(uv)$ is a constant map, by the Theorem 1. 2, we get vertex weight of the induced function f^+ is

Case(i): If $i = 0$. Here $1 \leq j \leq \lfloor n/2 \rfloor$.

$$f^+(v_0) = (n-1)a$$

Case(ii): If $i = 1, 2, \dots, n-1$. Here $1 \leq j \leq \lfloor n/2 \rfloor$.

$$f^+(v_0) = (n-3)a + 2a = (n-1)a$$

By the above cases, we get the constant value x is $(n-1)a$.

Theorem: 2. 2. Let $n \equiv 3 \pmod{4}$ and $n (> 3)$, $k (\geq 3)$ then the inverse graph $GS(Z_n)$ admits Z_k - magic labeling with magic constant $x = (n-1)a$.

Proof. Let $n \equiv 3 \pmod{4}$, then only the invertible element is 0. Let $G = GS(Z_n)$ be the Inverse graph associated with finite cyclic group. Let $V(G) = \{v_0, v_1, \dots, v_{n-1}\}$ and $E(G) = \{v_i v_{i+j} : 0 \leq i \leq n-1, 1 \leq j \leq \lfloor n/2 \rfloor \text{ and } j \neq n-2i\}$. Here the edges $v_i v_{n-i} \notin E(G)$ for all i ($0 \leq i \leq n-1$). Define the function $f: E(G) \rightarrow Z_k \setminus \{0\}$ by

$$f(v_i v_{i+1}) = \begin{cases} a & \text{if } i = 0, 2, 4, \dots, \lfloor n/2 \rfloor - 1 \\ 2a & \text{if } i = 1, 3, 5, \dots, \lfloor n/2 \rfloor - 2 \\ 2a & \text{if } i = \lfloor n/2 \rfloor + 1, \lfloor n/2 \rfloor + 3, \dots, n-3 \\ a & \text{if } i = \lfloor n/2 \rfloor + 2, \lfloor n/2 \rfloor + 4, \dots, n-2 \\ a & \text{if } i = n-1 \end{cases}$$

$$f(v_i v_{i+\lfloor n/2 \rfloor}) = \begin{cases} a & \text{if } i = 0, 1, 2, \dots, \lfloor n/2 \rfloor - 1 \\ 2a & \text{if } i = \lfloor n/2 \rfloor \\ 2a & \text{if } i = \lfloor n/2 \rfloor + 1, \lfloor n/2 \rfloor + 2, \dots, n-1 \end{cases}$$

$$f(v_i v_{i+j}) = a \quad \text{if } 0 \leq i \leq n-1, 2 \leq j \leq \lfloor n/2 \rfloor - 1$$

Then the induced vertex set labeling $f^+(v) = \sum_{uv \in E(G)} f(uv)$ is a constant map, by the Theorem 1. 2, we get vertex weight of the induced function is

Case(i): If $i = 0$. Here $1 \leq j \leq \lfloor n/2 \rfloor$.

$$f^+(v_0) = (n-1)a$$

Case(ii): If $i = 1, 2, \dots, n-1$. Here $1 \leq j \leq \lfloor n/2 \rfloor$.

$$f^+(v_0) = (n-3)a + 2a = (n-1)a$$

By the above cases, we get the constant value x is $(n-1)a$.

Theorem: 2. 3. Let $n \equiv 2 \pmod{4}$ and $k (\geq 3)$, then the inverse graph $GS(Z_n)$ admits Z_k - magic labeling with magic constant $x = (n-2)a$.

Proof. Let $n \equiv 2 \pmod{4}$, then only the invertible element is 0. Let $G = GS(Z_n)$ be the Inverse graph associated with finite cyclic group. Let $V(G) = \{v_0, v_1, \dots, v_{n-1}\}$ and $E(G) = \{v_i v_{i+j} : 0 \leq i \leq n-1, 1 \leq j \leq \lfloor n/2 \rfloor \text{ and } j \neq n-2i\}$. Here, the edges $v_i v_{n-i} \notin E(G)$ for all i ($0 \leq i \leq n-1$). Define the function $f: E(G) \rightarrow Z_k \setminus \{0\}$ by

$$f(v_i v_{i+1}) = a \quad \text{if} \quad 0 \leq i \leq n-1$$

$$f(v_i v_{i+\lfloor n/2 \rfloor}) = 2a \quad \text{if} \quad 0 \leq i \leq n-1$$

$$f(v_i v_{i+j}) = a \quad \text{if} \quad 0 \leq i \leq n-1, 2 \leq j \leq \lfloor n/2 \rfloor - 1$$

Then the induced vertex set labeling $f^+(v) = \sum_{uv \in E(G)} f(uv)$ is a constant map, by the Theorem 1. 2, we get vertex weight of the induced function is

Case(i): If $i = 0$ and $i = n-1$. Here $1 \leq j \leq \lfloor n/2 \rfloor$.

$$f^+(v_i) = (n-2)a$$

Case(ii): If $1 \leq i \leq n-2$. Here $1 \leq j \leq \lfloor n/2 \rfloor$

$$f^+(v_i) = (n-4)a + 2a = (n-2)a$$

By the above cases, we get the constant value x is $(n-2)a$.

Theorem: 2. 4. Let $n \equiv 0 \pmod{4}$ and $k (\geq 3)$, then the inverse graph $GS(Z_n)$ admits Z_k - magic labeling with magic constant $x = (n-2)a$.

Proof. Let $n \equiv 0 \pmod{4}$, then only the invertible element is 0. Let $G = GS(Z_n)$ be the Inverse graph associated with finite cyclic group. Let $V(G) = \{v_0, v_1, \dots, v_{n-1}\}$ and $E(G) = \{v_i v_{i+j} : 0 \leq i \leq n-1, 1 \leq j \leq \lfloor n/2 \rfloor \text{ and } j \neq n-2i\}$. Here the edges $v_i v_{n-1} \notin E(G)$ for all i ($0 \leq i \leq n-1$). Define the function $f: E(G) \rightarrow Z_k \setminus \{0\}$ by

$$f(v_i v_{i+1}) = a \quad \text{if} \quad 0 \leq i \leq n-1$$

$$f(v_i v_{i+\lfloor n/2 \rfloor}) = 2a \quad \text{if} \quad 0 \leq i \leq n-1$$

$$f(v_i v_{i+j}) = a \quad \text{if} \quad 0 \leq i \leq n-1, 2 \leq j \leq \lfloor n/2 \rfloor - 1$$

Then the induced vertex set labeling $f^+(v) = \sum_{uv \in E(G)} f(uv)$ is a constant map, by the Theorem 1. 2 we get vertex weight of the induced function is

Case(i): If $i = 0, n/4, 2n/4, 3n/4$. Here $1 \leq j \leq \lfloor n/2 \rfloor$.

$$f^+(v_i) = (n-2)a$$

Case(ii): If $1 \leq i \leq n-1$ and $i \neq n/4, 2n/4, 3n/4$. Here $1 \leq j \leq \lfloor n/2 \rfloor$.

$$f^+(v_i) = (n-4)a + 2a = (n-2)a$$

By the above cases, we get the constant value x is $(n-2)a$.

III. Concluding Remarks

In this paper, we proved that a class of inverse graphs associated with finite cyclic group admits Z_k - magic labeling. However, the problem remains open to find Z_k -magic labeling for the inverse graphs from arbitrary groups.

References

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