



Inclusion Properties for Certain Subclasses of Analytic Functions Defined by a Differential Operator

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Abstract: In present paper, we define a generalised differential operator and some corresponding subclasses of analytic functions. We discuss inclusion properties using the principle of subordination. Also inclusion properties of these classes involving integral operator are obtained.

Keywords: Analytic functions, p -valent functions, differential subordinations, Hadamard product, differential operator, integral operator.

I. Introduction

Let A_p denote the class of normalized analytic p -valent functions in the unit disk $\mathbb{U} = \{z: z \in \mathbb{C}, |z| < 1\}$ that have the form:

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n. \quad (1.1)$$

For functions $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$ and $g(z) = z^p + \sum_{n=p+1}^{\infty} b_n z^n$ in the class A_p , the Hadamard Product (or convolution) is defined as follows.

$$(f * g)(z) = z^p + \sum_{n=p+1}^{\infty} a_n b_n z^n, \quad z \in \mathbb{U} \quad (1.2)$$

For $\alpha_i, \beta_j \in \mathbb{C} - \{0, -1, -2, \dots\}$ ($j = 1, 2, \dots, m$) the generalised hypergeometric function ${}_l F_m(\alpha_1, \alpha_2, \dots, \alpha_l; \beta_1, \beta_2, \dots, \beta_m; z)$ is defined by the series

$${}_l F_m(\alpha_1, \alpha_2, \dots, \alpha_l; \beta_1, \beta_2, \dots, \beta_m; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n (\alpha_2)_n \dots (\alpha_l)_n}{(\beta_1)_n (\beta_2)_n \dots (\beta_m)_n} \frac{z^n}{n!} \quad (l \leq m+1; l, m \in \mathbb{N}_0)$$

where $(a)_n$ is the Pochhammer symbol defined by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = a(a+1)(a+2) \dots (a+n-1) \text{ for } n \in \mathbb{N}$$

and it is 1 for $n=0$.

Corresponding to the function

$${}_l F_m^p(\alpha_1, \alpha_2, \dots, \alpha_l; \beta_1, \beta_2, \dots, \beta_m; z) = z^p {}_l F_m(\alpha_1, \alpha_2, \dots, \alpha_l; \beta_1, \beta_2, \dots, \beta_m; z)$$

$$= \sum_{n=0}^{\infty} \frac{(\alpha_1)_n (\alpha_2)_n \dots (\alpha_l)_n}{(\beta_1)_n (\beta_2)_n \dots (\beta_m)_n} \frac{z^{n+p}}{n!}$$

We define ${}_l F_m^{p,\alpha}(\alpha_1, \alpha_2, \dots, \alpha_l; \beta_1, \beta_2, \dots, \beta_m; z)$ by using convolution $*$ as

$${}_l F_m^p(\alpha_1, \alpha_2, \dots, \alpha_l; \beta_1, \beta_2, \dots, \beta_m; z) * {}_l F_m^{p,\alpha}(\alpha_1, \alpha_2, \dots, \alpha_l; \beta_1, \beta_2, \dots, \beta_m; z) = z^p (1-z)^{-2(1-\alpha)},$$

where $\alpha < 1$.

Using this definition, we define $L_{l,m}^{p,\alpha}: A_p \rightarrow A_p$ as

$$\begin{aligned} L_{l,m}^{p,\alpha}(\alpha_1, \alpha_2, \dots, \alpha_l; \beta_1, \beta_2, \dots, \beta_m) f(z) &= {}_l F_m^{p,\alpha}(\alpha_1, \alpha_2, \dots, \alpha_l; \beta_1, \beta_2, \dots, \beta_m; z) * f(z) \\ &= z^p + \sum_{n=p+1}^{\infty} \frac{(\beta_1)_{n-p} (\beta_2)_{n-p} \dots (\beta_m)_{n-p}}{(\alpha_1)_{n-p} (\alpha_2)_{n-p} \dots (\alpha_l)_{n-p}} (2-2\alpha)_{n-p} a_n z^n \end{aligned} \quad (1.4)$$

For $l = m + 1$ and $\alpha_2 = \beta_1, \alpha_3 = \beta_2, \dots, \alpha_l = \beta_m$, we note that $L_{l,m}^{1,0}(1, \alpha_2, \dots, \alpha_l; \beta_1, \beta_2, \dots, \beta_m)f(z) = zf'(z)$ and $L_{l,m}^{1,0}(2, \alpha_2, \dots, \alpha_l; \beta_1, \beta_2, \dots, \beta_m)f(z) = f(z)$

On the lines of C.Selvaraj and K.R.Karthikeyan[12], we define the following generalized differential operator $D_{\lambda,\alpha}^{p,q}(\alpha_1, \beta_1): A_p \rightarrow A_p$ by

$$\begin{aligned} D_{\lambda,\alpha}^{p,0}(\alpha_1, \beta_1)f(z) &= L_{l,m}^{p,\alpha}[\alpha_1, \beta_1]f(z) \\ D_{\lambda,\alpha}^{p,1}(\alpha_1, \beta_1)f(z) &= (1 - \lambda)L_{l,m}^{p,\alpha}[\alpha_1, \beta_1]f(z) + \frac{\lambda}{p}z(L_{l,m}^{p,\alpha}[\alpha_1, \beta_1]f(z))' \\ D_{\lambda,\alpha}^{p,q}(\alpha_1, \beta_1)f(z) &= D_{\lambda,\alpha}^{p,1}(\alpha_1, \beta_1)\left(D_{\lambda,\alpha}^{p,q-1}(\alpha_1, \beta_1)f(z)\right) \end{aligned} \quad (1.5)$$

Where $q \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\alpha < 1$ and $\lambda \geq 0$.

If $f(z) \in A_p$, then we have

$$D_{\lambda,\alpha}^{p,q}(\alpha_1, \beta_1)f(z) = z^p + \sum_{n=p+1}^{\infty} \left(\frac{p+(n-p)\lambda}{p} \right)^q \frac{(\beta_1)_{n-p}(\beta_2)_{n-p}\dots(\beta_m)_{n-p}}{(\alpha_1)_{n-p}(\alpha_2)_{n-p}\dots(\alpha_l)_{n-p}} (2 - 2\alpha)_{n-p} a_n z^n \quad (1.6)$$

It can be seen that, by specializing the parameters the operator $D_{\lambda,\alpha}^{p,q}(\alpha_1, \beta_1)f(z)$ reduces to many known and new integral and differential operators. In particular, for $p = 1, l = 2, m = 1, \alpha_1 = \beta_1, \alpha_2 = 1$ and $\alpha = \frac{1}{2}$ it reduces to the operator introduced by F. Al-Oboudi[1]. Further we remark that, when $p = 1, l = 2, m = 1, \alpha_1 = \beta_1, \alpha_2 = 1, \alpha = \frac{1}{2}$ and $\lambda = 1$ the operator $D_{\lambda,\alpha}^{p,q}(\alpha_1, \beta_1)f(z)$ reduces to the operator introduced by G.S.Salagean[1]. It can be easily verified from (1.6) that

$$\lambda z \left(D_{\lambda,\alpha}^{p,q}(\alpha_1, \beta_1)f(z) \right)' = p D_{\lambda,\alpha}^{p,q+1}(\alpha_1, \beta_1)f(z) - p(1 - \lambda) D_{\lambda,\alpha}^{p,q}(\alpha_1, \beta_1)f(z) \quad (1.7)$$

And

$$\alpha_1 D_{\lambda,\alpha}^{p,q}(\alpha_1, \beta_1)f(z) = z \left(D_{\lambda,\alpha}^{p,q}(\alpha_1 + 1, \beta_1)f(z) \right)' + (\alpha_1 - p) D_{\lambda,\alpha}^{p,q}(\alpha_1 + 1, \beta_1)f(z) \quad (1.8)$$

If f and g are analytic in $\mathbb{U}$, we say that f is subordinate to g , written $f < g$, if there exists a Schwarz function $w(z)$, which is analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$, for $z \in \mathbb{U}$. Furthermore, if the function g is univalent in $\mathbb{U}$, then we have the following equivalence (See [3],[6],[7])

$$f < g \Leftrightarrow f(0) = g(0), f(\mathbb{U}) \subset g(\mathbb{U}).$$

For $0 \leq \eta < p, p \in \mathbb{N}$, we know that $S_p^*(\eta), K_p(\eta)$ and C_p are the subclasses of A_p consisting of all analytic functions which are respectively, p -valent starlike of order η , p -valent convex of order η and p -valent close-to-convex functions in $\mathbb{U}$ ([8], [10] and [13]).

Let Λ be the class of all functions ϕ which are analytic and univalent in $\mathbb{U}$ and for which $\phi(\mathbb{U})$ is convex with $\phi(0) = 1$ and $Re(\phi(z)) > 0, z \in \mathbb{U}$.

Making use of the principle of subordination between analytic functions, S.R.Swami [14] introduced the subclasses $S_p^*(\eta; \phi), K_p(\eta; \phi)$ and $C_p(\eta, \rho; \phi, \varphi)$ of the class $A_p, 0 \leq \eta < p, 0 \leq \rho < p$ and $\phi, \varphi \in \Lambda$, which are defined by

$$\begin{aligned} S_p^*(\eta; \phi) &= \left\{ f \in A_p : \frac{1}{p - \eta} \left(\frac{zf'(z)}{f(z)} - \eta \right) < \phi(z), z \in \mathbb{U} \right\}, \\ K_p(\eta; \phi) &= \left\{ f \in A_p : \frac{1}{p - \eta} \left(1 + \frac{zf''(z)}{f'(z)} - \eta \right) < \phi(z), z \in \mathbb{U} \right\} \end{aligned}$$

And

$$C_p(\eta, \rho; \phi, \varphi) = \left\{ f \in A_p : \frac{1}{p - \rho} \left(\frac{zf'(z)}{g(z)} - \rho \right) < \varphi(z), z \in \mathbb{U} \text{ where } g \in S_p^*(\eta; \phi) \right\}$$

Next, by using the operator $D_{\lambda,\alpha}^{p,q}(\alpha_1, \beta_1)$, we introduce the following subclasses of analytic functions for $\phi, \varphi \in \Lambda, p \in \mathbb{N}, q \in \mathbb{N}_0, 0 \leq \eta < p, 0 \leq \rho < p, \alpha < 1$ and $\lambda \geq 0$.

$$\begin{aligned} S_{p,\lambda}^{q,\alpha}(\alpha_1, \beta_1; \eta; \phi) &= \{ f \in A_p : D_{\lambda,\alpha}^{p,q}(\alpha_1, \beta_1)f(z) \in S_p^*(\eta; \phi) \}, \\ K_{p,\lambda}^{q,\alpha}(\alpha_1, \beta_1; \eta; \phi) &= \{ f \in A_p : D_{\lambda,\alpha}^{p,q}(\alpha_1, \beta_1)f(z) \in K(\eta; \phi) \} \end{aligned}$$

And

$$C_{p,\lambda}^{q,\alpha}(\alpha_1, \beta_1; \eta, \rho; \phi, \varphi) = \{ f \in A_p : D_{\lambda,\alpha}^{p,q}(\alpha_1, \beta_1)f(z) \in C_p(\eta, \rho; \phi, \varphi) \}$$

Clearly, we observe that, for $\lambda = 0, \alpha = \frac{1}{2}, l = m + 1, \alpha_1 = 1, \alpha_2 = \beta_1, \alpha_3 = \beta_2, \dots, \alpha_l = \beta_m$, we get

$$S_{p,0}^{q,\frac{1}{2}}(\alpha_1, \beta_1; \eta; \phi) = S_p^*(\eta; \phi), K_{p,0}^{q,\frac{1}{2}}(\alpha_1, \beta_1; \eta; \phi) = K_p(\eta; \phi) \text{ and } C_{p,0}^{q,\frac{1}{2}}(\alpha_1, \beta_1; \eta, \rho; \phi, \varphi) = C_p(\eta, \rho; \phi, \varphi).$$

We also note that,

$$f \in K_{p,\lambda}^{q,\alpha}(\alpha_1, \beta_1; \eta; \phi) \Leftrightarrow \frac{zf'(z)}{p} \in S_{p,\lambda}^{q,\alpha}(\alpha_1, \beta_1; \eta; \phi) \quad (1.10)$$

In particular, we set

$$S_{p,\lambda}^{q,\alpha} \left(\alpha_1, \beta_1; \eta; \frac{1+Az}{1+Bz} \right) = S_{p,\lambda}^{q,\alpha}(\alpha_1, \beta_1; \eta; A, B), -1 \leq B < A \leq 1, \quad (1.11)$$

And

$$K_{p,\lambda}^{q,\alpha}(\alpha_1, \beta_1; \eta; \frac{1+Az}{1+Bz}) = K_{p,\lambda}^{q,\alpha}(\alpha_1, \beta_1; \eta; A, B), -1 \leq B < A \leq 1 \quad (1.12)$$

In section 2, some preliminary results are mentioned. In section 3, we prove inclusion properties of above mentioned classes and in section 4, we study inclusion properties of above classes involving generalised Libra integral operator.

II. Preliminary Lemmas

The following lemmas will be required in our investigation.

Lemma 2.1. ([4]) Let ϕ be convex, univalent in $\mathbb{U}$ with $\phi(0) = 1$ and $Re(\kappa\phi(z) + \gamma) > 0$, $\kappa, \gamma \in \mathbb{C}$. If $p(z)$ is univalent in $\mathbb{U}$ with $p(0) = 1$, then

$$p(z) + \frac{zp'(z)}{\kappa p(z) + \gamma} < \phi(z), (z \in \mathbb{U})$$

implies

$$p(z) < \phi(z), (z \in \mathbb{U}).$$

Lemma 2.2. ([7]) Let ϕ be convex, univalent in $\mathbb{U}$ and ω be analytic in $\mathbb{U}$ with $Re(\omega(z)) \geq 0$. If $p(z)$ is analytic in $\mathbb{U}$ with $p(0) = \phi(0)$, then

$$p(z) + \omega(z)zp'(z) < \phi(z), (z \in \mathbb{U})$$

implies

$$p(z) < \phi(z), (z \in \mathbb{U}).$$

III. Inclusion Properties Involving The Operator $D_{\lambda,\alpha}^{p,q}(\alpha_1, \beta_1)$

Unless otherwise mentioned we shall assume that $p \in \mathbb{N}, q \in \mathbb{N}_0, \lambda \geq 0, 0 \leq \eta < p, 0 \leq \rho < p, \alpha < 1, \alpha_1 > p, \alpha_i, \beta_j \in \mathbb{C} - \{0, -1, -2, \dots\} (i = 1, 2, \dots, l; j = 1, 2, \dots, m)$.

Theorem 3.1. Let $f \in A_p$ and let $\phi \in \Lambda$ with $Re\left((p - \eta)\phi(z) + \eta - p + \frac{p}{\lambda}\right) > 0, Re((p - \eta)\phi(z) + \eta - p + \alpha_1) > 0$ and $\lambda > 0$. Then

$$S_{p,\lambda}^{q,\alpha}(\alpha_1, \beta_1; \eta; \phi) \subset S_{p,\lambda}^{q,\alpha}(\alpha_1 + 1, \beta_1; \eta; \phi)$$

And

$$S_{p,\lambda}^{q+1,\alpha}(\alpha_1, \beta_1; \eta; \phi) \subset S_{p,\lambda}^{q,\alpha}(\alpha_1, \beta_1; \eta; \phi).$$

Proof. To prove the first part of Theorem 3.1, let $f \in S_{p,\lambda}^{q,\alpha}(\alpha_1, \beta_1; \eta; \phi)$ and set

$$p(z) = \frac{1}{p-\eta} \left(\frac{z \left(D_{\lambda,\alpha}^{p,q}(\alpha_1+1, \beta_1)f(z) \right)'}{D_{\lambda,\alpha}^{p,q}(\alpha_1+1, \beta_1)f(z)} - \eta \right) \quad (3.1)$$

Where $p(z)$ is analytic in $\mathbb{U}$ and $\mathbb{U}$ with $p(0) = 1$.

From (1.8) and (3.1), we get

$$\alpha_1 \frac{D_{\lambda,\alpha}^{p,q}(\alpha_1, \beta_1)f(z)}{D_{\lambda,\alpha}^{p,q}(\alpha_1+1, \beta_1)f(z)} = (p - \eta)p(z) + \eta + \alpha_1 - p \quad (3.2)$$

By using the logarithmic differentiation on both sides of (3.2), we have

$$\frac{1}{p-\eta} \left(\frac{z \left(D_{\lambda,\alpha}^{p,q}(\alpha_1, \beta_1)f(z) \right)'}{D_{\lambda,\alpha}^{p,q}(\alpha_1, \beta_1)f(z)} - \eta \right) = p(z) + \frac{zp'(z)}{(p-\eta)p(z) + \eta - p + \alpha_1} \quad (3.3)$$

Applying Lemma (2.1) to equation (3.3), it will follow that $p(z) < \phi(z)$ in $\mathbb{U}$, that is $f \in S_{p,\lambda}^{q,\alpha}(\alpha_1 + 1, \beta_1; \eta; \phi)$.

To prove the second inclusion relationship asserted by the Theorem (3.1), let

$$f \in S_{p,\lambda}^{q+1,\alpha}(\alpha_1, \beta_1; \eta; \phi).$$

Assuming the function $p(z) = \frac{1}{p-\eta} \left(\frac{z \left(D_{\lambda,\alpha}^{p,q}(\alpha_1, \beta_1)f(z) \right)'}{D_{\lambda,\alpha}^{p,q}(\alpha_1, \beta_1)f(z)} - \eta \right)$ and by using arguments similar to those detailed

above with (1.7), it follows that $p(z) < \phi(z)$ in $\mathbb{U}$, that is

$$f \in S_{p,\lambda}^{q,\alpha}(\alpha_1, \beta_1; \eta; \phi).$$

Which completes the proof of the theorem (3.1).

Theorem 3.2. Let $f \in A_p$ and let $\phi \in \Lambda$ with $\operatorname{Re} \left((p - \eta)\phi(z) + \eta - p + \frac{p}{\lambda} \right) > 0$, $\operatorname{Re}((p - \eta)\phi(z) + \eta - p + \alpha_1) > 0$ and $\lambda > 0$. Then

$$K_{p,\lambda}^{q,\alpha}(\alpha_1, \beta_1; \eta; \phi) \subset K_{p,\lambda}^{q,\alpha}(\alpha_1 + 1, \beta_1; \eta; \phi)$$

And

$$K_{p,\lambda}^{q+1,\alpha}(\alpha_1, \beta_1; \eta; \phi) \subset K_{p,\lambda}^{q,\alpha}(\alpha_1, \beta_1; \eta; \phi).$$

Proof. Applying (1.10) and Theorem (3.1), we conclude that

$$\begin{aligned} f \in K_{p,\lambda}^{q,\alpha}(\alpha_1, \beta_1; \eta; \phi) &\Rightarrow \frac{zf'}{p} \in S_{p,\lambda}^{q,\alpha}(\alpha_1, \beta_1; \eta; \phi) \subset S_{p,\lambda}^{q,\alpha}(\alpha_1 + 1, \beta_1; \eta; \phi) \\ &\Leftrightarrow \frac{zf'}{p} \in S_{p,\lambda}^{q,\alpha}(\alpha_1 + 1, \beta_1; \eta; \phi) \\ &\Leftrightarrow f \in K_{p,\lambda}^{q,\alpha}(\alpha_1 + 1, \beta_1; \eta; \phi) \end{aligned}$$

and

$$\begin{aligned} f \in K_{p,\lambda}^{q+1,\alpha}(\alpha_1, \beta_1; \eta; \phi) &\Rightarrow \frac{zf'}{p} \in S_{p,\lambda}^{q+1,\alpha}(\alpha_1, \beta_1; \eta; \phi) \subset S_{p,\lambda}^{q,\alpha}(\alpha_1, \beta_1; \eta; \phi) \\ &\Leftrightarrow \frac{zf'}{p} \in S_{p,\lambda}^{q,\alpha}(\alpha_1, \beta_1; \eta; \phi) \\ &\Leftrightarrow f \in K_{p,\lambda}^{q,\alpha}(\alpha_1, \beta_1; \eta; \phi). \end{aligned}$$

Taking $\phi(z) = \frac{1+Az}{1+Bz}$, $-1 \leq B < A \leq 1$; $z \in \mathbb{U}$, in Theorem (3.1) and Theorem (3.2), we have the following corollary.

Corollary 3.3. Let $f \in A_p$. Then $S_{p,\lambda}^{q,\alpha}(\alpha_1, \beta_1; \eta; A, B) \subset S_{p,\lambda}^{q,\alpha}(\alpha_1 + 1, \beta_1; \eta; A, B)$,

$$S_{p,\lambda}^{q+1,\alpha}(\alpha_1, \beta_1; \eta; A, B) \subset S_{p,\lambda}^{q,\alpha}(\alpha_1, \beta_1; \eta; A, B),$$

$$K_{p,\lambda}^{q,\alpha}(\alpha_1, \beta_1; \eta; A, B) \subset K_{p,\lambda}^{q,\alpha}(\alpha_1 + 1, \beta_1; \eta; A, B)$$

And

$$K_{p,\lambda}^{q+1,\alpha}(\alpha_1, \beta_1; \eta; A, B) \subset K_{p,\lambda}^{q,\alpha}(\alpha_1, \beta_1; \eta; A, B).$$

By using Lemma 2.2, we obtain the following inclusion relation for the class $C_{p,\lambda}^{q,\alpha}(\alpha_1, \beta_1; \eta; \rho; \phi, \varphi)$.

Theorem 3.4. Let $f \in A_p$ and let $\phi, \varphi \in \Lambda$ with $\operatorname{Re} \left((p - \eta)\phi(z) + \eta - p + \frac{p}{\lambda} \right) > 0$, $\operatorname{Re}((p - \eta)\phi(z) + \eta - p + \alpha_1) > 0$, $\alpha < 1$ and $\lambda > 0$. Then

$$C_{p,\lambda}^{q,\alpha}(\alpha_1, \beta_1; \eta; \rho; \phi, \varphi) \subset C_{p,\lambda}^{q,\alpha}(\alpha_1 + 1, \beta_1; \eta; \rho; \phi, \varphi)$$

and

$$C_{p,\lambda}^{q+1,\alpha}(\alpha_1, \beta_1; \eta; \rho; \phi, \varphi) \subset C_{p,\lambda}^{q,\alpha}(\alpha_1, \beta_1; \eta; \rho; \phi, \varphi).$$

Proof. We begin by proving that,

$$C_{p,\lambda}^{q,\alpha}(\alpha_1, \beta_1; \eta; \rho; \phi, \varphi) \subset C_{p,\lambda}^{q,\alpha}(\alpha_1 + 1, \beta_1; \eta; \rho; \phi, \varphi).$$

Let $f \in C_{p,\lambda}^{q,\alpha}(\alpha_1, \beta_1; \eta; \rho; \phi, \varphi)$.

Then by definition, there exists a function $g \in S_{p,\lambda}^{q,\alpha}(\alpha_1, \beta_1; \eta; \phi)$ such that,

$$\frac{1}{p - \rho} \left(z \left(\frac{D_{\lambda,\alpha}^{p,q}(\alpha_1, \beta_1)f(z)}{D_{\lambda,\alpha}^{p,q}(\alpha_1, \beta_1)g(z)} \right)' - \rho \right) < \varphi(z), \quad z \in \mathbb{U}.$$

Now, let $p(z) = \frac{1}{p - \rho} \left(z \left(\frac{D_{\lambda,\alpha}^{p,q}(\alpha_1, \beta_1)f(z)}{D_{\lambda,\alpha}^{p,q}(\alpha_1, \beta_1)g(z)} \right)' - \rho \right)$, where $p(z)$ is analytic in $\mathbb{U}$ with $p(0) = 1$.

Using (1.8), we have

$$\alpha_1 D_{\lambda,\alpha}^{p,q}(\alpha_1, \beta_1)f(z) = (\alpha_1 - p) D_{\lambda,\alpha}^{p,q}(\alpha_1 + 1, \beta_1)f(z) + [(p - \rho)p(z) + \rho] D_{\lambda,\alpha}^{p,q}(\alpha_1 + 1, \beta_1)g(z) \quad (3.4)$$

Differentiating (3.4) with respect to z and multiplying by z , we get

$$\begin{aligned} \alpha_1 z \left(D_{\lambda,\alpha}^{p,q}(\alpha_1, \beta_1)f(z) \right)' &= (\alpha_1 - p) z \left(D_{\lambda,\alpha}^{p,q}(\alpha_1 + 1, \beta_1)f(z) \right)' \\ &+ [(p - \rho)p(z) + \rho] z \left(D_{\lambda,\alpha}^{p,q}(\alpha_1 + 1, \beta_1)g(z) \right)' + (p - \rho) z p'(z) D_{\lambda,\alpha}^{p,q}(\alpha_1 + 1, \beta_1)g(z) \end{aligned} \quad (3.5)$$

Since $g \in S_{p,\lambda}^{q,\alpha}(\alpha_1, \beta_1; \eta; \phi)$, then by Theorem 3.1, we have $g \in S_{p,\lambda}^{q,\alpha}(\alpha_1 + 1, \beta_1; \eta; \phi)$.

Let

$$h(z) = \frac{1}{p-\eta} \left(\frac{z \left(D_{\lambda, \alpha}^{p, q}(\alpha_1 + 1, \beta_1) g(z) \right)'}{D_{\lambda, \alpha}^{p, q}(\alpha_1 + 1, \beta_1) g(z)} - \eta \right)$$

Applying (1.8) again, we get

$$\alpha_1 D_{\lambda, \alpha}^{p, q}(\alpha_1, \beta_1) g(z) = [(p - \eta)h(z) + \eta + \alpha_1 - p] D_{\lambda, \alpha}^{p, q}(\alpha_1 + 1, \beta_1) g(z) \quad (3.6)$$

from (3.5) and (3.6), we have

$$\frac{1}{p-\rho} \left(\frac{z \left(D_{\lambda, \alpha}^{p, q}(\alpha_1, \beta_1) f(z) \right)'}{D_{\lambda, \alpha}^{p, q}(\alpha_1, \beta_1) g(z)} - \rho \right) = p(z) + \frac{zp'(z)}{(p-\eta)h(z) + \eta + \alpha_1 - p}, z \in \mathbb{U}.$$

So by taking $\omega(z) = \frac{1}{(p-\eta)h(z) + \eta + \alpha_1 - p}$ and applying Lemma 2.2, we can show that $p < \varphi$, so that

$$f \in C_{p, \lambda}^{q, \alpha}(\alpha_1 + 1, \beta_1; \eta, \rho; \phi, \varphi).$$

For the second inclusion relationship asserted by the Theorem, using arguments similar to those detailed above with equation (1.7), we obtain

$$C_{p, \lambda}^{q+1, \alpha}(\alpha_1, \beta_1; \eta, \rho; \phi, \varphi) \subset C_{p, \lambda}^{q, \alpha}(\alpha_1, \beta_1; \eta, \rho; \phi, \varphi).$$

IV. Inclusion Properties Involving The Integral Operator F_c

In this section we consider the generalised Libera integral operator $F_{p, c}$ (see [2], [5] and [9]), defined by

$$F_{p, c}(f)(z) = \frac{c+p}{z^c} \int_0^z t^{c-1} f(t) dt, \quad (c > -p; f \in A_p). \quad (4.1)$$

Theorem 4.1. Let $c > -p$ and let $\phi \in \Lambda$ with $\operatorname{Re}((p - \eta)\phi(z) + \eta + c) > 0$. If $f \in S_{p, \lambda}^{q, \alpha}(\alpha_1, \beta_1; \eta; \phi)$ then $F_{p, c}(f) \in S_{p, \lambda}^{q, \alpha}(\alpha_1, \beta_1; \eta; \phi)$.

Proof. Let $f \in S_{p, \lambda}^{q, \alpha}(\alpha_1, \beta_1; \eta; \phi)$ and set

$$p(z) = \frac{1}{p-\eta} \left(\frac{z \left(D_{\lambda, \alpha}^{p, q}(\alpha_1, \beta_1) F_{p, c}(f)(z) \right)'}{D_{\lambda, \alpha}^{p, q}(\alpha_1, \beta_1) F_{p, c}(f)(z)} - \eta \right) \quad (4.2)$$

where $p(z)$ is analytic in $\mathbb{U}$ with $p(0) = 1$. From (4.1), we have

$$z \left(D_{\lambda, \alpha}^{p, q}(\alpha_1, \beta_1) F_{p, c}(f)(z) \right)' = (c + p) D_{\lambda, \alpha}^{p, q}(\alpha_1, \beta_1) f(z) - c D_{\lambda, \alpha}^{p, q}(\alpha_1, \beta_1) F_{p, c}(f)(z) \quad (4.3)$$

From (4.2) and (4.3), we get

$$(c + p) \frac{D_{\lambda, \alpha}^{p, q}(\alpha_1, \beta_1) f(z)}{D_{\lambda, \alpha}^{p, q}(\alpha_1, \beta_1) F_{p, c}(f)(z)} = (p - \eta)p(z) + \eta + c \quad (4.4)$$

Differentiating (4.4) logarithmically with respect to z , we obtain

$$\frac{1}{p-\eta} \left(\frac{z \left(D_{\lambda, \alpha}^{p, q}(\alpha_1, \beta_1) f(z) \right)'}{D_{\lambda, \alpha}^{p, q}(\alpha_1, \beta_1) f(z)} - \eta \right) = p(z) + \frac{zp'(z)}{(p-\eta)p(z) + \eta + c} \quad (4.5)$$

Applying Lemma 2.1 to (4.5), we conclude that $F_{p, c}(f) \in S_{p, \lambda}^{q, \alpha}(\alpha_1, \beta_1; \eta; \phi)$.

Similarly applying (1.10) and Theorem 4.1, we have the following result.

Theorem 4.2. Let $c > -p$ and let $\phi \in \Lambda$ with $\operatorname{Re}((p - \eta)\phi(z) + \eta + c) > 0$. If $f \in K_{p, \lambda}^{q, \alpha}(\alpha_1, \beta_1; \eta; \phi)$ then $F_{p, c}(f) \in K_{p, \lambda}^{q, \alpha}(\alpha_1, \beta_1; \eta; \phi)$.

From Theorem 4.1 and Theorem 4.2, we have the following corollary.

Corollary 4.3. Let $f \in A_p$, $-1 \leq B < A \leq 1$ and $c > -p$. If $f \in S_{p, \lambda}^{q, \alpha}(\alpha_1, \beta_1; \eta; A, B)$ (or $K_{p, \lambda}^{q, \alpha}(\alpha_1, \beta_1; \eta; \phi)$) then $F_{p, c}(f) \in S_{p, \lambda}^{q, \alpha}(\alpha_1, \beta_1; \eta; A, B)$ (or $K_{p, \lambda}^{q, \alpha}(\alpha_1, \beta_1; \eta; \phi)$).

Theorem 4.4. Let $f \in A_p$ and let $\phi, \varphi \in \Lambda$ with $\operatorname{Re}((p - \eta)\phi(z) + \eta + c) > 0$. If $f \in C_{p, \lambda}^{q, \alpha}(\alpha_1, \beta_1; \eta, \rho; \phi, \varphi)$ then $F_{p, c}(f) \in C_{p, \lambda}^{q, \alpha}(\alpha_1, \beta_1; \eta, \rho; \phi, \varphi)$.

Proof. Let $f \in C_{p, \lambda}^{q, \alpha}(\alpha_1, \beta_1; \eta, \rho; \phi, \varphi)$, then by definition, there exists a function $g \in S_{p, \lambda}^{q, \alpha}(\alpha_1, \beta_1; \eta; \phi)$ such that,

$$\frac{1}{p-\rho} \left(\frac{z \left(D_{\lambda, \alpha}^{p, q}(\alpha_l, \beta_l) f(z) \right)'}{D_{\lambda, \alpha}^{p, q}(\alpha_l, \beta_l) g(z)} - \rho \right) < \varphi(z), \quad z \in \mathbb{U}.$$

We set,

$$p(z) = \frac{1}{p-\rho} \left(\frac{z \left(D_{\lambda, \alpha}^{p, q}(\alpha_l, \beta_l) F_{p, c}(f)(z) \right)'}{D_{\lambda, \alpha}^{p, q}(\alpha_l, \beta_l) F_{p, c}(g)(z)} - \rho \right)$$

where p is analytic in $\mathbb{U}$ and $p(0) = 1$. Since $g \in S_{p, \lambda}^{q, \alpha}(\alpha_l, \beta_l; \eta; \phi)$, we have from Theorem 4.1. that $F_{p, c}(g) \in S_{p, \lambda}^{q, \alpha}(\alpha_l, \beta_l; \eta; \phi)$.

From (4.3) and (4.6), we obtain

$$(c+p) D_{\lambda, \alpha}^{p, q}(\alpha_l, \beta_l) f(z) = c D_{\lambda, \alpha}^{p, q}(\alpha_l, \beta_l) F_{p, c}(f)(z) + [(p-\rho)p(z) + \rho] D_{\lambda, \alpha}^{p, q}(\alpha_l, \beta_l) F_{p, c}(g)(z).$$

Then by simple calculations, we get

$$(c+p)z \left(D_{\lambda, \alpha}^{p, q}(\alpha_l, \beta_l) f(z) \right)' = cz \left(D_{\lambda, \alpha}^{p, q}(\alpha_l, \beta_l) F_{p, c}(f)(z) \right)' + [(p-\rho)p(z) + \rho]z \left(D_{\lambda, \alpha}^{p, q}(\alpha_l, \beta_l) F_{p, c}(g)(z) \right)' + z(p-\rho)p'(z) D_{\lambda, \alpha}^{p, q}(\alpha_l, \beta_l) F_{p, c}(g)(z) \quad (4.7)$$

We have $g \in S_{p, \lambda}^{q, \alpha}(\alpha_l, \beta_l; \eta; \phi)$.

Then by Theorem 4.1, $F_{p, c}(g) \in S_{p, \lambda}^{q, \alpha}(\alpha_l, \beta_l; \eta; \phi)$.

Set

$$h(z) = \frac{1}{p-\eta} \left(\frac{z \left(D_{\lambda, \alpha}^{p, q}(\alpha_l, \beta_l) F_{p, c}(g)(z) \right)'}{D_{\lambda, \alpha}^{p, q}(\alpha_l, \beta_l) F_{p, c}(g)(z)} - \eta \right) \quad (4.8)$$

That is

$$\frac{z \left(D_{\lambda, \alpha}^{p, q}(\alpha_l, \beta_l) F_{p, c}(g)(z) \right)'}{D_{\lambda, \alpha}^{p, q}(\alpha_l, \beta_l) F_{p, c}(g)(z)} = (p-\eta)h(z) + \eta.$$

Applying (4.3), we get

$$(c+p) D_{\lambda, \alpha}^{p, q}(\alpha_l, \beta_l) g(z) = [(p-\eta)h(z) + \eta + c] D_{\lambda, \alpha}^{p, q}(\alpha_l, \beta_l) F_{p, c}(g)(z) \quad (4.9)$$

By (4.7) and (4.9), we get,

$$\frac{1}{p-\rho} \left(\frac{z \left(D_{\lambda, \alpha}^{p, q}(\alpha_l, \beta_l) f(z) \right)'}{D_{\lambda, \alpha}^{p, q}(\alpha_l, \beta_l) g(z)} - \rho \right) = p(z) + \frac{zp'(z)}{(p-\eta)h(z) + \eta + c}.$$

The remaining part of the proof is similar to that of Theorem 3.4 and so we omit it.

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