



Non Invariant Hypersurface of an ε -Framed Metric Manifold

Mayank Mittal, N. K. Joshi
Research Scholar¹, Faculty²
^{1,2}P.N.G. Govt. P.G.College, Ramnagar
Nainital, Uttarakhand, INDIA

Abstract: Hypersurface of a Manifold is a submanifold of dimension one less than the dimension of ambient manifold. The Submanifold of an ε -framed metric manifold was studied by Joshi & Dube. Non Invariant Hypersurface of Kenmotsu manifold was studied by Bhatt and others. In the present paper we define and study Non Invariant Hypersurface of an ε -framed metric manifold and obtain some interesting results.

Keywords: Hypersurface, Totally geodesic, Killing vector, Parallel tensor, Lie derivative

I. INTRODUCTION

Let us consider $\bar{M}$ be a $(2m+r)$ dimensional differentiable manifold of differentiability class C^∞ . Let there exist on $\bar{M}$, a tensor field J of type $(1,1)$, r C^∞ contravariant vector fields $U_1, U_2, U_3, \dots, U_r$ and r C^∞ 1-forms $u^1, u^2, u^3, \dots, u^r$ (r some finite integer) and a Riemannian metric g satisfying --

$$J^2 = \varepsilon(I - u^k \otimes U_k) \quad (1.1)$$

$$J U_k = 0, \quad u^k \circ J = 0, \quad u^k(U_l) = \delta_l^k \quad (1.2)$$

Where k, l take the values $1, 2, 3, \dots, r$ and δ_l^k denotes the kronecker delta and $\varepsilon^2 = 1$.

$$g(JX, JY) = g(X, Y) - \sum_{k=1}^r u^k(X) u^k(Y) \quad (1.3)$$

For all vector fields X and Y tangent to $\bar{M}$.

The manifold $\bar{M}$ satisfying conditions (1.1), (1.2) and (1.3) is said to possess an ε -framed metric structure.

The Manifold $\bar{M}$ with ε -framed metric structure is said to be an ε -framed metric manifold. This structure (resp. manifold) is a very general structure (resp. manifold) which in special cases reduces to several known structures (resp. manifold) given below :-

Structure/Manifold	r	ε	References
Framed metric		-1	[6]
Almost r -contact metric		-1	[4]
Almost contact metric	1	-1	[2]
Almost r -para contact metric		1	[1]
Almost para contact metric	1	1	[3]
$(J(2, \varepsilon), g)$	0		[5]
Almost hermitian	0	-1	[6]

A Hypersurface of ε -framed metric manifold $\bar{M}$ is called a non-invariant hypersurface, if the transform of tangent vector of the hypersurface under the action of $(1,1)$ tensor field defining ε -framed metric structure is never tangent to the hypersurface. Consider ε -framed metric manifold $\bar{M}$ and M be the hypersurface of $\bar{M}$ and let B be the differential of the immersion i of M into $\bar{M}$. Let X, Y, Z be tangent to M and N a unit normal vector, then we have,

$$JBX = BFX + \alpha(X)N \quad (1.4)$$

Where F is $(1,1)$ tensor field and α is a 1-form on M . If $\alpha \neq 0$ then M is called non-invariant hypersurface of $\bar{M}$. If α is identically zero, then M is said to be an invariant hypersurface that is, the tangent space of M is

invariant by J . The metric g of an ε -framed metric manifold induces a Riemannian metric G on the hypersurface M and is given by $G(X, Y) = g(BX, BY)$ (1.5)

$$g(BX, N) = 0 \quad (1.6)$$

$$g(N, N) = 1 \quad (1.7)$$

The Gauss and Weingarten formulae are given by

$$\bar{D}_{BX} BY = B(D_X Y) + h(X, Y)N \quad (1.8)$$

$$\bar{D}_{BX} N = -BX \quad (1.9)$$

For all $X, Y \in TM$, where $\bar{D}$ and D are the Riemannian and induced Riemannian connections on $\bar{M}$ and M respectively and N is the unit normal vector in the normal bundle $T^\perp M$. Here h is symmetric tensor of type $(0,2)$ called the second fundamental form of the hypersurface M and H is a $(1,1)$ tensor field on $\bar{M}$ such that,

$$h(X, Y) = g(HX, Y) \quad (1.10)$$

$$\text{Let } JN = \varepsilon BV \quad (1.11)$$

Where V is a vector field in M .

THEOREM 2.1 :- In a non-invariant hypersurface of ε -framed metric Manifold, we have $(F, \varepsilon, u^k, U_k, \alpha, V)$ structure.

$$F^2 = \varepsilon(I - u^k \otimes U_k - \alpha \otimes V) \quad (2.1)$$

$$\alpha \circ F = 0 \quad (2.2)$$

$$\alpha(V) = 1 \quad (2.3)$$

$$F(V) = 0 \quad (2.4)$$

PROOF:- Operating equation (1.4) by J

$$J^2 BX = J(BFX) + \alpha(X)JN$$

Now using equation (1.1) and (1.4) in above equation

$$\varepsilon(BX - u^k (BX)U_k) = BF^2 X + \alpha(FX)N + \alpha(X)JN$$

Using equation (1.11) in above equation,

$$\varepsilon(BX - u^k (BX)U_k) = BF^2 X + \alpha(FX)N + \alpha(X)\varepsilon BV$$

Equating tangential and normal parts we get,

$$\varepsilon(BX - u^k (BX)U_k) = BF^2 X + \alpha(X)\varepsilon BV$$

$$\text{and } \alpha(FX)N = 0 \Rightarrow \alpha(FX) = 0, \text{ since } N \text{ is a unit normal vector.}$$

or

$$F^2 = \varepsilon(I - u^k \otimes U_k - \alpha \otimes V)$$

$$\text{And } \alpha \circ F = 0$$

Operating equation (1.11) by J and using equation (1.1), we get

$$J^2 N = J(\varepsilon BV)$$

$$= \varepsilon J(BV)$$

$$\varepsilon(N - u^k (N)U_k) = \varepsilon(BFV + \alpha(V)N)$$

Or

$$N - u^k (N)U_k = (BFV + \alpha(V)N)$$

$$N = (BFV + \alpha(V)N) \quad (\ominus u^k (N) = 0)$$

Equating tangential and normal part, we get

$$BFV = 0 \Rightarrow FV = 0 \text{ and } \alpha(V) = 1$$

This completes the proof.

THEOREM 2.2 : -The metric g of ε -framed metric manifold induces a Riemannian metric G on the hypersurface M and is given by ,

$$G(FX, FY) = G(X, Y) - \alpha(X)\alpha(Y) - \sum_{k=1}^r u^k(BX)u^k(BY) \quad (2.5)$$

PROOF :- From equation (1.3) we have ,

$$g(JBX, JBY) = g(BX, BY) - \sum_{k=1}^r u^k(BX)u^k(BY)$$

Using equation (1.4), (1.5) in above equation

$$\begin{aligned} g(BFX + \alpha(X)N, BFY + \alpha(Y)N) &= G(X, Y) - \sum_{k=1}^r u^k(BX)u^k(BY) \\ g(BFX, BFY) + \alpha(Y)g(BFX, N) + \alpha(X)g(N, BFY) + \alpha(X)\alpha(Y)g(N, N) \\ &= G(X, Y) - \sum_{k=1}^r u^k(BX)u^k(BY) \end{aligned}$$

using equation (1.6), (1.7), we get,

$$g(BFX, BFY) + \alpha(X)\alpha(Y) = G(X, Y) - \sum_{k=1}^r u^k(BX)u^k(BY)$$

$$G(FX, FY) = G(X, Y) - \alpha(X)\alpha(Y) - \sum_{k=1}^r u^k(BX)u^k(BY)$$

This completes the proof.

COROLLARY :- Invariant hypersurface of ε -framed metric manifold is ε -framed metric manifold.

THEOREM 2.3 :- In an ε -Framed metric manifold, that is, ε -Framed metric structure in which tensor J is parallel i.e. $(\bar{D}_X J)Y = 0$, we have ,

$$(D_X F)Y = \alpha(Y)HX + h(X, Y)\varepsilon V \quad (2.6)$$

$$(D_X \alpha)Y = -h(X, FY) \quad (2.7)$$

PROOF :- We have $(\bar{D}_X J)Y = 0$

Therefore $(\bar{D}_{BX} J)BY = 0$

$$\bar{D}_{BX} J(BY) - J(\bar{D}_{BX} BY) = 0$$

Using equation (1.4), we get

$$\bar{D}_{BX}(BFY + \alpha(Y)N) - J(\bar{D}_{BX} BY) = 0$$

$$\bar{D}_{BX} BFY + \bar{D}_{BX} \alpha(Y)N - J(\bar{D}_{BX} BY) = 0$$

$$\bar{D}_{BX} BFY + (\bar{D}_{BX} \alpha(Y))N + \alpha(Y)\bar{D}_{BX} N - J(\bar{D}_{BX} BY) = 0$$

Using equation (1.8) and equation (1.9) in the above equation, we get

$$B(D_X FY) + h(X, FY)N + (\bar{D}_{BX} \alpha(Y))N - \alpha(Y)BHX - J(BD_X Y + h(X, Y)N) = 0$$

$$B(D_X FY) + h(X, FY)N + (\bar{D}_{BX} \alpha(Y))N - \alpha(Y)BHX - JB(D_X Y) - h(X, Y)JN = 0$$

$$B(D_X FY) + h(X, FY)N + (\bar{D}_{BX} \alpha(Y))N - \alpha(Y)BHX - BF(D_X Y) - \alpha(D_X Y)N - h(X, Y)\varepsilon BV = 0 \quad (2.8)$$

Tangential part from above equation (2.8) is given by

$$B(D_X FY) - \alpha(Y)BHX - BF(D_X Y) - h(X, Y)\varepsilon BV = 0$$

Or $B(D_X F)Y = \alpha(Y)BHX + h(X, Y)\varepsilon BV$

Or $(D_X F)Y = \alpha(Y)HX + h(X, Y)\varepsilon V$

Normal part from equation (2.8) is given by

$$h(X, FY)N + (\bar{D}_{BX} \alpha(Y))N - \alpha(D_X Y)N = 0$$

$$\text{Or } \bar{D}_{BX} \alpha(Y) - \alpha(D_X Y) = -h(X, FY)$$

$$\text{Or } (D_X \alpha)Y = -h(X, FY)$$

THEOREM (2.4) – Non-invariant hypersurface of a parallel ε -framed metric manifold is totally geodesic iff its induced tensor field F is parallel.

PROOF:- In view of (2.6), we get required result

$$(D_X F)Y = \alpha(Y)HX + h(X, Y)\varepsilon V$$

$$(D_X F)Y = 0 \Leftrightarrow \alpha(Y)HX + h(X, Y)\varepsilon V = 0$$

$$\Leftrightarrow HX = 0, h(X, Y) = 0$$

$$\Leftrightarrow h(X, Y) = 0$$

THEOREM (2.5) - If a Non invariant hypersurface of a parallel ε -Framed metric manifold is totally geodesic then V is Killing vector.

PROOF – If V is a Killing vector then Lie derivative of induced Riemannian metric G with respect to V is zero i.e.

$$(L_V G)(X, Y) = 0$$

which is equivalent to

$$(D_X \alpha)Y + (D_Y \alpha)X = 0 \tag{2.9}$$

The proof follows directly from equation (2.7) and (2.9).

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