



Some Properties of a Subclasses of Univalent Functions

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Abstract: In this paper we have defined classes $S(\lambda, \alpha, \beta)$ and $K(\lambda, \alpha, \beta)$ by using the class defined by B. Srutha Keerthi etc. (see [1]) and obtained conditions for univalence for a function to be in these classes and studied distortion theorems, radii of starlikeness and convexity.

Keywords: univalent functions, starlike functions, convex functions.

2010MSC: 30C45

I. Preliminaries

Let A denotes the class of functions of the form, $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ (1.1)

which are analytic and univalent in the unit disk $U = \{z : |z| < 1\}$. Let S be subclass of A , of functions univalent in U and T be subclass of S , consisting of functions of the form, $g(z) = z - \sum_{n=2}^{\infty} b_n z^n$ ($b_n \geq 0$) (1.2)

Goodman [2], [3] introduced classes UCV and UST of uniformly convex and uniformly starlike functions. Further Ronning studied classes UCV and UST (for more details see [4], [5], [6], [7], [8], [10] and many others).

For $-1 \leq \alpha < 1$, $\beta \geq 0$ the function $f(z)$ defined by Equation (1.1) is called as β - Uniformly starlike function of order α , which satisfies the condition, $Re \left\{ \frac{zf'(z)}{f(z)} - \alpha \right\} \geq \beta \left| \frac{zf'(z)}{f(z)} - 1 \right|$. The class of β - Uniformly starlike

function of order α is denoted by $\beta - S(\alpha)$. In particular, when $\beta = 0$, i.e. if $Re \left\{ \frac{zf'(z)}{f(z)} \right\} \geq \alpha$, the function $f(z)$ is called starlike of order α , when $\beta = 1$, the function $f(z) \in UST$ and the function $f(z)$ is called starlike, if both α and β are zero.

For $-1 \leq \alpha < 1$, $\beta \geq 0$ the function $f(z)$ defined by Equation (1.1) is called β - Uniformly Convex function of order α , which satisfies the condition, $Re \left\{ 1 + \frac{zf''(z)}{f'(z)} - \alpha \right\} \geq \beta \left| \frac{zf''(z)}{f'(z)} \right|$ and it is denoted by $\beta - K(\alpha)$. In

particular, when $\beta = 0$, i.e. if $Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} \geq \alpha$, the function $f(z)$ is called convex of order α . When $\beta = 1$, the function $f(z) \in UCV$ and if both α and β are zero, the function $f(z)$ is called convex. The classes $\beta - S(\alpha)$ and $\beta - K(\alpha)$ were studied by Goodman, Ronning and Minda and Ma. (see [2], [3], [4], [5], [8], [9], [10]).

II. Introduction

Definition 2.1 The function $f(z)$ defined by Equation (1.1), is said to be in class $S(\lambda, \alpha, \beta)$, ($\lambda \geq 0$, $\alpha, \beta \in \mathbf{R}$ and $z \in U$), if and only if,

$$Re \left\{ \frac{zf'(z) + \lambda z^2 f''(z)}{f(z)} - \alpha \right\} \geq \beta \left| \frac{zf'(z) + \lambda z^2 f''(z)}{f(z)} - 1 \right| \quad (2.1)$$

Definition 2.2 The function $f(z)$ defined by Equation (1.1), is said to be in class $K(\lambda, \alpha, \beta)$, ($\lambda \geq 0$, $\alpha, \beta \in \mathbf{R}$ and $z \in U$), if and only if,

$$Re \left\{ 1 + \frac{zf''(z) + \lambda z^2 f'''(z)}{f'(z)} - \alpha \right\} \geq \beta \left| \frac{zf''(z) + \lambda z^2 f'''(z)}{f'(z)} \right| \quad (2.2)$$

We should note that, when $\lambda = 0$, the class $S(0, \alpha, \beta)$ is same as $\beta - S(\alpha)$ and the class $K(0, \alpha, \beta) \equiv \beta - K(\alpha)$.

And for $\beta = 1$ the class $S(\lambda, \alpha, 1) \equiv S(\lambda, \alpha)$ as defined by B. Srutha Keerthi, B. Adolf Stephen, A. Gangadharan and S. Sivasubramanian [1]. For choice of $\lambda = 0$, $\beta = 1$, $S(0, \alpha, 1) \equiv S_p(\alpha)$ the subclass defined by N.E. Cho, S.Y. Woo and S. Owa [9].

Now we let, $S_T(\lambda, \alpha, \beta) = S(\lambda, \alpha, \beta) \cap T$, and $K_T(\lambda, \alpha, \beta) = K(\lambda, \alpha, \beta) \cap T$.

III. Main Result

In the present paper, we determine sufficient condition for the function $f(z)$ defined by Equation (1.1) is in the classes, $S(\lambda, \alpha, \beta)$ and $K(\lambda, \alpha, \beta)$. Then we will discuss the growth and distortion theorems. And finally obtain the results regarding the radii of starlikeness and convexity. For this we begin with the following theorem.

THEOREM 3.1 The function $f(z)$ defined by Equation (1.1) is in the class, $S(\lambda, \alpha, \beta)$, if and only if,

$$\sum_{n=2}^{\infty} [(\beta + 1)\lambda n(n-1) + n(\beta + 1) - \beta - \alpha] |a_n| \leq 1 - \alpha \quad (3.1)$$

PROOF: By using Equation (2.1), we can write,

$$\frac{zf'(z) + \lambda z^2 f''(z)}{f(z)} - 1 = \frac{\sum [\lambda n(n-1) + n-1] a_n z^n}{z + \sum a_n z^n}$$

Substituting right hand side of above Equation in Inequality (2.1),

$$\beta \left| \frac{\sum [\lambda n(n-1) + n-1] a_n z^n}{z + \sum a_n z^n} \right| \leq \operatorname{Re} \left\{ \frac{\sum [\lambda n(n-1) + n-1] a_n z^n}{z + \sum a_n z^n} \right\} + 1 - \alpha$$

Using the fact that, $\operatorname{Re}(z) \leq |z|$ and as $f(z)$ is defined in unit disc $U = \{z : |z| < 1\}$, letting $z \rightarrow 1$ we get,

$$\frac{2 \sum (n + \alpha n(n-1) + \beta n(n-1)(n-2) - 1) |a_n|}{1 + \sum |a_n|} \leq 1 - \lambda \quad (2.3)$$

Upon simplifying we get,

$$\sum_{n=2}^{\infty} [(\beta + 1)\lambda n(n-1) + n(\beta + 1) - \beta - \alpha] |a_n| \leq 1 - \alpha.$$

Which is the required result.

For $\beta = 1$ we get results of B. Srutha Keerthi, B. Adolf Stephen, A. Gangadharan and S. Sivasubramanian [1], [Lemma 2.1, p497].

COROLLARY 3.1 The function $f(z)$ defined by Equation (1.1) is in the class, $S(\lambda, \alpha, 1)$, $(-1 \leq \lambda \leq 1, \alpha \geq 0$ and $z \in U)$, if and only if,

$$\sum_{n=2}^{\infty} (2n + 2\alpha n(n-1) - 3 + \lambda) |a_n| \leq 1 - \alpha$$

THEOREM 3.2 The function $g(z)$ defined by Equation (1.2) is in the class, $S_T(\lambda, \alpha, \beta)$, if and only if,

$$\sum_{n=2}^{\infty} [(\beta + 1)\lambda n(n-1) + n(\beta + 1) - \beta - \alpha] b_n \leq 1 - \alpha \quad (3.2)$$

Prof of this theorem is easily obtained by appropriate rearrangements in Inequality (3.1) above.

As stated above, for $\beta = 1$ we get results of [1], [see Lemma 2.2, p497].

COROLLARY 3.2 The function $g(z)$ defined by Equation (1.2) is in the class $S_T(\alpha, 0, \lambda)$ $(-1 \leq \lambda \leq 1, \alpha$ and $z \in U)$, if and only if,

$$\sum_{n=2}^{\infty} (2n + 2\alpha n(n-1) - 1 - \lambda) b_n \leq 1 - \alpha$$

Theorem 3.3. The function $f(z)$ defined by equation (1.1) is in the class, $K(\lambda, \alpha, \beta)$ if and only if,

$$\sum_{n=2}^{\infty} n [n(\beta + 1)\lambda n(n-1) + \lambda(\beta + 1)(n-1)(n-2) - \beta - \alpha] |a_n| \leq 1 - \alpha \quad (3.3)$$

PROOF: From the function defined by Equation (1.1), we have,

$$\frac{zf''(z) + \lambda z^2 f'''(z)}{f'(z)} = \frac{\sum [1 + \lambda n(n-2)] n(n-1) a_n z^{n-1}}{1 + \sum n a_n z^{n-1}}$$

by using Equation (2.2) we can write,

$$\operatorname{Re} \left\{ \frac{\sum [1 + \lambda n(n-2)] n(n-1) a_n z^{n-1}}{1 + \sum n a_n z^{n-1}} \right\} + 1 - \alpha \geq \beta \left| \frac{\sum [1 + \lambda n(n-2)] n(n-1) a_n z^{n-1}}{1 + \sum n a_n z^{n-1}} \right|$$

Using the fact that, $\operatorname{Re}(z) \leq |z|$ and as $f(z)$ is defined in unit disc $U = \{z : |z| < 1\}$, letting $z \rightarrow 1$ we get equation (3.3), which completes the proof.

IV. Distortion and Growth Bounds for the classes $S(\lambda, \alpha, \beta)$ and $K(\lambda, \alpha, \beta)$.

Theorem 4.1. The function defined by Equation (1.1) belongs to $S(\lambda, \alpha, \beta)$ then, for $f|z| = r < 1$,

$$r - \frac{1 - \alpha}{2\lambda(\beta + 1) + \beta + 2 - \alpha} r^2 \leq |f(z)| \leq r + \frac{1 - \alpha}{2\lambda(\beta + 1) + \beta + 2 - \alpha} r^2 \quad (4.1)$$

$$1 - 2r \frac{1 - \alpha}{2\lambda(\beta + 1) + \beta + 2 - \alpha} \leq |f'(z)| \leq 1 + 2r \frac{1 - \alpha}{2\lambda(\beta + 1) + \beta + 2 - \alpha} \quad (4.2)$$

PROOF: For the function defined by Equation (1.1), using Theorem 3.1 we can write,

$$[2\lambda(\beta + 1) + 2(\beta + 1) - \beta - \alpha] \sum |a_n| \leq 1 - \alpha$$

That is,

$$\sum |a_n| \leq \frac{1-\alpha}{2\lambda(\beta+1)+\beta+2-\alpha} \quad (4.3)$$

For $|z| = r < 1$,

$$r - r^2 \sum |a_n| \leq |f(z)| \leq r + r^2 \sum |a_n|$$

Using inequality (4.3) in above, we get (4.1)

$$\text{Similarly for } |z| = r < 1, \quad 1 - 2r \sum |a_n| \leq |f'(z)| \leq 1 + 2r \sum |a_n|$$

We get (4.2).

Results are sharp for, $f(z) = z - \frac{1-\alpha}{2\lambda(\beta+1)+\beta+2-\alpha} z^2$ at $z = \pm r$.

Theorem 4.2. The function defined by Equation (1.1) belongs to $K(\lambda, \alpha, \beta)$ then, for $|z| = r < 1$,

$$r - \frac{1-\alpha}{2(\beta+2-\alpha)} r^2 \leq |f(z)| \leq r + \frac{1-\alpha}{2(\beta+2-\alpha)} r^2 \quad (4.4)$$

$$1 - r \frac{1-\alpha}{\beta+2-\alpha} \leq |f'(z)| \leq 1 + r \frac{1-\alpha}{\beta+2-\alpha} \quad (4.5)$$

PROOF: For the function defined by Equation (1.1), using Theorem 3.3 we can write,

$$(2(\beta+1) - \beta - \alpha) \sum |a_n| \leq 1 - \alpha$$

That is,

$$\sum |a_n| \leq \frac{1-\alpha}{2(\beta+2-\alpha)} \quad (4.6)$$

As we have seen above, for $|z| = r < 1$,

$$r - r^2 \sum |a_n| \leq |f(z)| \leq r + r^2 \sum |a_n|$$

Using inequality (4.6) in above, we get (4.4)

$$\text{Similarly, for } |z| = r < 1, \quad 1 - 2r \sum |a_n| \leq |f'(z)| \leq 1 + 2r \sum |a_n|$$

We get (4.5).

Results are sharp for, $f(z) = z - \frac{1-\alpha}{\beta+2-\alpha} z^2$ at $z = \pm r$.

V. Radius of starlikeness and convexity for the classes $S(\lambda, \alpha, \beta)$ and

$K(\lambda, \alpha, \beta)$

A. Radius of starlikeness and convexity for the class $S(\lambda, \alpha, \beta)$.

Theorem 5.1. Let the function $f(z)$ defined by Equation (1.1) belongs to $S(\lambda, \alpha, \beta)$, then $f(z)$ is starlike of order δ ($0 \leq \delta < 1$) in the disk $|z| < r_2$ where,

$$r_2 = \inf_{n \geq 2} \left[\left(\frac{1-\delta}{n-2+\delta} \right) \frac{[(\beta+1)\lambda n(n-1) + n(\beta+1) - \beta - \alpha]}{1-\alpha} \right]^{\frac{1}{n-1}} \quad (5.1)$$

PROOF: The function $f(z)$ defined by the Equation (1.1) is starlike of order δ ($0 \leq \delta < 1$) in

the unit disk $U = \{z : |z| < 1\}$, if $\left| \frac{zf'(z)}{f(z)} - 1 \right| < 1 - \delta$ that is,

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \frac{\sum_{n=2}^{\infty} (n-1) |a_n| |z|^{n-1}}{1 + \sum_{n=2}^{\infty} |a_n| |z|^{n-1}} < 1 - \delta$$

That is,

$$\sum_{n=2}^{\infty} \frac{(n-2+\delta) |a_n| |z|^{n-1}}{1-\delta} \leq 1 \quad (5.2)$$

Using the Theorem (3.1), we can say inequality (5.2) is true if,

$$\frac{(n-2+\delta)}{1-\delta} |z|^{n-1} \leq \frac{[(\beta+1)\lambda n(n-1) + n(\beta+1) - \beta - \alpha]}{1-\alpha}$$

The last inequality leads us to the disk $|z| < r_2$, where r_2 is given by (5.1) and the proof of this theorem is completed.

Theorem 5.2. Let the function $f(z)$ defined by Equation (1.1) belongs to $S(\lambda, \alpha, \beta)$, then $f(z)$ is convex of order δ ($0 \leq \delta < 1$) in the disk $|z| < r_3$ where,

$$r_3 = \inf_{n \geq 2} \left[\frac{(1-\delta)[(\beta+1)\lambda n(n-1) + n(\beta+1) - \beta - \alpha]}{n(n-2+\delta)(1-\alpha)} \right]^{\frac{1}{n-1}} \quad (5.3)$$

PROOF: The function $f(z)$ defined by the Equation (1.1) is convex of order δ ($0 \leq \delta < 1$) in

the unit disk $U = \{z : |z| < 1\}$, if $\left| \frac{zf''(z)}{f'(z)} \right| < 1 - \delta$

$$\left| \frac{zf''(z)}{f'(z)} \right| \leq \frac{\sum_{n=2}^{\infty} n(n-1)|a_n||z|^{n-1}}{1 + \sum_{n=2}^{\infty} n|a_n||z|^{n-1}} < 1 - \delta$$

That is,

$$\sum_{n=2}^{\infty} \frac{n(n-2+\delta)|a_n||z|^{n-1}}{1-\delta} \leq 1 \quad (5.4)$$

Using the Theorem (3.1), we can say inequality (5.4) is true if,

$$\frac{n(n-2+\delta)}{1-\delta} |z|^{n-1} \leq \frac{[(\beta+1)\lambda n(n-1) + n(\beta+1) - \beta - \alpha]}{1-\alpha}$$

The last inequality leads us to the disk $|z| < r_3$, where r_3 is given by (5.3) and the proof of this theorem is completed.

B. Radius of starlikeness and convexity for the class $K(\lambda, \alpha, \beta)$.

Theorem 5.3. Let the function $f(z)$ defined by Equation (1.1) belongs to $K(\lambda, \alpha, \beta)$, then $f(z)$ is starlike of order δ ($0 \leq \delta < 1$) in the disk $|z| < r_2$ where,

$$r_2 = \inf_{n \geq 2} \left[\left(\frac{1-\delta}{n-2+\delta} \right) \frac{n[n(\beta+1) + \lambda(n-1)(n-2)(\beta+1) - \beta - \alpha]}{1-\alpha} \right]^{\frac{1}{n-1}} \quad (5.5)$$

Theorem 5.4. Let the function $f(z)$ defined by Equation (1.1) belongs to $K(\lambda, \alpha, \beta)$, then $f(z)$ is convex of order δ ($0 \leq \delta < 1$) in the disk $|z| < r_3$ where,

$$r_3 = \inf_{n \geq 2} \left[\frac{(1-\delta)[n(\beta+1)\lambda(n-1)(n-2)(\beta+1) - \beta - \alpha]}{(n-2+\delta)(1-\alpha)} \right]^{\frac{1}{n-1}} \quad (5.6)$$

VI. References

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VII. Acknowledgements

The authors are thankful to referees and editors for their valuable suggestions.