



A Numerical Solution of Two-dimensional coupled Burgers' equation using Spline Explicit & Implicit Scheme.

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Abstract: In this paper, a numerical solution of the two dimensional nonlinear coupled viscous Burgers' equation is discussed with the appropriate initial and boundary conditions using the Spline explicit and implicit method. In two dimensional the Cole-Hopf procedure transforms Burgers' equation into the linear heat conduction equation. The accuracy of the scheme is illustrated numerical examples. Computed results are compared with the exact solutions and other results available in the literature. Numerical results show that the Spline explicit & implicit scheme is efficient and reliable scheme for solving the two dimensional coupled Burgers' equation.

Keywords: Burgers' equation, Cole-Hopf Transformation, Explicit Scheme, Implicit Scheme, Partial differential equation

I. Introduction

The Burgers model of turbulence is a very important fluid dynamic model and the study of this model and the theory of shock waves have been considered by many authors, both to obtain a conceptual understanding of a class of physical flows and for testing various Numerical methods. A well-known model was first introduced by Bateman, who found its steady solutions, descriptive of certain viscous flows. It was later proposed by Burgers as one of a class of equations describing mathematical models of turbulence. In the context of gas dynamics, it was discussed by Hopf and Cole.

Consider the two dimensional nonlinear unsteady coupled viscous Burgers' equations:

$$\begin{aligned}\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} &= \frac{1}{\text{Re}} \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \\ \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} &= \frac{1}{\text{Re}} \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) \dots\dots\dots (1)\end{aligned}$$

With the initial conditions

$$u(x, y, 0) = \psi_1(x, y); \quad (x, y) \in \Omega$$

$$v(x, y, 0) = \psi_2(x, y); \quad (x, y) \in \Omega$$

And Dirichlet boundary conditions

$$u(x, y, t) = \xi(x, y, t); \quad (x, y, t) \in \partial\Omega$$

$$v(x, y, t) = \zeta(x, y, t); \quad (x, y, t) \in \partial\Omega$$

Where $\Omega = \{(x, y) : a \leq x \leq b, c \leq y \leq d\}$ is the computational domain, where u & v are the velocity component to be determinant and $\partial\Omega$ is its boundary, ψ_1, ψ_2, ξ and ζ are known functions, $\frac{\partial u}{\partial t}$ is unsteady

term, $u \frac{\partial u}{\partial x}$ is the nonlinear convection term, $\frac{1}{\text{Re}} \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$ is the diffusion term, and Re is the Reynolds number.

Coupled viscous Burgers' equation is a more appropriate form of the Navier-Stokes equations having the exact solutions. It has the same convective and diffusion form as the incompressible Navier-Stokes equations, and is a simple model for understanding of various physical flows and problems, such as hydrodynamic turbulence.

Numerical solution of Burgers' equation is a natural and first step towards developing methods for the computation of complex flows. Thus, it has become customary to test new approaches in computational fluid dynamics by implementing novel and new approaches to the Burgers' equation yielding in various finite-differences, finite volume, finite-element and boundary element methods etc.

Analytic solution of two dimensional coupled Burgers' equations was first given by Fletcher using the Hopf-Cole transformation. The numerical solutions of coupled Burgers' equations are numerically solved by many researchers.

In this paper, we study the numerical simulation of the two-dimensional unsteady nonlinear coupled viscous Burgers' equations for different Reynolds number.

We have applied the Hopf-Cole transformation in equation (1.1)-(1.2).

$$u(x, y, t) = \frac{-2}{\text{Re}} \cdot \frac{1}{\phi} \frac{\partial \phi}{\partial x} \quad \& \quad v(x, y, t) = \frac{-2}{\text{Re}} \cdot \frac{1}{\phi} \frac{\partial \phi}{\partial y}$$

$$\frac{\partial u}{\partial t} = \frac{-2}{\text{Re}} \left(\frac{\phi_{xt} \phi - \phi_x \phi_t}{\phi^2} \right),$$

$$\frac{\partial u}{\partial x} = \frac{-2}{\text{Re}} \left(\frac{\phi_{xx} \phi - \phi_x^2}{\phi^2} \right), \quad \frac{\partial u}{\partial y} = \frac{-2}{\text{Re}} \left(\frac{\phi_{xy} \phi - \phi_x \phi_y}{\phi^2} \right),$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{-2}{\text{Re}} \left(\frac{\phi_{xxx} \phi - \phi_{xx} \phi_x}{\phi^2} - \frac{2\phi_{xx} \phi_x \phi^2 - 2\phi \phi_x^3}{\phi^4} \right)$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{-2}{\text{Re}} \left(\frac{\phi_{xyy} \phi - \phi_{xy} \phi_y}{\phi^2} - \frac{\phi_{xy} \phi_y \phi^2 + \phi^2 \phi_x \phi_{yy} - 2\phi \phi_x \phi_y^2}{\phi^4} \right)$$

Substitute all the above value in equation (1.1) leads to

$$\phi_{xt} \phi - \phi_x \phi_t = \frac{1}{\text{Re}} (\phi \phi_{xxx} - \phi_x \phi_{xx}) + (\phi \phi_{xyy} - \phi_x \phi_{yy}) \dots (1.5)$$

Dividing ϕ^2 both sides of above equation, the equation can be written as

$$\left(\frac{\phi_t}{\phi} \right)_x = \frac{1}{\text{Re}} \left[\left(\frac{\phi_{xx}}{\phi} \right)_x + \left(\frac{\phi_{yy}}{\phi} \right)_x \right] \dots (1.6)$$

Integrating both side w.r.t. x and simplifying it, we have

$$\phi_t = \frac{1}{\text{Re}} (\phi_{xx} + \phi_{yy}) + \alpha_1(y, t) \phi \dots (1.7)$$

Similarly using equation (1.2) we get another equation

$$\phi_t = \frac{1}{\text{Re}} (\phi_{xx} + \phi_{yy}) + \alpha_2(x, t) \phi \dots (1.8)$$

Where $\alpha_1(t)$ and $\alpha_2(t)$ are an arbitrary functions depending on t.

Combining equation (1.7) and (1.8), we can get

$$\phi_t = \frac{1}{\text{Re}} (\phi_{xx} + \phi_{yy}) + \alpha(t) \phi \dots (1.9)$$

Where $\alpha(t)$ is an arbitrary function depending on t only.

If we can choose $\alpha(t)=0$ in equation (1.9) to simplify the discussion and computation

$$\phi_t = \frac{1}{\text{Re}} (\phi_{xx} + \phi_{yy}) \dots (2.0)$$

Generalization of Spline Explicit & Implicit Formula

Now discretizing left side of equation (2.0) by forward difference formula and replacing right side by the second derivatives of S at j^{th} level like explicit scheme in finite difference, we get

$$\frac{\phi_{i,j,k+1} - \phi_{i,j,k}}{\Delta t} = \frac{1}{\text{Re}} (2S''_{i,j,k}) \dots \dots \dots (2.1)$$

Substitute values of $S''_{i,j}$ in the below equation

$$S''_{i-1,j,k} + 4S''_{i,j,k} + S''_{i+1,j,k} = (6/h^2)(\phi_{i-1,j,k} - 2\phi_{i,j,k} + \phi_{i+1,j,k}) \dots \dots \dots (2.2)$$

Then we get

$$\begin{aligned} \frac{(\phi_{i-1,j,k+1} - \phi_{i-1,j,k})}{2(1/\text{Re})\Delta t} + 4 \frac{(\phi_{i,j,k+1} - \phi_{i,j,k})}{2(1/\text{Re})\Delta t} + \frac{(\phi_{i+1,j,k+1} - \phi_{i+1,j,k})}{2(1/\text{Re})\Delta t} \\ = \frac{6}{h^2} [\phi_{i-1,j,k} - 2\phi_{i,j,k} + \phi_{i+1,j,k}] \dots \dots \dots (2.3) \end{aligned}$$

$$\begin{aligned} \phi_{i-1,j,k+1} + 4\phi_{i,j,k+1} + \phi_{i+1,j,k+1} \\ = (1 + 12 \frac{\Delta t}{h^2 \cdot \text{Re}}) \phi_{i-1,j,k} + (4 - 24 \frac{\Delta t}{h^2 \cdot \text{Re}}) \phi_{i,j,k} + (1 + 12 \frac{\Delta t}{h^2 \cdot \text{Re}}) \phi_{i+1,j,k} \dots \dots \dots (2.4) \end{aligned}$$

Equation (2.4) is known as Spline Explicit formula

Same as we discretizing left side of equation (2.0) by forward difference formula and replacing right side by the second derivatives of S at j^{th} and $j+1^{\text{th}}$ level like explicit scheme in finite difference, we get

$$\frac{\phi_{i,j,k+1} - \phi_{i,j,k}}{2(1/\text{Re})\Delta t} = S''_{i,j,k} + S''_{i,j,k+1} \dots \dots \dots (2.5)$$

Substitute above values in below two equations

$$S''_{i-1,j,k+1} + 4S''_{i,j,k+1} + S''_{i+1,j,k+1} = \frac{6}{h^2} (\phi_{i-1,j,k+1} - 2\phi_{i,j,k+1} + \phi_{i+1,j,k+1}) \dots \dots (2.6)$$

$$S''_{i-1,j,k} + 4S''_{i,j,k} + S''_{i+1,j,k} = \frac{6}{h^2} (\phi_{i-1,j,k} - 2\phi_{i,j,k} + \phi_{i+1,j,k}) \dots \dots \dots (2.7)$$

Then we get

$$\begin{aligned} \frac{\phi_{i-1,j,k+1} - \phi_{i-1,j,k}}{2(1/\text{Re})\Delta t} + 4 \left[\frac{\phi_{i,j,k+1} - \phi_{i,j,k}}{2(1/\text{Re})\Delta t} \right] + \frac{\phi_{i+1,j,k+1} - \phi_{i+1,j,k}}{2(1/\text{Re})\Delta t} \\ = \frac{6}{h^2} [(\phi_{i-1,j,k+1} + \phi_{i-1,j,k}) - 2(\phi_{i,j,k+1} + \phi_{i,j,k}) + (\phi_{i+1,j,k+1} + \phi_{i+1,j,k})] \dots \dots \dots (2.8) \end{aligned}$$

Equation (2.8) is known as Spline Implicit formula.

II. Research Method

Problem 1: In this problem, we take the computational domain

$$\Omega = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}$$

$$\left. \begin{aligned} u(x, y, t) &= \frac{3}{4} - \frac{1}{4[1 + \exp((-4x + 4y - t)\text{Re}/32)]} \\ v(x, y, t) &= \frac{3}{4} + \frac{1}{4[1 + \exp((-4x + 4y - t)\text{Re}/32)]} \end{aligned} \right\} (x, y) \in \Omega \dots \dots \dots (2.9)$$

With initial conditions

$$\left. \begin{aligned} u(x, y, 0) &= \frac{3}{4} - \frac{1}{4[1 + \exp((-4x + 4y)\text{Re}/32)]} \\ v(x, y, 0) &= \frac{3}{4} + \frac{1}{4[1 + \exp((-4x + 4y)\text{Re}/32)]} \end{aligned} \right\} (x, y) \in \Omega \dots \dots \dots (3.0)$$

In equation (2.4) we consider $\Delta t = 0.0001, h = 0.1$ & $\text{Re} = 100$

$$\left(1 + \frac{12\Delta t}{\text{Re} \cdot h^2}\right) = 1.0012 \quad \& \quad \left(4 - \frac{24\Delta t}{\text{Re} \cdot h^2}\right) = 3.9976$$

$$\phi_{i-1,j,k+1} + 4\phi_{i,j,k+1} + \phi_{i+1,j,k+1} = (1.0012)\phi_{i+1,j,k} + (3.9976)\phi_{i,j,k} + (1.0012)\phi_{i-1,j,k} \dots \dots \dots (3.1)$$

For $i = 1, j = 0$ and $k = 0$

$$\begin{aligned} \phi_{0,0,1} + 4\phi_{1,0,1} + \phi_{2,0,1} &= 3.11615 \\ \phi_{1,0,1} + 4\phi_{2,0,1} + \phi_{3,0,1} &= 3.35863 \\ \phi_{2,0,1} + 4\phi_{3,0,1} + \phi_{4,0,1} &= 3.13342 \\ \phi_{3,0,1} + 4\phi_{4,0,1} + \phi_{5,0,1} &= 3.04201 \\ \phi_{4,0,1} + 4\phi_{5,0,1} + \phi_{6,0,1} &= 3.01230 \\ \phi_{5,0,1} + 4\phi_{6,0,1} + \phi_{7,0,1} &= 3.00330 \\ \phi_{6,0,1} + 4\phi_{7,0,1} + \phi_{8,0,1} &= 3.00080 \\ \phi_{7,0,1} + 4\phi_{8,0,1} + \phi_{9,0,1} &= 3.00010 \\ \phi_{8,0,1} + 4\phi_{9,0,1} + \phi_{10,0,1} &= 3.00000 \end{aligned}$$

Solving above system of linear equations with the help of Matlab, we have got the solution as shown in below table and these solutions are compared with the exact solutions.

In equation (2.9) we consider $\Delta t = 0.0001, h = 0.1$ & $\text{Re} = 100$

$$1 - \frac{12\Delta t}{\text{Re} \cdot h^2} = 0.9999, \quad 4 + \frac{24\Delta t}{\text{Re} \cdot h^2} = 4.00015,$$

$$1 + \frac{12\Delta t}{\text{Re} \cdot h^2} = 1.00007 \quad \& \quad 4 - \frac{24\Delta t}{\text{Re} \cdot h^2} = 3.9998$$

$$(0.9999)\phi_{i-1,j,k+1} + (4.0015)\phi_{i,j,k+1} + (0.9999)\phi_{i+1,j,k+1} \\ = (1.0007)\phi_{i-1,j,k} + (3.9999)\phi_{i,j,k} + (1.0007)\phi_{i+1,j,k} \dots\dots\dots(3.2)$$

For $i = 1, j = 0$ and $k = 0$

$$(0.9999)\phi_{1,0,1} + (4.0015)\phi_{1,0,1} + (0.9999)\phi_{2,0,1} = 3.626589 \\ (0.9999)\phi_{1,0,1} + (4.0015)\phi_{2,0,1} + (0.9999)\phi_{3,0,1} = 3.131408 \\ (0.9999)\phi_{2,0,1} + (4.0015)\phi_{3,0,1} + (0.9999)\phi_{4,0,1} = 3.126608 \\ (0.9999)\phi_{3,0,1} + (4.0015)\phi_{4,0,1} + (0.9999)\phi_{5,0,1} = 3.501582 \\ (0.9999)\phi_{4,0,1} + (4.0015)\phi_{5,0,1} + (0.9999)\phi_{6,0,1} = 3.379726 \\ (0.9999)\phi_{5,0,1} + (4.0015)\phi_{6,0,1} + (0.9999)\phi_{7,0,1} = 3.119773 \\ (0.9999)\phi_{6,0,1} + (4.0015)\phi_{7,0,1} + (0.9999)\phi_{8,0,1} = 3.749889 \\ (0.9999)\phi_{7,0,1} + (4.0015)\phi_{8,0,1} + (0.9999)\phi_{9,0,1} = 3.379725 \\ (0.9999)\phi_{8,0,1} + (4.0015)\phi_{9,0,1} + (0.9999)\phi_{10,0,1} = 4.119779$$

Solving above system of linear equations with the help of Matlab, we have got the solution as shown in below table and these solutions are compared with the exact solutions.

In equation (2.4) we consider $\Delta t = 0.0001, h=0.1$ & $Re = 100$

$$(1 + \frac{12\Delta t}{Re.h^2}) = 1.0012 \quad \& \quad (4 - \frac{24\Delta t}{Re.h^2}) = 3.9976$$

$$\phi_{i-1,j,k+1} + 4\phi_{i,j,k+1} + \phi_{i+1,j,k+1} = (1.0012)\phi_{i+1,j,k} + (3.9976)\phi_{i,j,k} + (1.0012)\phi_{i-1,j,k} \dots\dots\dots(3.3)$$

For $i = 1, j = 0$ and $k = 0$

$$\phi_{0,0,1} + 4\phi_{1,0,1} + \phi_{2,0,1} = 4.79219 \\ \phi_{1,0,1} + 4\phi_{2,0,1} + \phi_{3,0,1} = 5.51863 \\ \phi_{2,0,1} + 4\phi_{3,0,1} + \phi_{4,0,1} = 5.86266 \\ \phi_{3,0,1} + 4\phi_{4,0,1} + \phi_{5,0,1} = 5.95633 \\ \phi_{4,0,1} + 4\phi_{5,0,1} + \phi_{6,0,1} = 5.98703 \\ \phi_{5,0,1} + 4\phi_{6,0,1} + \phi_{7,0,1} = 5.99621 \\ \phi_{6,0,1} + 4\phi_{7,0,1} + \phi_{8,0,1} = 5.99890 \\ \phi_{7,0,1} + 4\phi_{8,0,1} + \phi_{9,0,1} = 5.99967 \\ \phi_{8,0,1} + 4\phi_{9,0,1} + \phi_{10,0,1} = 5.99986$$

Solving above system of linear equations with the help of Matlab, we have got the solution as shown in below table and these solutions are compared with the exact solutions.

In equation (2.9) we consider $\Delta t = 0.0001$, $h=0.1$ & $Re = 100$

$$1 + \frac{12\Delta t}{Re \cdot h} = 1.00007 \text{ \& } 4 - \frac{24\Delta t}{Re \cdot h} = 3.9998$$

$$(0.9999)\phi_{i-1,j,k+1} + (4.0015)\phi_{i,j,k+1} + (0.9999)\phi_{i+1,j,k+1} \\ = (1.0007)\phi_{i-1,j,k} + (3.9999)\phi_{i,j,k} + (1.0007)\phi_{i+1,j,k} \dots\dots\dots(3.4)$$

For $i = 1$, $j = 0$ and $k = 0$

$$\begin{aligned} (0.9999)\phi_{0,0,1} + (4.0015)\phi_{1,0,1} + (0.9999)\phi_{2,0,1} &= 4.793048 \\ (0.9999)\phi_{1,0,1} + (4.0015)\phi_{2,0,1} + (0.9999)\phi_{3,0,1} &= 5.519934 \\ (0.9999)\phi_{2,0,1} + (4.0015)\phi_{3,0,1} + (0.9999)\phi_{4,0,1} &= 5.863948 \\ (0.9999)\phi_{3,0,1} + (4.0015)\phi_{4,0,1} + (0.9999)\phi_{5,0,1} &= 5.957636 \\ (0.9999)\phi_{4,0,1} + (4.0015)\phi_{5,0,1} + (0.9999)\phi_{6,0,1} &= 5.988335 \\ (0.9999)\phi_{5,0,1} + (4.0015)\phi_{6,0,1} + (0.9999)\phi_{7,0,1} &= 5.997518 \\ (0.9999)\phi_{6,0,1} + (4.0015)\phi_{7,0,1} + (0.9999)\phi_{8,0,1} &= 6.000209 \\ (0.9999)\phi_{7,0,1} + (4.0015)\phi_{8,0,1} + (0.9999)\phi_{9,0,1} &= 6.000979 \\ (0.9999)\phi_{8,0,1} + (4.0015)\phi_{9,0,1} + (0.9999)\phi_{10,0,1} &= 6.001169 \end{aligned}$$

Solving above system of linear equations with the help of Matlab, we have got the solution as shown in below table and these solutions are compared with the exact solutions.

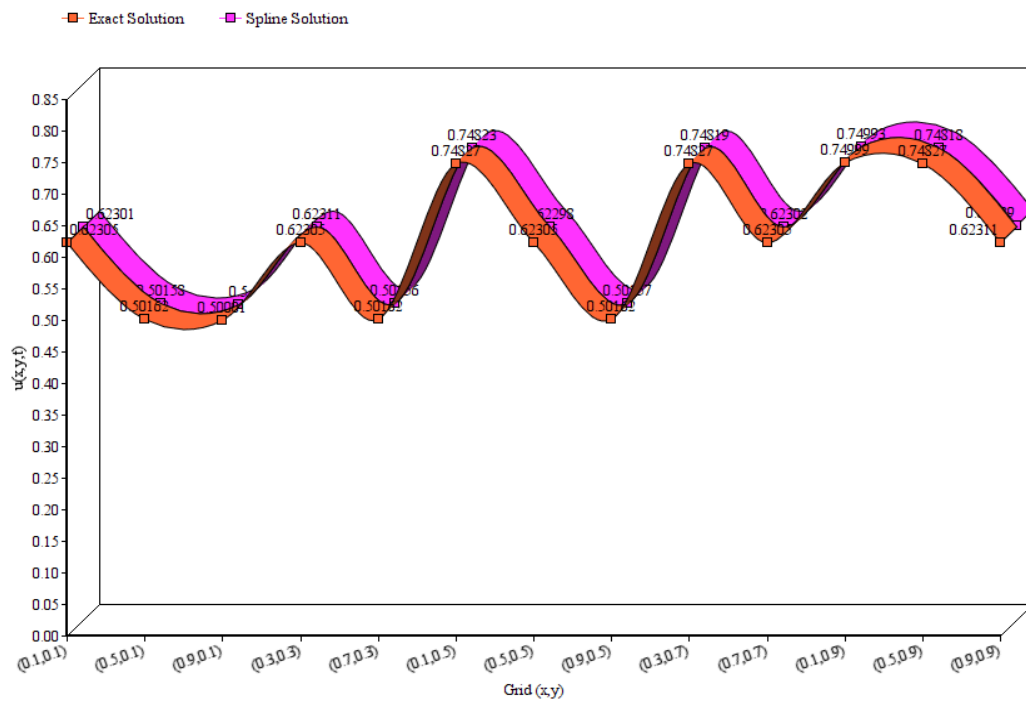
Computed Values of $u(\Delta t=0.0001 \text{ \& } Re = 100)$			
Grid(x,y)	Analytic	Explicit	Implicit
(0.1,0.1)	0.62305	0.62301	0.62299
(0.5,0.1)	0.50162	0.50158	0.50147
(0.9,0.1)	0.50001	0.50000	0.49995
(0.3,0.3)	0.62305	0.62311	0.62298
(0.7,0.3)	0.50162	0.50156	0.50143
(0.1,0.5)	0.74827	0.74823	0.74822
(0.5,0.5)	0.62305	0.62298	0.62302
(0.9,0.5)	0.50162	0.50157	0.50158
(0.3,0.7)	0.74827	0.74819	0.74816
(0.7,0.7)	0.62305	0.62302	0.62289
(0.1,0.9)	0.74999	0.74993	0.74987
(0.5,0.9)	0.74827	0.74818	0.74819
(0.9,0.9)	0.62311	0.62304	0.62307

Computed Values of $u(\Delta t=0.001 \text{ \& Re} = 100)$			
Grid(x,y)	Analytic	Explicit	Implicit
(0.1,0.1)	0.54332	0.53325	0.53323
(0.5,0.1)	0.50035	0.49976	0.49998
(0.9,0.1)	0.50000	0.49965	0.49974
(0.3,0.3)	0.54332	0.54314	0.54328
(0.7,0.3)	0.50035	0.49961	0.50023
(0.1,0.5)	0.74221	0.74134	0.74214
(0.5,0.5)	0.54332	0.54225	0.54316
(0.9,0.5)	0.50035	0.49994	0.50001
(0.3,0.7)	0.74221	0.74219	0.74218
(0.7,0.7)	0.54332	0.54304	0.54325
(0.1,0.9)	0.74995	0.74989	0.74989
(0.5,0.9)	0.74221	0.74219	0.74215
(0.9,0.9)	0.54332	0.54239	0.54326

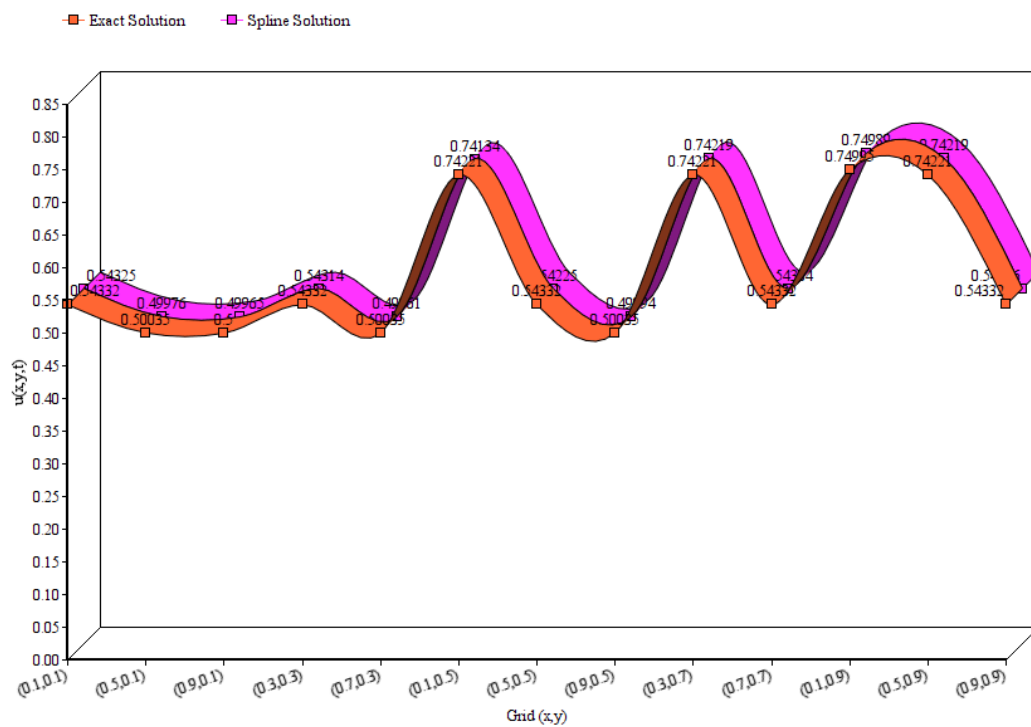
Computed Values of $v(\Delta t=0.0001 \text{ \& Re} = 100)$			
Grid(x,y)	Analytic	Explicit	Implicit
(0.1,0.1)	0.87695	0.87005	0.87088
(0.5,0.1)	0.99838	0.99789	0.99802
(0.9,0.1)	0.99999	0.99990	0.99989
(0.3,0.3)	0.87689	0.87687	0.87688
(0.7,0.3)	0.99838	0.99837	0.99838
(0.1,0.5)	0.75173	0.75170	0.75171
(0.5,0.5)	0.87695	0.87688	0.87691
(0.9,0.5)	0.99838	0.99835	0.99836
(0.3,0.7)	0.75173	0.75169	0.75171
(0.7,0.7)	0.87695	0.87684	0.87693
(0.1,0.9)	0.75001	0.74997	0.74999
(0.5,0.9)	0.75173	0.75169	0.75171
(0.9,0.9)	0.87695	0.87689	0.87691

Computed Values of $v(\Delta t=0.001 \text{ \& Re} = 100)$			
Grid(x,y)	Analytic	Explicit	Implicit
(0.1,0.1)	0.95668	0.94665	0.94661
(0.5,0.1)	0.99965	0.99857	0.99889
(0.9,0.1)	1.00000	0.99964	0.99969
(0.3,0.3)	0.95731	0.95695	0.95698
(0.7,0.3)	0.99968	0.99957	0.99958
(0.1,0.5)	0.75779	0.75589	0.75579
(0.5,0.5)	0.95668	0.95658	0.95661
(0.9,0.5)	0.99970	0.99969	0.99971
(0.3,0.7)	0.75779	0.75765	0.75773
(0.7,0.7)	0.95668	0.95654	0.95663
(0.1,0.9)	0.75005	0.74999	0.75001
(0.5,0.9)	0.75779	0.75774	0.75775
(0.9,0.9)	0.95668	0.95658	0.95663

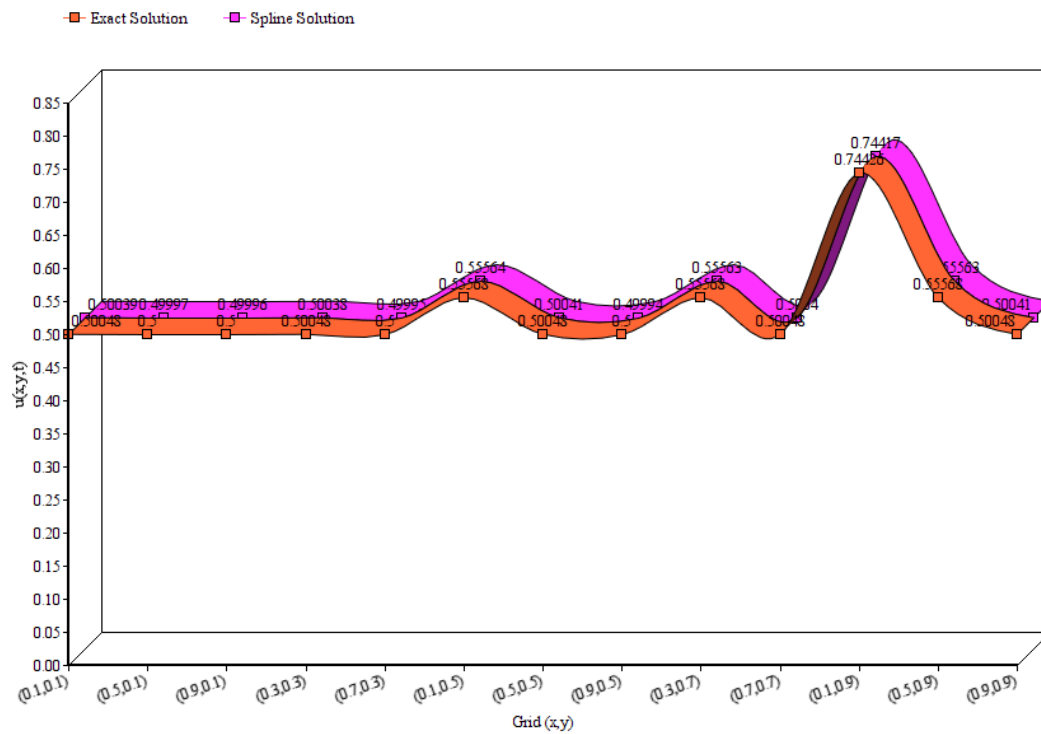
Computed values of u ($\Delta t = 0.0001$ & $Re = 100$)



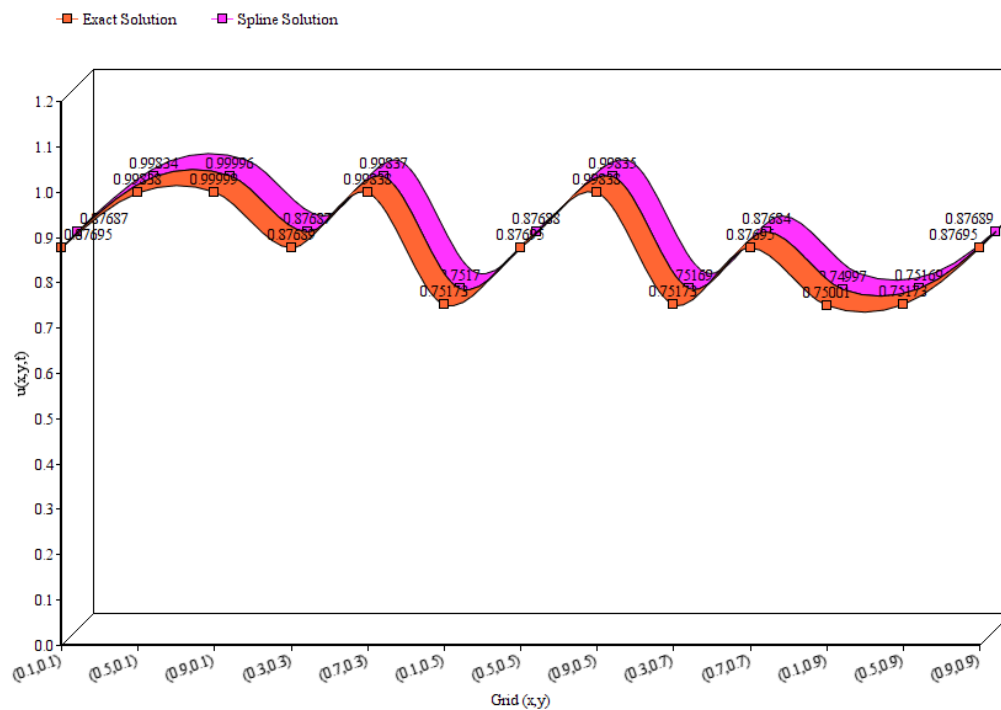
Computed values of u ($\Delta t = 0.001$ & $Re = 100$)



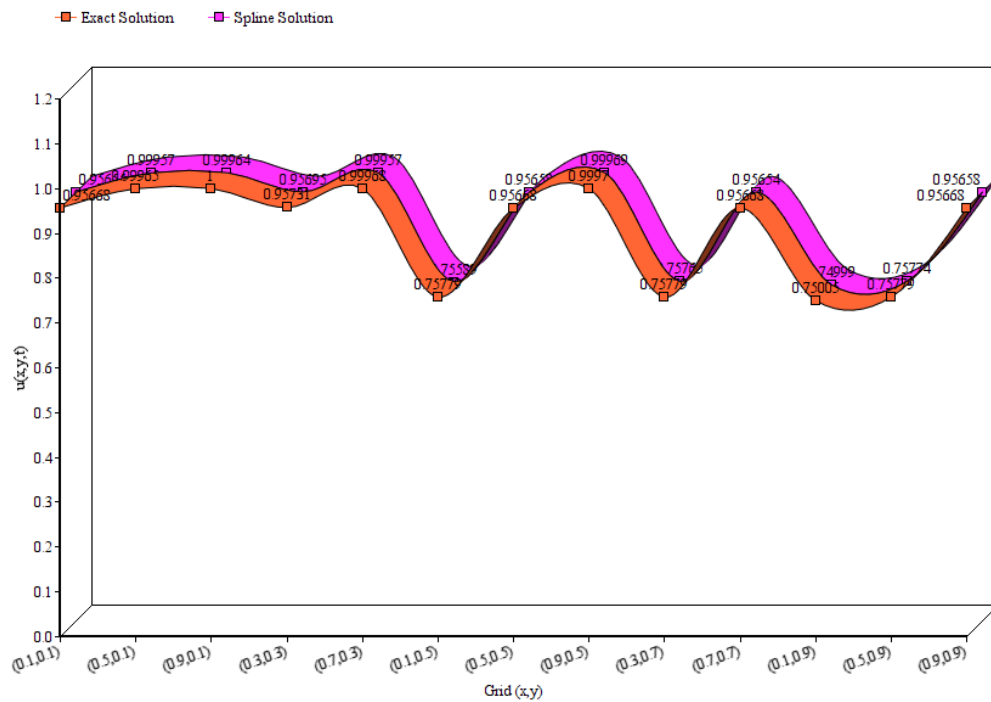
Computed values of u ($\Delta t = 0.01$ & $Re = 100$)



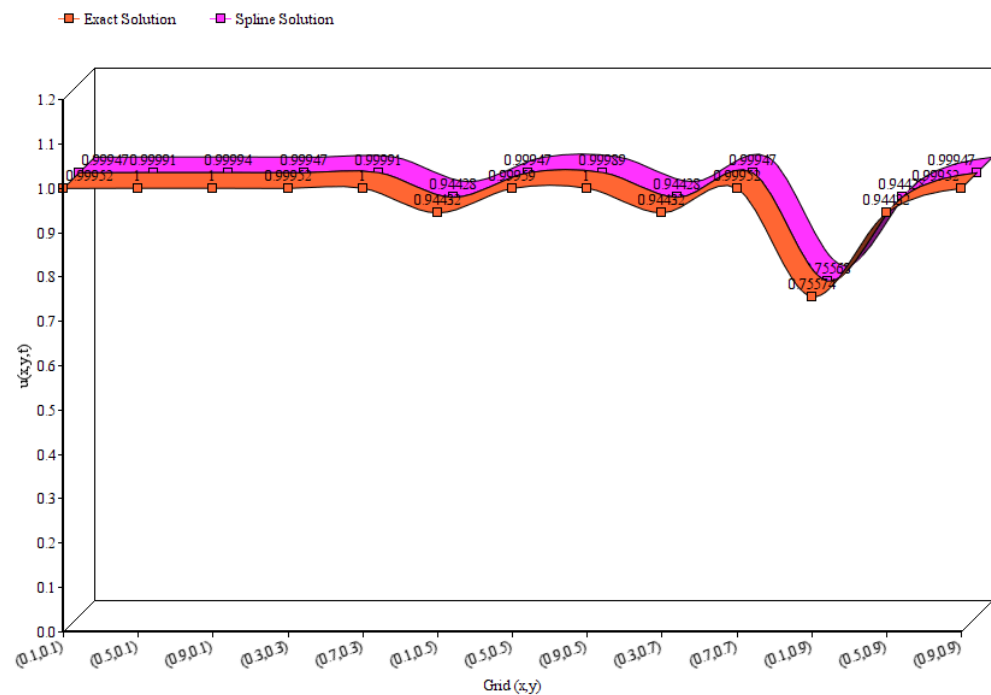
Computed values of v ($\Delta t = 0.0001$ & $Re = 100$)



Computed values of v ($\Delta t = 0.001$ & $Re = 100$)



Computed values of v ($\Delta t = 0.01$ & $Re = 100$)



In equation (1.9) $\alpha(t) \neq 0$ then we get

$$\phi_t = \frac{1}{\text{Re}} (\phi_{xx} + \phi_{yy}) + \phi$$

$$\phi_t - \phi = \frac{1}{\text{Re}} (\phi_{xx} + \phi_{yy}) \dots \dots \dots (3.5)$$

Now discretizing left side of equation (3.4) by forward difference formula and replacing right side by the second derivatives of S at j^{th} level like explicit scheme in finite difference, we get

$$\frac{\phi_{i,j,k+1} - \phi_{i,j,k}}{\Delta t} - \phi_{i,j,k} = \frac{1}{\text{Re}} (2S_{i,j,k}'')$$

$$\frac{\phi_{i,j,k+1} - \phi_{i,j,k}}{\Delta t(1/\text{Re})^2} - \frac{\phi_{i,j,k}}{(1/\text{Re})^2} = S_{i,j,k}''$$

Substitute values of $S_{i,j}''$ in the below equation

$$S_{i-1,j,k}'' + 4S_{i,j,k}'' + S_{i+1,j,k}'' = (6/h^2)(\phi_{i-1,j,k} - 2\phi_{i,j,k} + \phi_{i+1,j,k}) \dots \dots \dots (3.6)$$

Then we get

$$\frac{(\phi_{i-1,j,k+1} - \phi_{i-1,j,k})}{2(1/\text{Re})\Delta t} - \frac{\phi_{i-1,j,k}}{(1/\text{Re})^2} + 4 \frac{(\phi_{i,j,k+1} - \phi_{i,j,k})}{2(1/\text{Re})\Delta t} - 4 \frac{\phi_{i,j,k}}{(1/\text{Re})^2}$$

$$+ \frac{(\phi_{i+1,j,k+1} - \phi_{i+1,j,k})}{2(1/\text{Re})\Delta t} - \frac{\phi_{i+1,j,k}}{(1/\text{Re})^2} = \frac{6}{h^2} [\phi_{i-1,j,k} - 2\phi_{i,j,k} + \phi_{i+1,j,k}] \dots \dots \dots (3.7)$$

$$\phi_{i-1,j,k+1} + 4\phi_{i,j,k+1} + \phi_{i+1,j,k+1}$$

$$= (1 + 12 \frac{\Delta t}{h^2 \cdot \text{Re}} + \Delta t)\phi_{i-1,j,k} + (4 - 24 \frac{\Delta t}{h^2 \cdot \text{Re}} + 4\Delta t)\phi_{i,j,k} + (1 + 12 \frac{\Delta t}{h^2 \cdot \text{Re}} + \Delta t)\phi_{i+1,j,k} \dots \dots \dots (3.8)$$

Same as we discretizing left side of equation (2.0) by forward difference formula and replacing right side by the second derivatives of S at j^{th} and $j+1^{\text{th}}$ level like explicit scheme in finite difference, we get

$$\frac{\phi_{i,j,k+1} - \phi_{i,j,k}}{\Delta t} - \phi_{i,j,k} = \frac{2}{\text{Re}} (S_{i,j,k}'' + S_{i,j,k+1}'')$$

$$\frac{\phi_{i,j,k+1} - \phi_{i,j,k}}{2(1/\text{Re})\Delta t} - \frac{\phi_{i,j,k}}{2(1/\text{Re})} = S_{i,j,k}'' + S_{i,j,k+1}'' \dots \dots \dots (3.9)$$

Substitute above values in below two equations

$$S_{i-1,j,k+1}'' + 4S_{i,j,k+1}'' + S_{i+1,j,k+1}'' = \frac{6}{h^2} (\phi_{i-1,j,k+1} - 2\phi_{i,j,k+1} + \phi_{i+1,j,k+1}) \dots \dots (4.0)$$

$$S_{i-1,j,k}'' + 4S_{i,j,k}'' + S_{i+1,j,k}'' = \frac{6}{h^2} (\phi_{i-1,j,k} - 2\phi_{i,j,k} + \phi_{i+1,j,k}) \dots \dots \dots (4.1)$$

Equation (4.4) is known as Spline Implicit formula

Now we consider $\Delta t = 0.0001$, $h=0.1$ & $\text{Re} = 100$ in equation (3.9)

$$(1 + \frac{12\Delta t}{\text{Re} \cdot h^2} + \Delta t) = 1.0013 \quad \& \quad (4 - \frac{24\Delta t}{\text{Re} \cdot h^2} + 4\Delta t) = 3.9977$$

$$\phi_{i-1,j,k+1} + 4\phi_{i,j,k+1} + \phi_{i+1,j,k+1} = (1.0013)\phi_{i+1,j,k} + (3.9977)\phi_{i,j,k} + (1.0013)\phi_{i-1,j,k} \dots \dots \dots (4.2)$$

For $i = 1, j = 0$ and $k = 0$

$$\begin{aligned}\phi_{0,0,1} + 4\phi_{1,0,1} + \phi_{2,0,1} &= 3.05569 \\ \phi_{1,0,1} + 4\phi_{2,0,1} + \phi_{3,0,1} &= 3.36650 \\ \phi_{2,0,1} + 4\phi_{3,0,1} + \phi_{4,0,1} &= 3.13708 \\ \phi_{3,0,1} + 4\phi_{4,0,1} + \phi_{5,0,1} &= 3.04375 \\ \phi_{4,0,1} + 4\phi_{5,0,1} + \phi_{6,0,1} &= 3.01305 \\ \phi_{5,0,1} + 4\phi_{6,0,1} + \phi_{7,0,1} &= 3.00387 \\ \phi_{6,0,1} + 4\phi_{7,0,1} + \phi_{8,0,1} &= 3.00118 \\ \phi_{7,0,1} + 4\phi_{8,0,1} + \phi_{9,0,1} &= 3.00041 \\ \phi_{8,0,1} + 4\phi_{9,0,1} + \phi_{10,0,1} &= 3.00022\end{aligned}$$

Solving above system of linear equations with the help of Matlab, we have got the solution as shown in below table and these solutions are compared with the exact solutions.

In equation (4.4) we consider $\Delta t = 0.0001, h=0.1$ & $Re = 100$

$$\begin{aligned}1 - \frac{12\Delta t}{Re \cdot h} &= 0.9999, 4 + \frac{24\Delta t}{Re \cdot h} = 4.00015, \\ 1 + \frac{12\Delta t}{Re \cdot h} + \Delta t &= 1.00017 \text{ \& } 4 - \frac{24\Delta t}{Re \cdot h} + 4\Delta t = 3.9999 \\ (0.9999)\phi_{i-1,j,k+1} + (4.0015)\phi_{i,j,k+1} + (0.9999)\phi_{i+1,j,k+1} \\ &= (1.00017)\phi_{i-1,j,k} + (3.9999)\phi_{i,j,k} + (1.00017)\phi_{i+1,j,k} \dots\dots\dots(4.3)\end{aligned}$$

For $i = 1, j = 0$ and $k = 0$

$$\begin{aligned}(0.9999)\phi_{0,0,1} + (4.0015)\phi_{1,0,1} + (0.9999)\phi_{2,0,1} &= 3.751211 \\ (0.9999)\phi_{1,0,1} + (4.0015)\phi_{2,0,1} + (0.9999)\phi_{3,0,1} &= 3.366802 \\ (0.9999)\phi_{2,0,1} + (4.0015)\phi_{3,0,1} + (0.9999)\phi_{4,0,1} &= 3.137403 \\ (0.9999)\phi_{3,0,1} + (4.0015)\phi_{4,0,1} + (0.9999)\phi_{5,0,1} &= 3.043735 \\ (0.9999)\phi_{4,0,1} + (4.0015)\phi_{5,0,1} + (0.9999)\phi_{6,0,1} &= 3.013037 \\ (0.9999)\phi_{5,0,1} + (4.0015)\phi_{6,0,1} + (0.9999)\phi_{7,0,1} &= 3.003855 \\ (0.9999)\phi_{6,0,1} + (4.0015)\phi_{7,0,1} + (0.9999)\phi_{8,0,1} &= 3.001192 \\ (0.9999)\phi_{7,0,1} + (4.0015)\phi_{8,0,1} + (0.9999)\phi_{9,0,1} &= 3.000425 \\ (0.9999)\phi_{8,0,1} + (4.0015)\phi_{9,0,1} + (0.9999)\phi_{10,0,1} &= 3.000471\end{aligned}$$

Solving above system of linear equations with the help of Matlab, we have got the solution as shown in below table and these solutions are compared with the exact solutions.

Computed Values of $u(\Delta t=0.0001 \text{ \& Re} = 100)$			
Grid(x,y)	Analytic	Explicit	Implicit
(0.1,0.1)	0.62305	0.64001	0.63998
(0.5,0.1)	0.50162	0.51232	0.50785
(0.9,0.1)	0.50001	0.46422	0.48997
(0.3,0.3)	0.62305	0.64543	0.63905
(0.7,0.3)	0.50162	0.51337	0.50003
(0.1,0.5)	0.74827	0.77001	0.76991
(0.5,0.5)	0.62305	0.65857	0.65012
(0.9,0.5)	0.50162	0.52213	0.51473
(0.3,0.7)	0.74827	0.76019	0.75225
(0.7,0.7)	0.62305	0.63645	0.63025
(0.1,0.9)	0.74999	0.76017	0.75644
(0.5,0.9)	0.74827	0.76878	0.75376
(0.9,0.9)	0.62311	0.64405	0.64028

III. Results and Analysis

Tables 1, 2, 3 and 4 display results obtained by explicit and Implicit Spline Scheme of the problem, respectively. It is observed that both results are reasonably in good agreement with the exact solution, and exhibit the expected convergence as the mesh size is refined.

IV. Conclusion

An Implicit and explicit Spline Scheme has been proposed for the numerical solutions of two dimensional nonlinear coupled viscous Burgers' equations. The computed results show that the solution obtained by the method is closed to analytic solution. The proposed method is an effective and reliable method for solving a wide range of engineering problems.

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