



Non-successive Occurrence of Digit 0 in All Base b Natural Numbers Less Than b^n

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Abstract: Consecutive base b positive integers less than b^n , for every positive integer n , are under scrutiny. Digit 0 is the candidate being searched for its non-successive occurrences in this range. The number of times different number of non-successive 0's occurs in all base b numbers in this range, the first number containing them and the last one, captured in formulae, generalized many such earlier findings. Illustration with $b = 16$ gives hints of how the procedure goes on.

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I. Introduction

For the occurrence of non-successive 0's, all positive integers

$$1, 2, 3, \dots$$

are being inspected. But the view is quite general in the sense that the base in which the numbers are considered is allowed to be any positive integer greater than 1, so that in this representation system, the occurrence is analyzed for formulations. The requirement of inspecting all general base b numbers having specific n number of or fewer digits makes the range $1 \leq m < b^n$.

It goes without saying that only significant 0's coming after some non-zero digit are considered.

II. Non-successive Occurrence of Digit 0

Digit 0 is of paramount importance in any place value number system [1]; in fact, the availability of digit 0 has revolutionized the way we use number systems to such an effective extent that we consider it as one of the biggest inventions in mathematics. In this work, the non-successive occurrences of digit 0 in base b numbers are analyzed in the above range. All occurrences and successive occurrences of 0's in base b numbers in similar ranges have already been dealt with in [5] and [6], respectively.

Theorem 1 : If r , n and b are positive integers with $1 < r < n$ and $b > 1$, then the number of base b numbers containing exactly r number of significant digit 0's in the range $1 \leq m < b^n$ is

$${}_0^A O_r^n = \sum_{j=r+1}^n {}^{j-1}C_r (b-1)^{j-r}$$

where the notation ${}_0^A O_r^n$ means number of base b numbers less than b^n with r number of 0's.

Theorem 2 : If r , n and b are positive integers with $1 < r < n$ and $b > 1$, then the number of base b numbers containing exactly r number of significant successive digit 0's in the range $1 \leq m < b^n$ is

$${}_0^S O_r^n = \sum_{j=2}^{n-(r-1)} {}^{j-1}C_1 (b-1)^{j-1}$$

where the notation ${}_0^S O_r^n$ means number of base b numbers less than b^n with r number of successive 0's.

For well-known base $b = 16$, we have determined the count of non-successive occurrence of multiple 0's in hexadecimal numbers till 16^{15} , which is little more than one quintillion (10^{18}), by computer and Java program and are given below. While the base 16 numbers are being analyzed below, their count is provided in usual base 10.

Sr. No.	Numbers Range <	Number of Base 16 Numbers with single (Non-successive) 0	Number of Base 16 Numbers with Non-successive 2 0's
1.	16^1	0	0
2.	16^2	0	0
3.	16^3	0	0
4.	16^4	0	225
5.	16^5	0	10,350
6.	16^6	0	314,100
7.	16^7	0	7,907,850
8.	16^8	0	178,767,225
9.	16^9	0	3,766,814,100
10.	16^{10}	0	75,527,751,600
11.	16^{11}	0	1,459,488,689,100
12.	16^{12}	0	27,408,756,267,225
13.	16^{13}	0	503,145,328,532,850
14.	16^{14}	0	9,066,403,629,314,100
15.	16^{15}	0	160,869,618,961,345,350

Formal definition of non-successive occurrences of any digit is that any two instances of its occurrence are to be separated by presence of another digit in-between them. Under this, single occurrence of any digit in any number cannot be treated as non-successive and hence by default becomes successive! So there can be no non-successive single 0 in any number in any range whatsoever.

We start analyzing ranges from lowest and proceed ahead.

The first range $1 \leq m < 16^1 = 16$, doesn't contain any 0, there is no question of non-successive 0's coming in it.

In the second range $1 \leq m < 16^2 = 256$, only single 0 occurs 15 times in number mentioned in [5] and, being single, is to be ignored for search of non-successive ones.

In the third range, $1 \leq m < 16^3 = 4,096$, after neglecting single 0's instances, there are higher number of 0's coming in numbers. 2 0's also occur 15 times in

100, 200, 300, $\dots$, F00,

but of none of them are non-successive! Till 16^3 , non-successive 0's are yet to appear!

For the next fourth range, $1 \leq m < 16^4 = 65,536$, for study of non-successive 0's, first we rule out all 10,590 single occurrences mentioned in [5].

For the first time, in this range 2 non-successive 0's occur in 225 numbers

1010, 1020, $\dots$, 10F0, 2010, 2020, $\dots$, 20F0, $\dots$, F010, F020, $\dots$, F0F0.

This count now rapidly increases for higher ranges. As a straightforward relation between all occurrences, successive occurrences & non-successive occurrences and the results developed so far [4] and [5], we get formulation of this count quite directly.

Notation: We introduce the notation ${}_b^N O_r^n$ for number of base b numbers less than b^n with r number of non-successive 0's.

Theorem 3: If $r > 1$, $n > 3$ and b are positive integers with $r < n - 1$ and $b > 1$, then the number of base b numbers containing exactly r number of significant non-successive digit 0's in the range $1 \leq m < b^n$ is

$${}_b^N O_r^n = \sum_{j=r+1}^n {}^{j-1}C_r (b-1)^{j-r} - \sum_{j=2}^{n-(r-1)} {}^{j-1}C_1 (b-1)^{j-1}$$

Proof. Let $n > 3$, $r > 1$, and $b > 1$ be positive integers with $r < n - 1$. Numbers with base b will have b number of digits. If $b \leq 10$, these digits are denoted by numeral symbols 0, 1, 2, $\dots$, $b - 1$ and for $b > 10$, for their representation, in addition to the 10 usual numeral symbols, successive alphabets A, B, C, $\dots$, X are used, where X is not actual alphabet X but a symbol to designate the alphabet at number equal to 9 more than its own number in the list of alphabets.

The reason for taking $n > 3$ is that there cannot occur any number of non-successive 0's in numbers $< b^3$.

By Theorem 1, the number of base b numbers in the range $1 \leq m < b^n$ with all kinds of r number of 0's is given by

$${}_b^A O_r^n = \sum_{j=r+1}^n {}^{j-1}C_r (b-1)^{j-r}$$

By Theorem 2, the number of base b numbers in the range $1 \leq m < b^n$ with r number of successive 0's is given by

$${}_b^S O_r^n = \sum_{j=2}^{n-(r-1)} j^{-1} C_1(b-1)^{j-1}$$

Clearly number of base b numbers in the range $1 \leq m < b^n$ with r number of non-successive 0's will be difference of these two values and so,

$${}_b^N O_r^n = {}_b^A O_r^n - {}_b^S O_r^n = \sum_{j=r+1}^n j^{-1} C_r(b-1)^{j-r} - \sum_{j=2}^{n-(r-1)} j^{-1} C_1(b-1)^{j-1}$$

This completes the proof.

It is interesting to extend the first table for higher occurrences of 0's.

Sr. No.	Number Range <	Number of Base 16 Numbers with		
		3 Non-successive 0's	4 Non-successive 0's	5 Non-successive 0's
1.	16^5	450	0	0
2.	16^6	24,075	675	0
3.	16^7	834,075	41,175	900
4.	16^8	23,615,325	1,610,550	61,650
5.	16^9	593,146,575	50,969,925	2,694,150
6.	16^{10}	13,749,318,450	1,417,844,925	94,578,525
7.	16^{11}	300,793,068,450	36,102,298,050	2,896,672,275
8.	16^{12}	6,297,957,130,950	861,353,079,300	80,637,687,900
9.	16^{13}	127,394,539,162,200	19,544,825,735,550	2,089,943,937,900
10.	16^{14}	2,506,077,400,490,325	426,083,351,126,175	51,220,557,219,150
11.	16^{15}	48,176,788,337,990,325	8,989,341,651,907,425	1,199,908,135,344,150

Sr. No.	Number Range <	Number of Base 16 Numbers with		
		6 Non-successive 0's	7 Non-successive 0's	8 Non-successive 0's
1.	16^8	1,125	0	0
2.	16^9	85,500	1,350	0
3.	16^{16}	4,135,500	112,725	1,575
4.	16^{10}	159,807,375	5,985,225	143,325
5.	16^{12}	5,353,932,375	252,782,100	8,293,950
6.	16^{13}	162,031,979,250	9,205,813,350	380,387,700
7.	16^{14}	4,539,449,166,750	301,204,485,225	14,971,778,325
8.	16^{15}	119,638,867,135,500	9,076,541,985,225	526,866,465,825

Sr. No.	Number Range <	Number of Base 16 Numbers with		
		9 Non-successive 0's	10 Non-successive 0's	11 Non-successive 0's
1.	16^{10}	1,800	0	0
2.	16^{12}	177,300	2,025	0
3.	16^{13}	11,112,300	214,650	2,250
4.	16^{14}	550,268,550	14,490,900	255,375
5.	16^{15}	23,285,956,050	770,828,400	18,480,375

Sr. No.	Number Range <	Number of Base 16 Numbers with		
		12 Non-successive 0's	13 Non-successive 0's	14 Non-successive 0's
1.	16^{14}	2,475	0	0
2.	16^{15}	299,475	2,700	0

III. First Occurrence of Non-successive Digit 0's

For the numbers written in corresponding base b , the first numbers with r number of non-successive 0's are as given in following.

Formula 1: If n, r and b are natural numbers with $b > 1$, then the first occurrence of r number of non-successive significant 0's in base b numbers in range $1 \leq m < b^n$ is

$$f = \begin{cases} - & , \text{ if } r < 2 \text{ or } r \geq n-1 \\ b(b^r + 1), & \text{ if } r > 1 \text{ and } r < n-1 \end{cases}$$

For any base $b > 1$, the following figures in their own base b come out as per expectation.

Sr. No.	Number Range $\leftarrow$ First number with Non-successive $\downarrow$	b^4	b^5	b^6	b^7	b^8	b^9	b^{10}
1.	2 0's	1,010	1,010	1,010	1,010	1,010	1,010	1,010
2.	3 0's	-	10,010	10,010	10,010	10,010	10,010	10,010
3.	4 0's	-	-	100,010	100,010	100,010	100,010	100,010
4.	5 0's	-	-	-	1,000,010	1,000,010	1,000,010	1,000,010
5.	6 0's	-	-	-	-	10,000,010	10,000,010	10,000,010
6.	7 0's	-	-	-	-	-	100,000,010	100,000,010
7.	8 0's	-	-	-	-	-	-	1,000,000,010

Last Occurrence of Non-successive Digit 0's

Finally, here are the last occurrences of multiple number of non-successive 0's for choice of hexadecimal system of numbers.

Sr. No.	Number Range $\leftarrow$ First number with Non-successive $\downarrow$	16^4	16^5	16^6	16^7	16^8	16^9	16^{10}
1.	2 0's	F,0F0	FF,0F0	FFF,0F0	F,FFF,0F0	FF,FFF,0F0	FFF,FFF,0F0	F,FFF,FFF,0F0
2.	3 0's	-	F0,F00	FF0,F00	F,FF0,F00	FF,FF0,F00	FFF,FF0,F00	F,FFF,FF0,F00
3.	4 0's	-	-	F0F,000	F,F0F,000	FF,F0F,000	FFF,F0F,000	F,FFF,F0F,000
4.	5 0's	-	-	-	F,0F0,000	FF,0F0,000	FFF,0F0,000	F,FFF,0F0,000
5.	6 0's	-	-	-	-	F0,F00,000	FF0,F00,000	F,FF0,F00,000
6.	7 0's	-	-	-	-	-	F0F,000,000	F,F0F,000,000
7.	8 0's	-	-	-	-	-	-	F,0F0,000,000

And next follows the rule giving rise to them.

Formula 2: If n , r and b are natural numbers with $b > 1$, then the last occurrence of r number of non-successive 0's in base b numbers in range $1 \leq m < b^n$ is

$$l = \begin{cases} - & , \text{ if } r < 2 \text{ or } r \geq n-1 \\ b^n - (b(b-1)+1)b^{r-1} & , \text{ if } r > 1 \text{ and } r < n-1 \end{cases}$$

Remark: All the results here have generalized the those in [4].

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