



Three Variants of Coloring and Clique number of graphs

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Abstract: In this paper, we prove that complementary chromatic number of a graph G cannot be more than clique number of the graph. We also prove that these two numbers are equal if and only if the graph is complete k – partite. Additionally, we prove that complementary chromatic number, chromatic number, clique number and b – chromatic numbers are equal if and only if the graph is a complete k – partite.

I. Introduction

We begin with simple, finite, connected and undirected graph. We are all accustomed to chromatic number and clique number of a graph. In [2], D. K. Thakkar and A.B. Kothiya introduced the concept of complementary coloring of a graph and defined complementary chromatic number of the graph. In [3], R. W. Irwing and D. F. Manlove introduced the concept of b – coloring of a graph and defined the b – chromatic number of the graph. In [2], it is proved that complementary chromatic number and chromatic number of a graph are equal if and only if the graph is a complete k – partite graph. In this paper, we consider all these three variants of coloring and clique number of a graph and determined some relations between them.

II. Definitions

Definition 2.1[1]: (Clique)

Let $G = (V, E)$ be a graph. Then a subset S of $V(G)$ is said to be a clique in G , if whenever x and y are distinct vertices in S , they are adjacent.

Definition 2.2[1]: (Clique number)

The number of vertices in a maximum clique in a graph G is known as clique number. It is denoted by $\omega(G)$ or just by ω .

Definition 2.3 [1]: (Proper n - coloring)

An n – colouring of a graph G is a function $f: V(G) \rightarrow \{1, 2, \dots, n\}$ (for some $n \geq 1$). This is called a proper n – colouring if whenever u and v are adjacent then $f(u) \neq f(v)$. A graph G is said to be n – colourable if it admits n – colouring and it is called a proper n – colourable graph if it admits a proper n – colouring.

Definition 2.4 [1]: (Chromatic number)

The chromatic number of a graph G is the smallest value of n such that G admits a proper n – colouring. This number is denoted as $\text{chr}(G)$ or by $\chi(G)$ or by χ .

Definition 2.5[2]: (complementary n - coloring)

A coloring of a graph G is called a complementary coloring of G if whenever two vertices u and v have distinct colors then u and v are adjacent. An n – coloring which is also complementary is called complementary n – coloring.

Definition 2.6[2]: (Complementary chromatic number)

Complementary chromatic number of a graph G is the largest integer k such that G admits a complementary k – coloring. This number is denoted by $\text{cchr}(G)$. (We will denote Complementary chromatic number of G by $\chi^c(G)$ or by χ^c)

Definition 2.7[3]: (b - coloring)

A proper coloring of a graph G is called a b – coloring if for every color class, Say V_i , there is a vertex which is has at least one neighbour in every other color classes.

Definition 2.8[3]: (b – chromatic number)

The b – chromatic, denoted by $\phi(G)$, of a graph G , is the largest integer k for which G admits a b – coloring using k colors. (We will denote b – chromatic number of G by $\chi^b(G)$ or by χ^b)

Definition 2.9[1]: (k – partite graph/complete k – partite graph)

A graph G is said to be a k – partite graph if its vertex $V(G)$ can be partitioned into k independent sets $V_1, V_2, \dots, V_k$ such that each edge joins a vertex of V_i with a vertex of V_j for $j \neq i$. A k – partite graph G is said to be a complete k – partite graph if for each i , each vertex of V_i is adjacent to each vertex of V_j for every $j \neq i$.

III. Main results

Remark 3.1: For any graph $G = (V, E)$, by

(1) χ – partition of G , we mean a partition of vertex set V of G by proper coloring of vertices of the graph using minimum number of colors.

(2) χ^c – partition of G , we mean a partition of vertex set V of G by complementary coloring of vertices the graph using maximum number of colors.

(3) χ^b – partition of G , we mean a partition of vertex set V of G by b – coloring of vertices the graph using maximum number of colors.

Theorem 3.1: Let G is a graph of order n . $\chi^b(G) = n$ if and only if $\chi(G) = \omega(G) = n$.

Proof: Let $\chi^b(G) = n$. Then definitely G admits a unique χ^b – partition and each vertex of G is a color class. Hence $\omega(G) = \chi(G) = n$.

Converse is obvious.

Theorem 3.2: Let G is a graph of order n . $\chi^b(G) = n - 1$ if and only if $\chi(G) = \omega(G) = n - 1$.

Proof: Let $\chi^b(G) = n - 1$. Then definitely G admits a unique χ^b – partition in which $n-2$ color classes have single vertex and one color class contains two vertices. Obviously, $\omega(G) = n - 1$. Hence $\chi(G) = n - 1$.

Converse is obvious.

Theorem 3.3: Let G is a graph of order n . If $\chi(G) = n - 2$ then $\chi^b(G) = n - 2$.

Proof: Assume that $\chi(G) = n - 2$. Then $\chi^b(G) \geq n - 2$. Hence by earlier theorems 3.1 and 3.2, $\chi(G) = n - 2$.

Remark 3.4: $\chi^b(G) = n - 2$ does not imply $\chi(G) = n - 3$. We know that $\chi^b(P_n) = 3, \forall n \geq 4$ by [4]. In particular, $\chi^b(P_5) = 3 = n - 2$ but $\chi(P_5) = 2 = n - 3$.

Theorem 3.5[2]: For a graph G , $\chi^c(G) = \chi(G) = k$ if and only if G is a complete k – partite graph.

Theorem 3.6: Let $G = (V, E)$ be a graph. Then $\chi^c(G) \leq \omega(G)$.

Proof: Let $\Pi = \{V_1, V_2, \dots, V_{\chi^c}\}$ be a χ^c – partition of a graph G . Then definitely one vertex from each color class of Π forms a clique. Hence $\chi^c(G) \leq \omega(G)$.

Corollary 3.7: Let $G = (V, E)$ be a graph. Then $\chi^c(G) \leq \omega(G) \leq \chi(G)$.

Corollary 3.8: Let $G = (V, E)$ be a graph. If G is a complete k – partite graph then $\omega(G) = \chi(G)$.

Remark 3.9: Converse of the above corollary is not true, in general. For example consider path graph P_5 . $\omega(P_5) = \chi(P_5) = 2$ but P_5 is not complete k – partite graph.

Theorem 3.10: Let $G = (V, E)$ be a graph. Then $\chi^c(G) = \omega(G)$ if and only if G is a complete k – partite graph.

Proof: Assume $\chi^c(G) = \omega(G)$. Let $\Pi = \{V_1, V_2, \dots, V_{\chi^c}\}$ be the χ^c – partition of a graph G .

Claim: Each color class of Π is independent.

Suppose not. Then there exists two adjacent vertices, say u and v , that belong to V_i . Now collection of these two vertices together with one vertex from each remaining color classes of Π will be a clique. Hence $\chi^c(G) < \omega(G)$, which is contradiction.

So each color class of Π is independent. Therefore Π is actually χ – partition of G , as

$\chi^c(G) \leq \chi(G)$. Here we note that as Π is a χ^c – partition of the graph G , any two vertices that belong to distinct color class are adjacent. Hence G is a complete k – partite graph.

Conversely, assume that G is a complete k – partite graph. By theorem 3.5 and corollary 3.7, $\chi^c(G) = \omega(G)$.

Theorem 3.11: Let $G = (V, E)$ be a graph. If G is a complete k – partite graph then $\chi(G) = \chi^b(G)$.

Proof: If G is a complete k – partite graph. Let $\Pi = \{V_1, V_2, \dots, V_k\}$ be the χ – partition of G . Consider a χ^b – partition, $\Pi^b = \{W_1, W_2, \dots, W_{\chi^b}\}$ of the graph G .

Consider $W_p \cap V_q \neq \emptyset$, for some p and q .

Claim: $W_p \cap V_r = \emptyset, \forall r \neq q$.

Suppose not. Then $W_p \cap V_r \neq \emptyset$, for some $r \neq q$. Then there exists $u \in V_q$ and $v \in V_r$ such that u and v belong to color class W_p , which is not possible as u and v are adjacent and b – coloring is a proper coloring. Hence $W_p \cap V_r = \emptyset, \forall r \neq q$ and so $W_p \subseteq V_q$.

Hence each color class of Π^b is a subset of some color class of Π .

Let $W_p \subseteq V_q$, for some p and q .

Claim: $W_p = V_q$.

Suppose not. Then $W_p \subsetneq V_q$. Consider $S = V_q \setminus W_p$. Then there exists $x \in S$ such that $x \in W_r$, for some $r \neq p$. Note that $W_r \subseteq V_q$, which is a contradiction as no two color classes of Π^b can be a subset of same color class of Π . Therefore $W_p = V_q$ and so $\Pi^b \subseteq \Pi$. Hence $\chi^b(G) \leq \chi(G)$. By $\chi(G) \leq \chi^b(G)$, we have $\chi^b(G) = \chi(G)$.

Remark 3.12: Converse of above theorem is not true, in general. For example consider cycle graph C_5 . By [4], $\chi^b(C_5) = 3$ and we know that $\chi(C_5) = 3$. Hence $\chi^b(C_5) = \chi(C_5)$ but C_5 is not complete k – partite graph.

Theorem 3.13: Let $G = (V, E)$ be a graph. Then $\chi^c(G) = \omega(G) = \chi(G) = \chi^b(G)$ if and only if G is a complete k – partite graph.

Proof: Follows above results.

References

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