



Improved Interactive Fuzzy Programming Approach to Fractional Programming Problem through Weight Generation

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Abstract: Decision making problems in hierarchical managerial or public organizations are often formulated as multi-level (or bi-level) mathematical programming problems. This paper deals with interactive fuzzy programming through weight generation to solve Fractional programming problem (FPP). Here we propose an interactive fuzzy programming method for obtaining a satisfactory solution to a “bi-level quadratic fractional programming problem” with two decision makers (DMs) interacting with their optimal solutions by allotting due weight. After determining the fuzzy goals of the DMs at both levels, a satisfactory solution is efficiently derived by updating the satisfactory level of the DM at the upper level with consideration of overall satisfactory balance between both levels. Furthermore, the feasibility and efficiency of the proposed approach is shown by applying it to illustrative numerical example.

Keywords: Bi-level Programming Problems, Fractional Programming Problems, Fuzzy Membership Function, satisfactory solution.

I. Introduction

In the context of Bi-level or two-level programming, the decision maker at the upper level first specifies a strategy, and then the decision maker at the lower level specifies a strategy so as to optimize the objective with full knowledge of the action of the decision maker at the upper level. Bi-level programming is a tool for modeling decentralized decisions that consists of the objective of the leader at its first level and that of the follower at the second level. A Bi-level programming problem (BLPP) is a special case of multi level programming problem (MLPP). Multi level programming problem can be defined as a p-person, non-zero sum game with perfect information in which each player moves sequentially from top to bottom. This problem is a nested hierarchical structure. When $p=2$, we call the system a bi-level programming problem. In conventional multi-level mathematical programming models employing the solution concept of Stackelberg equilibrium, it is assumed that there is no communication among decision makers, or they do not make any binding agreement even if there exists such communication. Compared with this, for decision making problems in such as decentralized large firms with divisional independence, it is quite natural to suppose that there exists communication and some cooperative relationship among the decision makers. This paper deals with interactive fuzzy programming through weight generation to solve Fractional programming problem (FPP). Here we propose an interactive fuzzy programming method for obtaining a satisfactory solution to a “Bi-level quadratic fractional programming problem” with two decision makers (DMs) interacting with their optimal solutions by allotting due weight. After determining the fuzzy goals of the DMs at both levels, a satisfactory solution is efficiently derived by updating the satisfactory level of the DM at the upper level with consideration of overall satisfactory balance between both levels.

Bi-level programming is a powerful and robust technique for solving hierarchical decision making problem. The bi-level programming problem (BLPP) has received increasing attention in the literature. The formulation and different version of BLPP are given by Bard [1,2,3], Candler [6,7], Herminia and Carmen [8] and Bialas and Karwan [4,5]. Bialas and Karwan [5] are the pioneers for linear BLPP. Most of the developments on BLPPs are based on vertex enumeration method and transformation approaches which are effective only for very simple types of problems. In these methods the DMs have no cooperating attitude with each other so these approaches are unable to give a satisfactory solution which would be acceptable to both the DMs. To overcome such difficulties Zimmermann [18] first applied fuzzy set theory in decision making problems with several conflicting objectives. Lai [9,10] introduced an effective fuzzy approach using the concept of tolerance membership functions for solving MLPPs. Shih et al. [16,15] extended Lai's concept using a non-compensatory maximin aggregation operator for solving MLPPs. Mishra and Ghosh [11] has introduced an interactive fuzzy programming method for obtaining a satisfactory solution to a “bi-level quadratic fractional programming problem” with two decision makers (DMs) interacting with their optimal solutions. After determining the fuzzy goals of the DMs at both levels, a satisfactory

solution is efficiently derived by updating the satisfactory level of the DM at the upper level with consideration of overall satisfactory balance between both levels. Sakawa, Kato and Katagiri [14] proposed the same for Multi-level non-linear integer programming problem through Genetic algorithm for obtaining a satisfactory solution for all DMs. Again Mishra [12] has introduced the solution of a bi-level linear fractional programming problem (BLFFPP) by weighting method. They convert the hierarchical system into scalar optimization problem (SOP) by finding proper weights using the analytic hierarchy process (AHP) then the objective functions of both levels was combined into one objective function. A non-dominated solution set was obtained by this method. Mishra and Verma [13] proposed Analytic Hierarchy Process for Solutions to Bi-Level Quadratic Fractional Programming Problems.

In this paper, an interactive fuzzy decision making method with weighting approach was proposed for solving Bi-level Quadratic Fractional Programming problem. Introducing a new balance function, we consider the overall satisfactory balance between the leader and the follower. Then, a satisfactory solution can be obtained by the proposed method.

II. A bi-level quadratic fractional programming problem

A bi-level quadratic fractional programming problem is mathematically formulated as:

$$\begin{aligned} & \underset{x_1}{\text{Minimize}} && z_1(x_1, x_2) && \text{Where } x_2 \text{ solves} \\ & \underset{x_2}{\text{Minimize}} && z_2(x_1, x_2) && \text{subject to } A_1 x_1 + A_2 x_2 \leq b, \quad x_1 \geq 0, x_2 \geq 0. \end{aligned} \quad (1)$$

Where objective functions $z_i(x_1, x_2), (i = 1, 2)$ are represented by a quadratic fractional function

$$z_i(x_1, x_2) = \frac{p_i(x_1, x_2)}{q_i(x_1, x_2)} = \frac{x_1' Q_{i1} x_1 + x_2' Q_{i2} x_2 + c_{i1} x_1 + c_{i2} x_2 + c_{i3}}{x_1' R_{i1} x_1 + x_2' R_{i2} x_2 + d_{i1} x_1 + d_{i2} x_2 + d_{i3}} \quad (2)$$

$x_i, (i = 1, 2)$ is an n_i -dimensional decision variables;

Q_{i1} and $R_{i1}, (i = 1, 2)$ are $n_1 \times n_1$ positive definite matrix.

Q_{i2} and $R_{i2}, (i = 1, 2)$ are $n_2 \times n_2$ positive definite matrix;

c_{i1} and $d_{i1}, (i = 1, 2)$ are n_1 -dimensional row vectors.

c_{i2} and $d_{i2}, (i = 1, 2)$ are n_2 -dimensional row vectors;

c_{i3} and $d_{i3}, (i = 1, 2)$ are constants.

b is an m -dimensional constant column vector; $A_i, (i = 1, 2)$ is an $m \times n_i$ constant matrix; and it is assumed that the denominators are positive i.e. $q_i(x_1, x_2) > 0, (i = 1, 2)$. For the sake of simplicity, we use the following notations:-

$x = (x_1, x_2) \in \mathbb{R}^{n_1+n_2}, c_i = (Q_{i1}, Q_{i2}, c_{i1}, c_{i2}, c_{i3}), d_i = (R_{i1}, R_{i2}, d_{i1}, d_{i2}, d_{i3}), (i = 1, 2)$.

Also let DM1 denote the DM at the upper level and DM2 denote the DM at the lower level. In the bi-level quadratic fractional programming problem (1), $Z_1(x_1, x_2)$ and $Z_2(x_1, x_2)$ respectively represent objective functions of DM1 and DM2, and x_1 and x_2 represent decision variables of DM1 and DM2 respectively.

III. Interactive fuzzy programming

It is natural that the DMs have fuzzy goal of their objective functions when they take fuzziness of human judgments into consideration. For each of the objective functions $Z_i(x), i = 1, 2$, in (1), it seems natural to introduce such fuzzy goals for objective functions as “ $Z_i(x)$ should be subjectively less than or equal to a certain value”.

First, we solve problems to obtain the individual minimum

$$Z_i^{\min} = \min_{x \in X} Z_i(x), (i = 1, 2) \quad (3)$$

and the individual maximum

$$Z_i^{\max} = \max_{x \in X} Z_i(x), (i = 1, 2) \quad (4)$$

Of the objective functions are referred to when the DMs elicit membership functions prescribing the fuzzy goals of the objective functions $Z_i(x), i = 1, 2$.

The DMs determine the membership functions $\mu_i(Z_i(x))$ for $i = 1, 2$ which are strictly monotone decreasing for $Z_i(x)$. The domain of the membership functions is in the interval $[Z_i^{\max}, Z_i^{\min}]$ and each of the DMs

specifies the value Z_i^0 of the objective function for which the satisfactory degree is 0 and the value Z_i^1 of the objective function for which the degree of satisfaction is 1.

The corresponding linear membership function $\mu_i(Z_i(x))$ which characterizes the fuzzy goal of DM is defined as:

$$\mu_i[z_i(x)] = \begin{cases} 1, & z_i(x) \leq z_i^1 \\ [z_i(x) - z_i^0] / [z_i^1 - z_i^0], & z_i^1 < z_i(x) \leq z_i^0 \\ 0, & z_i(x) > z_i^0 \end{cases} \quad (5)$$

Where Z_i^0 and Z_i^1 denoted the value of objective function $Z_i(x)$, such that the degree of the membership function are 0 and 1, respectively. The DM's determine the linear membership function as in (5) by choosing $Z_i^1 = Z_i^{\min}$, $Z_i^0 = Z_i^{\max}$.

For deriving an overall satisfactory solution to the formulated problem (1), first the maximizing decision of the fuzzy decision proposed by Bellman and Zadeh is found. Namely, the following problem is solved for obtaining a solution which maximizes the smallest degree of satisfaction between the two DMs.

$$\max_{x \in X} \min\{\mu_1(Z_1(x)), \mu_2(Z_2(x))\} \quad (6)$$

Let us denote an optimal solution to the problem (6) by x^* . Then, we define the satisfactory degree of both the DMs under the constraints as:

$$\lambda = \min\{\mu_1(Z_1(x^*)), \mu_2(Z_2(x^*))\} \quad (7)$$

The maximin problem (7) can be written as an equivalent maximization problem:

Maximize λ

$$\text{Subject to } \left. \begin{aligned} A_1 x_1 + A_2 x_2 &\leq b, \\ \mu_1(Z_1(x)) &\geq \lambda \\ \mu_2(Z_2(x)) &\geq \lambda, \\ 0 \leq \lambda \leq 1, x_1, x_2 &\geq 0 \end{aligned} \right\} \quad (8)$$

To take into account the overall satisfactory balance between both the levels, DM1 needs to compromise with DM2 on DM1's own minimal satisfactory level. To do so, a ratio of satisfaction degree between both the DMs is defined as:

$$\Delta = \frac{\mu_2(Z_2(x^*))}{\mu_1(Z_1(x^*))} \quad (9)$$

Let Δ_L and Δ_U denote the lower and upper bound of Δ specified by DM1. If $\Delta > \Delta_U$ i.e., $\mu_2(Z_2(x^*)) > \Delta_U \mu_1(Z_1(x^*))$, then DM1 updates the minimal satisfactory level S by increasing S . Then DM1 obtains a large satisfactory degree and DM2 accepts a smaller satisfactory degree.

Conversely if, $\Delta < \Delta_L$ i.e., $\mu_2(Z_2(x^*)) < \Delta_L \mu_1(Z_1(x^*))$, DM1 updates the minimal satisfactory level S by decreasing S . Then DM1 accepts a smaller satisfactory degree and DM2 obtains a larger satisfactory degree.

At an iteration K , let $\mu_1(Z_1(x^k)), \mu_2(Z_2(x^k)), \lambda^k$ and $\Delta^k = \frac{\mu_2(Z_2^k)}{\mu_1(Z_1^k)}$ denotes DM1's and DM2's

satisfactory degree, a satisfactory degrees of both the levels and the ratio of satisfactory degrees between both DMs, respectively, and let a corresponding solution be x^k . The iterated interactive process terminates if the following two conditions are satisfied and DM1 concludes the solution as a satisfactory solution.

Termination conditions of the interactive process for BLQFPP

Condition 1: DM1's satisfactory degree is larger than or equal to the minimal satisfactory level S specified by DM1 i.e. $\mu_1(Z_1^k) \geq S$. **Condition 2:** the ratio Δ^k of satisfactory degrees lies in the interval between the lower and the upper bounds specified by DM1.

Condition 1 is DM1's required condition for solutions, and condition2 is provided in order to keep overall satisfactory balance between both the levels. Unless the conditions are satisfied simultaneously, some of DM1, needs to update his minimal satisfactory level S.

IV. Generation of non-dominated solution by weighting using AHP

A. Method for generating non-dominated solutions

The concept of non-dominated solution was introduced by *Pareto*, an economist in 1896. A non-dominated solution is one for which there is no other solution giving equal or greater values of each and every objective function. A feasible solution $\bar{x}^* \in X$ (decision space) is a non-dominated solution to the VMP if and only if there does not exist any other feasible solution $\bar{x} \in X$ such that $z_p(\bar{x}^*) \leq z_p(\bar{x})$, $p = 1, 2$ and $z_p(\bar{x}^*) < z_p(\bar{x})$ for at least one p .

B. Weighting method

The basic idea of assigning weights to the various objective functions, combining these into a single objective function and parametrically varying the weights to generate the non-dominated set was first proposed by Zadeh in 1963. Mathematically, the weighting method can be stated as follows:

$$\max/\min w_1 z_1(\bar{x}) + w_2 z_2(\bar{x}) + \dots + w_p z_p(\bar{x}) \quad (10)$$

Subject to $\bar{x} \in X$ where X is the feasible region. Thus, a multiple objective problem has been transformed into a single objective optimization problem for which solution methods exist. The coefficient w_p operating on the p^{th} objective function, $z_p(\bar{x})$, is called the weight and can be interpreted as "the relative weight or worth" of that objective function when compared to the other objectives. These weights can be obtained by Analytic Hierarchy Process (AHP).

Weighting method has been widely studied, experimented and applied in many fields in engineering worlds. Not only does weighting method provide an alternate method to solving problem, it consistently outperforms other traditional methods in the most of the problem link. Weighting method has no special requirement for the characters and differentiability of the function. Perhaps the most creative task in making a decision with the hierarchical decision making with the hierarchical decision making situations is to choose the factors that are important for that decision.

V. Interactive Fuzzy with weighting method

Here we are going to use the concept of Interactive Fuzzy and the concept of weighting method to yield a new optimization technique, where the basic idea of assigning weights to the various membership functions, combining these into a single objective function and parametrically varying the weights to generate the non-dominated set of solution. Then the weighting problem for BLQFPP is formulated as follows:

$$P(w) = \max_{x \in X} \sum_{i=1}^2 w_i \mu_i(z_i(x)) \quad (11)$$

Subject to,

$$g_i(x) \leq b, i = 1, 2, \dots, m \quad (12)$$

$$\left. \begin{array}{l} L_1 \leq x_1 \leq U_1 \\ L_2 \leq x_2 \leq U_2 \end{array} \right\} \quad (13)$$

Where, $w \in W = \{w : w \in R^p, w_n \geq 0, i = 1, 2 \text{ and } \sum_{i=1}^2 w_i = 1\}$,

$$\text{and } z_i(x_1, x_2) = \frac{p_i(x_1, x_2)}{q_i(x_1, x_2)} = \frac{x_1' Q_{i1} x_1 + x_2' Q_{i2} x_2 + c_{i1} x_1 + c_{i2} x_2 + c_{i3}}{x_1' R_{i1} x_1 + x_2' R_{i2} x_2 + d_{i1} x_1 + d_{i2} x_2 + d_{i3}}$$

U_i and L_i are the upper and lower bounds of decision vector provided by the respective DM. Finally the quadratic fractional programming problem (11) - (13), with a single objective function is solved. Here the weighting coefficients convey the importance attached to an objective function. Suppose that the relative importance of the both objective functions is known and is constant. Then the preferred solution is obtained by solving $P(w)$ where

$w_i \geq 0$ the weighting coefficients. The w_i 's are normalized since, $\sum_{i=1}^2 w_i = 1$. This method can be used to generate non-dominated solutions by utilizing various values of w . In such a case the weighting coefficients w do not reflect the relative importance of the objective functions in the proportional sense, but are only parameters varied to locate the non-dominated points.

VI. Numerical example

In this section we present numerical example to demonstrate the solution procedures by proposed approach to solve bi-level quadratic fractional programming problem (BLQFPP). The following example considered by Mishra [11] is again used to demonstrate the solution procedures and clarify the effectiveness of the proposed approach: Consider the following BLQFPP:

$$\min_{x_1} Z_1(x_1, x_2) = \frac{4x_1^2 + x_2^2 + 1}{2x_1^2 + 5x_2^2 + 1}$$

Where x_2 solves,

$$\min_{x_2} Z_2(x_1, x_2) = \frac{3x_1^2 + 5x_2^2 + 1}{4x_1^2 + 3x_2^2 + 1}$$

Subject to

$$-5x_1 + 3x_2 \leq 15, 4x_1 + 3x_2 \leq 45, x_1, x_2 \geq 0$$

The optimal solution to the above problem obtained by Mishra [11] through interactive fuzzy approach is given below:

x_1^2	x_2^2	Z_1^2	Z_2^2	$\mu_1(Z_1^2)$	$\mu_2(Z_2^2)$	Δ^2
0.940894	2.171512	0.329758	0.829242	0.70000	0.571602	0.816574

The pair wise comparison matrix $\tilde{A}$ of order 2 and its normalized matrix $\tilde{N}$ for the hierarchical objective function are given by:

$$\tilde{A} = \begin{bmatrix} 1 & 4 \\ 1/4 & 1 \end{bmatrix}$$

$$\tilde{N} = \begin{bmatrix} 1/.25 & 4/5 \\ .25/1.25 & 1/5 \end{bmatrix} = \begin{bmatrix} 0.80 & 0.80 \\ 0.20 & 0.20 \end{bmatrix}$$

Thus the normalized relative weights are $\tilde{w}_1 = (0.80 + 0.80)/2 = 0.80$ and

$\tilde{w}_2 = (0.20 + 0.20)/2 = 0.20$ Matrix $\tilde{A}$ is consistent. The above problem is therefore formulated as:

$$P(\bar{w}) = \max(\tilde{w}_1 \mu_1 + \tilde{w}_2 \mu_2)$$

$$P(\bar{w}) = \max[0.80 \times 0.700 + 0.20 \times 0.571]$$

Subject to

$$-5x_1 + 3x_2 \leq 15, 4x_1 + 3x_2 \leq 45, x_1, x_2 \geq 0$$

The non-dominated solution set is generated by parametrically varying the weights and is tabulated below:

$\tilde{w}_1, \tilde{w}_2$	x_1, x_2	μ_1, μ_2	$P(\bar{w})$
0.7, 0.3	0.940, 2.171	0.651, 0.651	0.65
0.8, 0.2	0.940, 2.171	0.700, 0.571	0.674

Here, we see that even we vary the weight vector, the solution remains more or less the same. This approach determines a subset of the complete set of non-dominated solutions and unique characteristics of a BLQFPP is reflected by allowing each DM to assign upper and lower bounds for the decision variables under his control.

VII. Conclusion

The objective of this paper is to present an improved interactive fuzzy programming approach through weight generation to solve Bi-level quadratic fractional programming problems. The proposed approach is easy to

implement. The procedure is not excessively interactive, which most DMs prefer. This method can be used to generate non-dominated solutions by utilizing various values of w . AHP gives the relative weights to form a super objective function. Thus the non-dominated solution set reduces to a point. The main advantage of the proposed approach is that the possibility of rejecting the solution again and again by interactive fuzzy approach and re-evaluation of the problem repeatedly, by redefining the elicited membership functions, needed to reach to the satisfactory decision does not arise. The problem can never be infeasible and unbounded. Proposed method facilitates computation to reduce the complexity in solving problem and is much more effective.

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