



On weighted arithmetic aggregation of intuitionistic fuzzy information with intuitionistic fuzzy weights

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Abstract: Aggregation operators play a crucial role in modelling real-life problems. In this paper, the definition of normalization of IFSs is proposed and related properties are studied. The normalized intuitionistic fuzzy criteria weights are used to define an intuitionistic fuzzy arithmetic aggregation operator with respect to intuitionistic fuzzy weights and their properties are analyzed. Novel accuracy function is revised for easy handling of ranking of IFSs through accuracy degrees. The effectiveness of the proposed method is illustrated through a multicriteria decision making problem in intuitionistic fuzzy environment with criteria as IFSs. The ranking of alternatives through both the proposed approach and the existing approach are compared yielding identical ranking of alternatives and thus validating the proposed approach.

Keywords: Intuitionistic fuzzy sets; Intuitionistic fuzzy aggregation operator; Novel accuracy function; Normalization of IFSs; Multicriteria decision making

I. Introduction

Multicriteria decision making (MCDM) has a great impact on the human progress and prosperity due to making selections among some courses of action in the presence of multiple conflicting attributes in greatly uncertain world and it needed to employ the available scientific tools and techniques to accurately model unstructured real-life problems for better and effective solutions. Fuzzy set theory [3] based on the membership function revolutionized the approach towards modelling uncertainty and incompleteness in unstructured problems occurred as a result of human judgment. As a further generalization of fuzzy sets, Atanassov [1, 2] introduced the concept of nonmembership function in the fuzzy system and gave the nomenclature intuitionistic fuzzy sets (IFSs). The concepts of vague sets as another generalization of fuzzy sets were proposed by Gau and Buehrer [4]. Later Burillo and Bustince [5] established that the notion of vague sets were synonymous with IFSs. Thus, the theory of IFSs remained a superior tool for modelling imprecision and incompleteness in numerous decision contexts in comparison to fuzzy sets. Atanassov and Gargov [16] extended the concepts of IFSs to interval-valued intuitionistic fuzzy sets (IVIFSs). In MCDM problems, the aggregation operators play indispensable roles. Xu and Da [8] made a survey of existing aggregation operators and introduced some aggregation operators like induced ordered weighted geometric averaging (IOWGA) operator, generalized induced ordered weighted averaging (GIOWA) operator, hybrid weighted averaging (HWA) operator. Aggregation operators like intuitionistic fuzzy weighted averaging (IFWA) operator, intuitionistic fuzzy ordered weighted averaging (IFOWA) operator, and intuitionistic fuzzy hybrid aggregation (IFHA) operator for aggregating intuitionistic fuzzy values were proposed by Xu [9]. Dynamic intuitionistic fuzzy weighted averaging (DIFWA) operator and uncertain dynamic intuitionistic fuzzy weighted averaging (UDIFWA) operator were developed by Xu and Yager [17] based on which two respective procedures were developed to solve the dynamic intuitionistic fuzzy multi-attribute decision making (DIF-MADM) problems. Some techniques based on vague set theory for handling multicriteria decision making problems were proposed by Chen and Tan [11] and Hong and Choi [12]. Mardani et al. [18] reviewed the approaches and applications of fuzzy MCDM during the period 1994 to 2014.

All the weighted aggregation operators developed so far employed crisp normalized weights with arguments as IFSs and thus ignore the inherent uncertainty in human judgment of criteria expressible in IFSs. The ominous presence of crisp weights in weighted aggregation operators thus puts some hindrance towards appropriate modelling of real-life problems. Thus, the existing weighted aggregation operators need extension with keeping weights as IFSs. In this pace, normalized weights summing to unity should be taken care of in the intuitionistic fuzzy (IF) environment as well. Normalization of IFSs was defined by De et al. [6]. However, Zeng et al. [7] pointed out the inconsistency in the definition of normalization of IFSs through an example. No effort is ever seen in the vast literature on IFS towards defining normalization over the last decade. Probably this could be the chief reason for standstill in this direction of consideration of IFS weights in the weighted IF aggregation operator. In this paper, we propose a definition of normalization of IFSs and introduce IF arithmetic aggregation operator relative to IF weights and study their properties.

This paper is organized as follows. Section II reviews the concepts of IFSs, score function, accuracy function, novel accuracy function and IF weighted aggregation operator relative to crisp weight vector. Section III defines normalization of IFSs and IF arithmetic aggregation operator relative to IF weights is proposed and their properties are studied. Novel accuracy function is also revised. In section IV, the usefulness of the approach developed in the previous section to the problems of multicriteria decision making in IF environment is presented. The application of the proposed method is illustrated through a MCDM problem and the result is validated through the existing approach in section V. The concluding remarks are made in section VI.

II. Preliminaries

Here, we present a brief review of some existing results to be used for future development.

Definition 2.1 ([1, 2]) In a universe of discourse X , an IFS, A , is defined as an object of the form $A = \{\langle x, \mu_A(x), \nu_A(x) \rangle : x \in X\}$, where the functions $\mu_A : X \rightarrow [0,1]$ and $\nu_A : X \rightarrow [0,1]$ represent respectively the membership degree and the non-membership degree of the element $x \in X$, satisfying $0 \leq \mu_A(x) + \nu_A(x) \leq 1$, $\forall x \in X$.

An associated function $\pi_A : X \rightarrow [0,1]$ defined by $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$ represents the hesitancy degree of the element $x \in X$ to the set IFS A .

The fuzzy set thus becomes a special case of IFS which may be expressed as $\{\langle x, \mu_A(x), 1 - \mu_A(x) \rangle : x \in X\}$.

For fuzzy sets, $\pi_A(x) = 0$, $\forall x \in X$. This implies that the concept of fuzzy sets is devoid of the idea of hesitancy degree making IFS more realistic tool to model real-life problems.

The ranking of IFSs had numerous approaches. Chen and Tan [11] proposed a score function S for IFS $A = \langle \mu, \nu \rangle$ which is defined as follows:

$$S(A) = \mu - \nu$$

Clearly, $S(A) \in [-1, 1]$. The more the value of score $S(A)$ the greater would be the IFS A .

Due to limitations of score function in ranking of IFSs, an accuracy function H was proposed by Hong and Choi [12] for the IFS $A = \langle \mu, \nu \rangle$ which is defined as follows:

$$H(A) = \mu + \nu$$

Clearly, $H(A) \in [0, 1]$. The more the degree of accuracy $H(A)$ the greater would be the IFS A .

Thus, the measure of IFS A is proportional to the score $S(A)$ as well as accuracy $H(A)$.

An IFSA $A = \langle \mu, \nu \rangle$ is also called an intuitionistic fuzzy number (IFN) while an IVIFSA $A = \langle [a, b], [c, d] \rangle$ is called an interval-valued intuitionistic fuzzy number (IVIFN) [19].

It is found that the score function and accuracy function represent respectively the difference and sum of the membership and nonmembership degrees of an element of the IFS A . The characteristics of the score function and accuracy function have resemblance to the mean and variance in statistics respectively. These two approaches in unison effectively rank the IFNs by first utilizing the score functional values until equality occurs and then employing accuracy degrees for breaking the tie. The equality of two score functional values and two accuracy degrees of two distinct IFNs cannot occur at the same time.

Definition 2.2 (Novel accuracy function [14])

Let $A = \langle [a, b], [c, d] \rangle$ be an IVIFN. Then, a novel accuracy function M of the IVIFNA, based on the unknown degree, is defined by

$$M(A) = a + b - 1 + \frac{c + d}{2}$$

Consider $a = b = \mu$ and $c = d = \nu$. The IVIFNA reduces to the IFN $\langle \mu, \nu \rangle$. Then, the novel accuracy function is transformed to $M(A) = 2\mu - 1 + \nu$

Obviously, $M(A) \in [0, 1]$.

Definition 2.3 ([10]) Let $A_j = \langle \mu_j, \nu_j \rangle \in \Omega$ ($j = 1, 2, \dots, n$) be IFNs. Assume that $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$ is an associated weight vector with $\omega_j \in [0, 1]$ ($j = 1, 2, \dots, n$) satisfying $\sum_{j=1}^n \omega_j = 1$. Then, an IF weighted aggregation operator $f_\omega : \Omega^n \rightarrow \Omega$ relative to weight vector ω is defined by

$$f_\omega(A_1, A_2, \dots, A_n) = \sum_{j=1}^n \omega_j A_j = \langle 1 - \prod_{j=1}^n (1 - \mu_j)^{\omega_j}, \prod_{j=1}^n \nu_j^{\omega_j} \rangle$$

III. Intuitionistic fuzzy aggregation operator relative to intuitionistic fuzzy weights

As clear evidence towards defining the normalization of IFSs, De et al. [6] introduced a definition of normalization of IFSs. However, Zeng et al. [7] pointed out the inconsistency crept in that definition through an example. Over the last decade no effort is ever seen in the vast literature on IFS defining normalization there. The failed definition of normalization [6] when minutely observed the inherent invalidity appears in the idea of defining μ_{NORM} and ν_{NORM} independently giving no scope to check the satisfiability of the inequality $\mu_{NORM}(x) + \nu_{NORM}(x) \leq 1$, $\forall x \in X$ (universe of discourse).

For extending the concept of normalization of fuzzy sets to the theory of IFS, the necessary condition of sum of IFS weights being unity should be satisfied.

Let $F_j = \langle \mu_j, \nu_j \rangle$, ($j = 1, 2, \dots, n$) be a set of normalized IFSs.

Then, $\sum_{j=1}^n F_j = F_1 \oplus F_2 \oplus \dots \oplus F_n = \langle 1, 0 \rangle$

$$\Rightarrow \langle 1 - \prod_{j=1}^n (1 - \mu_j), \prod_{j=1}^n \nu_j \rangle = \langle 1, 0 \rangle$$

$$\Rightarrow 1 - \prod_{j=1}^n (1 - \mu_j) = 1 \text{ and } \prod_{j=1}^n \nu_j = 0$$

$$\Rightarrow \prod_{j=1}^n (1 - \mu_j) = 0 \text{ and } \prod_{j=1}^n \nu_j = 0$$

$$\Rightarrow \mu_l = 1, \text{ for some } j = l \text{ and } \mu_j + \nu_j \leq 1 \text{ implies that } \nu_l = 0$$

$$\Rightarrow F_l = \langle 1, 0 \rangle$$

Hence, the set of normalized IFSs should necessarily contain $\langle 1, 0 \rangle$ at least once which is the greatest IFS. Normalization of IFSs should also preserve the order or ranking of the given set of IFSs and therefore any mechanism of normalization of IFSs should be able to convert the greatest IFS into the IFS $\langle 1, 0 \rangle$. In this context, for the correct normalization process it is imperative to know the ranking of IFSs in advance or the process should be so complex that it maintains the order of IFSs while the largest IFS being converted into IFS $\langle 1, 0 \rangle$.

Let $A = \{ \langle x_j, \mu_A(x_j), \nu_A(x_j) \rangle \mid x_j \in X \}$ be an IFS in the universe of discourse $X = \{x_1, x_2, \dots, x_n\}$. For convenience, let us denote $\langle x_j, \mu_A(x_j), \nu_A(x_j) \rangle$ by $\langle \mu_j, \nu_j \rangle$, where $\mu_j = \mu_A(x_j)$ and $\nu_j = \nu_A(x_j)$. Then, the IFS A reduces to $A = \{ \alpha_j \mid j = 1, 2, \dots, n \}$, where $\alpha_j = \langle \mu_j, \nu_j \rangle$.

In light of the above discussion, we propose the following definition of normalization of IFSs.

Definition 3.1: Let $A = \{ \alpha_j \mid j = 1, 2, \dots, n \}$, where $\alpha_j = \langle \mu_j, \nu_j \rangle$, be an IFS in the universe of discourse $X = \{x_1, x_2, \dots, x_n\}$ with $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_n$. The normalization of A , denoted by $NORM(A)$, is defined by

$$NORM(A) = \{ \langle \mu_{Aj}^N, \nu_{Aj}^N \rangle \mid j = 1, 2, \dots, n \},$$

where

$$\mu_{Aj}^N = \min \left\{ \max_{k \geq j} \left\{ \frac{\mu_k}{\mu_1} \right\}, 1 - \min_{k \geq j} |\nu_k - \nu_1| \right\} \text{ and } \nu_{Aj}^N = \min_{k \geq j} |\nu_k - \nu_1|$$

Theorem 3.1: Let $A = \{ \alpha_j \mid j = 1, 2, \dots, n \}$, where $\alpha_j = \langle \mu_j, \nu_j \rangle$, be an IFS in the universe of discourse $X = \{x_1, x_2, \dots, x_n\}$ with $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_n$.

If $NORM(A) = \{ \alpha_j^N \mid j = 1, 2, \dots, n \}$, where $\alpha_j^N = \langle \mu_j^N, \nu_j^N \rangle$ then

$$(i) \alpha_1^N = \langle 1, 0 \rangle$$

$$(ii) \alpha_1^N \geq \alpha_2^N \geq \dots \geq \alpha_n^N$$

$$(iii) \sum_{j=1}^n \alpha_j^N = \alpha_1^N \oplus \alpha_2^N \oplus \dots \oplus \alpha_n^N = \langle 1, 0 \rangle$$

Proof:

$$(i) \text{ By Definition 3.1, } \nu_1^N = \min_{1 \leq k \leq n} |\nu_k - \nu_1| = 0$$

$$\mu_1^N = \min \left\{ \max_{1 \leq k \leq n} \left\{ \frac{\mu_k}{\mu_1} \right\}, 1 - \min_{1 \leq k \leq n} |\nu_k - \nu_1| \right\} = \min \{1, 1 - 0\} = 1$$

$$\text{Thus, } \alpha_1^N = \langle \mu_1^N, \nu_1^N \rangle = \langle 1, 0 \rangle.$$

This completes the proof.

$$(ii) \text{ Let } p < q \text{ then } \alpha_p \geq \alpha_q.$$

$$\alpha_p \geq \alpha_q \Rightarrow \mu_p \geq \mu_q \wedge \nu_p \leq \nu_q$$

$$\text{Also, } \min_{p \leq k \leq n} |\nu_k - \nu_1| \leq \min_{q \leq k \leq n} |\nu_k - \nu_1| (\because p < q)$$

$$\Rightarrow \nu_p^N \leq \nu_q^N$$

$$\min_{p \leq k \leq n} |\nu_k - \nu_1| \leq \min_{q \leq k \leq n} |\nu_k - \nu_1| \Rightarrow 1 - \min_{p \leq k \leq n} |\nu_k - \nu_1| \geq 1 - \min_{q \leq k \leq n} |\nu_k - \nu_1|$$

$$\text{Now, } \mu_p \geq \mu_q \Rightarrow \frac{\mu_p}{\mu_1} \geq \frac{\mu_q}{\mu_1}$$

$$\Rightarrow \max_{p \leq k \leq n} \left\{ \frac{\mu_p}{\mu_1} \right\} \geq \max_{q \leq k \leq n} \left\{ \frac{\mu_q}{\mu_1} \right\} (\because p < q)$$

$$\text{Thus, } \min \left\{ \max_{p \leq k \leq n} \left\{ \frac{\mu_p}{\mu_1} \right\}, 1 - \min_{p \leq k \leq n} |\nu_k - \nu_1| \right\} \geq \min \left\{ \max_{q \leq k \leq n} \left\{ \frac{\mu_q}{\mu_1} \right\}, 1 - \min_{q \leq k \leq n} |\nu_k - \nu_1| \right\}$$

$$\Rightarrow \mu_p^N \geq \mu_q^N \text{ (By Definition 3.1)}$$

$$\mu_p^N \geq \mu_q^N \text{ and } \nu_p^N \leq \nu_q^N \Rightarrow \alpha_p^N \geq \alpha_q^N$$

$$\text{Hence, } \alpha_1^N \geq \alpha_2^N \geq \dots \geq \alpha_n^N$$

This completes the proof.

$$(iii) \sum_{j=1}^n \alpha_j^N = \alpha_1^N \oplus \alpha_2^N \oplus \dots \oplus \alpha_n^N$$

$$= \langle 1 - \prod_{j=1}^n (1 - \mu_j^N), \prod_{j=1}^n \nu_j^N \rangle = \langle 1, 0 \rangle (\because \mu_1^N = 1, \nu_1^N = 0)$$

Definition 3.2: Let $\Omega = \{\alpha_j = \langle \mu_j, \nu_j \rangle | (j = 1, 2, \dots, n)\}$ be a collection of IFNs and $w = (w_1, w_2, \dots, w_n)$, where $w_j = \langle \rho_j, \sigma_j \rangle (j = 1, 2, \dots, n)$, be the normalized IF weight vector of $\alpha_j (j = 1, 2, \dots, n)$. Then, the weighted IF arithmetic aggregation operator relative to the IF weights $IFWA_w: \Omega^n \rightarrow \Omega$ is defined by

$$IFWA_w(\alpha_1, \alpha_2, \dots, \alpha_n) = \sum_{j=1}^n (w_j \otimes \alpha_j) = \left(1 - \prod_{j=1}^n (1 - \rho_j \mu_j), \prod_{j=1}^n (1 - (1 - \sigma_j)(1 - \nu_j)) \right)$$

Theorem 3.2: (Properties of $IFWA_w$)

Let $\alpha_j = \langle \mu_j, \nu_j \rangle (j = 1, 2, \dots, n)$, be a collection of IFNs. Assume that $w = (w_1, w_2, \dots, w_n)$, where $w_j = \langle \rho_j, \sigma_j \rangle (j = 1, 2, \dots, n)$, is a set of normalized IF weight vector.

(1) The aggregation operator $IFWA_w$ satisfies boundary conditions.

Proof:

Case (i): Let $\alpha_j = \langle 0, 1 \rangle$, for $j = 1, 2, \dots, n$.

$$\begin{aligned} \text{Here, } IFWA_w(\alpha_1, \alpha_2, \dots, \alpha_n) &= \sum_{j=1}^n (w_j \otimes \alpha_j) \\ &= (w_1 \otimes \alpha_1) \oplus (w_2 \otimes \alpha_2) \oplus \dots \oplus (w_n \otimes \alpha_n) \end{aligned}$$

$$\begin{aligned} \text{Thus, } IFWA_w(\langle 0, 1 \rangle, \langle 0, 1 \rangle, \dots, \langle 0, 1 \rangle) &= (\langle \rho_1, \sigma_1 \rangle \otimes \langle 0, 1 \rangle) \oplus (\langle \rho_2, \sigma_2 \rangle \otimes \langle 0, 1 \rangle) \oplus \dots \oplus (\langle \rho_n, \sigma_n \rangle \otimes \langle 0, 1 \rangle) \\ &= \langle 0, 1 \rangle \oplus \langle 0, 1 \rangle \oplus \dots \oplus \langle 0, 1 \rangle \\ &= \langle 0, 1 \rangle \end{aligned}$$

Case (ii): Let $\alpha_j = \langle 1, 0 \rangle$, for $j = 1, 2, \dots, n$.

$$\begin{aligned} \text{Here, } IFWA_w(\alpha_1, \alpha_2, \dots, \alpha_n) &= \sum_{j=1}^n (w_j \otimes \alpha_j) \\ &= (w_1 \otimes \alpha_1) \oplus (w_2 \otimes \alpha_2) \oplus \dots \oplus (w_n \otimes \alpha_n) \end{aligned}$$

$$\begin{aligned} \text{Thus, } IFWA_w(\langle 1, 0 \rangle, \langle 1, 0 \rangle, \dots, \langle 1, 0 \rangle) &= (\langle \rho_1, \sigma_1 \rangle \otimes \langle 1, 0 \rangle) \oplus (\langle \rho_2, \sigma_2 \rangle \otimes \langle 1, 0 \rangle) \oplus \dots \oplus (\langle \rho_n, \sigma_n \rangle \otimes \langle 1, 0 \rangle) \\ &= \langle \rho_1, \sigma_1 \rangle \oplus \langle \rho_2, \sigma_2 \rangle \oplus \dots \oplus \langle \rho_n, \sigma_n \rangle \end{aligned}$$

$= \langle 1, 0 \rangle$ (By Theorem 3.1 (iii))

(2) The aggregation operator $IFWA_w$ is monotonic increasing.

Proof:

Let $A_j = \langle \mu_{Aj}, \nu_{Aj} \rangle$ and $B_j = \langle \mu_{Bj}, \nu_{Bj} \rangle$ with $A_j \leq B_j (j = 1, 2, \dots, n)$ be IFNs. Assume that $w_j = \langle \rho_j, \sigma_j \rangle (j = 1, 2, \dots, n)$ be IF weights.

We show that $IFWA_w(A_1, A_2, \dots, A_n) \leq IFWA_w(B_1, B_2, \dots, B_n)$

$A_j \leq B_j \Rightarrow \mu_{Aj} \leq \mu_{Bj}$ and $\nu_{Aj} \geq \nu_{Bj}$, for $j = 1, 2, \dots, n$

Now, $\mu_{Aj} \leq \mu_{Bj} \Rightarrow \rho_j \mu_{Aj} \leq \rho_j \mu_{Bj}$

$\Rightarrow 1 - \rho_j \mu_{Aj} \geq 1 - \rho_j \mu_{Bj}$, for $j = 1, 2, \dots, n$

$$\Rightarrow \prod_{j=1}^n (1 - \rho_j \mu_{Aj}) \geq \prod_{j=1}^n (1 - \rho_j \mu_{Bj})$$

$$\Rightarrow 1 - \prod_{j=1}^n (1 - \rho_j \mu_{Aj}) \leq 1 - \prod_{j=1}^n (1 - \rho_j \mu_{Bj}) \quad (1)$$

Again, $\nu_{Aj} \geq \nu_{Bj} \Rightarrow 1 - \nu_{Aj} \leq 1 - \nu_{Bj}$

$$\Rightarrow (1 - \sigma_j)(1 - \nu_{Aj}) \leq (1 - \sigma_j)(1 - \nu_{Bj})$$

$\Rightarrow 1 - (1 - \sigma_j)(1 - \nu_{Aj}) \geq 1 - (1 - \sigma_j)(1 - \nu_{Bj})$, for $j = 1, 2, \dots, n$

$$\Rightarrow \prod_{j=1}^n (1 - (1 - \sigma_j)(1 - \nu_{Aj})) \geq \prod_{j=1}^n (1 - (1 - \sigma_j)(1 - \nu_{Bj})) \quad (2)$$

By Definition 3.2 and Eqns. (1) and (2), it is found that

$$IFWA_w(A_1, A_2, \dots, A_n) \leq IFWA_w(B_1, B_2, \dots, B_n)$$

(2) The aggregation operator $IFWA_w$ is continuous which is obvious from the Definition 3.2.

Remark 3.1: The essential properties of aggregation operator [13] are thus satisfied as is evident from Theorem 3.2. Hence, the Definition 3.2 rightly introduces an IF aggregation operator relative to IF weights.

As in Definition 2.2, the novel accuracy function M of the IFN $A (= \langle \mu, \nu \rangle)$, based on the unknown degree, is represented by $M(A) = 2\mu - 1 + \nu \in [0, 1]$

If IF weights $\omega_j = \langle \rho_j, \sigma_j \rangle (j = 1, 2, \dots, n)$ be transformed to novel accuracy degrees $M(\omega_j)$ then the possibility of occurrence of zero novel accuracy degree for some IF weight makes it unsuitable for being used as the respective crisp weights in a multicriteria decision making problems otherwise there would be significant loss of information.

In this context, we propose the modification of novel accuracy function (Definition 2.2) as follows:

Definition 3.3 Let $A = \langle [a, b], [c, d] \rangle$ be an interval-valued IFN. Then, a revised novel accuracy function L of the IVIFN A , based on the unknown degree, is defined by

$$L(A) = M(A) + 1 = a + b + \frac{c + d}{2}$$

Consider $a = b = \mu$ and $c = d = \nu$. Then, the revised novel accuracy function of the IFN $A = \langle \mu, \nu \rangle$ is expressed as

$$L(A) = M(A) + 1 = 2\mu + \nu \quad (3)$$

Obviously, $L(A) \in [1, 2]$.

The monotonicity of $M(A)$ is extended to $L(A)$ and the revised novel accuracy function $L(A)$ eliminates the possibility of occurrence of zero novel accuracy degree for some IF weight enabling it for being considered as the respective crisp weights in a MCDM problems and there would be no loss of information.

IV. Multicriteria fuzzy decision-making method based on the intuitionistic fuzzy aggregation operator relative to intuitionistic fuzzy weights

In this section, we present the usefulness of the approach developed in the previous section to the problems of multicriteria decision making in IF environment.

Let $A = \{A_1, A_2, \dots, A_m\}$ be a set of alternatives which are evaluated by decision maker(s) on the set of attributes or criteria $C = \{C_1, C_2, \dots, C_n\}$ in IF environment generating IF decision matrix. If the performance value of i^{th} alternative $A_i \in A$ on j^{th} criterion $C_j \in C$ be given by a decision maker as an IFN $p_{ij} = \langle \mu_{ij}, \nu_{ij} \rangle$, where $0 \leq \mu_{ij} \leq 1$, $0 \leq \nu_{ij} \leq 1$, $\mu_{ij} + \nu_{ij} \leq 1$, then the IF decision matrix is expressed as

$$D = (d_{ij})_{m \times n} = (\langle \mu_{ij}, \nu_{ij} \rangle)_{m \times n} = \begin{matrix} & \begin{matrix} C_1 & C_2 & \dots & C_n \end{matrix} \\ \begin{matrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{matrix} & \begin{pmatrix} \langle \mu_{11}, \nu_{11} \rangle & \langle \mu_{12}, \nu_{12} \rangle & \dots & \langle \mu_{1n}, \nu_{1n} \rangle \\ \langle \mu_{21}, \nu_{21} \rangle & \langle \mu_{22}, \nu_{22} \rangle & \dots & \langle \mu_{2n}, \nu_{2n} \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle \mu_{m1}, \nu_{m1} \rangle & \langle \mu_{m2}, \nu_{m2} \rangle & \dots & \langle \mu_{mn}, \nu_{mn} \rangle \end{pmatrix} \end{matrix}$$

Now, assume that the decision maker evaluates criterion $C_j (j = 1, 2, \dots, n)$ also in IF environment generating IF weight vector $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$, where $\omega_j = \langle \rho_j, \sigma_j \rangle$ satisfying $0 \leq \rho_j \leq 1$, $0 \leq \sigma_j \leq 1$, $\rho_j + \sigma_j \leq 1 (j = 1, 2, \dots, n)$.

Thus the weight vector of all attributes with IFN as elements can be expressed as

$$\omega = (\langle \rho_1, \sigma_1 \rangle, \langle \rho_2, \sigma_2 \rangle, \dots, \langle \rho_n, \sigma_n \rangle)^T$$

We propose the following steps for solving the abovementioned intuitionistic fuzzy MCDM problems.

- Step 1:** The ranking of weights $\omega_j = \langle \rho_j, \sigma_j \rangle (j = 1, 2, \dots, n)$ is obtained using the revised novel accuracy function expressed by Eqn. (3). Let $\omega_{p(1)} \geq \omega_{p(2)} \geq \dots \geq \omega_{p(n)}$, where p is a permutation on $\{1, 2, \dots, n\}$.
- Step 2:** The IF weights $\omega_{p(j)} (j = 1, 2, \dots, n)$ are normalized using Definition 3.1 resulting in the normalized IF weights $\omega_{p(j)}^N (j = 1, 2, \dots, n)$ satisfying $\sum_{j=1}^n \omega_{p(j)}^N = \langle 1, 0 \rangle$. Hence, the given IF weight vector $\omega = (\langle \rho_1, \sigma_1 \rangle, \langle \rho_2, \sigma_2 \rangle, \dots, \langle \rho_n, \sigma_n \rangle)^T$ is transformed to the normalized IF weight vector $\omega^N = (\langle \rho_1^N, \sigma_1^N \rangle, \langle \rho_2^N, \sigma_2^N \rangle, \dots, \langle \rho_n^N, \sigma_n^N \rangle)^T$.
- Step 3:** The IF performance score of an alternative $A_j (j = 1, 2, \dots, n)$ is estimated using IF arithmetic aggregation operator relative to the IF weights $IFWA_w$ given by Definition 3.2.
- Step 4:** The crisp performance scores of alternatives $A_j (j = 1, 2, \dots, n)$ are obtained by applying the revised novel accuracy function given by Eqn. (3) to their respective IF performance scores.
- Step 5:** The alternatives $A_j (j = 1, 2, \dots, n)$ are ranked in the decreasing order of their crisp performance scores.

V. Illustrative example

In this section, an example for illustration has been adapted from ([14], [15]) for demonstrating the application of proposed IF decision making method in real situation and validating the effectiveness of the proposed method.

The problem is related to an investment company wishing to invest the money in the four possible companies (alternatives) A_1 (car company), A_2 (food company), A_3 (computer company), and A_4 (arms company). These alternatives are to be evaluated with respect to three criteria C_1 (risk analysis), C_2 (growth analysis), and C_3 (environmental impact analysis) by an expert generating IF decision matrix as

$$D = (d_{ij})_{4 \times 3} = \begin{matrix} & \begin{matrix} C_1 & C_2 & C_3 \end{matrix} \\ \begin{matrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{matrix} & \begin{pmatrix} \langle 0.45, 0.35 \rangle & \langle 0.5, 0.3 \rangle & \langle 0.2, 0.55 \rangle \\ \langle 0.65, 0.25 \rangle & \langle 0.65, 0.25 \rangle & \langle 0.6, 0.15 \rangle \\ \langle 0.45, 0.35 \rangle & \langle 0.55, 0.35 \rangle & \langle 0.45, 0.2 \rangle \\ \langle 0.75, 0.15 \rangle & \langle 0.65, 0.2 \rangle & \langle 0.35, 0.15 \rangle \end{pmatrix} \end{matrix}$$

The weight vector of the criteria C_1, C_2 and C_3 with IFNs as elements is expressed as

$$\omega = (\omega_j)_{1 \times 3} = (\langle 0.5, 0.1 \rangle, \langle 0.4, 0.2 \rangle, \langle 0.7, 0.3 \rangle)$$

Using Eqn. (3), the criteria weights are ranked as $\omega_3 \geq \omega_1 \geq \omega_2$.

By Definition 3.1, the normalized IF weight vector is given by

$$\omega^N = (\omega_j^N)_{1 \times 3} = \langle (0.71, 0.1), (0.57, 0.1), (1, 0) \rangle$$

Here, $\omega_1^N + \omega_2^N + \omega_3^N = \langle 1, 0 \rangle$

Employing IF arithmetic aggregation operator relative to the IF weights $IFWA_w$ given by Definition 3.2, the IF performance score S_i of alternative A_i ($i = 1, 2, 3, 4$) is estimated as

$$S_1 = \langle 0.61, 0.08 \rangle, S_2 = \langle 0.86, 0.02 \rangle, S_3 = \langle 0.74, 0.03 \rangle, S_4 = \langle 0.81, 0.01 \rangle$$

Now, applying Eqn. (3), the crisp performance scores of alternatives as revised novel accuracy functional degrees are computed as $L(S_1) = 1.3, L(S_2) = 1.74, L(S_3) = 1.51, L(S_4) = 1.63$

Thus, the ranking order of alternatives becomes $A_2 > A_4 > A_3 > A_1$ which agrees with that of Ye [14].

Let us now estimate the revised novel accuracy functional degrees of the IF weights ω_j ($j = 1, 2, 3$) using Eqn. (3) which are given as $L(\omega_1) = 1.1, L(\omega_2) = 1.0, L(\omega_3) = 1.7$

Treating these accuracy degrees as the weights of respective criteria, the normalized crisp weight vector is expressed as $w = (w_j)_{1 \times 3} = (0.29, 0.26, 0.45)$.

Using IF weighted aggregation operator relative to the normalized crisp criteria weights as given by Definition 2.3, the IF performance score R_i of alternative A_i ($i = 1, 2, 3, 4$) is estimated as

$$R_1 = \langle 0.36, 0.41 \rangle, R_2 = \langle 0.63, 0.20 \rangle, R_3 = \langle 0.48, 0.27 \rangle, R_4 = \langle 0.58, 0.16 \rangle$$

Applying Eqn. (3), the crisp performance scores of alternatives as revised novel accuracy functional degrees are computed as

$$L(R_1) = 1.13, L(R_2) = 1.46, L(R_3) = 1.23, L(R_4) = 1.32.$$

It is found that the ranking order of alternatives remains $A_2 > A_4 > A_3 > A_1$ which again agrees with that of Ye [14] validating the effectiveness of the proposed method.

VI. Conclusion

For modelling the inherent uncertainty in human judgment of criteria expressible in IFs, definition of normalization of IFs is proposed and related properties are studied. The normalized IFs as criteria weights are used to define an IF arithmetic aggregation operator relative to IF weights and their properties are analyzed establishing the veracity of the proposed definition. Novel accuracy function is revised for easy handling of ranking of IFs through accuracy degrees. The effectiveness of the proposed method is illustrated through an MCDM problem in IF environment with criteria as IFs. The IF criteria weights are transformed to accuracy degrees by using revised novel accuracy function that are normalized resulting in the normalized crisp weights and an existing IF weighted aggregation operator relative to these crisp weights is used to get performance scores. The ranking of alternatives through both the proposed approach and the existing approach are compared and it is found that both yield the same ranking of alternatives coinciding with that of Ye [14] validating the rationality of the proposed approach. The developed aggregation operator should model the imprecision in more reliable manner as the expert has freedom to express his judgment relating to criteria also in IFs just like his choices for alternatives in IF settings.

References

- [1] K. T. Atanassov, "Intuitionistic fuzzy sets", Fuzzy Sets and Systems, vol. 20, 1986, pp. 87-96.
- [2] K. T. Atanassov, Intuitionistic fuzzy sets, Heidelberg, Springer-Verlag, 1999.
- [3] L. A. Zadeh, "Fuzzy sets", Information and Control, vol. 8, 1965, pp. 338-356.
- [4] W. L. Gau and D. J. Buehrer, "Vague sets", IEEE Transactions on systems, Man and Cybernetics, vol. 23, 1993, pp. 610-614.
- [5] P. Burillo and H. Bustince, "Vague sets are intuitionistic fuzzy sets", Fuzzy Sets and Systems, vol. 79, 1996, pp. 403-405.
- [6] S. K. De, R. Biswas, and A. R. Roy, "Some operations on intuitionistic fuzzy sets", Fuzzy Sets and Systems, vol. 114, 2000, pp. 477- 484.
- [7] W. Zeng and H. Li, "Note on "Some operations on intuitionistic fuzzy sets" [Fuzzy Sets and Systems 114 (2000) 477]", Fuzzy Sets and Systems, vol. 157, 2006, pp. 990 – 991.
- [8] Z. S. Xu and Q. L. Da, (2003), "An overview of operators for aggregating information", International Journal of Intelligent Systems, vol. 18, 2003, pp. 953-969.
- [9] Z. S. Xu, "Intuitionistic fuzzy aggregation operators", IEEE Transactions on Fuzzy Systems, vol. 15(6), December 2007, pp. 1179 – 1187.
- [10] D. F. Li, Decision and Game Theory in Management with Intuitionistic Fuzzy Sets, Studies in Fuzziness and Soft Computing, vol. 308, Springer-Verlag Berlin Heidelberg, 2014.
- [11] S. M. Chen and J. M. Tan, (1994), "Handling multi-criteria fuzzy decision-making problems based on vague set theory", Fuzzy Sets and Systems, vol. 67(2), 1994, pp. 163-172.
- [12] D. H. Hong and C. H. Choi, "Multi-criteria fuzzy decision-making problems based on vague set theory", Fuzzy Sets and Systems, vol. 114(1), 2000, pp. 103-113.
- [13] G. J. Klir and B. Yuan, Fuzzy Sets and Fuzzy Logic: Theory and Applications, Pearson Education Inc., New Jersey, USA, 1995.
- [14] J. Ye, "Multicriteria fuzzy decision-making method based on a novel accuracy function under interval-valued intuitionistic fuzzy environment", Expert Systems with Applications, vol. 36, 2009, pp. 6899–6902.
- [15] F. Herrera and E. Herrera-Viedma, "Linguistic decision analysis: Steps for solving decision problems under linguistic information", Fuzzy Sets and Systems, vol. 115, 2000, pp. 67–82.
- [16] K. T. Atanassov and G. Gargov, "Interval-valued intuitionistic fuzzy sets", Fuzzy Sets and Systems, vol. 31, 1989, pp. 343-349.
- [17] Z. S. Xu and R. Yager, (2008), "Dynamic intuitionistic fuzzy multi-attribute decision making", International Journal of Approximate Reasoning, vol. 48, 2008, pp. 246-262.
- [18] A. Mardani, A. Jusoh and E. K. Zavadskas, "Fuzzy multiple criteria decision-making techniques and applications - Two decades review from 1994 to 2014", Expert Systems with Applications, vol. 42, 2015, pp. 4126–4148.
- [19] Z. S. Xu, "Methods for aggregating interval-valued intuitionistic fuzzy information and their application to decision making", Control and Decision, vol. 22(2), 2007, pp. 215-219