



Some Abelian Theorems for the Classical Sumudu Transform

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Abstract: Initial value theorem and final value theorem for classical Sumudu transform are proved.

Keywords: Sumudu Transform, Initial value theorem, Final value theorem

I. Introduction

We recall that the sumudu transform which is similar to the Laplace transform is defined over the set of functions which has exponential growth by

$$F(u) = S\{f(t)\} = \frac{1}{u} \int_0^\infty e^{-\frac{t}{u}} f(t) dt \quad (1)$$

Provided that the improper integration in R.H.S. converges. In the case of complex variable u , the sumudu transform can be defined by

$$F(u) = S\{f(t)\} = \frac{1}{u} \int_0^\infty e^{-\frac{t}{u}} f(t) dt, \text{ for } \operatorname{Re}\left(\frac{1}{u}\right) > 0 \quad (2)$$

For existence and uniqueness of sumudu transform see [1], for further properties of sumudu transform see [2],[3].

The objective of present work is to relate the behavior of the sumudu transform $F(u) = S\{f(t)\}$ as u approaches zero or infinity to behavior of original function $f(t)$ as t approaches infinity or zero respectively. Theorems of this nature are usually called Abelian and Tauberian theorems see [4]. In this paper we have proved initial value and final value theorems for the classical Sumudu transform.

II. Initial Value Theorem for classical sumudu transform

This theorem relates the asymptotic behavior of $f(t)$ as $t \rightarrow 0+$ to the asymptotic behavior of $F(u)$ as $u \rightarrow \infty$.

Theorem 1: If the locally integrable function $f(t)$ whose support is in $[0, \infty[$ and satisfies,

- 1) There exist a real number 'c' such that $f(t)e^{-\frac{t}{c}}$ is absolutely integrable over $-\infty < t < \infty$.
- 2) There exist a complex number α and a real number $n > -1$ such that $\lim_{t \rightarrow 0+} \frac{f(t)}{t^n} = \alpha$ then

$$\lim_{v \rightarrow \infty} \frac{F(v)}{v^n \Gamma(n)} = \alpha.$$

Where $F(u)$ is the sumudu transform of $f(t)$, v is real number and $\Gamma(n)$ is Gamma function of n .

Proof: Assuming that $v > 0, y > 0$, we may write

$$\left| \frac{F(v)}{v^n} - \alpha \Gamma(n) \right| = \frac{1}{v^n} |F(v) - \alpha v^n \Gamma(n)|$$

$$= \frac{1}{v^n} \left| \frac{1}{v} \int_0^\infty e^{-\frac{t}{v}} (f(t) - \alpha t^n) dt \right|$$

Since $0 < y < \infty$ therefore,

$$\leq \frac{1}{v^{n+1}} \left\{ \left| \int_0^y e^{-\frac{t}{v}} (f(t) - \alpha t^n) dt \right| + \left| \int_y^\infty e^{-\frac{t}{v}} (f(t) - \alpha t^n) dt \right| \right\}$$

Let,

$$I_1 = \frac{1}{v^{n+1}} \int_0^y e^{-\frac{t}{v}} |f(t) - \alpha t^n| dt \quad (3)$$

$$I_2 = \frac{1}{v^{n+1}} \int_y^\infty e^{-\frac{t}{v}} |f(t) - \alpha t^n| dt \quad (4)$$

An estimate of I_1 is as follows,

$$I_1 \leq \int_0^y e^{\frac{-t}{v}} \left(\frac{t}{v}\right)^n d\left(\frac{t}{v}\right) \sup_{0 \leq t \leq y} \left| \frac{f(t)}{t^n} - \alpha \right|$$

$$\leq \gamma(n) \sup_{0 \leq t \leq y} \left| \frac{f(t)}{t^n} - \alpha \right|$$

Since $\lim_{t \rightarrow 0^+} \frac{f(t)}{t^n} = \alpha$ therefore we can choose y so small that $\left| \frac{f(t)}{t^n} - \alpha \right| < \frac{\epsilon}{2 \cdot \gamma(n)}$, Hence

$$I_1 < \frac{\epsilon}{2}$$

Now consider,

$$I_2 = \frac{1}{v^{n+1}} \int_y^\infty e^{\frac{-t}{v}} |f(t) - \alpha t^n| dt$$

By the given condition in (1), we can choose $\frac{1}{c} > 0$ as large as possible, so that $e^{\frac{-t}{c}} |f(t) - \alpha t^n|$ is absolutely integrable over $(0, \infty)$. Therefore for $\frac{1}{v} > \frac{1}{c}$

$$I_2 = \frac{1}{v^{n+1}} \int_0^\infty e^{-(\frac{1}{v}-\frac{1}{c})t} e^{\frac{-t}{c}} |f(t) - \alpha t^n| dt$$

$$\leq K \frac{1}{v^{n+1}} e^{-(\frac{1}{v}-\frac{1}{c})y}$$

Where K is a constant and $K = \int_0^\infty e^{\frac{-t}{c}} |f(t) - \alpha t^n| dt$

Choose V so large that $I_2 < \frac{\epsilon}{2}$, therefore for sufficiently large V , we must have

$$\lim_{v \rightarrow \infty} \frac{F(v)}{\gamma(n) \cdot v^n} = \alpha$$

Hence the proof.

III. Final Value Theorem of the Classical Sumudu Transform

Now we shall relate the asymptotic behavior of $f(t)$ as $t \rightarrow \infty$ to the asymptotic behavior of $F(u)$ as $u \rightarrow 0^+$.

Theorem 2:

If the locally integrable function $f(t)$ whose support is in $[0, \infty[$ and satisfies,

- 1) There exist a real number 'c' such that $f(t)e^{\frac{-t}{c}}$ is absolutely integrable over $-\infty < t < \infty$.
- 2) There exist a complex number α and a real number $n > -1$ such that $\lim_{t \rightarrow \infty} \frac{f(t)}{t^n} = \alpha$ then

$$\lim_{v \rightarrow 0^+} \frac{F(v)}{v^n \cdot \gamma(n)} = \alpha.$$

Where $F(u)$ is the sumudu transform of $f(t)$, v is real number.

Proof: Assuming that $v > 0, y > 0$, we may write

$$\left| \frac{F(v)}{v^n} - \alpha \cdot \gamma(n) \right| = \frac{1}{v^n} |F(v) - \alpha \cdot v^n \gamma(n)|$$

$$= \frac{1}{v^n} \left| \frac{1}{v} \int_0^\infty e^{\frac{-t}{v}} (f(t) - \alpha t^n) dt \right|$$

Since $0 < y < \infty$ therefore,

$$\leq \frac{1}{v^{n+1}} \left\{ \left| \int_0^y e^{\frac{-t}{v}} (f(t) - \alpha t^n) dt \right| + \left| \int_y^\infty e^{\frac{-t}{v}} (f(t) - \alpha t^n) dt \right| \right\}$$

Let,

$$I_1 = \frac{1}{v^{n+1}} \int_0^y e^{\frac{-t}{v}} |f(t) - \alpha t^n| dt \quad (5)$$

$$I_2 = \frac{1}{v^{n+1}} \int_y^\infty e^{\frac{-t}{v}} |f(t) - \alpha t^n| dt \quad (6)$$

An estimate of I_1 is as follows,

$$I_1 \leq \int_0^y e^{\frac{-t}{v}} \left(\frac{t}{v}\right)^n d\left(\frac{t}{v}\right) \sup_{0 \leq t \leq y} \left| \frac{f(t)}{t^n} - \alpha \right|$$

$$\leq \text{gamma}(n) \sup_{0 \leq t \leq y} \left| \frac{f(t)}{t^n} - \alpha \right|$$

Since $\lim_{t \rightarrow \infty} \frac{f(t)}{t^n} = \alpha$ therefore we can choose y so large that $\left| \frac{f(t)}{t^n} - \alpha \right| < \frac{\epsilon}{2 \cdot \text{gamma}(n)}$, Hence

$$I_1 < \frac{\epsilon}{2}$$

Now consider,

$$I_2 = \frac{1}{v^{n+1}} \int_y^\infty e^{-\frac{t}{v}} |f(t) - \alpha t^n| dt$$

By the given condition in (1), we can choose $\frac{1}{c} > 0$ as large as possible, so that $e^{-\frac{t}{c}} |f(t) - \alpha t^n|$ is absolutely integrable over $(0, \infty)$. Therefore for $\frac{1}{v} > \frac{1}{c}$

$$I_2 = \frac{1}{v^{n+1}} \int_0^\infty e^{-(\frac{1}{v} - \frac{1}{c})t} e^{-\frac{t}{c}} |f(t) - \alpha t^n| dt$$

$$\leq K \frac{1}{v^{n+1}} e^{-(\frac{1}{v} - \frac{1}{c})y}$$

Where K is a constant and $K = \int_0^\infty e^{-\frac{t}{c}} |f(t) - \alpha t^n| dt$

Choose V so small that $I_2 < \frac{\epsilon}{2}$, therefore for sufficiently small positive V , we must have

$$\lim_{v \rightarrow 0^+} \frac{F(v)}{\text{gamma}(n) \cdot v^n} = \alpha$$

Hence the proof.

References

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