



Mathematical solution of Bi-level Quadratic Fractional Programming Problems with Fuzzy Goal Programming Approach

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Abstract: In this paper we will apply fuzzy goal programming (FGP) approach to solve a bi-level quadratic fractional programming problem. In the proposed approach, the individual optimal decision of each of the decision makers (DMs) located at different hierarchical levels is determined first. In the decision process, the fuzzy goal-programming solution approach is used for achieving the highest degree (utility) of each of the defined membership goals to the extent possible by minimizing their under deviational variables and there by obtaining the most satisfactory solution for all the decision makers in the decision making environment. The main advantage of the proposed FGP approach presented here is that the computational load with re-evaluation of the problem again and again by re-defining the elicited membership values of the DMs for searching higher degree of satisfaction does not arise in the solution search process. Theoretical results are illustrated with the help of a numerical example.

Key words: Bi-level programming problems, Quadratic Programming, Fuzzy goal programming, Fuzzy membership functions, satisfactory solution.

I. INTRODUCTION

In this paper we deal with the bi-level quadratic fractional programming problem (BLQFPP) with the essentially co-operative decision makers (DMs) and propose an algorithm to solve bi-level quadratic fractional programming problem via fuzzy goal programming.

BI-LEVEL PROGRAMMING PROBLEM (BLPP)

A bi-level programming problem (BLPP) consists of two levels, namely, the first level and the second level. The first level decision maker (DM) is called the centre. The second level DM called follower, executes its policies after the decision of higher level DM called leader (centre) and then the leader optimizes its objective independently but may be affected by the reaction of the follower i.e. BLPP is a sequence of two optimization problems in which the constraints region of one is determined by the solution of second. Bi-level programming structure is used for central economic planning at the regional or national level to create model problems concerning organizational design, facility location, signal optimization, traffic assignment etc. In a decentralized firm, top management or an executive of headquarters makes a decision such as budget of the firm, and then each division determines production plan in the full knowledge of the budget.

Hierarchical optimization or Bi-level programming problems (BLPPs) have the following common characteristics: interactive decision making units exist within predominantly hierarchical structures; the execution of decision is sequential from higher level to lower level; each decision making unit independently controls a set of decision variables and is interested in maximizing its own objective but is affected by the reaction of lower level decision makers (DMs). Due to their dissatisfaction with the decision of the higher level DMs, decision deadlock arises frequently in the decision-making situation.

A bi-level programming problem (BLPP) is a special case of multi level programming problem (MLPP). Multi level programming problem can be defined as a p -person, non-zero sum game with perfect information in which each player moves sequentially from top to bottom. This problem is a nested hierarchical structure. When $p = 2$, we call the system a bi-level programming problem. Hierarchical optimization or Bi-level programming

techniques are extension of Stackelberg games for solving decentralized planning problem with multiple DMs in a hierarchical organization. The Stackelberg solution has been employed as a solution concept to bi-level programming problems, and a considerable number of algorithms for obtaining the solution have been employed. Bi-level programming is a powerful and robust technique for solving hierarchical decision making problem. The concept of BLPP was first introduced by Candler and Townsley[1]. The bi-level programming problem (BLPP) has received increasing attention in the literature. Candler and Townsley[1], as well as Fortuny-Amat and McCarl [3] presented the formal formulation of BLPP. Most of the developments on BLPPs are based on vertex enumeration method and transformation approaches which are effective only for very simple types of problems. In these methods the DMs have no cooperating attitude with each other so these approaches unable to give a satisfactory solution which would be acceptable to both the DMs. To overcome such difficulties Zimmermann [22] first applied fuzzy set theory in decision making problems with several conflicting objectives.

Lai[4] introduced an effective fuzzy approach using the concept of tolerance membership functions for solving MLPPs. Shih et al. [19,20] extended Lai's concept using a non-compensatory maximum-minimum aggregation operator for solving MLPPs. Shih and Lee[20] further extended Lai's concept by introducing the compensatory fuzzy operator for solving MLPPs. The main difficulty of fuzzy programming (FP) approach is that the objectives of the DMs are conflicting. So there is a possibility of rejecting the solution again and again which make the solution process very lengthy and tedious one. To overcome such difficulty fuzzy goal programming (FGP) procedure introduced by Mohamed [8] is applied in multi decision making problems. In the recent past FGP approach to bi-level linear programming problems have been discussed by Sinha and Biswal[18]. In fuzzy environment, Pal and Moitra[6,12,17] proposed fuzzy goal programming (FGP) approach for solving bi-level programming problem. Here, Pal and Moitra formulated BLPP in two phases by using the notion of distance function. At the first phase of the decision process, Pal and Moitra transform the linear programming model into a equivalent fuzzy goal model to achieve the highest degree (unity) of each of the defined membership function goals to the extent possible by minimizing their deviational variables and thereby obtaining the most satisfactory solution for both the decision maker

S.Pramanik, and P. Dey [13,14,15,16] has introduced Fuzzy Goal Programming approach in Multi-level and Bi-level programming problems. In the proposed approach, after formulating the quadratic membership functions it is linearized into equivalent linear membership functions at the best solution point by using first order Taylor polynomial series. Then fuzzy goal programming technique was used for solving the problem by minimizing only the negative deviational variables.

M. Saraj and s. Sadeghi [21] has propose a fuzzy goal programming method for obtaining a satisfactory solution to a bi-level multi-objective absolute value fractional programming problems. The method of variable change on the under- and over-deviational variables of the membership goals associated with the fuzzy goals of the model was introduced to solve the problem efficiently by using linear goal programming methodology. K. Lachhwani [5] has introduced fuzzy goal programming (FGP) procedure for multi objective quadratic programming problem. In the FGP model formulation, firstly the objectives are transformed into fuzzy goals (membership functions) then achievement of the highest membership value of each of fuzzy goals was formulated by minimizing the negative deviational variables. P. Dey, S. Pramanik and Bibhas C. Giri[2] extended the work on bi-level multi-objective linear fractional programming problems. S. Mishra, A.Verma.[9,10] has introduced analytic hierarchy process for Solving bi-level quadratic fractional programming problems and also presents an algorithm based on the fitness function for solving bi-level linear fractional programming problems.

In this paper we deal with the bi-level quadratic fractional programming problem (BLQFPP) with the essentially cooperative DMs and propose an algorithm to solve bi-level quadratic fractional programming problem via. Fuzzy Goal Programming (GP). It is a simple method to apply to the bi-level systems compared to the other transformation method. In this paper, the FGP approach is extended to solve the proposed BLQFPP. In the model formulation of the problem, the fuzzy goal levels of the objectives as well as the decision vectors controlled by the upper-level DMs are defined first. The fuzzy goals are then characterized by the associated membership functions. In the solution process, first the membership functions are defined as flexible membership goals by introducing under- and over-deviational variables and assigning highest membership value (unity) as aspiration level to each of them. Then in the achievement function, minimization of the under-deviational variables for achieving the membership goals to the highest degree to the extent possible on the basis of their importance is taken into account.

II. BI-LEVEL QUADRATIC FRACTIONAL PROGRAMMING PROBLEMS

A bi-level quadratic fractional programming problem (BLQFPP) consists of two levels, namely, the first level and the second level and each has quadratic fractional objective function. A multi level programming problem (MLPP) can be defined as a p -person, nonzero sum game with perfect information in which each player moves sequentially from top to bottom. This problem is a nested hierarchical structure. When $p = 2$, we call the system a bi-level programming problem.

For instance, by adopting a criterion with respect to finance or corporate planning as an objective function at the upper level and employing a criterion regarding production planning as an objective function at the lower level, a bi-level linear or non-linear fractional programming problem can be formulated for hierarchical decision problems in firms.

A bi-level quadratic fractional programming problem is mathematically formulated as:

$$\begin{aligned} \underset{x_1}{\text{Max}} \quad & z_1(x_1, x_2) \quad \text{Where } x_2 \text{ solves} \\ \underset{x_2}{\text{Max}} \quad & z_2(x_1, x_2) \quad \text{subject to} \quad A_1 x_1 + A_2 x_2 \leq b, \quad x_1 \geq 0, x_2 \geq 0. \end{aligned}$$

Where objective functions $z_i(x_1, x_2), (i = 1, 2)$ are represented by a quadratic fractional function

$$z_k(x_1, x_2) = \frac{n_k(x_1, x_2)}{d_k(x_1, x_2)} = \frac{x_1' Q_{k1} x_1 + x_2' Q_{k2} x_2 + c_{k1} x_1 + c_{k2} x_2 + c_{k3}}{x_1' R_{k1} x_1 + x_2' R_{k2} x_2 + d_{k1} x_1 + d_{k2} x_2 + d_{k3}} \quad (1)$$

$x_k, k = 1, 2$ are n_k – dimensional decision variables.

Q_{k1} and $R_{k1}, k = 1, 2$ are $n_1 \times n_1$ positive definite matrix.

Q_{i2} and $R_{i2}, k = 1, 2$ are $n_2 \times n_2$ positive definite matrix.

c_{i1} and $d_{i1}, k = 1, 2$ are n_1 – dimensional row vectors.

c_{i2} and $d_{i2}, k = 1, 2$ are n_2 – dimensional row vectors.

c_{i3} and $d_{i3}, k = 1, 2$ are constants.

b is a m – dimensional constant column vector.

$n_k(x_1, x_2)$ is the numerator of the quadratic fractional objective function.

$d_k(x_1, x_2)$ is the denominator of the quadratic fractional objective function.

$A_i, k = 1, 2$ is a $m \times n_k$ constant matrix; and it is assumed that the denominators are positive i.e.

$d_k(x_1, x_2) > 0, k = 1, 2.$

For the sake of simplicity, we use the following notations:-

$$x = (x_1, x_2) \in R^{n_1+n_2}; \quad z_1(x_1, x_2) = f(x_1, x_2); \quad z_2(x_1, x_2) = f(x_1, x_2)$$

Also let DM1 denote the DM at the upper level and DM2 denote the DM at the lower level.

In the bi-level quadratic fractional programming problem “(1)”, $z_1(x_1, x_2)$ and $z_2(x_1, x_2)$ respectively represent objective functions of DM1 and DM2 and x_1, x_2 represent decision variables of DM1 and DM2 respectively.

III. FUZZY GOAL PROGRAMMING APPROACH TO BLQFPP

To formulate the fuzzy programming model of BLQFPP under consideration, the objective functions z_1, z_2 and decision vectors x_1, x_2 are required to be transformed into fuzzy goals by means of assigning an aspiration level to each of them. Then they are characterized by their membership functions by defining the tolerance limits for achievement of the aspired levels of the goals.

IV. CONSTRUCTION OF MEMBERSHIP FUNCTION

Since the DMs of both the levels are interested in optimizing their individual benefit over some feasible region, the optimal solution of each of them calculated in isolation would be the aspiration levels of their associated fuzzy goals.

Let (x_1^u, x_2^u, z_1^u) be the solution for the upper level problem where $z_1^u = z_1(x_1^u, x_2^u)$ and (x_1^L, x_2^L, z_2^L) be the solution for the lower level problem where $z_2^L = z_2(x_1^L, x_2^L)$. Then the decision maker at upper level gives some tolerance to the decision vector x_1^u and the objective function z_1 . Tolerance on the decision vector x_1^u enables the decision maker at lower level to search for his optimum in a wider feasible range. By giving some tolerance to his objective function the decision maker at upper level directs the decision maker at lower level to search for his solutions in the right direction. Let p_1^+ and p_1^- be the tolerance on x_1^u .

Let the membership function for decision vector x_1 be defined as :

$$\begin{aligned} \mu_{x_1}(x_1) &= [x_1 - (x_1^u - p_1^-)] / p_1^-, (x_1^u - p_1^-) \leq x_1 \leq x_1^u \\ &[(x_1^u + p_1^+) - x_1] / p_1^+, x_1^u \leq x_1 \leq x_1^u + p_1^+ \\ &0, \text{otherwise} \end{aligned} \quad (2)$$

Then the membership function [Zimmermann (1991)] for the defined fuzzy goals appear as follows:

$$\begin{aligned} 1, z_1(x) &\geq z_1^u \\ \mu_{z_1}[z_1(x)] &= [z_1(x) - z_1^L] / [z_1^u - z_1^L], z_1^L \leq z_1(x) \leq z_1^u \\ 0, z_1(x) &\leq z_1^L \end{aligned} \quad (3)$$

$$\begin{aligned} 1, z_2(x) &\geq z_2^u \\ \mu_{z_2}[z_2(x)] &= [z_2(x) - z_2^L] / [z_2^u - z_2^L], z_2^L \leq z_2(x) \leq z_2^u \\ 0, z_2(x) &\leq z_2^L \end{aligned} \quad (4)$$

Now, in the decision making situation, the aim of each DM is to achieve the highest membership value of the associated fuzzy goal. But in actual practice, achievement of all membership values to the highest degree is not possible due to limitation of the resources. In such a case, the FGP solution technique, as a robust and most flexible technique for solving single objective or multi-objective decision analysis, is used to solve the problem of achieving the highest degree (unity) of the defined membership functions and thereby obtaining the most satisfactory decision.

V. FUZZY GOAL PROGRAMMING SOLUTION APPROACH

In decision making situation, the aim of each DM is to achieve highest membership value (unity) of the associated fuzzy goal in order to obtain the absolute satisfactory solution.

However, in real practice, achievement of all membership values to the highest degree (unity) is not possible due to conflicting objectives. Therefore, decision policy for minimizing the regrets of the DMs for all the levels should be taken into consideration. Hence, each DM should try to maximize his or her membership function by making them as close as possible to unity by minimizing its negative deviational variables. Therefore, in effect, we are simultaneously optimizing all the objective functions. So, for the defined membership functions in (2), (3) and (4), the flexible membership goals having the aspired level unity can be represented as:

$$\begin{aligned} [z_1(x) - z_1^L] / [z_1^u - z_1^L] + d_1^- - d_1^+ &= 1, \\ [z_2(x) - z_2^L] / [z_2^u - z_2^L] + d_2^- - d_2^+ &= 1 \\ [x_1 - (x_1^u - p_1^-)] / p_1^- + d_1^{L-} - d_1^{L+} &= I \\ [(x_1^u + p_1^+) - x_1] / p_1^+ + d_1^{R-} - d_1^{R+} &= I \\ \text{Where } d_1^-, d_1^+ \geq 0, \quad d_2^-, d_2^+ \geq 0, & \text{-----(a)} \end{aligned}$$

$$d_1^{R-}, d_1^{R+} \geq 0, \quad d_1^{L-}, d_1^{L+} \geq 0 \quad \text{-----}(b)$$

Here (a) represents the over and under deviational variables and (b) represents the vectors of over and under deviational variables associated with the respective goals. I is the column vector having all components equal to 1. Now, FGP approach to the problem can be presented as:

$$\text{Minimize } Z = w_1^- d_1^- + w_2^- d_2^- + w_1^{L-} d_1^{L-} + w_1^{R-} d_1^{R-}$$

Subject to

$$[z_1(x) - z_1^L] / [z_1^U - z_1^L] + d_1^- - d_1^+ = 1,$$

$$[z_2(x) - z_2^L] / [z_2^U - z_2^L] + d_2^- - d_2^+ = 1$$

$$[x_1 - (x_1^u - p_1^-)] / p_1^- + d_1^{L-} - d_1^{L+} = I$$

$$[(x_1^u + p_1^+) - x_1] / p_1^+ + d_1^{R-} - d_1^{R+} = I$$

In the present formulation, numerical weights $w_1^-, w_2^- (\geq 0)$, and the vectors of the numerical weight $w_1^{L-}, w_1^{R-} (\geq 0)$, represents the relative importance of the goals for achieving their aspired levels, and they are determined as:

$$w_1^- = 1 / [z_1^U - z_1^L]$$

$$w_2^- = 1 / [z_2^U - z_2^L]$$

$$w_1^{L-} = 1 / p_1^-, \quad w_1^{R-} = 1 / p_1^+$$

The above FGP model provides the most satisfactory decision by achieving the aspired levels of the membership goals to the extent possible in the decision making environment.

VI. NUMERICAL EXAMPLE

In this section we present numerical example to demonstrate the solution procedures by proposed approach to solve bi-level quadratic fractional programming problem (BLQFPP). The following example considered by Mishra [11] is again used to demonstrate the solution procedures and clarify the effectiveness of the proposed approach:

Consider the following BLQFPP :

$$\min_{x_1} Z_1(x_1, x_2) = \frac{4x_1^2 + x_2^2 + 1}{2x_1^2 + 5x_2^2 + 1}$$

Where x_2 solves,

$$\min_{x_2} Z_2(x_1, x_2) = \frac{3x_1^2 + 5x_2^2 + 1}{4x_1^2 + 3x_2^2 + 1}$$

Subject to

$$-5x_1 + 3x_2 \leq 15, 4x_1 + 3x_2 \leq 45, x_1, x_2 \geq 0$$

By solving the above FGP model, the compromise optimal solution of the BLQFPP is obtained as

X_1	X_2	Z_1	Z_2	μ_{z_1}	μ_{z_2}
1.87	1.9	0.71	1.1	0.72	0.58

In the proposed FGP approach, a compromised decision is achieved by maintaining hierarchy of decision

powers of the DMs for overall benefit of the organization.

VII. CONCLUSION

An effort has been made to solve BLQFPP based on fuzzy set theory and goal programming approach. The main advantage of the FGP approach presented here is that the computational load with re-evaluation of the problem again and again by re-defining the elicited membership values of the DMs for searching higher degree of satisfaction does not arise in the solution search process. The proposed approach can be extended to solve fuzzy multi-objective MLPPs. An extension of the approach for fuzzy multi-level decentralized is one of the current research problems. However, it is hoped that proposed approach can contribute to future study in the field of practical hierarchical decision making problems.

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