



Solving the Column Buckling Equation through the Application of Volterra Equations

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Abstract: Along with development of nonlinear sciences, an increasing trend has also been developed towards solving equations through analytic methods. Solving nonlinear equations has always been a challenge and finding a solution for solving nonlinear equations has always been a major goal for engineers. In applied mathematics, a solid instrument for solving normal and partial differential equations is integral equations. The theory of integral equations is one of the most valuable assets of the whole math science and engineering. Its main importance lies in marginal value problems in partial differential equations theory. In these methods, unknown functions are transformed into integrals through special conversions and then calculated through numerical or analytic methods. One of the most conventional integral equations is the Fredholm integral belonging to the family of Volterra equations. This research elaborates on solvation of column buckling equation under different marginal conditions. The nonlinear equation of column buckling was solved through the second type of Fredholm integral equation under centralized axial load and different marginal conditions. In addition, buckling was compared with the finite element method through solvation of linear equations.

Keywords: column buckling, critical load, integral equations, Volterra, Fredholm type two

I. Introduction

The stability of a structure is defined as the ability of structure for bearing of a specific load without occurrence of any instant deformation in the structure. Investigation of stability of columns and determination of their buckling critical load can be considered as oldest engineering problems. Buckling is a physical phenomenon in which an element which is relatively straight and thin undergoes an instant side bending due to compressive loads. Components of a structure under tensile, cutting or torsion pressure are damaged at a certain level of pressure or stress. On this basis, determining the maximum tolerance of structural components easily allows us to calculate the structure's overall loading capacity. In structural components buckling occurs before pressures exceed from the tolerance of components. The pressure that results in occurrence of buckling depends on several parameters including geometric dimensions, marginal conditions and characteristics of material of structural components. Therefore, determination of buckling pressure in structural components has remained as a relatively-complex problem. In order to investigate a system's stability, one can assume the picture of a sphere seeking balance. In any deviation from original balance, there are three situations that might occur: similar to what you can see in Figure 1, (a), if the sphere is able to return to its previous condition, the system has a stable balance. However, if no change was resulted in the system's stability, like what you can see in Figure 1 (b), the system is of neutral balance. In addition, if as you can see in Figure 1 (c), the system was unable to return to its previous condition, then it is assumed to have an unstable balance. The stability of a structure is defined as the ability of structure for bearing of a specific load without occurrence of any instant deformation in the structure. In a structure, there are three types of balance which are schematically shown in Figure 1. As you can see, in addition to stable and unstable balances, a structure can also have a neutral balance. The load which results in neutral balance is called the critical load.

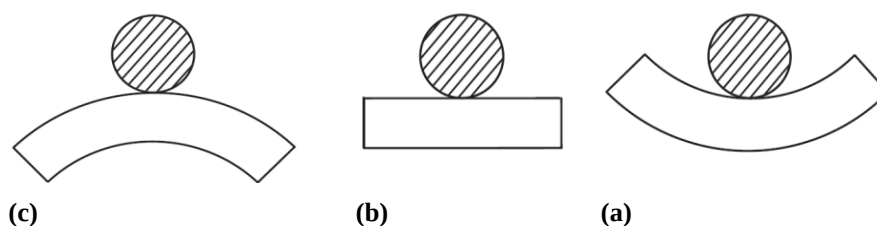


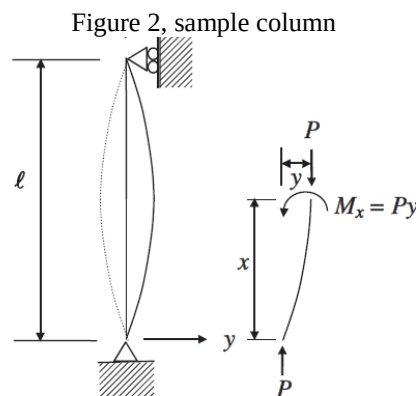
Figure 1, notions of balance and stability

Those structural components which are referred to as columns are one-dimensional components which are exposed to axial compressive loads and are generally under the simultaneous effects of compressive and moment forces. Investigation of stability of columns is considered as an introduction to investigation of stability of frames and is highly important. If we presume that the completely straight component shown in figure two maintains a similar section under the effect of axial load, then we will have Relationship (1).

$$(1) \quad \frac{M}{EI} = \frac{y''}{[1 + (y')^2]^{3/2}}$$

In the upper relation M resembles moment, E resembles elasticity module, I resembles inertia and Y resembles movement in column's side direction. Assuming that column's side movements are extremely small to the extent that the Y^2 expression could be omitted from the equation, then columns' curvature can be expressed as Y'' . Therefore, the internal resistant moment (M_x) is equal to relationship (2) at the distance of x from the centre.

$$(2) \quad EIy'' = -M_x = -Py \quad \text{or} \quad EIy'' + Py = 0$$



The equation shown in relationship (2) is a second type differential equation with fixed coefficients. In many occasions, solving these problems is rather difficult or impossible. For this purpose, an approximation method must be used. On the other hand, numerical operations in approximation methods are rather extensive and they are hence, more suited for computers. In each approximation method, the continuous actual system is substituted with a system with limited freedom degree. The behavior of the continuous system with infinite freedom degree is expressible through one or more differential equations. On the other hand, the behavior of a system with a finite freedom degree is expressible through one or more algebraic equations. In this regard, the approximation method which is a type of algebraic equations and is easier to handle is substituted for differential equations that might be harder to solve.

II. Solving the buckling equation through the Euler's method

The first attempt for solvation of buckling in static columns was made by Euler in 1744. The only objection made to this approach is because of usage of approximation functions as deformation functions.

For solving the buckling equation (relationship 2), Euler represents the deformation equation as the relationship (3) in an approximated fashion in which A and b coefficients are Integral constants.

$$(3) \quad A \sin(kx) + B \cos(kx) = 0$$

For the column shown in figure 3, the marginal conditions are defined in form of the relationship (4):

$$(4) \quad x = 0, y = 0, \quad x = L, y = 0$$

Considering the above mentioned marginal conditions, we have:

$$(x = 0, y = 0) \rightarrow B = 0$$

$$(5) \quad (x = L, Y = 0) \rightarrow A \sin(kL) = 0 \rightarrow (A = 0, KL = n\pi)$$

In the upper relation, $B = 0$ is the absolute answer to the relationship. Therefore, the rich equation is obtained in form of the relationship (6):

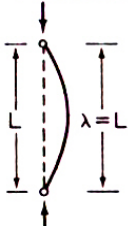
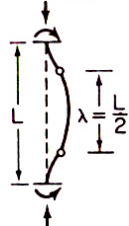
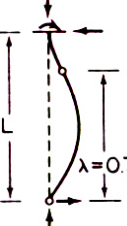
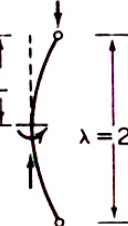
$$(6) \quad y = A \sin\left(\frac{n\pi}{L} x\right)$$

Finally, by inserting the above answer in the buckling equation, the amount of critical load or in other words, the least amount of load that results in buckling is obtained in form of the relationship (7):

$$(7) \quad p_{cr} = \frac{\pi^2 EI}{L^2}$$

This solution reduces the system with an infinite freedom degree into a system with a finite freedom degree through assuming a sinusoidal curve as the column buckling curve. Therefore, instead of being obligated to use a differential equation for accurate calculation of critical load, the same results are obtained through application of series of algebraic equations. In order to obtain the suitable answer, the assumed deformation curve must satisfy the maximum amount of different marginal conditions as possible. Table 1 shows the effective length of column buckling and Euler's critical buckling load for a column with different marginal conditions.

Table 1, results of Euler's method

TWO JOINTED	TWO CLAMPED	ONE JOINTED	ONE CLAMPED	
				Effective buckling length
$\frac{\pi^2 EI}{L^2}$	$\frac{\pi^2 EI}{(L/2)^2}$	$\frac{\pi^2 EI}{(0.7L)^2}$	$\frac{\pi^2 EI}{(2L)^2}$	Critical buckling load

III. Solving the buckling equation through finite elements method

Through generalization of the buckling problem to the finite elements condition, we also consider for the effects of internal forces of column. The exact equation of buckling of a column exposed to compressive axial load includes several different parameters. For this purpose, while simplifying the finite elements analysis, it should always be tried to neglect the effects of certain parameters while maintaining a suitable overall accuracy. For this purpose, since the stresses and strains occurred due to compressive loads are very small in comparison with buckling stresses, they are neglected. However, the effect of external work of axial compressive load is considered for under side movements. Therefore, the finite elements equation for the problem of buckling for the component number M of the column is as follows:

$$(8) \quad ([k_m] - P[k_{g,m}])\{u_c\} = \{p_c\}$$

In this relation, $[k_m]$ is the hardness matrix of the number M component of the column and is expressed in the following way:

$$(9) \quad [k_m] = \frac{E_m I_m}{l_m^3} \begin{bmatrix} 12 & 6l_m & -12 & 6l_m \\ 6l_m & 4l_m^2 & -6l_m & 2l_m^2 \\ -12 & -6l_m & 12 & -6l_m \\ 6l_m & 2l_m^2 & -6l_m & 4l_m^2 \end{bmatrix}$$

The $[k_{g,m}]$ matrix has different names including geometric hardness matrix, differential hardness matrix or the initial stress matrix. This matrix is in the form of relationship (10):

$$(10) \quad [k_{g,m}] = \frac{1}{30l_m} \begin{bmatrix} 36 & -3l_m & -36 & -3l_m \\ -3l_m & 4l_m^2 & 3l_m & -l_m^2 \\ -36 & 3l_m & 36 & 3l_m \\ -3l_m & -l_m^2 & 3l_m & 4l_m^2 \end{bmatrix}$$

By installation of hardness matrixes of column components, the relationship (11) will be yielded:

$$(11) \quad [K] = \sum [k_m] \quad [K_G] = \sum [k_{g,m}]$$

Ultimately, the final column buckling equation is as follows:

$$(12) \quad ([K] - P[K_G])\{U_c\} = \{P_c\}$$

Under compressive axial load, when the outcome of the overall hardness and geometrical hardness of column elements are equal to zero, the column loses its resistance against the applied load and resultantly, buckling instability occurs. Determination of critical buckling load and different buckling modes are ready to be calculated through the special value problem in form of relationship (13).

$$(13) \quad ([K] - \lambda[K_G])\{U\} = 0$$

In the upper relation, λ is the coefficient of buckling load and is calculated corresponding to the level of column's freedom degree. In addition $\{U\}$ is the special vector related to special values. Therefore, under the applied load of P_i , the critical buckling load (P_{cr}) calculated according to the smallest obtained special value (λ_{cr}) will be in form of relationship (14):

$$P_{cr} = P_i \times \lambda_{cr} \quad (14)$$

In figures 3-6, four different buckling modes have been shown for a single column under different anchor conditions. Results obtained from the finite elements method are largely consistent with results obtained from the Euler's method.

Figure 3, first to fourth modes for the two jointed column; finite elements method

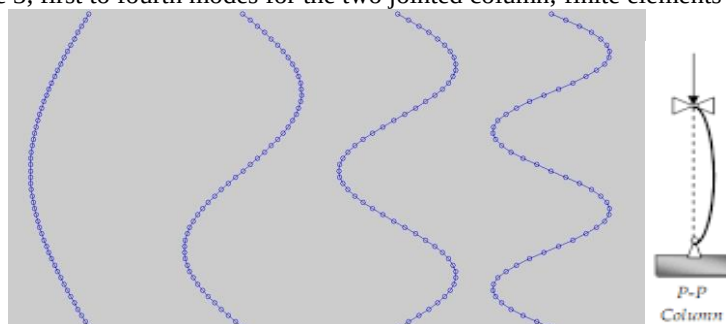


Figure 4, first to fourth modes for the one buckled and one jointed column; finite elements method

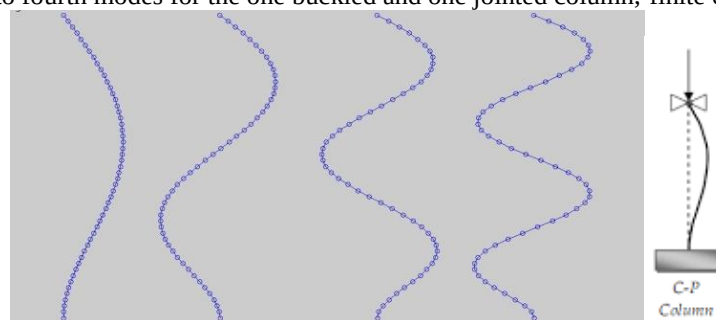


Figure 5, first to fourth modes for the two buckled column; finite elements method

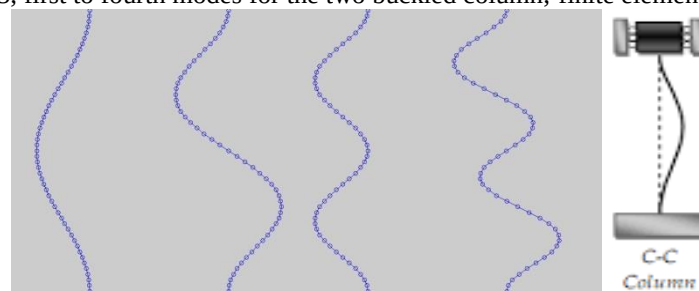


Figure 6, first to fourth modes for the consular column; finite elements method

IV. Integral equations

An integral equation is an equation in which the unknown function of $u(x)$ is appeared at least under one integral sign. A sample integral equation in which $u(x)$ is an unknown function and is in form the relationship (15):

$$(15) \quad u(x) = f(x) + \int_{a(x)}^{\beta(x)} k(x, t)u(t)dt.$$

In the upper relationship, the expression of $k(x, t)$ is the nucleus of the integral equation and $a(x)$ and $b(x)$ are integration limits. In addition, one is a constant coefficient. In relationship (15) the expression $u(x)$ is appeared under an integral sign and in other cases, it might also appear outside the boundary of integral sign. It should be noted that the nucleus and the $f(x)$ functions are predetermined and the goal is to find the value of unknown function of $u(x)$. There are different methods for doing so. Most conventional integral equations can be divided into two groups of Fredholm and Volterra equations. The standard form of Fredholm equations in which the upper and lower integral limits are respectively the constant values of a , and b is the following relationship:

$$(16) \quad \varphi(x)u(x) = f(x) + \mu \int_a^b k(x, t)u(t)dt, \quad a \leq x, t \leq b$$

Fredholm integral equations are divided into two subcategories based on the choice of $\varphi(x)$. When the choice is zero, the equation is referred to as a first type Fredholm equation. For expression of this relation, the relationship (16) is reformed as the following relationship:

$$(17) \quad f(x) + \mu \int_a^b k(x, t)u(t)dt = 0.$$

In addition, when the choice is 1, the equation is referred to as a second type Fredholm equation and therefore, we will have relationship (18):

$$(18) \quad u(x) = f(x) + \mu \int_a^b k(x, t)u(t)dt.$$

Considering the type of problems that result in integral equations, there are different ideas and methods proposed for determination of the answer of the integral equation.

V. Transforming the differential equation into integral equation

In this section, the author elaborates on the issue of transforming a differential problem into a second type Fredholm integral equation. In general, a second type Fredholm equation is assumed as follows:

$$(19) \quad y''(x) + P(x)y'(x) + Q(x)y(x) = f(x)$$

In this equation, functions P , Q and F are known. The marginal condition of this problem is also in form of the relationship (20):

$$(20) \quad \begin{aligned} x = a : y(a) &= \alpha \\ x = b : y(b) &= \beta \end{aligned}$$

Alpha and beta are constant values. Initially, the transformation is assumed in form of the relationship (21):

$$(21) \quad y''(x) = u(x)$$

Relationships (22) and (23) are resulted from integration from both sides of the relationship (21) from a , to x and applying the marginal condition at $x = a$.

$$(22) \quad y'(x) = y'(a) + \int_a^x u(t)dt$$

$$(23) \quad y(x) = y(a) + (x-a)y'(a) + \int_a^x \int_a^x u(t)dt dt = \alpha + (x-a)y'(a) + \int_a^x \int_a^x u(t)dt dt$$

In these relationships, $y'(a)$ is unknown and it should be determined. Through the application of $x = b$ marginal condition, we will have:

$$(24) \quad y(b) = \beta = \alpha + (b-a)y'(a) + \int_a^b \int_a^b u(t)dt dt$$

Therefore the unknown function of $y'(a)$ will be in form of relationship (25):

$$(25) \quad y'(a) = \frac{\beta - \alpha}{b - a} - \frac{1}{b - a} \int_a^b \int_a^b u(t)dt dt$$

By using the unknown function of $y'(a)$ in the relationship (23), we will have relationship (26):

$$(26) \quad y(x) = \alpha + (x-a) \left\{ \frac{\beta - \alpha}{b-a} - \frac{1}{b-a} \int_a^b \int_a^b u(t) dt dt \right\} + \int_a^x \int_a^x u(t) dt dt$$

Finally, by substituting the obtained unknown functions in the relationship (19), we will have:

$$(27) \quad u(x) = f(x) - p(x) \left\{ y'(a) + \int_a^x u(t) dt \right\} - Q(x) \left\{ \alpha + (x-a)y'(a) + \int_a^x \int_a^x u(t) dt dt \right\}$$

In the upper relation we have: $u(x) = y''(x)$ and $y(x)$ is to be calculated from relationship (26).

VI. Solving the buckling equation through integral equation method

In this part of the paper, the problem of column buckling is investigated in form of the relationship (28)

$$(28) \quad y'' + \frac{P}{EI} y = 0$$

In order to assume a state of no-dimensions, the length of the column is assumed as equal to 1 and the integration range is set as $[0, 1]$. For the two jointed column, the marginal conditions are assumed as follows:

$$\begin{aligned} x=0: y(0) &= 0 \\ x=1: y(1) &= 0 \end{aligned} \quad (29)$$

Therefore, according to relationship (26), we will have:

$$(30) \quad y(x) = xy'(0) + \int_0^x \int_0^x u(t) dt dt = xy'(0) + \int_0^x (x-t)u(t) dt$$

For the unknown parameter of $y'(0)$ we will have relation 31:

$$(31) \quad y'(0) = -\int_0^1 (1-t)u(t) dt = -\int_0^x (1-t)u(t) dt - \int_x^1 (1-t)u(t) dt$$

By substituting the unknown parameter, we will have:

$$\begin{aligned} (32) \quad y(x) &= x \left(-\int_0^x (1-t)u(t) dt - \int_x^1 (1-t)u(t) dt \right) + \int_0^x (x-t)u(t) dt \\ &= \int_0^x t(x-1)u(t) dt + \int_x^1 x(t-1)u(t) dt \end{aligned}$$

Relationship (32) is an integral equation and is expressible through the following:

$$\begin{aligned} (33) \quad y(x) &= \int_0^1 K(x,t)u(t) dt \\ K(x,t) &= \begin{cases} t(x-1), & 0 \leq t \leq x \\ x(t-1), & x < t \leq 1 \end{cases} \end{aligned}$$

Atkinson *et al.* carried out a research and proposed algorithms for second type Fredholm integral equations using the software of MATLAB. This research uses the algorithm proposed in the present paper for the purpose of solving the mentioned integral equation. Therefore, for a sample column with a unified length and different marginal conditions, the buckling form was calculated both through the finite elements method and the integral equations method. Results are summarized in figures 7-10. In addition, table two presents the results of calculation of critical buckling load for investigated columns.

Figure 7, comparing the finite elements method and the integral equations method for two jointed column

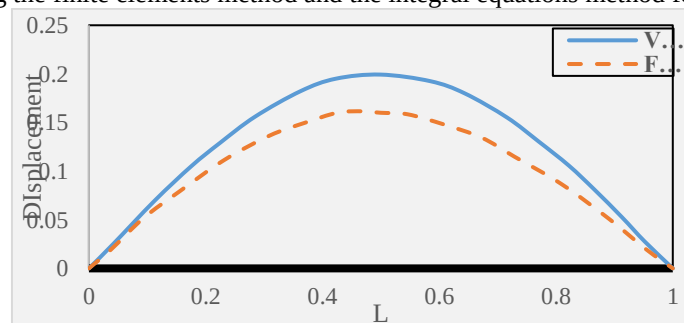


Figure 8, comparing the finite elements method and the integral equations method for two jointed column

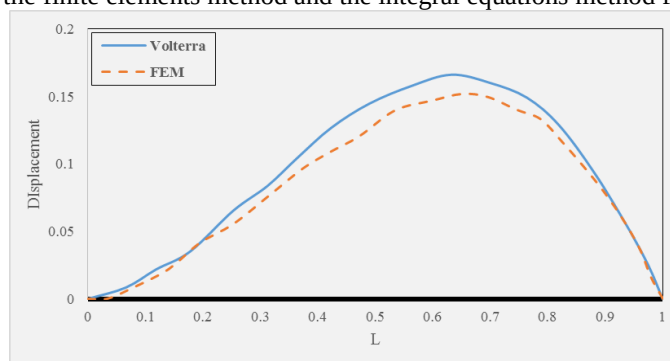


Figure 9, comparing the finite elements method and the integral equations method for two jointed column

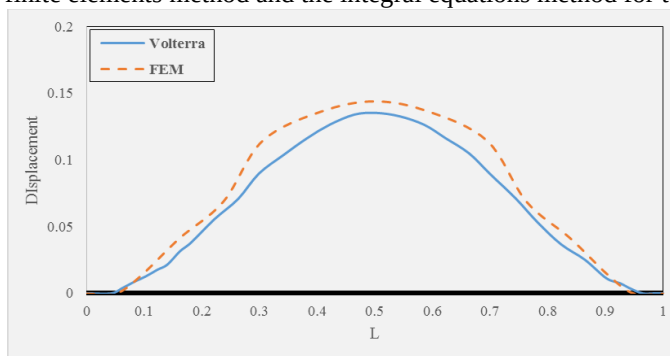


Figure 10, comparing the finite elements method and the integral equations method for two jointed column

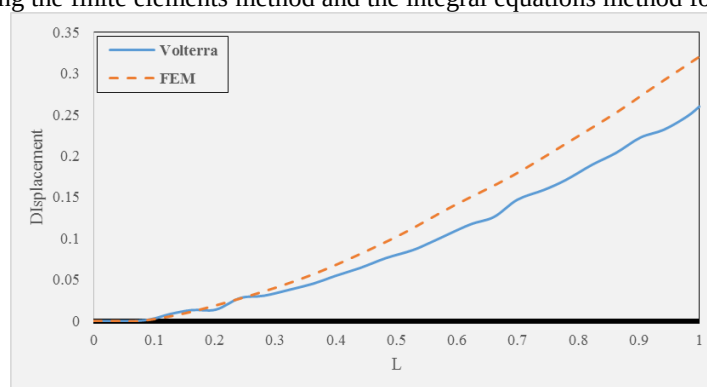


Table 2, results of integral equations method for a column under different marginal conditions

TWO JOINTED	TWO CLAPMED	ONE END JOINTED	ONE END FREE	
$1.043 \frac{\pi^2 EI}{L^2}$	$1.048 \frac{\pi^2 EI}{(L/2)^2}$	$1.039 \frac{\pi^2 EI}{(0.7L)^2}$	$1.073 \frac{\pi^2 EI}{(2L)^2}$	Buckling critical load

VII. Result and Discussion

The author of this study has tried to make use of integral equations for solvation of differential equations. For this purpose, the differential equation of column buckling was used. Four columns under different marginal conditions which were previously solved through Euler's equation and the limited elements method were selected as the cases of study. The integral equations method of Volterra can be applied to a major variety of differential problems that exist in different engineering fields. Since there are no approximations in functions, therefore the results are more trustworthy. Obtained answers for deformities are in a 5-20% difference with the limited elements method and this difference depends on the anchor condition. In addition, in Volterra method the buckling load was approximately 4% for columns with no side movement and 7% for columns with side movement. Therefore a huge difference exists for the column with side movement.

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